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## ON RIBBON 2-KNOTS III ON THE UNKNOTTING RIBBON 2-KNOTS IN $S^4$

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### 1. Introduction

Concerning the unknotting theorem for the pair  $(S^n, M^{n+2})$  with the codimension 2, there are several remarkable results; by T. Homma in the case  $n=1$  and  $M^3=S^3$ , by C.D. Papakyriakopoulos in the case  $n=1$  and any 3-manifold  $M^3$ , by J. Stallings in the case  $n \geq 3$  and  $M^{n+2}=S^{n+2}$  in the topological sense and by J. Levine in the case  $n \geq 4$  and  $M^{n+2}=S^{n+2}$  in the combinatorial sense, see [1], [2], [3], [4] and [5]. Confining ourselves to the case  $M^{n+2}=S^{n+2}$ , the unknotting theorem has not been solved in the case  $n=2$ .

In this paper, we will prove the following theorem which is an answer under an additional condition to the unknotting theorem in the case  $n=2$  and  $M^4=S^4$ :

**Theorem (2, 2).** *For a ribbon 2-knot  $K^2$  in  $S^4$ ,  $K^2$  is unknotted in  $S^4$  if and only if  $\pi_1(S^4 - K^2) = Z^{13}$ .*

In this paper, everything will be considered from the combinatorial point of view.

### 2. Proof of Theorem

**Lemma (2, 1)<sup>2)</sup>.** *Let  $M^4$  be a combinatorial 4-manifold and let  $\gamma$  be a simple closed curve in  $\overset{\circ}{M}^4$  which is contractible to a point in  $\overset{\circ}{M}^4$ . Then,  $\gamma$  bounds a non-singular, locally flat 2-ball in  $\overset{\circ}{M}^4$ <sup>3)</sup>.*

**Proof.** Since  $\gamma$  is contractible to a point in  $\overset{\circ}{M}^4$ , there is a PL-map  $\varphi$  of a 2-ball  $D^2$  into  $\overset{\circ}{M}^4$  satisfying the following (1), (2) and (3):

- (1)  $\varphi(D^2) \subset \overset{\circ}{M}^4$ ,  $\varphi(\partial D^2) = \gamma$ ,
- (2)  $\varphi(D^2)$  is in a general position in  $\overset{\circ}{M}^4$  so that the self-intersection consists of a finite number of double points,

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1) See [6] for the definition of the *ribbon 2-knots* in  $R^4$ .  
2) Cf. the result in [7], the proof of Lemma (2, 7).  
3)  $\overset{\circ}{X}$  and  $\partial X$  mean the interior and the boundary of  $X$  respectively.

- (3) there are at most a finite number of *locally knotted points* on  $\varphi(D^2)$  which are different from the double points in (2).

Here, in (3), a point  $x$  of  $\varphi(D^2)$  is called a *locally knotted point*<sup>4)</sup> if the pair  $(Lk(x, \varphi(D^2)), Lk(x, M^4))$  is a *knotted sphere-pair* for the combinatorial triangulation of  $M^4$  for which  $\varphi(D^2)$  is a subcomplex and the point  $x$  is a vertex. If there is a locally knotted point  $x$  of  $\varphi(D^2)$ , it is possible to exchange a non-singular 2-ball  $St(x, \varphi(D^2))$ , which may be not locally flat, for an immersed 2-ball  $\rho(B^2)$  in a 4-ball  $B^4 = St(x, M^4)$  by an immersion  $\rho$  of a 2-ball  $B^2$  such that  $\rho(\partial B^2) = \varphi(D^2) \cap \partial B^4$ ,  $\rho(\mathring{B}^2) \subset \mathring{B}^4$  and that each pair  $(\rho Lk(y, B^2), Lk(\rho(y), B^4))$  is unknotted for a fine subdivision of  $B^4$  and each vertex  $y$  of  $B^2$ . Perform the exchange for all locally knotted points of  $\varphi(D^2)$ . By making use of the general position theory, we have a PL-map  $\varphi'$  of  $D^2$  into  $\mathring{M}^4$  satisfying the following (1'), (2') and (3'):

- (1')  $\varphi'(D^2) \subset M^4, \varphi'(\partial D^2) = \gamma,$
- (2')  $\varphi'(D^2)$  is in a general position in  $M^4$  so that the self-intersection consists of a finite number of double points,
- (3')  $\varphi'(D^2)$  has no locally knotted point.

