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SOME STRUCTURE THEOREMS ON PSEUDO-SYMMETRIC SETS

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1. Introduction

In [2], a pseudo-symmetric set is defined as a binary system (S, \circ) satisfying (1) $a \circ a = a$, (2) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ and (3) a mapping $\sigma_a: x \mapsto x \circ a$ is a permutation on S . The object of this paper is to develop a structure theory for pseudo-symmetric sets. Denote σ_a by $\sigma(a)$. Then σ is considered as a mapping of S to the group of permutations on S satisfying the fundamental identity $\sigma(a^{\sigma(b)}) = \sigma(b)^{-1} \sigma(a) \sigma(b)$, which results from (2). The mapping σ is called a pseudo-symmetric structure on S . The group of automorphisms generated by all σ_a is denoted by $G(S)$ or simply by G . The subgroup of G generated by all $\sigma_a^{-1} \sigma_b$ is denoted by $H(S)$. $H(S)$ is called the group of displacements of S according to the terminology in the theory of symmetric sets. It was found in previous works (See [1] and [2]) that there is a close connection between the structure of S and that of $H(S)$. In this paper, we shall investigate this connection more closely to find some structure theorems on S . To develop structure theory, we start with the concept of homomorphisms of our sets, which can be defined in a natural way. However, the concept of the kernel of a homomorphism is not available. To replace it, we introduce the concept of normal decompositions, which was already used in previous works (it was called coset-decompositions). Then, a more important concept is introduced. It is that of the group of displacements for a normal decomposition. As in the usual structure theory of groups, then, we proceed to consider sub- and factor-normal decompositions and the group of displacements for them. In the previous works, the structure of simple pseudo-symmetric sets was discussed. In this paper, we shall obtain a structure theorem on solvable (or nilpotent) pseudo-symmetric sets: S is solvable (or nilpotent) if and only if $H(S)$ is so. Lastly we remark that the theory developed here is more general. We do not need the conditions "effectiveness" and "transitivity". Also the condition (1) of the pseudo-symmetric set is not needed except for the main theorem of simple pseudo-symmetric sets. It should be also noted that the concept of normal decompositions can be applied for more general binary sys-

tems, for example, for groupoids.

2. Homomorphisms and normal decompositions

A homomorphism f of a pseudo-symmetric set S to a pseudo-symmetric set T is a mapping f of S to T such that $f(a \circ b) = f(a) \circ f(b)$. The pseudo-symmetric structure σ is a homomorphism of S to G , the latter being considered as a pseudo-symmetric set by definition: $x \circ y = y^{-1}xy$. ($x, y \in G$.) This is clear from the fundamental identity given in 1. Denote the image of S by σ by $I(S)$ or simply by I , i.e., $I = \{\sigma_a \mid a \in S\}$. Let K and L be groups, which are also considered as pseudo-symmetric sets as above. A group-homomorphism of K to L is naturally a pseudo-symmetric set-homomorphism. Especially, the natural homomorphism $\nu: G \rightarrow G/N$ for a normal subgroup N is a (pseudo-symmetric set-)homomorphism. More generally, let T be a pseudo-symmetric subset of G . Then, the mapping $T \xrightarrow{\nu} TN/N$ is a homomorphism. Thus, we have a homomorphism of S to IN/N through $S \xrightarrow{\sigma} I \xrightarrow{\nu} IN/N$.

