<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Injective pairs in perfect rings</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Hoshino, Mitsuo; Sumioka, Takeshi</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 35(3) P.501-P.508</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1998</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/7051">https://doi.org/10.18910/7051</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/7051</td>
</tr>
</tbody>
</table>
Throughout this note, rings are associative rings with identity and modules are unitary modules. Sometimes, we use the notation $AX$ (resp. $XA$) to signify that the module $X$ considered is a left (resp. right) $A$-module. For each pair of subsets $X$ and $M$ of $A$, we set $\ell_X(M) = \{a \in X | aM = 0\}$ and $r_M(X) = \{a \in M | Xa = 0\}$.

Following Baba and Oshiro [1], we call a pair $(eA, Af)$ of a right ideal $eA$ and a left ideal $Af$ in a ring $A$ an i-pair if (a) $e$ and $f$ are local idempotents; (b) $eA$ and $Af$ have essential socles; and (c) $\text{soc}(eA) \cong fAf/Je$ and $\text{soc}(Af) \cong Ae/Jf$.

Generalizing a result of Fuller [3], Baba and Oshiro [1] showed that for a local idempotent $e$ in a semiprimary ring $A$, $eA$ is injective if and only if there exists a local idempotent $f$ in $A$ such that $(eA, Af)$ is an i-pair in $A$ and $rAf(\ell_{eA}(M)) = M$ for every submodule $M$ of $Af/Jf$, and that for an i-pair $(eA, Af)$ in a semiprimary ring $A$ the following are equivalent: (1) $eAeA$ is artinian; (2) $AfAf$ is artinian; and (3) both $eA$ and $Af$ are injective.

Our aim is to extend the results mentioned above to perfect rings. Following Harada [4], we call a module $L_A$ $M$-simple-injective if for any submodule $N$ of $M_A$ every $\theta : N_A \to L_A$ with $\text{Im} \theta$ simple can be extended to some $\phi : M_A \to L_A$. For a local idempotent $e$ in a left perfect ring $A$, we will show that $eA$ is $A$-simple-injective if and only if there exists a local idempotent $f$ in $A$ such that $(eA, Af)$ is an i-pair in $A$ and $rAf(\ell_{eA}(M)) = M$ for every submodule $M$ of $Af/Jf$, and that $eA$ is injective if it is $A$-simple-injective and has finite Loewy length. We will show also that for an i-pair $(eA, Af)$ in a left perfect ring $A$ the following are equivalent: (1) $eAeA$ is artinian; (2) $AfAf$ is artinian; and (3) both $eA$ and $Af$ are injective.

1. Localization and injective objects

Let $A$ and $B$ be abelian categories, $F : A \to B$ and $G : B \to A$ covariant functors, and $\varepsilon : 1_A \to GF$ and $\delta : FG \to 1_B$ homomorphisms of functors, where $1_A : A \to A$ and $1_B : B \to B$ are identity functors. We assume the conditions: (a) $\delta F \circ \varepsilon F = \text{id}_F$; (b) $G \delta \circ \varepsilon G = \text{id}_G$; (c) $F$ is exact; and (d) $\delta$ is an isomorphism.

Remark 1. (1) By the conditions (a) and (b), for each pair of $X \in \text{Ob}(A)$ and
$M \in \text{Ob}(B)$ we have a natural isomorphism
\[ \theta_{X,M} : \text{Hom}_B(FX, M) \to \text{Hom}_A(X, GM), \beta \mapsto G\beta \circ \varepsilon_X \]
with $\theta_{X,M}^{-1}(\alpha) = \delta_M \circ F\alpha$ for $\alpha \in \text{Hom}_A(X, GM)$. Namely, $G$ is a right adjoint of $F$. In particular, $G$ is left exact.

(2) By the conditions (a), (b) and (d), $G : B \to A$ is fully faithful.

(3) By the conditions (a) and (d), $F\varepsilon : F \to FGF$ is an isomorphism with $F\varepsilon^{-1} = \delta_F$.