Let  $x$  be a double point of  $\varphi'(D^2)$  and  $x = \varphi'(y) = \varphi'(y')$  for just two points  $y, y'$  of  $D^2$ . Then there is an arc  $\alpha$  spanning  $y$  and a point  $y''$  on  $\partial D^2$  such that the image  $\varphi'(\alpha)$  does not cross any double point of  $\varphi'(D^2)$  except for  $x$ . Let  $V^4$  be a regular neighborhood of  $\varphi'(\alpha)$  in  $M^4$ . Since  $V^4$  is a 4-ball, there is a PL-homeomorphism  $\phi$  of  $V^4$  onto a standard 4-cube  $\Delta^4$  such that

- (i)  $\Delta^4; |x_1|, |x_2|, |x_3|, |x_4| \leq 2$
- (ii)  $\phi\varphi'(\alpha); 0 \leq x_1 \leq 1, x_2 = x_3 = x_4 = 0$
- (iii)  $\phi\varphi'(x); x_1 = x_2 = x_3 = x_4 = 0$
- (iv)  $\phi(V^4 \cap \varphi'(D^2)); \begin{cases} -2 \leq x_1 \leq 1, x_2 = x_3 = 0, |x_4| \leq 2 \\ x_1 = 0, |x_2|, |x_3| \leq 2, x_4 = 0 \dots (*) \end{cases}$

Let  $N^2$  be the 2-ball in  $\phi\varphi'(D^2)$  defined by the equation (\*) and let  $N'^2$  be the 2-ball in  $\Delta^4$  defined as follows:

$$(v) \quad N'^2; \begin{cases} 0 \leq x_1 < 2, |x_2| \leq 2, |x_3| = 2, x_4 = 0, \\ 0 \leq x_1 < 2, |x_2| = 2, |x_3| \leq 2, x_4 = 0, \\ x_1 = 2, |x_2|, |x_3| \leq 2, x_4 = 0. \end{cases}$$

If we consider a singular 2-ball  $B'^2 = \phi^{-1}((\phi\varphi'(D^2) - N^2) \cup N'^2)$ , then this 2-ball  $B'^2$  is not only locally flat but also has a number of the double points less

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4) See [8] p. 34.

than the number of those of  $\varphi'(D^2)$ . Moreover we have that  $\partial B'^2 = \gamma$ . Repeating this process, we have finally a *locally flat, non-singular 2-ball*  $B^2$  such that  $B^2 \subset \dot{M}^4$  and that  $\partial B^2 = \gamma$ . The proof is thus complete.

Let  $K^2$  be a ribbon 2-knot in  $R^4$ , then there is a 3-manifold  $W^3$  satisfying the following properties:

- (1)  $W^3 \approx B^3$  or  $W^3 \approx \#(S^1 \times S^2) - \dot{B}^3$ <sup>5)</sup>,
- (2) If  $W^3 \approx B^3$ ,  $W^3$  has a trivial system of 2-spheres  $\{S_1^2, \dots, S_{2n}^2\}$  satisfying that
  - (i) a 2-link  $\{S_1^2, \dots, S_{2n}^2\}$  is trivial in  $R^4$ ,
  - (ii)  $S_i^2 \cup S_{n+i}^2$  bounds a spherical-shell  $N_i^3$  in  $W^3$  ( $i=1, \dots, n$ )<sup>6)</sup>,
  - (iii)  $W^3 - \dot{N}_1^3 \cup \dots \cup \dot{N}_n^3 \approx B^3 - \dot{\Delta}_1^3 \cup \dots \cup \dot{\Delta}_{2n}^3$ ,

see (3, 5) and (3, 6) in [6].

