Yanagawa, T.
Osaka J. Math.
7 (1970), 165-172

ON RIBBON 2-KNOTS III
ON THE UNKNOTTING RIBBON 2—KNOTS IN S^4

Takaaki YANAGAWA

(Received September 8, 1969)

1. Introduction

Concerning the unknotting theorem for the pair (S^n, M^{n+2}) with the codimension 2, there are several remarkable results; by T. Homma in the case $n=1$ and $M^3=S^3$, by C.D. Papakyriakopoulos in the case $n=1$ and any 3-manifold M^3, by J. Stallings in the case $n\geqq 3$ and $M^{n+2}=S^{n+2}$ in the topological sense and by J. Levine in the case $n\geqq 4$ and $M^{n+2}=S^{n+2}$ in the combinatorial sense, see [1], [2], [3], [4] and [5]. Confining ourselves to the case $M^{n+2}=S^{n+2}$, the unknotting theorem has not been solved in the case $n=2$.

In this paper, we will prove the following theorem which is an answer under an additional condition to the unknotting theorem in the case $n=2$ and $M^4=S^4$:

Theorem (2, 2). *For a ribbon 2-knot K^2 in S^4, K^2 is unknotted in S^4 if and only if $\pi_1(S^4-K^2)=Z$*[1].

In this paper, everything will be considered from the combinatorial point of view.

2. Proof of Theorem

Lemma (2, 1)[2]. *Let M^4 be a combinatorial 4-manifold and let γ be a simple closed curve in $\mathring{M}^4$ which is contractible to a point in $\mathring{M}^4$. Then, γ bounds a non-singular, locally flat 2-ball in $\mathring{M}^4$*[3].

Proof. Since γ is contractible to a point in $\mathring{M}^4$, there is a PL-map φ of a 2-ball D^2 into $\mathring{M}^4$ satisfying the following (1), (2) and (3):

(1) $\varphi(D^2)\subset\mathring{M}^4$, $\varphi(\partial D^2)=\gamma$,

(2) $\varphi(D^2)$ is in a general position in $\mathring{M}^4$ so that the self-intersection consists of a finite number of double points,

1) See [6] for the definition of the *ribbon* 2-*knots* in R^4.

2) Cf. the result in [7], the proof of Lemma (2, 7).

3) $\mathring{X}$ and ∂X mean the interior and the boundary of X respectively.

(3) there are at most a finite number of *locally knotted points* on $\varphi(D^2)$ which are different from the double points in (2).

Here, in (3), a point x of $\varphi(D^2)$ is called a *locally knotted point*[4)] *if the pair* $(Lk(x, \varphi(D^2)), Lk(x, M^4))$ *is a knotted sphere-pair* for the combinatorial triangulation of M^4 for which $\varphi(D^2)$ is a subcomplex and the point x is a vertex. If there is a locally knotted point x of $\varphi(D^2)$, it is possible to exchange a non-singular 2–ball $St(x, \varphi(D^2))$, which may be not locally flat, for an immersed 2–ball $\rho(B^2)$ in a 4–ball $B^4=\mathrm{St}(x, M^4)$ by an immersion ρ of a 2–ball B^2 such that $\rho(\partial B^2)=\varphi(D^2)\cap \partial B^4$, $\rho(\mathring{B}^2)\subset \mathring{B}^4$ and that each pair $(\rho Lk(y, B^2), Lk(\rho(y), B^4))$ is unknotted for a fine subdivision of B^4 and each virtex y of B^2. Perform the exchange for all locally knotted points of $\varphi(D^2)$. By making use of the general position theory, we have a PL-map φ' of D^2 into $\mathring{M}^4$ satisfying the following (1′), (2′) and (3′):

(1′) $\varphi'(D^2)\subset M^4$, $\varphi'(\partial D^2)=\gamma$,

(2′) $\varphi'(D^2)$ is in a general position in M^4 so that the self-intersection consists of a finite number of double points,

(3′) $\varphi'(D^2)$ has no locally knotted point.