Let $f: S \rightarrow T$ be a homomorphism. We define an equivalent relation on S : $a \sim b$ if and only if $f(a) = f(b)$. Then, S is decomposed into equivalent classes: $S = \cup S_i$, where $S_i = f^{-1}(t_i)$ for some $t_i \in T$. It satisfies that (i) G induces a group of permutations on $X = \{S_i\}_i$, the set of classes S_i , and (ii) σ_a and σ_b induce the same permutation on X if a and b belong to a same class S_i . Conversely, suppose that $S = \cup S_i$ with $S_i \cap S_j = \phi$ if $i \neq j$ and that the above conditions (i) and (ii) are satisfied. Then, we can define a pseudo-symmetric structure $\bar{\sigma}$ on X such that $\bar{\sigma}(S_i)$ is the permutation on X induced by $\sigma(a_i)$ where $a_i \in S_i$. In this case, the mapping $a_i \mapsto S_i$ is a homomorphism of S to X , and the decomposition $S = \cup S_i$ coincides with the decomposition of S relative to the homomorphism $S \rightarrow X$. We call a decomposition of S satisfying (i) and (ii) a *normal decomposition* of S and denote it by $D: S = \cup S_i$. The pseudo-symmetric set $X = \{S_i\}$ will be denoted by S/D . Thus, there is a homomorphism of S onto S/D and the decomposition associated with it is D . Two trivial examples: Let $\{e\}$ be a one-point trivial pseudo-symmetric set and consider the trivial homomorphism $S \rightarrow \{e\}$. The normal decomposition associated with it is $S = S$. We denote this decomposition by S , which will give no complication. Thus, $S: S = S$. The factor pseudo-symmetric set in this case is trivial; $S/S \cong \{e\}$. The second trivial example comes from the identity homomorphism $S \rightarrow S$. We denote the normal decomposition associated with it by E . Thus, $E: S = \cup \{a\}$, i.e., each class is a one-point set $\{a\}$ ($a \in S$). We have $S/E \simeq S$.

Let N be a normal subgroup of G . As was given before, there is the homomorphism $S \rightarrow IN/N$. We denote the normal decomposition associated with it by D_N . We have

$$(1) \quad S/D_N \cong IN/N.$$

Note that D_N is given by the equivalence: $a \sim b$ if and only if $\sigma_a \equiv \sigma_b \pmod N$. If $N=G$, then $D_G=S$ (the trivial decomposition). If $N=(1)$ (the identity), then the decomposition $D_{(1)}$ is given by $S = \cup S_i$ with $S_i = \{a \in S \mid \sigma_a = \sigma_{a_i}\}$ for fixed a_i . We have $S/D_{(1)} \cong I$.

Let $f: S \rightarrow T$ be a homomorphism of S onto T . Then $S/D \cong T$, where D is the normal decomposition associated with f . $G(S)$ induces a group of automorphisms on S/D in a natural way. So, we have a homomorphism $f^*: G(S) \rightarrow G(T)$. f^* is onto because $f^*(\sigma_a) = \sigma(f(a))$ (if $a \in S$) by definition and $G(T)$ is generated by $\sigma(f(a))$ (note that f is onto.).

Lemma 1. *Let N be a normal subgroup of G , and f the homomorphism $S \rightarrow S/D_N$. Then, the kernel K of $f^*: G \rightarrow G(S/D_N)$ is a normal subgroup of G containing N and $K/N \subseteq Z(G/N)$, the center of G/N .*

Proof. We have the homomorphism $f': S \rightarrow IN/N$. Naturally, K coincides with the kernel of $(f')^*$. If $\rho \in G$, then $(f')^*(\rho)$ is the conjugation by ρ on IN/N . (This is seen by taking $\rho = \sigma_a$.) It is clear that K contains N . Since IN/N generates G/N , the latter part of Lemma 1 is also clear.

3. Subdecompositions, factor decompositions and groups of displacements of normal decompositions

Let $D_1: S = \cup S_i$ and $D_2: S = \cup T_j$ be two normal decompositions of S . When the decomposition D_2 is finer than D_1 , i.e., $S_i \cap T_j = T_j$ or ϕ , we say that D_2 is a (normal) subdecomposition of D_1 and denote $D_2 \leq D_1$. So, we have $E \leq D \leq S$ for every normal decomposition D . Let M and N be normal subgroups of G . Then,