Let  $\Delta_0^3$  be a 3-ball in  $W^3 - N_1^3 \cup \dots \cup N_n^3$ , let  $S_0^2$  be a boundary 2-sphere of  $\Delta_0^3$  and let  $\beta_1, \dots, \beta_{2n}$  be a collection of mutually disjoint arcs spanning  $S_0^2$  and  $S_1^2, \dots, S_{2n}^2$  in  $W^3 - N_1^3 \cup \dots \cup N_n^3 \cup \dot{\Delta}_0^3$  respectively. Moreover, let  $U_\lambda^3$  be a regular neighborhood of the arc  $\beta_\lambda$  in  $W^3 - \dot{N}_1^3 \cup \dots \cup \dot{N}_n^3 \cup \dot{\Delta}_0^3$  where  $U_\lambda^3 \cap S_\lambda^2 = e_\lambda^2$  and  $U_\lambda^3 \cap S_0^2 = e_\lambda'^2$  are 2-balls such that  $e_\lambda'^2 \cap e_\mu'^2 = \emptyset$  ( $\lambda \neq \mu, \lambda, \mu = 1, \dots, 2n$ ). Since the 2-link  $\{S_0^2, \dots, S_{2n}^2\}$  is trivial in  $R^4$ , there is an isotopy  $\xi$  of  $R^4$  by which  $\xi(S_\lambda^2)$  ( $\lambda=0, \dots, 2n$ ) are moved into the position given by the equations below:

$$\begin{aligned} \xi(S_0^2); & x_1^2 + x_2^2 + x_3^2 = 1, \quad x_4 = 0 \\ \xi(S_i^2); & (x_1 - 4i)^2 + x_2^2 + x_3^2 = 1, \quad x_4 = 0 \\ \xi(S_{n+i}^2); & (x_1 - 4i)^2 + x_2^2 + x_3^2 = 2, \quad x_4 = 0 \\ \xi(N_i^3); & 1 \leq (x_1 - 4i)^2 + x_2^2 + x_3^2 \leq 2, \quad x_4 = 0 \quad (i = 1, \dots, n). \end{aligned}$$

Moreover, we may suppose that the center line  $\xi(\beta_\lambda)$  of the tube  $\xi(U_\lambda^3)$  is given by the equations below:

$$\begin{aligned} x_1 = 4i, \quad x_2 = 0, \quad x_3 = 1 - x_4, \quad x_4 \geq 0 \\ \text{in the neighborhood of } \xi(\beta_i \cap S_i^2), \\ x_1 = 4i, \quad x_2 = 0, \quad x_3 = \sqrt{2} - x_4, \quad x_4 \leq 0 \\ \text{in the neighborhood of } \xi(\beta_{n+i} \cap S_{n+i}^2) \\ (i = 1, 2, \dots, n). \end{aligned}$$

**Theorem (2. 2)'. For a ribbon 2-knot  $K^2$  in  $R^4$ ,  $K^2$  is unknotted in  $R^4$ , if and only if  $\pi_1(R^4 - K^2) = Z$ .**

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5)  $B^3$  means a 3-ball and  $\approx$  means to be homeomorphic to.  
 6)  $N_i^3 \approx S^2 \times [0, 1]$ .

The proof of this theorem is divided into two steps, and the second-step of the proof will be given later after we have proved two lemmas (2, 3) and (2, 4). Since  $K^2$  is a ribbon 2-knot in  $R^4$ , it bounds a 3-manifold  $W^3$  previously described, therefore if  $W^3 \approx B^3$ , we have nothing particular to say. Hence, in the following discussion we will consider the case that  $W^3 \not\approx B^3$ . Consider the trivial system  $\{S^2_1, \dots, S^2_{2n}\}$  and the isotopy  $\xi$  of  $R^4$  as before. Let  $\tilde{K}^2$  be a 2-knot in  $R^4$  such that

$$\tilde{K}^2 = \xi\left(\bigcup_{\lambda=0}^{2n} S^2_\lambda - \bigcup_{\lambda=1}^{2n} (e^2_\lambda \cup e'^2_\lambda)\right) \cup \xi\left(\bigcup_{\lambda=1}^{2n} \partial U^3_\lambda - \bigcup_{\lambda=1}^{2n} (e^2_\lambda \cup e'^2_\lambda)\right).$$

Then, since two 2-spheres  $\xi(K^2)$  and  $\tilde{K}^2$  bound a 3-manifold which is a sub-complex of  $\xi(W^3)$  and which is homeomorphic to  $S^2 \times [0, 1]$  in  $R^4$ ,  $\tilde{K}^2$  belongs to the 2-knot-type  $\{\xi(K^2)\}$  which coincides with the 2-knot-type  $\{K^2\}$ .