(4) By the conditions (b) and (d), $\varepsilon_G : G \to GFG$ is an isomorphism with $\varepsilon_G^{-1} = G\delta$.

Though the following lemmas are well known and more or less obvious, we include proofs for completeness.

**Lemma 1.1.** Let $X \in \text{Ob}(A)$ be simple with $FX \neq 0$. Then $FX \in \text{Ob}(B)$ is simple.

**Proof.** Let $\beta : FX \to M$ be a nonzero morphism in $B$. We claim $\beta$ monic. Note that $\beta = \delta_M \circ F(G\beta \circ \varepsilon_X)$. Thus $G\beta \circ \varepsilon_X : X \to GM$ is nonzero and monic, so is $\beta = \delta_M \circ F(G\beta \circ \varepsilon_X)$. \qed

**Lemma 1.2.** Let $\mu : Y \to X$ be an essential monomorphism in $A$ with $\varepsilon_Y$ monic. Then $F\mu : FY \to FX$ is an essential monomorphism in $B$.

**Proof.** Let $\beta : FX \to M$ be a morphism in $B$ with $\beta \circ F\mu$ monic. We claim $\beta$ monic. Since $(G\beta \circ \varepsilon_X) \circ \mu = G\beta \circ GF\mu \circ \varepsilon_Y = G(\beta \circ F\mu) \circ \varepsilon_Y$ is monic, $G\beta \circ \varepsilon_X$ is monic and so is $\beta = \delta_M \circ F(G\beta \circ \varepsilon_X)$. \qed

**Lemma 1.3.** Let $X \in \text{Ob}(A)$ be injective with $\varepsilon_X$ monic. Then $\varepsilon_X : X \to GFX$ is an isomorphism and $FX \in \text{Ob}(B)$ is injective.

**Proof.** Since $F\varepsilon_X$ is an isomorphism, $F(\text{Cok}\varepsilon_X) \cong \text{Cok}F\varepsilon_X = 0$ and $\text{Hom}_A(\text{Cok}\varepsilon_X, GFX) \cong \text{Hom}_B(F(\text{Cok}\varepsilon_X), FX) = 0$. Thus, since $\varepsilon_X : X \to GFX$ is a split monomorphism, Cok$\varepsilon_X = 0$. Hence for each $M \in \text{Ob}(B)$ we have a natural isomorphism
\[ \eta_M : \text{Hom}_B(M, FX) \to \text{Hom}_A(GM, X), \beta \mapsto \varepsilon_X^{-1} \circ G\beta. \]
Let $\nu : N \to M$ be a monomorphism in $B$. Since $G\nu$ is monic, $\text{Hom}_A(G\nu, X)$ is epic and so is $\text{Hom}_B(\nu, FX) = \eta_N^{-1} \circ \text{Hom}_A(G\nu, X) \circ \eta_M$. \qed

**Remark 2.** (1) An object $M \in \text{Ob}(B)$ is injective if and only if so is $GM \in \text{Ob}(A)$. 

Ob(A).

(2) The canonical monomorphism $\text{Im}\, \varepsilon_X \to G\text{FX}$ is an essential monomorphism for every $X \in \text{Ob}(A)$ with $FX \neq 0$.

(3) If $\nu : N \to M$ is an essential monomorphism in $B$, so is $G\nu : GN \to GM$.

(4) For $X \in \text{Ob}(A)$ with $\varepsilon_X$ monic, a monomorphism $\mu : Y \to X$ in $A$ is an essential monomorphism if and only if so is $F\mu : FY \to FX$.

2. Injective pairs

Throughout the rest of this note, $A$ stands for a ring with Jacobson radical $J$. For an $i$-pair $(eA, Af)$ in $A$, we denote by $\mathcal{A}_e(eA, Af)$ the lattice of submodules $X$ of $eAeA$ with $\ell_{eA}(r_{Af}(X)) = X$ and by $\mathcal{A}_r(eA, Af)$ the lattice of submodules $M$ of $AfAf$ with $r_{Af}(\ell_{eA}(M)) = M$.