$$(2) \quad D_M \leq D_N \quad \text{if } M \subseteq N.$$

Suppose that $D_2 \leq D_1$. Then, there exists a homomorphism $S/D_2 \rightarrow S/D_1$, where an element T_j of S/D_2 is mapped to an element S_i of S/D_1 if $T_j \subseteq S_i$. The normal decomposition of S/D_2 associated with the homomorphism is denoted by D_1/D_2 and is called a factor normal decomposition. Thus, $D_1/D_2: S/D_2 = \cup \bar{S}_i$, where $\bar{S}_i = \{T_j \mid T_j \subseteq S_i\}$. We have

$$(3) \quad (S/D_2)/(D_1/D_2) \cong S/D_1.$$

Now we define the group of displacements of a normal decomposition $D: S = \cup S_i$ by $H(D) = \langle \sigma_a^{-1} \sigma_b \mid a, b \in S_i, i = 1, 2, \dots \rangle$. $H(D)$ is a normal subgroup of G contained in $H(S)$. Note that $H(S)$ for the decomposition S coincides with the group of displacements $H(S)$ of the pseudo-symmetric set S . Let $f: S \rightarrow S/D$ and $f^*: G \rightarrow G(S/D)$ for a normal decomposition D . Then, $H(D)$

is contained in the kernel of f^* . Besides the trivial example $H(S)$, we have another trivial example: $H(E)=(1)$. There is another interesting example. Let S' denote the orbit-decomposition of S ; $S': S=\cup(a^G)$. It is easy to see that it is a normal decomposition. S/S' is the set of G -orbits and is a trivial pseudo-symmetric set. In this case, we have $H(S')=G'$, the commutator subgroup of G . For, $\sigma_a^{-1}\sigma_b^{-1}\sigma_a\sigma_b=\sigma_a^{-1}\sigma(a^{\sigma(b)})\in H(S')$. Clearly we have

$$(4) \quad H(D_2)\subseteq H(D_1) \quad \text{if } D_2\leq D_1.$$

Especially, $G'=H(S')\subseteq H(S)$, which is also clear from definition. Let N be a normal subgroup and D a normal decomposition. Then,

$$(5) \quad H(D_N)\subseteq N.$$

$$(6) \quad D\leq D_{H(D)}.$$

Generally, D is not equal to $D_{H(D)}$. But, we have that $H(D_{H(D)}/D)=(1)$. For, $H(D_{H(D)}/D)$ is a subgroup of $G(S/D)$ generated by $f^*(\sigma_a^{-1})f^*(\sigma_b)$ with $\sigma_a\equiv\sigma_b \pmod{H(D)}$, where $f: S\rightarrow S/D$, and hence $\sigma_a^{-1}\sigma_b\in H(D)$ and $f^*(\sigma_a^{-1})f^*(\sigma_b)=1$.

4. Some structure theorems

Theorem 1. *Let N be a normal subgroup of G and D a normal decomposition of S . Then, $D_{H(D_N)}=D_N$ and $H(D_{H(D)})=H(D)$.*

Proof. (5) and (2) imply $D_{H(D_N)}\leq D_N$. Conversely, $D_N\leq D_{H(D_N)}$ by (6). Therefore, the first identity follows. (6) and (4) imply $H(D)\subseteq H(D_{H(D)})$. Conversely, $H(D_{H(D)})\subseteq H(D)$ by (5). Therefore, the second identity follows.

Theorem 2. *Let K be the kernel of $f^*: G\rightarrow G(S/D)$. Then, K contains $H(D)$ and $K/H(D)\subseteq Z(G/H(D))$.*

Proof. Apply Lemma 1 for $N=H(D)$, and we have $K'/H(D)\subseteq Z(G/H(D))$, where K' is the kernel of the homomorphism $G\rightarrow G(S/D_N)$. But, $K\subseteq K'$ since $D\leq D_N$ by (6).

Corollary 1. *Let N be a normal subgroup of G . Then, $N/H(D_N)\subseteq Z(G/H(D_N))$.*

When σ is one to one, we say that S is effective. This means $S\cong I$, or $D_{(1)}=E$. In this case, $Z(G)=(1)$. For, if $\rho\in Z(G)$, ρ induces the trivial automorphism on I (by conjugation). $I\cong S$ implies $\rho=1$. When $S=a^G$, we say that S is transitive. This means $S=S'$. So, in this case, $H(S)=G'$.