Let  $B^3_i$  be a 3-ball bounded by the 2-sphere  $\xi(S^2_i)$ :

$$B^3_i; (x_1 - 4i)^2 + x_2^2 + x_3^2 \leq 1, \quad x_4 = 0 \quad (i = 0, 1, 2, \dots, n).$$

**First-step of the proof of (2, 2)'**: Each 3-ball  $B^3_i$  bounded by the 2-sphere  $\xi(S^2_i)$  ( $i=1, \dots, n$ ) in  $R^3_0$  does not meet any arc  $\xi(\beta_\lambda)$  except for the end points ( $\lambda=1, \dots, 2n$ ).

Since we can find a regular neighborhood  $U^3_\lambda$  of  $\beta_\lambda$  so fine that  $\xi(U^3_\lambda) \cap B^3_i = \emptyset$  because  $\xi(\beta_\lambda) \cap B^3_i = \emptyset$  for all  $i$  and  $\lambda$  ( $i=1, \dots, n, \lambda=1, \dots, 2n$ ), the 2-knot  $\tilde{K}^2$  bounds a 3-ball  $B^3_0 \cup B^3_1 \cup \dots \cup B^3_n \cup B^3_{n+1} \cup \dots \cup B^3_{2n} \cup \xi(U^3_1) \cup \dots \cup \xi(U^3_{2n})$ , where the 3-ball  $B^3_{n+i}$  is bounded by  $\xi(S^2_{n+i})$  in the neighborhood of  $\xi(N^3_i) \cup B^3_i$  in  $R^4$  so that  $B^3_{n+i} \cap B^3_j = \emptyset$ ,  $B^3_{n+i} \cap \xi(U^3_\lambda) = \emptyset$  ( $\lambda \neq n+i$ ) and  $= \xi(e^2_\lambda)$  ( $\lambda = n+i$ ) ( $i, j=1, \dots, n, \lambda=1, \dots, 2n$ ): for a sufficiently small  $\varepsilon (< 0)$ ,

$$B^3_{n+i}; \begin{cases} (x_1 - 4i)^2 + x_2^2 + x_3^2 = 2, & -\varepsilon \leq x_4 \leq 0 \\ (x_1 - 4i)^2 + x_2^2 + x_3^2 \leq 2, & x_4 = -\varepsilon \end{cases} \\ (i = 1, 2, \dots, n).$$

If there is a 3-ball  $B^3_i$  which meets some arcs  $\xi(\beta_\lambda)$  ( $1 \leq i \leq n, 1 \leq \lambda \leq 2n$ ), we will consider how to remove the intersection of the 3-ball  $B^3_i$  and the arcs  $\xi(\beta_\lambda)$  without changing the 2-knot-type of  $\tilde{K}^2$ . We need following two lemmas (2, 3) and (2, 4) to remove the intersection.

**Lemma (2, 3).** *If there are an arc  $b$  in  $R^4 - \tilde{K}^2$  and a subarc  $\beta'_\lambda$  of the arc  $\beta_\lambda$  ( $1 \leq \lambda \leq 2n$ ) such that the simple closed curve  $\gamma = b \cup \xi(\beta'_\lambda)$  is contractible in  $R^4 - \tilde{K}^2$ , then there exists an isotopy  $\eta$  of  $R^4$  by which  $\eta(\tilde{K}^2) = \tilde{K}^2$  and  $\eta\xi(\beta_\lambda) = \xi(\beta_\lambda - \beta'_\lambda) \cup b$ .*

**Proof.** Since  $\beta_\lambda$  is contained in  $\tilde{U}^3_\lambda$  except two end points, we can triangulate  $R^4$  so that the regular neighborhood  $N(\tilde{K}^2)$  of  $\tilde{K}^2$  in  $R^4$  does not meet  $\xi(\beta_\lambda)$ . If we apply (2, 1) to the 4-manifold  $M^4 = R^4 - \overset{\circ}{N}(\tilde{K}^2)$  and the simple closed

curve  $\gamma = b \cup \xi(\beta'_\lambda)$ , the simple closed curve  $\gamma$  bounds a locally flat 2-ball  $B^2$  in  $\dot{M}^4$ . Therefore there exists a combinatorial 4-ball  $B^4$  containing  $B^2$  in its interior and contained in  $R^4 - \tilde{K}^2$ . Now, we have easily an isotopy which is identical on  $\partial B^4$  and transfers the subarc  $\xi(\beta'_\lambda)$  onto the arc  $b$ . Hence, the proof is complete.