**Remark 3.** Let $(eA, Af)$ be an $i$-pair in $A$. Let $X$ be a submodule of $eAeA$. Then $Xr_{Af}(X) = 0$ implies $X \subseteq \ell_{eA}(r_{Af}(X))$ and thus $r_{Af}(\ell_{eA}(r_{Af}(X))) \subseteq r_{Af}(X)$. Also, $\ell_{eA}(r_{Af}(X))r_{Af}(X) = 0$ implies $r_{Af}(X) \subseteq r_{Af}(\ell_{eA}(r_{Af}(X)))$. Thus $r_{Af}(X) \in \mathcal{A}_e(eA, Af)$. Similarly, $\ell_{eA}(M) \in \mathcal{A}_e(eA, Af)$ for every submodule $M$ of $AfAf$. It follows that $\mathcal{A}_e(eA, Af)$ is anti-isomorphic to $\mathcal{A}_r(eA, Af)$.

The following lemmas have been established in [5], [3], [1], [8], [6] and so on. However, for the benefit of the reader, we provide direct proofs.

**Lemma 2.1.** Let $e, f \in A$ be idempotents and assume $\ell_{eA}(Af) = 0 = r_{Af}(eA)$. Then the following hold.

1. For a two-sided ideal $I$ of $A$, $eI = 0$ if and only if $If = 0$.
2. $\ell_{eA}(I) = \ell_{eA}(If)$ for every right ideal $I$ of $A$.
3. $r_{Af}(I) = r_{Af}(eI)$ for every left ideal $I$ of $A$.

Proof. (1) Assume $eI = 0$. Then $eAf = eIf = 0$ and $If \subseteq r_{Af}(eA) = 0$. By symmetry, $If = 0$ implies $eI = 0$.

(2) Since $If \subseteq I$, $\ell_{eA}(I) \subseteq \ell_{eA}(If)$. For any $x \in \ell_{eA}(If)$, since $xIAf = xIf = 0$, $xI \subseteq \ell_{eA}(Af) = 0$ and $x \in \ell_{eA}(I)$. Thus $\ell_{eA}(I) \subseteq \ell_{eA}(I)$.

(3) Similar to (2).

**Lemma 2.2.** Let $(eA, Af)$ be an $i$-pair in $A$. Then the following hold.

1. $\ell_{eA}(Af) = 0 = r_{Af}(eA)$.
2. $eAfAf$ and $eAeAf$ have simple essential socles and $\text{soc}(eA_A)f = \text{soc}(eAfAf) = \text{soc}(eAeAf) = e(\text{soc}(A_Af))$.

Proof. (1) For any $0 \neq x \in eA$, since $\text{soc}(eA_A) \subseteq xA$, $0 \neq \text{soc}(eA_A)f \subseteq xAf$ and $x \notin \ell_{eA}(Af)$. Thus $\ell_{eA}(Af) = 0$. Similarly $r_{Af}(eA) = 0$. 


(2) Since by Lemma 1.1 soc(eA)f_{Af} and \( eAe(soc(\Lambda Af)) \) are simple, and since by Lemma 1.2 \( soc(eA)f_{Af} \subseteq eAf_{Af} \) and \( eAe(soc(\Lambda Af)) \subseteq eAeAf \) are essential extensions, the assertion follows.

**Lemma 2.3.** Let \((eA, Af)\) be an i-pair in \( A \). Then for any \( n \geq 1 \) \( eJ^n = 0 \) if and only if \( J^n f = 0 \), so that \( eA_A \) and \( AAf \) have the same Loewy length.

**Proof.** By Lemmas 2.2(1) and 2.1(1).

**Lemma 2.4.** Let \((eA, Af)\) be an i-pair in \( A \). Let \( N, M \) be submodules of \( Af_{Af} \) with \( N \subseteq M \) and \( M/N \) simple. Assume \( N \in A_r(eA, Af) \). Then the following hold.