Corollary 2. (1) *Suppose that S is effective. If N is a normal subgroup of G such that $(1)\subset N\subset H(S)$, then $E<D_N<S$. (2) *Suppose that S is transi-**

tive. If D is a normal decomposition of S such that $D_{(1)} < D < S$, then $(1) \subset H(D) \subset H(S)$.

Proof. (1) $D_N \neq S$, since $H(S) \supset N \supseteq H(D_N)$. Suppose $D_N = E$. Then, $N = N/H(D_N)$ is contained in $Z(G) = Z(G/H(D_N))$ by Corollary 1. But $Z(G) = (1)$, and hence $N = (1)$, which is a contradiction. (2) First, we remark that $S = a^G$ implies $S = a^{H(S)}$. For, by definition of $H(S)$, $G = \langle \sigma_b, H(S) \rangle$ for any element b . Let b be any element in S , and we have $b = a^\tau$ with $\tau \in G$. Then, we can replace τ by ρ (if necessary) such that $a^\tau = a^\rho$ and $\rho \in H(S)$, because $a^\tau = a^{\tau\sigma^i}$ (i is any integer) and $\tau\sigma^i_b (= \rho)$ can be an element in $H(S)$ for a suitable i . Now we prove (2). $H(D) \neq (1)$, since $D_{H(D)} \geq D > D_{(1)}$. Suppose that $H(D) = H(S)$. Then, $S = a^{H(D)}$, which implies that D is the trivial decomposition $S = S$. This contradicts the assumption $D \neq S$.

From Corollary 2, we can derive the main theorem on a simple pseudo-symmetric set: Suppose that S is effective and transitive. Then, S has no normal decomposition D such that $E < D < S$ if and only if there is no normal subgroup N such that $(1) \subset N \subset H(S)$. We also note that in the above discussion we used the condition (1) $a \circ a = a$ (actually $b^{\sigma_b} = b$), and this is the only place we use (1).

When $D = D_N$, Theorem 2 can be strengthen:

Theorem 3. *Let K be the kernel of $f^*: G \rightarrow G(S/D_N)$ for a normal subgroup N . Then, $K/H(D_N) = Z(G/H(D_N))$.*

Proof. Let $\tau \in G$ be such that $\tau H(D_N) \in Z(G/H(D_N))$. τ induces the trivial automorphism on $IH(D_N)/H(D_N)$. Since $D_N = D_{H(D_N)}$ by Theorem 1, we have $S/D_N = S/D_{H(D_N)}$. But, the latter is isomorphic with $IH(D_N)/H(D_N)$. So, $f^*(\tau) = 1$ and hence $\tau \in K$. This with Theorem 2 implies Theorem 3.

Theorem 4. *Let M and N be normal subgroups of G such that $N \subseteq M$. Then, there exist a subgroup M_1/N of M/N and an onto-homomorphism $M_1/N \rightarrow H(D_M/D_N)$.*

Proof. Let K be the kernel of $f^*: G \rightarrow G(S/D_N)$, and J the kernel of $h^*: H(D_M) \rightarrow H(D_M/D_N)$, where h^* is the restriction of f^* on $H(D_M)$. Then, $J = K \cap H(D_M)$. Since $K \supseteq N$, we have $J \supseteq N \cap H(D_M)$. On the other hand, $H(D_M)/H(D_M) \cap N \cong H(D_M)N/N$. Let $M_1 = H(D_M)N$. Clearly, $M_1 \subseteq M$. We have the canonical homomorphism $M_1/N \rightarrow H(D_M)/J \cong H(D_M/D_N)$.