**Lemma (2, 4).** *If  $\pi_1(R^4 - \tilde{K}^2) = Z$  and an arc  $\xi(\beta_\lambda)$  ( $1 \leq \lambda \leq 2n$ ) pierces through a 3-ball  $B_i^3$  ( $1 \leq i \leq n$ ) at a point  $A$ , there are an arc  $b$  in  $R^4 - \tilde{K}^2$  and a subarc  $\beta'_\lambda$  on the arc  $\beta_\lambda$  containing  $A$  such that the simple closed curve  $b \cup \xi(\beta'_\lambda)$  is contractible in  $R^4 - \tilde{K}^2$ .*

Proof. For convenience's sake, we may suppose that  $\xi(\beta_\lambda)$  is given in the neighborhood of the point  $A$  as follows:

$$\xi(\beta_\lambda); x_1 = 4i, \quad x_2 = x_3 = 0, \quad -1 \leq x_4 \leq 1.$$

Consider the cross-sections of  $\xi(S_i^2)$ ,  $\xi(S_{n+i}^2)$  and  $\xi(N_i^3)$  by the hyperplane  $P$ ;  $x_3 = 0$ . Then, we have the following figure Fig. (1).

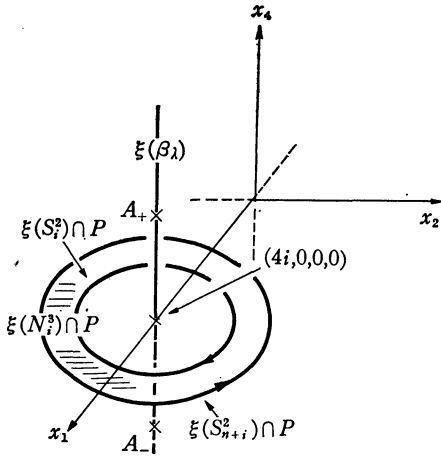


Fig. 1

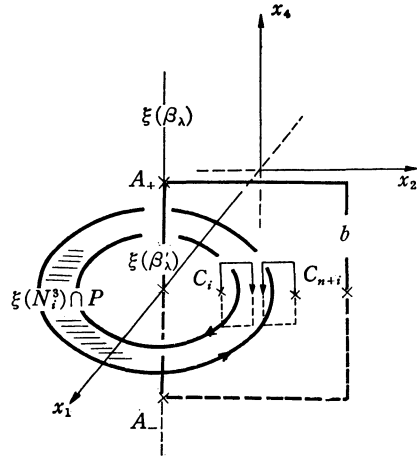


Fig. 2

Place  $\xi(U_\lambda^3)$  in a general position with respect to the hyperplane  $P$ , then the cross-section  $\xi(U_\lambda^3) \cap P$  is at most 2-dimensional, and we can find an arc  $b$  spanning two points  $A_+$  and  $A_-$  in  $P - P \cap \tilde{K}^2$  as follows:

$$b; \begin{cases} x_1 = 4i, & 0 \leq x_2 \leq 2, & x_3 = 0, & x_4 = \varepsilon \\ x_1 = 4i, & x_2 = 2, & x_3 = 0, & -\varepsilon \leq x_4 \leq \varepsilon \\ x_1 = 4i, & 0 \leq x_2 \leq 2, & x_3 = 0, & x_4 = -\varepsilon, \end{cases}$$

see Fig. (2).

Since the 2-knot  $\tilde{K}^2$  bounds the orientable 3-manifold  $\tilde{W}^3 = \xi(N_1^3 \cup \dots \cup N_n^3 \cup U_1^3 \cup \dots \cup U_{2n}^3 \cup \Delta_0^3)$  in  $R^4$ , we will give an orientation induced from the orientation of  $\tilde{W}^3$  for  $\tilde{K}^2$ . Then, the trivial link  $\xi(S_i^2 \cup S_{n+i}^2) \cap P$ , which bounds an annulus  $\xi(N_i^3) \cap P$  in  $P$ , can be given the orientation induced from that of  $\xi(N_i^3) \cap P$ , see Fig. (1) again. Since  $\tilde{K}^2 \cap P$  is a cross-section of a 2-knot  $\tilde{K}^2$ , the simple closed curves  $c_i$  and  $c_{n+i}$  represent the generator of  $H_1(R^4 - \tilde{K}^2)$ , see Fig. (2) again. Therefore, the loop  $w\gamma w^{-1}$  represents an element of the commutator subgroup of  $\pi_1(R^4 - \tilde{K}^2)$  for any arc  $w$  from the base-point to a point on  $\gamma$ , where the simple closed curve  $\gamma$  is  $b \cup \xi(\beta'_\lambda)$  for the segment  $\xi(\beta'_\lambda)$  between  $A_+$  and  $A_-$  on  $\xi(\beta_\lambda)$ . Now,  $w\gamma w^{-1} \sim 0$  for any arc  $w$ , because  $\pi_1(R^4 - \tilde{K}^2) = Z$ ; that is,  $\gamma$  is contractible in  $R^4 - \tilde{K}^2$ .