1. \( eAe(N)/eAe(M) \) is simple.
2. \( M \in A_r(eA, Af) \).

**Proof.** (1) Let \( a \in M \) with \( a \not\in N \). Then \( M = N + afAf \). Also, since \( M \neq N = r_{Af}(eA(N)) \), \( eA(N) \subseteq eA(N)/eA(M) \) with \( eA(N)/eA(M) \neq 0 \). Since \( 0 \neq eA(N)M = eA(N)afAf \) and \( eA(N)afAf = soc(eAf_{Af}) \). Thus by Lemma 2.2(2) \( eA(N)a = soc(eAeAf) \) and, since \( eAe(Af) = 0 \), \( eAe(N)/eA(M) \cong soc(eAeAf) \).

(2) Since \( eAe(M) \subseteq eA(N) \subseteq eAeAf \) with \( eAe(M) \in A_r(eA, Af) \) and \( eAe(M)/eAe(M) \) simple, we can apply the part (1) to conclude that \( r_{Af}(eA(M)) \) is simple. Thus \( r_{Af}(eA(N)) = N \subseteq M \subseteq r_{Af}(eA(M)) \) with both \( r_{Af}(eA(M)) \) simple, so that \( M = r_{Af}(eA(M)) \).

**Lemma 2.5.** Let \((eA, Af)\) be an i-pair in \( A \). Then \( M \in A_r(eA, Af) \) for every submodule \( M \) of \( Af_{Af} \) of finite composition length.

**Proof.** Lemma 2.4(2) together with Lemma 2.2(1) enables us to make use of induction on the composition length.

**Lemma 2.6.** Let \((eA, Af)\) be an i-pair in \( A \). Then \( eAeA \) and \( Af_{Af} \) have the same composition length.

**Proof.** By symmetry, we may assume \( Af_{Af} \) has finite composition length. Let \( 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = Af \) be a composition series of \( Af_{Af} \). Put \( X_i = eAe(M_i) \) for \( 0 \leq i \leq n \). Since by Lemma 2.5 \( M_i \in A_r(eA, Af) \) for all \( 0 \leq i \leq n \), by Lemmas 2.4(1) and 2.2(1) we have a composition series \( 0 = X_n \subseteq \cdots \subseteq X_1 \subseteq X_0 = eA \) of \( eAeA \).
Lemma 2.7. Let \((eA, Af)\) be an i-pair in \(A\). Then the following are equivalent.

1. \(eA_A\) is \(A\)-simple-injective.
2. \(\ell_{\ell}(M) = \ell_{\ell}(N)\) implies \(N = M\) for submodules \(N, M\) of \(Af\) with \(N \subseteq M\).
3. \(M \in A_{\ell}(eA, Af)\) for every submodule \(M\) of \(Af\).

Proof. (1) \(\Rightarrow\) (2). Let \(N, M\) be submodules of \(Af\) with \(N \subseteq M\) and \(M/N \neq 0\). Since \((MA/NA)f \cong M/N \neq 0\), there exist submodules \(K, I\) of \(MA\) such that \(NA \subseteq K \subseteq I\) and \(I/K \cong fA/fJ\). Let \(\mu : I_A \rightarrow A_A\) denote the inclusion. Since we have \(\theta : I_A \rightarrow eA_A\) with \(\text{Im} \theta = \text{soc}(eA_A)\) and \(\text{Ker} \theta = K\), there exists \(\phi : A_A \rightarrow eA_A\) with \(\phi \circ \mu = \theta\). Then \(\phi(1)I = \phi(I) = \theta(I) \neq 0\) and \(\phi(1)K = \phi(K) = \theta(K) = 0\). Thus \(\phi(1) \in \ell_{\ell}(K)\) and \(\phi(1) \notin \ell_{\ell}(I)\). Since \(\ell_{\ell}(M) = \ell_{\ell}(MA) \subseteq \ell_{\ell}(I) \subseteq \ell_{\ell}(K) \subseteq \ell_{\ell}(NA) = \ell_{\ell}(N), \ell_{\ell}(I) \neq \ell_{\ell}(K)\) implies \(\ell_{\ell}(M) \neq \ell_{\ell}(N)\).