Let x be a double point of $\varphi'(D^2)$ and $x=\varphi'(y)=\varphi'(y')$ for just two points y, y' of D^2. Then there is an arc α spanning y and a point y'' on ∂D^2 such that the image $\varphi'(\alpha)$ does not cross any double point of $\varphi'(D^2)$ except for x. Let V^4 be a regular neighborhood of $\varphi'(\alpha)$ in M^4. Since V^4 is a 4–ball, there is a PL-homeomorphism ϕ of V^4 onto a standard 4–cube Δ^4 such that

(i) Δ^4; $|x_1|, |x_2|, |x_3|, |x_4|\leqq 2$

(ii) $\phi\varphi'(\alpha)$; $0\leqq x_1\leqq 1$, $x_2=x_3=x_4=0$

(iii) $\phi\varphi'(x)$; $x_1=x_2=x_3=x_4=0$

(iv) $\phi(V^4\cap\varphi'(D^2))$; $\begin{cases}-2\leqq x_1\leqq 1,\ x_2=x_3=0,\ |x_4|\leqq 2\\ x_1=0,\ |x_2|,\ |x_3|\leqq 2,\ x_4=0\cdots(*)\,.\end{cases}$

Let N^2 be the 2–ball in $\phi\varphi'(D^2)$ defined by the equation $(*)$ and let N'^2 be the 2–ball in Δ^4 defined as follows:

$$\text{(v)}\quad N'^2;\ \begin{cases}0\leqq x_1<2,\ |x_2|\leqq 2,\ |x_3|=2,\ x_4=0\,,\\ 0\leqq x_1<2,\ |x_2|=2,\ |x_3|\leqq 2,\ x_4=0\,,\\ x_1=2,\ |x_2|,\ |x_3|\leqq 2,\ x_4=0.\end{cases}$$

If we consider a singular 2–ball $B'^2=\phi^{-1}((\phi\varphi'(D^2)-N^2)\cup N'^2)$, then this 2–ball B'^2 is not only locally flat but also has a number of the double points less

4) See [8] p. 34.

than the number of those of $\varphi'(D^2)$. Moreover we have that $\partial B'^2=\gamma$. Repeating this process, we have finally *a locally flat, non-singular 2-ball* B^2 such that $B^2\subset\mathring{M}^4$ and that $\partial B^2=\gamma$. The proof is thus complete.

Let K^2 be a ribbon 2-knot in R^4, then there is a 3-manifold W^3 satisfying the following properties:

(1) $W^3\approx B^3$ or $W^3\approx\#(S^1\times S^2)-\mathring{B}^3$ [5],

(2) If $W^3\not\approx B^3$, W^3 has a trivial system of 2-spheres $\{S_1^2, \cdots, S_{2n}^2\}$ satisfying that

(i) a 2-link $\{S_1^2, \cdots, S_{2n}^2\}$ is trivial in R^4,

(ii) $S_i^2\cup S_{n+i}^2$ bounds a spherical-shell N_i^3 in W^3 $(i=1, \cdots, n)$[6],

(iii) $W^3-\mathring{N}_1^3\cup\cdots\cup\mathring{N}_n^3\approx B^3-\mathring{\Delta}_1^3\cup\cdots\cup\mathring{\Delta}_{2n}^3$,

see (3, 5) and (3, 6) in [6].

Let Δ_0^3 be a 3-ball in $W^3-N_1^3\cup\cdots\cup N_n^3$, let S_0^2 be a boundary 2-sphere of Δ_0^3 and let $\beta_1, \cdots, \beta_{2n}$ be a collection of mutually disjoint arcs spanning S_0^2 and $S_1^2, \cdots, S_{2n}^2$ in $W^3-N_1^3\cup\cdots\cup N_n^3\cup\mathring{\Delta}_0^3$ respectively. Moreover, let U_λ^3 be a regular neighborhood of the arc β_λ in $W^3-\mathring{N}_1^3\cup\cdots\cup\mathring{N}_n^3\cup\mathring{\Delta}_0^3$ where $U_\lambda^3\cap S_\lambda^2=e_\lambda^2$ and $U_\lambda^3\cap S_0^2=e_\lambda'^2$ are 2-balls such that $e_\lambda'^2\cap e_\mu'^2=\emptyset$ $(\lambda\neq\mu, \lambda, \mu=1, \cdots, 2n)$. Since the 2-link $\{S_0^2, \cdots, S_{2n}^2\}$ is trivial in R^4, there is an isotopy ξ of R^4 by which $\xi(S_\lambda^2)$ $(\lambda=0, \cdots, 2n)$ are moved into the position given by the equations below:

$$\begin{aligned}
\xi(S_0^2);\ & x_1^2+x_2^2+x_3^2=1, \quad x_4=0\\
\xi(S_i^2);\ & (x_1-4i)^2+x_2^2+x_3^2=1, \quad x_4=0\\
\xi(S_{n+i}^2);\ & (x_1-4i)^2+x_2^2+x_3^2=2, \quad x_4=0\\
\xi(N_i^3);\ & 1\leqq(x_1-4i)^2+x_2^2+x_3^2\leqq 2, \quad x_4=0 \quad (i=1, \cdots, n).
\end{aligned}$$

Moreover, we may suppose that the center line $\xi(\beta_\lambda)$ of the tube $\xi(U_\lambda^3)$ is given by the equations below:

$$\begin{aligned}
x_1=4i,\quad & x_2=0,\quad x_3=1-x_4,\quad x_4\geqq 0\\
& \text{in the neighborhood of } \xi(\beta_i\cap S_i^2),\\
x_1=4i,\quad & x_2=0,\quad x_3=\sqrt{2}-x_4,\quad x_4\leqq 0\\
& \text{in the neighborhood of } \xi(\beta_{n+i}\cap S_{n+i}^2)\\
& (i=1, 2, \cdots, n).
\end{aligned}$$

Theorem (2. 2)′. *For a ribbon 2-knot K^2 in R^4, K^2 is unknotted in R^4, if and only if $\pi_1(R^4-K^2)=Z$.*

---

5) B^3 means a 3-ball and $\approx$ means *to be homeomorphic to.*

6) $N_i^3\approx S^2\times[0, 1]$.

The proof of this theorem is divided into two steps, and the second-step of the proof will be given later after we have proved two lemmas (2, 3) and (2, 4). Since K^2 is a ribbon 2–knot in R^4, it bounds a 3–manifold W^3 previously described, therefore if $W^3 \approx B^3$, we have nothing particular to say. Hence, in the following discussion we will consider the case that $W^3 \not\approx B^3$. Consider the trivial system $\{S_1^2, \cdots, S_{2n}^2\}$ and the isotopy ξ of R^4 as before. Let $\tilde{K}^2$ be a 2–knot in R^4 such that

$$\tilde{K}^2 = \xi(\bigcup_{\lambda=0}^{2n} S_\lambda^2 - \bigcup_{\lambda=1}^{2n} (\mathring{e}_\lambda^2 \cup \mathring{e}_\lambda'^2)) \cup \xi(\bigcup_{\lambda=1}^{2n} \partial U_\lambda^3 - \bigcup_{\lambda=1}^{2n} (\mathring{e}_\lambda^2 \cup \mathring{e}_\lambda'^2)) .$$

Then, since two 2–spheres $\xi(K^2)$ and $\tilde{K}^2$ bound a 3–manifold which is a subcomplex of $\xi(W^3)$ and which is homeomorphic to $S^2 \times [0, 1]$ in R^4, $\tilde{K}^2$ belongs to the 2–knot-type $\{\xi(K^2)\}$ which coincides with the 2–knot-type $\{K^2\}$.

Let B_i^3 be a 3–ball bounded by the 2–sphere $\xi(S_i^2)$:

$$B_i^3;\ (x_1-4i)^2+x_2^2+x_3^2 \leqq 1, \quad x_4 = 0 \qquad (i = 0, 1, 2, \cdots, n) .$$

First-step of the proof of (2, 2)′: Each 3–ball B_i^3 bounded by the 2–sphere $\xi(S_i^2)$ $(i=1, \cdots, n)$ in R_0^3 does not meet any arc $\xi(\beta_\lambda)$ except for the end points $(\lambda=1, \cdots, 2n)$.