**Second-step of the proof of (2, 2)'**: There is a 3-ball  $B_i^3$  which meets some arcs  $\xi(\beta_\lambda)$ .

Since the 2-knot  $\tilde{K}^2$  constructed by making use of  $W^3$ ,  $S_\lambda^2$  and  $U_\lambda^3$  and bounding the 3-manifold  $\tilde{W}^3 = \xi(N_1^3 \cup \dots \cup N_n^3 \cup U_1^3 \cup \dots \cup U_{2n}^3 \cup \Delta_0^3)$  in  $R^4$  belongs to the 2-knot-type  $\{K^2\}$ , it is sufficient to prove that  $\tilde{K}^2$  is unknotted. On the other hand, by making use of (2, 4) and (2, 5), there exists an isotopy  $\eta$  of  $R^4$  such that  $\eta(\tilde{K}^2) = \tilde{K}^2$  and that  $\eta\xi(\beta_\lambda)$  ( $\lambda = 1, \dots, 2n$ ) does not meet any 3-ball  $B_i^3$  ( $i = 1, \dots, n$ ). Since  $\xi(\hat{\beta}_\lambda) \subset \xi(\hat{U}_\lambda^3)$ , so  $\eta\xi(\hat{\beta}_\lambda) \subset \eta\xi(\hat{U}_\lambda^3)$ . Take a sufficiently fine tube (a regular neighborhood in  $U_\lambda^3$ )  $\tilde{U}_\lambda^3$  of the arc  $\beta_\lambda$  in  $U_\lambda^3$  so that  $\eta\xi(\tilde{U}_\lambda^3)$  does not meet any  $\hat{B}_i^3$  ( $i = 1, \dots, n$ ) and that  $\tilde{U}_\lambda^3 \cap S_\lambda^2 = f_\lambda^2$  and  $\tilde{U}_\lambda^3 \cap S_0^2 = f_\lambda'^2$  are 2-balls in  $e_\lambda^2$  and  $e_\lambda'^2$  respectively ( $\lambda = 1, \dots, 2n$ ). Then, the fusion  $K^{*2} = \eta\xi(\bigcup_{\lambda=0}^{2n} S_\lambda^2 - \bigcup_{\lambda=1}^{2n} (f_\lambda^2 \cup f_\lambda'^2)) \cup \eta\xi(\bigcup_{\lambda=1}^{2n} \partial\tilde{U}_\lambda^3 - \bigcup_{\lambda=1}^{2n} (f_\lambda^2 \cup f_\lambda'^2))$  not only belongs to  $\{\tilde{K}^2\}$  which coincides with  $\{K^2\}$ , but also the tubes  $\eta\xi(\tilde{U}_\lambda^3)$  ( $\lambda = 1, \dots, 2n$ ) does not meet any 3-ball  $B_i^3$  ( $i = 1, \dots, n$ ). Since we can construct a 3-ball bounded by the 2-knot  $K^{*2}$  in  $R^4$  as we have done in the first-step of the proof, the 2-knot  $K^{*2}$  is unknotted in  $R^4$ . This implies that  $\tilde{K}^2$  is unknotted, and the proof is thus complete.