Corollary 3. *In Theorem 4, if M/N is abelian, so is $H(D_M/D_N)$.*

Theorem 5. *Let M and N be as in Theorem 4, and assume that $N \subseteq H(S)$. If $M/N \subseteq Z(H(S)/N)$, then $H(D_M/D_N) \subseteq Z(H(S/D_N))$.*

Proof. Let $\tau \in H(D_M) \subseteq M$, and $\rho \in H(S)$. We want to show $f^*(\tau)f^*(\rho)$

$=f^*(\rho)f^*(\tau)$, where $f^*: G \rightarrow G(S/D_N)$. Let $\sigma = \tau^{-1}\rho^{-1}\tau\rho$. Then, $\sigma \in N$ because $M/N \subseteq Z(H(S)/N)$. Hence, $f^*(\sigma) = 1$, proving the above identity.

Theorem 6. *If $D_2 \leq D_1$ and $H(D_2) = H(D_1)$, then $H(D_1/D_2) = (1)$. If $N \subseteq M$ and $D_N = D_M$, then $M/N \subseteq Z(G/N)$.*

Proof. Let $D = D_{H(D_1)} (= D_{H(D_2)})$. Then, $D_2 \leq D_1 \leq D$ by (6) and Theorem 1. By the remark given after (6), $H(D/D_2) = (1)$. Hence, $H(D_1/D_2) = (1)$. Let $T = H(D_M) (= H(D_N))$. Then, $T \subseteq N \subseteq M$ by (2) and Theorem 1. By Corollary 1, $M/T \subseteq Z(G/T)$. Considering the canonical homomorphism $G/T \rightarrow G/N$, we can conclude that $M/N \subseteq Z(G/N)$.

Although the correspondences $D \mapsto H(D)$ and $N \mapsto D_N$ are not one to one, Theorem 6 tells that their discrepancies are within trivial (or, abelian).

5. Solvable and nilpotent pseudo-symmetric sets

A normal decomposition D is called abelian if $H(D)$ is abelian. A pseudo-symmetric set S is called solvable if there exists a sequence of normal decompositions

$$(7) \quad S = D_0 \geq D_1 \geq \dots \geq D_{n-1} \geq D_n = E$$

such that D_i/D_{i+1} are abelian ($i=0, 1, \dots, n-1$).

Theorem 7. *S is solvable if and only if $H(S)$ is solvable.*

Proof. Suppose that S is solvable as above. Then, we have a sequence of normal subgroups of G : $H(S) = H(D_0) \supseteq H(D_1) \supseteq \dots \supseteq H(D_{n-1}) \supseteq H(E) = (1)$. Let J_i be the kernel of the homomorphism $h_i^*: H(D_i) \rightarrow H(D_i/D_{i+1})$. J_i contains $H(D_{i+1})$ and $J_i/H(D_{i+1})$ is abelian by Theorem 2. $H(D_i)/J_i$ ($\cong H(D_i/D_{i+1})$) is also abelian by assumption. Thus, $H(S)$ is solvable. Conversely, suppose that $H(S)$ is solvable. Let $H(S) = N_0 \supseteq N_1 \supseteq \dots \supseteq N_{n-1} \supseteq N_n = (1)$ be a sequence of normal subgroups of $H(S)$ such that N_i/N_{i+1} are all abelian. We may also assume that N_i are G -normal. Let $D_i = D_{N_i}$, and we have $S = D_0 \geq D_1 \geq \dots \geq D_n = D_{(1)} \geq D_{n+1} = E$. By Corollary 3, D_i/D_{i+1} are abelian ($i=0, 1, \dots, n-1$). Also, $H(D_n/D_{n+1}) = (1)$ implies D_n/D_{n+1} is abelian. Thus, S is solvable.

A normal decomposition D is called central if $H(D) \subseteq Z(H(S))$. S is called nilpotent if there exists (7) such that D_i/D_{i+1} are all central, i.e., $H(D_i/D_{i+1}) \subseteq Z(H(S/D_{i+1}))$.