Since we can find a regular neighborhood U_λ^3 of β_λ so fine that $\xi(U_\lambda^3) \cap B_i^3 = \emptyset$ because $\xi(\beta_\lambda) \cap B_i^3 = \emptyset$ for all i and λ $(i=1, \cdots, n, \lambda=1, \cdots, 2n)$, the 2–knot $\tilde{K}^2$ bounds a 3–ball $B_0^3 \cup B_1^3 \cup \cdots \cup B_n^3 \cup B_{n+1}^3 \cup \cdots \cup B_{2n}^3 \cup \xi(U_1^3) \cup \cdots \cup \xi(U_{2n}^3)$, where the 3–ball B_{n+i}^3 is bounded by $\xi(S_{n+i}^2)$ in the neighborhood of $\xi(N_i^3) \cup B_i^3$ in R^4 so that $B_{n+i}^3 \cap B_j^3 = \emptyset$, $B_{n+i}^3 \cap \xi(U_\lambda^3) = \emptyset$ $(\lambda \neq n+i)$ and $=\xi(e_\lambda^2)$ $(\lambda=n+i)$ $(i, j=1, \cdots, n, \lambda=1, \cdots, 2n)$: for a sufficiently small $\varepsilon(<0)$,

$$B_{n+i}^3;\ \begin{cases} (x_1-4i)^2+x_2^2+x_3^2 = 2, & -\varepsilon \leqq x_4 \leqq 0 \\ (x_1-4i)^2+x_2^2+x_3^2 \leqq 2, & x_4 = -\varepsilon \end{cases} \qquad (i = 1, 2, \cdots, n) .$$

If there is a 3–ball B_i^3 which meets some arcs $\xi(\beta_\lambda)$ $(1 \leqq i \leqq n, 1 \leqq \lambda \leqq 2n)$, we will consider how to remove the intersection of the 3–ball B_i^3 and the arcs $\xi(\beta_\lambda)$ without changing the 2–knot-type of $\tilde{K}^2$. We need following two lemmas (2, 3) and (2, 4) to remove the intersection.

Lemma (2, 3). *If there are an arc b in $R^4-\tilde{K}^2$ and a subarc β_λ' of the arc β_λ $(1 \leqq \lambda \leqq 2n)$ such that the simple closed curve $\gamma = b \cup \xi(\beta_\lambda')$ is contractible in $R^4-\tilde{K}^2$, then there exists an isotopy η of R^4 by which $\eta(\tilde{K}^2)=\tilde{K}^2$ and $\eta\xi(\beta_\lambda) = \xi(\beta_\lambda - \beta_\lambda') \cup b$.*

Proof. Since β_λ is contained in $\mathring{U}_\lambda^3$ except two end points, we can triangulate R^4 so that the regular neighborhood $N(\tilde{K}^2)$ of $\tilde{K}^2$ in R^4 does not meet $\xi(\beta_\lambda)$. If we apply (2, 1) to the 4–manifold $M^4 = R^4 - \mathring{N}(\tilde{K}^2)$ and the simple closed curve $\gamma=b\cup\xi(\beta'_\lambda)$, the simple closed curve γ bounds a locally flat 2-ball B^2 in $\mathring{M}^4$. Therefore there exists a combinatorial 4-ball B^4 containning B^2 in its interior and contained in $R^4-\tilde{K}^2$. Now, we have easily an isotopy which is identical on ∂B^4 and transfers the subarc $\xi(\beta'_\lambda)$ onto the arc b. Hence, the proof is complete.

Lemma (2, 4). *If $\pi_1(R^4-\tilde{K}^2)=Z$ and an arc $\xi(\beta_\lambda)$ $(1\leqq\lambda\leqq 2n)$ pierces through a 3-ball B^3_i $(1\leqq i\leqq n)$ at a point A, there are an arc b in $R^4-\tilde{K}^2$ and a subarc β'_λ on the arc β_λ containning A such that the simple closed curve $b\cup\xi(\beta'_\lambda)$ is contractible in $R^4-\tilde{K}^2$.*

Proof. For convenience's sake, we may suppose that $\xi(\beta_\lambda)$ is given in the neighborhood of the point A as follows:

$$\xi(\beta_\lambda);\ x_1=4i,\quad x_2=x_3=0\,,\quad -1\leqq x_4\leqq 1\,.$$

Consider the cross-sections of $\xi(S^2_i)$, $\xi(S^2_{n+i})$ and $\xi(N^3_i)$ by the hyperplane P; $x_3=0$. Then, we have the following figure Fig. (1).