(2) \(\Rightarrow\) (3). Let \(M\) be a submodule of \(Af\) and put \(L = r_{\ell}(eA(M))\). Then \(M \subseteq L\) and \(\ell_{\ell}(L) = \ell_{\ell}(r_{\ell}(eA(M))) = \ell_{\ell}(M)\). Thus \(M = L\).

(3) \(\Rightarrow\) (1). Let \(I\) be a nonzero right ideal and \(\mu : I_A \rightarrow A_A\) the inclusion. Let \(\theta : I_A \rightarrow eA_A\) with \(\text{Im} \theta = \text{soc}(eA_A)\) and put \(K = \text{Ker} \theta\). Then by Lemma 1.1 \(If/Kf = (I/K)f\) is simple, so is \(eA_{\ell}eA(Kf)/eA_{\ell}(If)\) by Lemma 2.4(1). Let \(a \in If\) with \(a \notin Kf\). Then, since \(\ell_{\ell}(Kf)a \neq 0\) and \(\ell_{\ell}(If)a = 0, eA_{\ell}eA(Kf)a\) is simple. Thus by Lemma 2.2(2) \(\ell_{\ell}(Kf)a = \text{soc}(eA_A)f, \text{so that} \theta(a) = \theta(af) = \theta(a)f = ba\) with \(b \in \ell_{\ell}(Kf)\). Define \(\phi : A_A \rightarrow eA_A\) by \(1 \mapsto b\). Then, since by Lemmas 2.2(1) and 2.1(2) \(b \in \ell_{\ell}(K)\), and since \(I = K + aA\), we have \(\phi \circ \mu = \theta\).

Lemma 2.8. Let \((eA, Af)\) be an i-pair in \(A\). Assume \(eA_A\) is injective. Then the canonical homomorphism \(eAeA_A \rightarrow eA\text{Hom}_AfAf, a \mapsto (b \mapsto ab)\), is an isomorphism and \(eAf\) is injective.

Proof. By Lemmas 2.2(1) and 1.3.

3. Injective pairs in perfect rings

In this section, we extend results of Baba and Oshiro [1] to left perfect rings. We refer to [2] for perfect rings. We abbreviate the ascending (resp. descending) chain condition as the ACC (resp. DCC).

Remark 4. (1) Let \((eA, Af)\) be an i-pair in \(A\). Then, since \(A_{\ell}(eA, Af)\) is anti-isomorphic to \(A_{\ell}(eA, Af), A_{\ell}(eA, Af)\) satisfies the ACC (resp. DCC) if and only if \(A_{\ell}(eA, Af)\) satisfies the DCC (resp. ACC).

(2) Let \(e \in A\) be an idempotent. Then, since \(eAeAe\) appears as a direct sum-
mand in $e_AeA$, $e_AeA$ is artinian if and only if it has finite composition length.

(3) Every module $L_A$ with soc($L_A$) = 0 is $A$-simple-injective.

**Lemma 3.1** (cf. [1, Proposition 5]). Let $(eA, Af)$ be an i-pair in $A$. Assume
$A_r(eA, Af)$ satisfies the ACC and $fAf$ is a left perfect ring. Then $AfAf$ is artinian
and $M \in A_r(eA, Af)$ for every submodule $M$ of $AfAf$.

Proof. It follows by Lemma 2.5 that there exists a maximal element $M$ in the
set of submodules of $AfAf$ of finite composition length. We claim $M = AfAf$. Otherwise,
there exists a submodule $L$ of $AfAf$ with $M \subset L$ and $L/M$ simple, a
contradiction. Thus $AfAf$ has finite composition length and again by Lemma 2.5 the last
assertion follows.