From (2, 2)', we have easily the main theorem of this paper:

**Theorem (2, 2).** *For a ribbon 2-knot  $K^2$  in  $S^4$ ,  $K^2$  is unknotted in  $S^4$ , if and only if  $\pi_1(S^4 - K^2) = Z$ .*

**Corollary (2, 3).** *Let  $K^2$  be a 2-knot in  $R^4$  satisfying the following (1), (2) and (3). Then  $K^2$  is unknotted in  $R^4$ ;*

- (1) *a 2-node  $K^2 \cap H_+^4$  contains no minimum,*
- (2) *the 2-nodes  $K^2 \cap H_+^4$  and  $K^2 \cap H_-^4$  are symmetric each other with respect to the hyperplane  $R_0^3$ ,*
- (3) *the knot  $k = K^2 \cap R_0^3$  is unknotted in  $R_0^3$ .*

**Proof.** This follows from (2, 2)'. Since  $K^2$  satisfies (1) and (2),  $K^2$  is a

ribbon 2-knot, see [6]. Moreover there is a homomorphism of  $\pi_1(R_0^3 - k)$  onto  $\pi_1(R^4 - K^2)$ , cf. p. 132-6 in [9]. Then, it is easy to see that  $\pi_1(R^4 - K^2) = Z$  as  $\pi_1(R_0^3 - k) = Z$  by the condition (3). (2, 3) is a proposition analogous to the theorem in [11].

The converse of (2, 3) is not always true, see the remark below:

REMARK. *There is an unknotted 2-knot  $K^2$  which satisfies (1) and (2) in (2, 3) but does not satisfy (3) in (2, 3), see the following example.*

The knot  $k$  in  $R_0^3$ , described in Fig. (3), is knotted in  $R_0^3$ , although its Alexander polynomial  $\Delta(t) = 1$ , see Fig. (13) on p. 151 in [10].

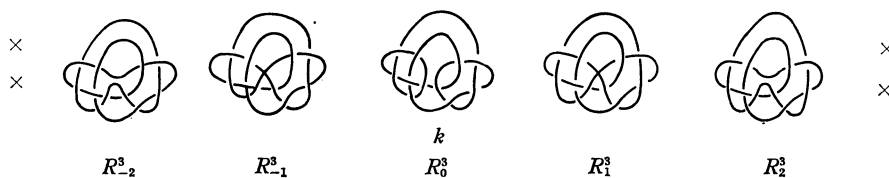


Fig. 3

### 3. A generalization to the higher dimensional case

Let  $K^m$  be a locally flat  $m$ -sphere in  $R^{m+2}$  and let  $W^{m+1}$  be a  $(m+1)$ -manifold satisfying the following (1), (2) and (3):

- (1)  $W^{m+1} \subset R^{m+2}, \partial W^{m+1} = K^m,$
- (2)  $W^{m+1} \approx B^{m+1}$  or  $W^{m+1} \approx \#(S^1 \times S^m) - \dot{B}^{m+1},$
- (3) if  $W^{m+1} \approx B^{m+1}, W^{m+1}$  has a trivial system of  $m$ -spheres  $\{S_1^m, \dots, S_n^m\}$  such that
  - (i) the locally flat  $m$ -link  $\{S_1^m, \dots, S_n^m\}$  is combinatorially trivial in  $R^{m+2},$
  - (ii)  $S_i^m \cup S_{n+i}^m$  bounds a spherical-shell  $N_i^{m+1}$  in  $W^{m+1}$  8),
  - (iii)  $W^{m+1} - \dot{N}_1^{m+1} \cup \dots \cup \dot{N}_n^{m+1} \approx B^{m+1} - \dot{\Delta}_1^{m+1} \cup \dots \cup \dot{\Delta}_{2n}^{m+1}$  9)

Then, we have the following theorem in the same way as (2, 2).

**Theorem (3, 1).** *Let  $K^m$  be a locally flat  $m$ -sphere in  $S^{m+2}$  and  $(m+1)$ -manifold  $W^{m+1}$  satisfying the above conditions. Then,  $K^m$  is unknotted<sup>10)</sup> in  $S^{m+2}$ , if and only if  $\pi_1(S^{m+2} - K^m) = Z.$*

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7) Prof. R.H. Fox named this 2-knot *Terasaka-Kinoshita 2-sphere*.  
 8)  $N_i^{m+1} \approx S^m \times [0, 1].$   
 9)  $\Delta_1^{m+1}, \dots, \Delta_{2n}^{m+1}$  are disjoint  $(m+1)$ -simplices in a  $(m+1)$ -ball  $B^{m+1}.$   
 10) At least *topologically* unknotted.



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