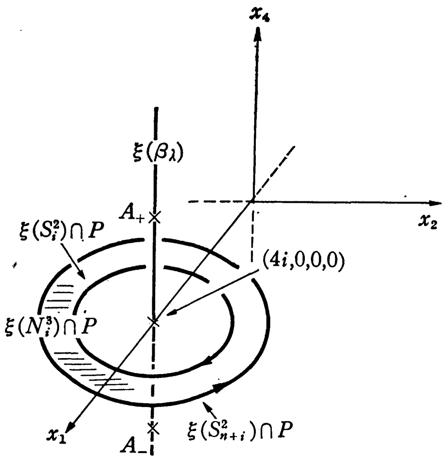

Fig. 1

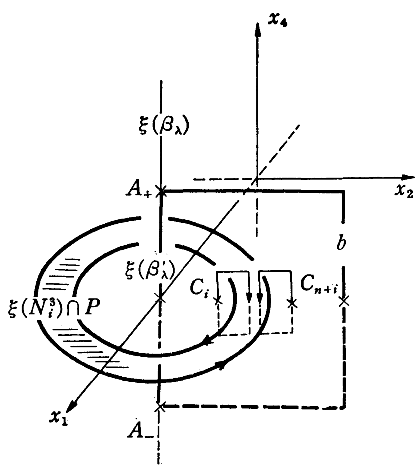

Fig. 2

Place $\xi(U^3_\lambda)$ in a general position with respect to the hyperplane P, then the cross-section $\xi(U^3_\lambda)\cap P$ is at most 2-dimensional, and we can find an arc b spanning two points A_+ and A_- in $P-P\cap\tilde{K}^2$ as follows:

$$b;\ \begin{cases} x_1=4i, & 0\leqq x_2\leqq 2,\quad x_3=0,\quad x_4=\varepsilon \\ x_1=4i, & x_2=2,\quad x_3=0,\quad -\varepsilon\leqq x_4\leqq\varepsilon \\ x_1=4i, & 0\leqq x_2\leqq 2,\quad x_3=0,\quad x_4=-\varepsilon\,, \end{cases}$$

see Fig. (2).

Since the 2-knot $\tilde{K}^2$ bounds the orientable 3-manifold $\tilde{W}^3=\xi(N_1^3\cup\cdots\cup N_n^3\cup U_1^3\cup\cdots\cup U_{2n}^3\cup\Delta_0^3)$ in R^4, we will give an orientation induced from the orientation of $\tilde{W}^3$ for $\tilde{K}^2$. Then, the trivial link $\xi(S_i^2\cup S_{n+i}^2)\cap P$, which bounds an annulus $\xi(N_i^3)\cap P$ in P, can be given the orientation induced from that of $\xi(N_i^3)\cap P$, see Fig. (1) again. Since $\tilde{K}^2\cap P$ is a cross-section of a 2-knot $\tilde{K}^2$, the simple closed curves c_i and c_{n+i} represent the generator of $H_1(R^4-\tilde{K}^2)$, see Fig. (2) again. Therefore, the loop $w\gamma w^{-1}$ represents an element of the commutator subgroup of $\pi_1(R^4-\tilde{K}^2)$ for any arc w from the base-point to a point on γ, where the simple closed curve γ is $b\cup\xi(\beta'_\lambda)$ for the segment $\xi(\beta'_\lambda)$ between A_+ and A_- on $\xi(\beta_\lambda)$. Now, $w\gamma w^{-1}\sim 0$ for any arc w, because $\pi_1(R^4-\tilde{K}^2)=Z$; that is, γ is contractible in $R^4-\tilde{K}^2$.

Second-step of the proof of (2, 2)′: There is a 3-ball B_i^3 which meets some arcs $\xi(\beta_\lambda)$.

