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DECAY OF MASS FOR A SEMILINEAR PARABOLIC SYSTEM: THE CRITICAL CASE

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1. Introduction and main result

In the recent paper [1], Amour & Raoux have studied the large-time behaviour of the L^1 -norm of nonnegative and integrable solutions (u, v) to

$$(1) \quad \begin{cases} u_t - \Delta u + |\nabla v|^q = 0 \\ v_t - \Delta v + |\nabla u|^p = 0 \end{cases} \quad \text{in } (0, +\infty) \times \mathbb{R}^N,$$

with initial data

$$(2) \quad u(0) = u_0, \quad v(0) = v_0 \quad \text{in } \mathbb{R}^N,$$

where p and q are real numbers satisfying $1 \leq p \leq q$ and N is a positive integer. Assuming that u_0 and v_0 are nonnegative functions in $L^1(\mathbb{R}^N)$ with

$$\int |x| u_0(x) dx < \infty$$

and that (u, v) is a solution to (1)–(2) with $u \geq 0$ and $v \geq 0$, they show that

$$(3) \quad \lim_{t \rightarrow +\infty} \|u(t) + v(t)\|_{L^1} > 0 \quad \text{if } q > q_p,$$

while

$$(4) \quad \lim_{t \rightarrow +\infty} \|u(t) + v(t)\|_{L^1} = 0 \quad \text{if } q < q_p,$$

where

$$(5) \quad q_p = \frac{1}{N+1} + \frac{1}{p} \frac{N+2}{N+1},$$

the critical case $q = q_p$ being left opened [1]. It is the purpose of this note to fill this gap and prove that (4) also holds true if $q = q_p$. More precisely, we assume that

$$(6) \quad 1 \leq p \leq q \leq q_p,$$

and observe that (6) implies that

$$(7) \quad 1 \leq p \leq \frac{N+2}{N+1}.$$

We next assume that

$$(8) \quad \begin{cases} u_0 \text{ and } v_0 \text{ are nonnegative functions in } L^1(\mathbb{R}^N) \text{ with} \\ \int |x|^{((N+2)-p(N+1))/p} u_0(x) \, dx < \infty. \end{cases}$$

Our result then reads as follows.

Theorem 1. *Assume that p, q, u_0 and v_0 fulfil the conditions (6) and (8), and let (u, v) be a nonnegative solution to (1)–(2), that is, u and v are nonnegative functions satisfying*

$$\begin{aligned} u &\in \mathcal{C}([0, +\infty); L^1(\mathbb{R}^N)) \text{ with } \nabla u \in L^p((0, +\infty) \times \mathbb{R}^N), \\ v &\in \mathcal{C}([0, +\infty); L^1(\mathbb{R}^N)) \text{ with } \nabla v \in L^q((0, +\infty) \times \mathbb{R}^N), \end{aligned}$$

and u and v are mild solutions to the first and the second equation of (1), respectively, with $(u, v)(0) = (u_0, v_0)$. Then

$$\lim_{t \rightarrow +\infty} t^{((N+2)-p(N+1))/(2p)} \|u(t)\|_{L^1} = \lim_{t \rightarrow +\infty} \|v(t)\|_{L^1} = 0,$$

and thus

$$\lim_{t \rightarrow +\infty} \|u(t) + v(t)\|_{L^1} = 0.$$

Let us stress here that only the case $q = q_p$ is new in Theorem 1. However our proof works for the whole range of parameters (p, q) given by (6) and differs from the one used in [1] to handle the case $q < q_p$. We will thus give it in the general case described by (6). As in [1], the first step towards the proof of Theorem 1 is the following properties enjoyed by (u, v) which follow at once from (1) and the nonnegativity of u and v :

$$(9) \quad t \mapsto \|u(t)\|_{L^1} \text{ and } t \mapsto \|v(t)\|_{L^1} \text{ are nonincreasing functions on } [0, +\infty),$$

$$(10) \quad \int_0^\infty \int (|\nabla u(t, x)|^p + |\nabla v(t, x)|^q) \, dx dt < \infty,$$

and

$$(11) \quad u(t, x) \leq (e^{t\Delta} u_0)(x) \text{ and } v(t, x) \leq (e^{t\Delta} v_0)(x)$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^N$, where $(e^{t\Delta})_{t \geq 0}$ denotes the linear heat semigroup in \mathbb{R}^N . The second step, which is the main contribution of this work, is to deduce that

$$(12) \quad \lim_{t \rightarrow +\infty} t^{((N+2)-p(N+1))/(2p)} \|u(t)\|_{L^1} = 0$$

by a careful use of (1) and (10). Notice that, for $p < (N + 2)/(N + 1)$, (12) improves [1, Lemma 2] where the weaker bound

$$\sup_{t \geq 0} t^{((N+2)-p(N+1))/(2p)-\varepsilon} \|u(t)\|_{L^1} < \infty$$

is proved for each $\varepsilon > 0$. The estimate (12) is actually the cornerstone of the proof of Theorem 1. Combining (10) and (12) then leads us to the expected result.