**Proposition 3.2.** Let $(eA, Af)$ be an i-pair in a left perfect ring $A$. Then the
following are equivalent.

(1) $e_AeA$ is artinian.

(2) $A_r(eA, Af)$ satisfies both the ACC and the DCC.

(3) $A_r(eA, Af)$ satisfies the ACC.

Proof. The implications (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) are obvious.

(3) \(\Rightarrow\) (1). Since the ascending chain $\ell_{eA}(Af) \subset \ell_{eA}(Jf) \subset \ell_{eA}(J^2f) \subset \cdots$ in $A_r(eA, Af)$ terminates, $\ell_{eA}(J^n f) = \ell_{eA}(J^{n+1} f)$ for some $n \geq 0$. We claim $\ell_{eA}(J^n f) = eA$. Suppose otherwise. Then there exists a submodule $M$ of $eA$ with $\ell_{eA}(J^n f) \subset M$ and $M/\ell_{eA}(J^n f)$ simple. Since $MJ \subset \ell_{eA}(J^n f)$, $MJ^{n+1}f \subset \ell_{eA}(J^n f)$, $J^n f = 0$ and $M \subset \ell_{eA}(J^{n+1} f) = \ell_{eA}(J^n f)$, a contradiction. Thus $\ell_{eA}(J^n f) = eA$ and by Lemma 2.2(1) $J^n f \subset rAf(\ell_{eA}(J^n f)) = 0$. Then by Lemma 2.3 $eJ^n = 0$ and $eAe$ is a semiprimary ring. Thus by Lemma 3.1 $e_AeA$ is artinian.

**Lemma 3.3.** Let $e \in A$ be a local idempotent. Assume $eA_A$ is $A$-simple-
injective and has nonzero socle. Then soc($eA_A$) is simple.

Proof. Let $S$ be a simple submodule of soc($eA_A$)$_A$. We claim $S = $ soc($eA_A$).
Suppose otherwise. Let $\pi : \text{soc}(eA_A) \rightarrow S_A$ be a projection and $\mu : \text{soc}(eA_A) \rightarrow eA_A$,
$\nu : S_A \rightarrow eA_A$ inclusions. There exists $\phi : eA_A \rightarrow eA_A$ with $\phi \circ \mu = \nu \circ \pi$. Since $\pi$ is not monic, $\phi$ is not an isomorphism. Thus $\phi(e) \in eJe$ and $(e - \phi(e))$ is a unit in $eAe$. For any $x \in S$, since $\phi(e)x = \phi(x) = \pi(x) = x$, $(e - \phi(e))x = 0$ and thus $x = 0$, a contradiction.

**Lemma 3.4** (cf. [1, Proposition 2]). Let $A$ be a semiperfect ring and $e \in A$ a
local idempotent. Assume $eA_A$ is $A$-simple-injective and has finite Loewy length. Then
Proof. Let $I$ be a nonzero right ideal and $\mu : I_A \to A_A$ the inclusion. Let $\theta : I_A \to eA_A$. We make use of induction on the Loewy length of $\theta(I)$ to show the existence of $\phi : A_A \to eA_A$ with $\theta = \phi \circ \mu$. Let $n = \min\{k \geq 0|\theta(I)J^k = 0\}$. We may assume $n > 0$. Since $eA_A$ has nonzero socle, by Lemma 3.3 $\text{soc}(eA_A)$ is simple and $\text{soc}(eA_A) = \theta(I)J^{n-1} = \theta(I)J^n$. Let $\mu_1$ and $\theta_1$ denote the restrictions of $\mu$ and $\theta$ to $IJ^n$, respectively. Then $\text{Im} \theta_1 = \text{soc}(eA_A)$ and there exists $\phi_1 : A_A \to eA_A$ with $\phi_1 \circ \mu_1 = \theta_1$. Since $(\theta - \phi_1 \circ \mu)(IJ^n) = 0$, by induction hypothesis there exists $\phi_2 : A_A \to eA_A$ with $\phi_2 \circ \mu = \theta - \phi_1 \circ \mu$. Then $\theta = (\phi_1 + \phi_2) \circ \mu$. □

Lemma 3.5 (cf. [1, Proposition 4]). Let $A$ be a semiperfect ring and $e \in A$ a local idempotent. Assume $eA_A$ is $A$-simple-injective and has essential socle. Then there exists a local idempotent $f \in A$ such that $(eA_A, Af)$ is an i-pair in $A$.