Since the 2-knot $\tilde{K}^2$ constructed by making use of W^3, S_λ^2 and U_λ^3 and bounding the 3-manifold $\tilde{W}^3=\xi(N_1^3\cup\cdots\cup N_n^3\cup U_1^3\cup\cdots\cup U_{2n}^3\cup\Delta_0^3)$ in R^4 belongs to the 2-knot-type $\{K^2\}$, it is sufficient to prove that $\tilde{K}^2$ is unknotted. On the other hand, by making use of (2, 4) and (2, 5), there exists an isotopy η of R^4 such that $\eta(\tilde{K}^2)=\tilde{K}^2$ and that $\eta\xi(\beta_\lambda)$ $(\lambda=1,\cdots,2n)$ does not meet any 3-ball B_i^3 $(i=1,\cdots,n)$. Since $\xi(\mathring{\beta}_\lambda)\subset\xi(\mathring{U}_\lambda^3)$, so $\eta\xi(\mathring{\beta}_\lambda)\subset\eta\xi(\mathring{U}_\lambda^3)$. Take a sufficiently fine tube (a regular neighborhood in U_λ^3) $\tilde{U}_\lambda^3$ of the arc β_λ in U_λ^3 so that $\eta\xi(\tilde{U}_\lambda^3)$ does not meet any $\mathring{B}_i^3$ $(i=1,\cdots,n)$ and that $\tilde{U}_\lambda^3\cap S_\lambda^2=f_\lambda^2$ and $\tilde{U}_\lambda^3\cap S_0^2=f_\lambda'^2$ are 2-balls in e_λ^2 and $e_\lambda'^2$ respectively $(\lambda=1,\cdots,2n)$. Then, the fusion $K^{*2}=\eta\xi(\bigcup_{\lambda=0}^{2n}S_\lambda^2-\bigcup_{\lambda=1}^{2n}(\mathring{f}_\lambda^2\cup\mathring{f}_\lambda'^2))\cup\eta\xi(\bigcup_{\lambda=1}^{2n}\partial\tilde{U}_\lambda^3-\bigcup_{\lambda=1}^{2n}(\mathring{f}_\lambda^2\cup\mathring{f}_\lambda'^2))$ not only belongs to $\{\tilde{K}^2\}$ which coincides with $\{K^2\}$, but also the tubes $\eta\xi(\tilde{U}_\lambda^3)$ $(\lambda=1,\cdots,2n)$ does not meet any 3-ball B_i^3 $(i=1,\cdots,n)$. Since we can construct a 3-ball bounded by the 2-knot K^{*2} in R^4 as we have done in the first-step of the proof, the 2-knot K^{*2} is unknotted in R^4. This implies that $\tilde{K}^2$ is unknotted, and the proof is thus complete.

From (2, 2)′, we have easily the main theorem of this paper:

Theorem (2, 2). *For a ribbon 2-knot K^2 in S^4, K^2 is unknotted in S^4, if and only if $\pi_1(S^4-K^2)=Z$.*

Corollary (2, 3). *Let K^2 be a 2-knot in R^4 satisfying the following* (1), (2) *and* (3). *Then K^2 is unknotted in R^4;*

(1) *a 2-node $K^2\cap H_+^4$ containes no minimum,*

(2) *the 2-nodes $K^2\cap H_+^4$ and $K^2\cap H_-^4$ are symmetric each other with respect to the hyperplane R_0^3,*

(3) *the knot $k=K^2\cap R_0^3$ is unknotted in R_0^3.*

Proof. This follows from (2, 2)′. Since K^2 satisfies (1) and (2), K^2 is a ribbon 2-knot, see [6]. Moreover there is a homomorphism of $\pi_1(R_0^3-k)$ onto $\pi_1(R^4-K^2)$, cf. p. 132–6 in [9]. Then, it is easy to see that $\pi_1(R^4-K^2)=Z$ as $\pi_1(R_0^3-k)=Z$ by the condition (3). (2, 3) is a proposition analogous to the theorem in [11].

The converse of (2, 3) is not always true, see the remark below:

REMARK. *There is an unknotted 2-knot K^2* [7] *which satisfies* (1) *and* (2) *in* (2, 3) *but does not satisfy* (3) *in* (2, 3), *see the following example.*

The knot k in R_0^3, described in Fig. (3), is knotted in R_0^3, although its Alexander polynomial $\Delta(t)=1$, see Fig. (13) on p. 151 in [10].