REMARK 2. Since $0 \leq (N + 2) - p(N + 1) \leq p$ by (7) (with equality only if $p = 1$) the additional integrability property in (8) on u_0 is weaker than the one required in [1].

Let us finally mention that we do not consider here the question of the existence of nonnegative solutions to (1)–(2) and refer to [2] for results in that direction. Moreover, the techniques developed in [5, 6] could possibly give further results.

2. Proof of Theorem 1

From now on, we fix p, q, u_0 and v_0 fulfilling the conditions (6) and (8) and let (u, v) be a nonnegative solution to (1)–(2). We recall that (1)–(2) and the nonnegativity of u and v yield that

$$(1) \quad \int_0^\infty \int (|\nabla u(t, x)|^p + |\nabla v(t, x)|^q) \, dx dt \leq \|u_0\|_{L^1} + \|v_0\|_{L^1}$$

after integration of (1) over $(0, +\infty) \times \mathbb{R}^N$. We then put

$$(2) \quad \omega(t) = \left(\int_{t/2}^\infty \int |\nabla u(s, x)|^p \, dx ds \right)^{1/p}$$

for $t \geq 0$ and notice that $\omega \in \mathcal{C}([0, +\infty))$ is a nonincreasing function which satisfies

$$(3) \quad \lim_{t \rightarrow +\infty} \omega(t) = 0,$$

thanks to (1). Observe that we may assume that $\omega(t) > 0$ for each $t \geq 0$. Indeed, if $\omega(t_0) = 0$ for some $t_0 \geq 0$, we realize that $\nabla u(t) \equiv 0$ for $t \geq t_0$, whence $u(t) \equiv 0$ for $t \geq t_0$ by the integrability of $u(t)$. By (1), this also implies that $\nabla v(t) \equiv 0$ for $t \geq t_0$ and thus $v(t) \equiv 0$ for $t \geq t_0$. Theorem 1 is then obvious in that case.

We now state some preliminary estimates which will be used throughout the paper. We first recall a Morrey-type inequality established in [4, Eq. (2.1)].

Lemma 3 ([4]). *If $w \in W^{1,1}(\mathbb{R}^N)$ and $R > 0$, there holds*

$$(4) \quad \|w\|_{L^1} \leq 2R \int_{\{|x| \leq 3R\}} |\nabla w(x)| \, dx + 2 \int_{\{|x| > R\}} |w(x)| \, dx.$$

Next, since both u and v are subsolutions to the linear heat equation, a control of $u(t)$ and $v(t)$ for large values of x and t is available and is a consequence of [4, Lemma 2.1].

Lemma 4 ([4]). *If $r \in \mathcal{C}([0, +\infty))$ is a nonnegative function such that*

$$(5) \quad \lim_{t \rightarrow +\infty} r(t) t^{-1/2} = +\infty,$$

then

$$(6) \quad \lim_{t \rightarrow +\infty} \int_{\{|x| \geq r(t)\}} (u(t, x) + v(t, x)) \, dx = 0.$$

We finally adapt a technique from the proof of [3, Proposition 14] to obtain another estimate on $u(t)$ for large values of x . We fix a function $\varrho \in \mathcal{C}^\infty(\mathbb{R}^N)$ satisfying $0 \leq \varrho \leq 1$ with

$$\varrho(x) = 0 \text{ if } |x| \leq 1 \text{ and } \varrho(x) = 1 \text{ if } |x| \geq 2.$$

For $R > 0$ and $x \in \mathbb{R}^N$ we put $\varrho_R(x) = \varrho(x/R)$. In the following, we denote by C any positive constant depending only on N, p, q, u_0, v_0 and ϱ .