Proof. By Lemma 3.3 $S_A = \text{soc}(eA_A)$ is simple. Let $f \in A$ be a local idempotent with $Sf \neq 0$. We claim that $(eA_A, Af)$ is an i-pair in $A$. Let $0 \neq a \in Sf$. It suffices to show $a \in Ab$ for all $0 \neq b \in Af$. Let $0 \neq b \in Af$. Define $\alpha : fA_A \to aA_A$ by $x \mapsto ax$ and $\beta : fA_A \to bA_A$ by $x \mapsto bx$. Since $\text{Ker} \beta = r_{fA}(b) \subset fJ = r_{fA}(a) = \text{Ker} \alpha$, we have $\theta : bA_A \to aA_A = S_A$ with $\alpha = \theta \circ \beta$. Let $\mu : S_A \to eA_A$, $\nu : bA_A \to A_A$ be inclusions. Then there exists $\phi : A_A \to eA_A$ with $\phi \circ \nu = \mu \circ \theta$ and $a = \alpha(f) = \theta(\beta(f)) = \theta(b) = \phi(b) = \phi(1)b \in Ab$. □

Theorem 3.6 (cf. [1, Theorem 1]). Let $A$ be a left perfect ring and $e \in A$ a local idempotent. Then the following are equivalent.

1. $eA_A$ is $A$-simple-injective.
2. There exists a local idempotent $f \in A$ such that $(eA_A, Af)$ is an i-pair in $A$ and $M \in \mathcal{A}_e(eA_A, Af)$ for every submodule $M$ of $Af/Af$.

Proof. By Lemmas 3.5 and 2.7.

Theorem 3.7 (cf. [1, Theorem 2]). Let $(eA_A, Af)$ be an i-pair in a left perfect ring $A$. Then the following are equivalent.

1. $eAi$ is artinian.
2. $Af$ is artinian.
3. Both $eA_A$ and $Af$ are injective.

Proof. (1) $\Leftrightarrow$ (2). By Lemma 2.6.

(2) $\Rightarrow$ (3). By Lemmas 2.6, 2.5 and 2.7 both $eA_A$ and $Af$ are $A$-simple-injective. Also, by Lemma 2.3 both $eA_A$ and $Af$ have finite Loewy length. Thus by Lemma 3.4 both $eA_A$ and $Af$ are injective.
(3) ⇒ (1). By Lemma 2.8 the canonical homomorphism

\[ e_A eAe \rightarrow e_A eA \rightarrow \text{Hom}_{Af}(Af,eAf)A \]

is an isomorphism and \( eAf \) is injective. Similarly, the canonical homomorphism \( eAf \rightarrow \text{Hom}_{eAe}(eA,eAf)A \) is an isomorphism and \( eAeAf \) is injective. It follows that \( eAeAf \) defines a Morita duality. Thus by [7, Theorem 3] \( eAe \) is left artinian and \( eAeA \) has finite Loewy length. Since the canonical homomorphism \( eAeA \rightarrow eAeA \) is an isomorphism, it follows by [7, Lemma 13] that \( eAeA \) has finite composition length.

**Remark 5.** In Theorem 3.7 the assumption that \( A \) is left perfect cannot be replaced by a weaker condition that \( A \) is semiperfect (see [7, Example 1]).

**References**


M. Hoshino
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-0006
Japan

T. Sumioka
Department of Mathematics
Osaka City University
Osaka, 558-8585
Japan