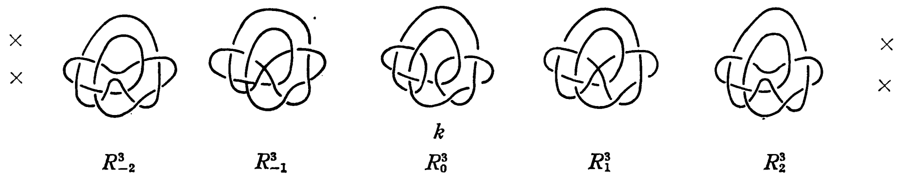

Fig. 3

3. A generalization to the higher dimensional case

Let K^m be a locally flat m-sphere in R^{m+2} and let W^{m+1} be a $(m+1)$-manifold satisfying the following (1), (2) and (3):

(1) $W^{m+1}\subset R^{m+2}, \quad \partial W^{m+1}=K^m$,

(2) $W^{m+1}\approx B^{m+1}$ or $W^{m+1}\approx \#(S^1\times S^m)-\mathring{B}^{m+1}$,

(3) if $W^{m+1}\not\approx B^{m+1}$, W^{m+1} has a trivial system of m-spheres $\{S_1^m, \cdots, S_{2n}^m\}$ such that

(i) the locally flat m-link $\{S_1^m, \cdots, S_{2n}^m\}$ is combinatorially trivial in R^{m+2},

(ii) $S_i^m\cup S_{n+i}^m$ bounds a spherical-shell N_i^{m+1} in W^{m+1} [8],

(iii) $W^{m+1}-\mathring{N}_1^{m+1}\cup\cdots\cup\mathring{N}_n^{m+1}\approx B^{m+1}-\mathring{\Delta}_1^{m+1}\cup\cdots\cup\mathring{\Delta}_{2n}^{m+1}$. [9]

Then, we have the following theorem in the same way as (2, 2).

Theorem (3, 1). *Let K^m be a locally flat m-sphere in S^{m+2} and $(m+1)$-manifold W^{m+1} satisfying the above conditions. Then, K^m is unknotted*[10] *in S^{m+2}, if and only if $\pi_1(S^{m+2}-K^m)=Z$.*

KOBE UNIVERSITY

7) Prof. R.H. Fox named this 2-knot *Terasaka-Kinoshita 2-sphere.*

8) $N_i^{m+1}\approx S^m\times[0, 1]$.

9) $\Delta_1^{m+1}, \cdots, \Delta_{2n}^{m+1}$ are disjoint $(m+1)$-simplices in $a(m+1)$-ball B^{m+1}.

10) At least *topologically* unknotted.

References

[1] T. Homma: *On Dehn's Lemma for S^3*, Yokohama Math. J. **5** (1957), 223–244.

[2] C.D. Papakyriakopoulos: *On Dehn's lemma and the asphericity of knots*, Ann. of Math. **66** (1957), 1–26.

[3] A. Shapiro and J.H.C. Whitehead: *A proof and extension of Dehn's lemma*, Bull. Amer. Math. Soc. **64** (1958), 174–178.

[4] J. Stallings: *On topologically unknotted spheres*, Ann. of Math. **77** (1963), 490–503.

[5] J. Levine: *Unknotting spheres in codimension two*, Topology **4** (1965), 9–16.

[6] T. Yanagawa: *On ribbon 2-knots*, Osaka J. Math. **6** (1969), 447–464.

[7] R. Penrose, E.C. Zeeman and J.H.C. Whitehead: *Imbeddings of manifolds in Euclidean space*, Ann. of Math. **73** (1961), 154–212.

[8] S. Suzuki: *Local knots of 2-spheres in 4-manifolds*, Proc. Japan Acad. **45** (1969), 34–38.

[9] R.H. Fox: *A quick trip through knot theory*, Top. of 3-manifolds and related topics., edited by M.K. Fort Jr. Prentice Hall, 1962.

[10] S. Kinoshita and H. Terasaka: *On unions of knots*, Osaka Math. J. **9** (1957), 131–153.

[11] F. Hosokawa: *On trivial 2-spheres in 4-space*, The Quart. J. Math. (2) **19** (1968), 249–256.