Lemma 5. *For $t \geq 0$ and $R > 0$ we have*

$$(7) \quad \int_{\{|x| \geq 2R\}} u(t, x) \, dx \leq \int_{\{|x| \geq R\}} u_0(x) \, dx + C R^{(p(N-1)-N)/p} t^{(p-1)/p}.$$

Proof. We multiply the first equation of (1) by ϱ_R and integrate over $(0, t) \times \mathbb{R}^N$ to obtain

$$\int \varrho_R(x) u(t, x) \, dx \leq \int \varrho_R(x) u_0(x) \, dx + \frac{1}{R} \int_0^t \int \nabla \varrho \left(\frac{x}{R} \right) \cdot \nabla u(s, x) \, dx ds.$$

Owing to the properties of ϱ , we infer from the Hölder inequality that

$$\int_{\{|x| \geq 2R\}} u(t, x) \, dx \leq \int_{\{|x| \geq R\}} u_0(x) \, dx + C R^{N(1-1/p)-1} \int_0^t \|\nabla u(s)\|_{L^p} \, ds$$

$$\begin{aligned} &\leq \int_{\{|x| \geq R\}} u_0(x) \, dx \\ &\quad + C R^{(p(N-1)-N)/p} t^{(p-1)/p} \left(\int_0^t \|\nabla u(s)\|_{L^p}^p \, ds \right)^{1/p}, \end{aligned}$$

from which (7) follows, thanks to (1). □

REMARK 6. Observe that, if we take $R = r(t)/2$ in (7) with r as in Lemma 4, (7) yields a stronger decay estimate than (6) if $p < (N + 2)/(N + 1)$.

We next prove the temporal decay estimate for $\|u(t)\|_{L^1}$ claimed in the Introduction.

Proposition 7. *There exists $\sigma \in \mathcal{C}([0, +\infty))$, positive and nonincreasing, such that*

$$(8) \quad t^\alpha \|u(t)\|_{L^1} \leq \sigma(t) \text{ for } t > 0 \text{ with } \lim_{t \rightarrow +\infty} \sigma(t) = 0,$$

where $\alpha := ((N + 2) - p(N + 1))/(2p) \geq 0$.

Proof. Consider $t > 0$, $R > 0$ and $s \in [t/2, t]$. On the one hand, we infer from (8) that

$$\int_{\{|x| \geq R\}} u_0(x) \, dx \leq R^{-2\alpha} \int_{\{|x| \geq R\}} |x|^{2\alpha} u_0(x) \, dx.$$

Inserting this estimate in (7) yields

$$\int_{\{|x| \geq 2R\}} u(s, x) \, dx \leq C R^{-2\alpha} \left(I_R + (s R^{-2})^{(p-1)/p} \right),$$

where

$$I_R := \int_{\{|x| \geq R\}} |x|^{2\alpha} u_0(x) \, dx.$$

After integrating this inequality with respect to s over $(t/2, t)$, we obtain

$$(9) \quad \int_{t/2}^t \int_{\{|x| \geq 2R\}} u(s, x) \, dx ds \leq C R^{-2\alpha} t \left(I_R + (t R^{-2})^{(p-1)/p} \right).$$

On the other hand, it follows from the Hölder inequality and (2) that

$$\int_{t/2}^t \int_{\{|x| \leq 6R\}} |\nabla u(s, x)| \, dx ds$$

$$\begin{aligned}
 &\leq C (t R^N)^{(p-1)/p} \left(\int_{t/2}^t \|\nabla u(s)\|_{L^p}^p ds \right)^{1/p} \\
 (10) \quad &\leq C (t R^N)^{(p-1)/p} \omega(t).
 \end{aligned}$$

Combining (9), (10) and Lemma 3, we end up with

$$\int_{t/2}^t \|u(s)\|_{L^1} ds \leq C R^{-2\alpha} t \left(I_R + (t R^{-2})^{(p-1)/p} + \omega(t) (t R^{-2})^{-1/p} \right).$$

We take $R = R(t) := t^{1/2} \omega(t)^{-1/2}$ in the previous inequality to conclude that

$$\frac{2}{t} \int_{t/2}^t \|u(s)\|_{L^1} ds \leq C \left(\frac{\omega(t)}{t} \right)^\alpha \left(I_{R(t)} + \omega(t)^{(p-1)/p} \right).$$