Theorem 8. *S is nilpotent if and only if $H(S)$ is nilpotent.*

Proof. Suppose S is nilpotent as above. We have the sequence: $H(S)$

$\supseteq H(D_1) \supseteq \dots$ Again, let J_i be the kernel of $h_i^*: H(D_i) \rightarrow H(D_i/D_{i+1})$. Then, $J_i/H(D_{i+1}) \subseteq Z(G/H(D_{i+1}))$ by Theorem 2. We must show that $H(D_i)/J_i \subseteq Z(H(S)/J_i)$. For it, recall the assumption $H(D_i/D_{i+1}) \subseteq Z(H(S)/D_{i+1})$ and the two isomorphisms $H(D_i/D_{i+1}) \cong H(D_i)/J_i$ and $H(S/D_{i+1}) \cong H(S)/J'_i$, where $J_i = K \cap H(D_i)$ and $J'_i = K \cap H(S)$ (K is the kernel of the homomorphism $G \rightarrow G(S/D_{i+1})$). Hence, $J_i = J'_i \cap H(D_i)$. Let $\tau \in H(D_i)$ and $\rho \in H(S)$. We must show that $\tau\rho \equiv \rho\tau \pmod{J_i}$. Let $\sigma = \tau^{-1}\rho^{-1}\tau\rho$. Then, $\sigma \in J'_i$. σ is naturally contained in $H(D_i)$. Therefore, $\sigma \in J_i$, and hence $\tau\rho \equiv \rho\tau \pmod{J_i}$. We have proven that $H(D_i)/J_i \subseteq Z(H(S)/J_i)$, which finishes the proof that $H(S)$ is nilpotent. Conversely, suppose that $H(S)$ is nilpotent and let $H(S) = N_0 \supseteq N_1 \supseteq \dots \supseteq N_n = (1)$ be such that $N_i/N_{i+1} \subseteq Z(N_0/N_{i+1})$. We may assume that N_i are all G -normal. Then, let $D_i = D_{N_i}$. By Theorem 5, $H(D_i/D_{i+1}) \subseteq Z(H(S)/D_{i+1})$. Thus, S is nilpotent.

EXAMPLE 1. Let $G = S_4$, the symmetric group of degree 4. Let S be the symmetric set consisting of transpositions in S_4 : $S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. We have a normal decomposition D : $S = S_1 \cup S_2 \cup S_3$, where $S_1 = \{(1, 2), (3, 4)\}$, $S_2 = \{(1, 3), (2, 4)\}$, and $S_3 = \{(1, 4), (2, 3)\}$. Then, $H(D) = \{\text{the identity}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$, which is an abelian group. It is also easy to see that $H(S/D)$ is an abelian group of order 3. Thus, $S \geq D \geq E$ is a solvable sequence of normal decompositions. So, S is solvable. $H(S) = A_4$ is naturally solvable.

EXAMPLE 2. Let L be a nilpotent Lie algebra of finite dimension. Define $\sigma(a) = \exp(ad a)$ for $a \in L$, and L is considered as a pseudo-symmetric set. We can show that L is a nilpotent pseudo-symmetric set as follows. Let C be the center of L . The (Lie algebra) homomorphism $L \rightarrow L/C$ is seen to be a pseudo-symmetric set homomorphism. Let D be the normal decomposition associated with the homomorphism. D is the usual cosets decomposition $L = \cup S_i$, where $S_i = \{a_i + C\}$. If $z \in C$, then we see that $\sigma(a+z) = \sigma(a)$. Therefore, $H(D) = (1)$. Next, we consider L/C in place of L and apply the above argument. We obtain a normal decomposition D' such that $D \leq D'$ and $H(D'/D) = (1)$. Repeating this, we obtain a nilpotent sequence of normal decompositions $L \geq \dots \geq D' \geq D \geq E$. Thus, L is nilpotent.

References

- [1] H. Nagao: *A remark on simple symmetric sets*, Osaka J. Math. **16** (1979), 349-352.

- [2] N. Nobusawa: *A remark on conjugacy classes in simple groups*, Osaka J. Math. **18** (1981), 749–754.

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