Owing to (9), the left-hand side of the above inequality is bounded from below by $\|u(t)\|_{L^1}$ and we finally obtain that

$$\|u(t)\|_{L^1} \leq t^{-\alpha} \sigma(t)$$

where

$$\sigma(t) := C \omega(t)^\alpha \left(I_{R(t)} + \omega(t)^{(p-1)/p} \right).$$

Now the monotonicity of ω and (3) warrant that $R(t)$ increases to $+\infty$ as $t \rightarrow +\infty$ which implies that $t \mapsto I_{R(t)}$ is a nonincreasing function which converges to zero as $t \rightarrow +\infty$ by (8). Using once more the monotonicity of ω , (3) and (7) as well, it is straightforward to conclude that σ is a nonincreasing function which converges to zero as $t \rightarrow +\infty$, whence (8). \square

We are now in a position to complete the proof of Theorem 1.

Proof of Theorem 1. Consider $t > 0$, $R > 0$ and $s \in (t/2, t)$. By Lemma 3 and the Hölder inequality, we have

$$\begin{aligned}
 \|v(s)\|_{L^1} &\leq 2R \int_{\{|x| \leq 3R\}} |\nabla v(s, x)| dx + 2 \int_{\{|x| > R\}} |v(s, x)| dx \\
 \|v(s)\|_{L^1} &\leq C R^{(q(N+1)-N)/q} \|\nabla v(s)\|_{L^q} + 2 \int_{\{|x| > R\}} |v(s, x)| dx.
 \end{aligned}$$

We integrate the previous inequality with respect to s over $(t/2, t)$ and use again the Hölder inequality to obtain

$$\int_{t/2}^t \|v(s)\|_{L^1} ds \leq C R^{(q(N+1)-N)/q} t^{(q-1)/q} \left(\int_{t/2}^t \|\nabla v(s)\|_{L^q}^q ds \right)^{1/q}$$

$$+ 2 \int_{t/2}^t \int_{\{|x|>R\}} |v(s, x)| \, dx ds .$$

On the one hand, the monotonicity (9) of $s \mapsto \|v(s)\|_{L^1}$ entails that

$$\|v(t)\|_{L^1} \leq \frac{2}{t} \int_{t/2}^t \|v(s)\|_{L^1} \, ds .$$

On the other hand, integrating the first equation of (1) over $(t/2, t) \times \mathbb{R}^N$ yields

$$\int_{t/2}^t \|\nabla v(s)\|_{L^q}^q \, ds \leq \left\| u\left(\frac{t}{2}\right) \right\|_{L^1} .$$

Combining the previous three inequalities and (8), we end up with

$$\begin{aligned} \|v(t)\|_{L^1} &\leq C R^{(q(N+1)-N)/q} t^{-(1+\alpha)/q} \sigma(t)^{1/q} \\ (11) \quad &+ \frac{C}{t} \int_{t/2}^t \int_{\{|x|>R\}} |v(s, x)| \, dx ds , \end{aligned}$$

where α and σ are defined in Proposition 7.

Finally, since $q \geq 1$, let δ be a positive real number such that

$$0 < \delta < \frac{1}{q(N+1) - N}$$

and take $R = R(t) := t^{1/2} \sigma(t)^{-\delta}$ in (11). Owing to the monotonicity of σ , $s \mapsto R(s)$ is an nondecreasing function and we deduce from (11) that

$$\begin{aligned} \|v(t)\|_{L^1} &\leq C t^{(N+1)(q-q_p)/(2q)} \sigma(t)^{(1-\delta(q(N+1)-N))/q} \\ (12) \quad &+ \frac{C}{t} \int_{t/2}^t \int_{\{|x|>R(s)\}} |v(s, x)| \, dx ds . \end{aligned}$$

Now, by (6) and Proposition 7, we have

$$\lim_{t \rightarrow +\infty} t^{(N+1)(q-q_p)/(2q)} \sigma(t)^{(1-\delta(q(N+1)-N))/q} = 0 .$$

Consequently, since $R(s) s^{-1/2} \rightarrow +\infty$ as $s \rightarrow +\infty$, we have

$$\lim_{t \rightarrow +\infty} \frac{C}{t} \int_{t/2}^t \int_{\{|x|>R(s)\}} |v(s, x)| \, dx ds = 0$$

by Lemma 4. We may then let $t \rightarrow +\infty$ in (12) and conclude that

$$\lim_{t \rightarrow +\infty} \|v(t)\|_{L^1} = 0 .$$

Theorem 1 follows at once from this last assertion and Proposition 7. \square

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