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INJECTIVE PAIRS IN PERFECT RINGS

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Throughout this note, rings are associative rings with identity and modules are unitary modules. Sometimes, we use the notation ${}_AX$ (resp. X_A) to signify that the module X considered is a left (resp. right) A-module. For each pair of subsets X and M of a ring A, we set $\ell_X(M) = \{a \in X | aM = 0\}$ and $r_M(X) = \{a \in M | Xa = 0\}$.

Following Baba and Oshiro [1], we call a pair (eA, Af) of a right ideal eA and a left ideal Af in a ring A an i-pair if (a) e and f are local idempotents; (b) eA_A and ${}_AAf$ have essential socles; and (c) $\mathrm{soc}(eA_A) \cong fA/fJ$ and $\mathrm{soc}({}_AAf) \cong Ae/Je$, where J is the Jacobson radical of A.

Generalizing a result of Fuller [3], Baba and Oshiro [1] showed that for a local idempotent e in a semiprimary ring A, eA_A is injective if and only if there exists a local idempotent f in A such that (eA, Af) is an i-pair in A and $r_{Af}(\ell_{eA}(M)) = M$ for every submodule M of Af_{fAf}, and that for an i-pair (eA, Af) in a semiprimary ring A the following are equivalent: (1) ${}_{eAe}eA$ is artinian; (2) Af_{fAf} is artinian; and (3) both eA_A and ${}_AAf$ are injective.

Our aim is to extend the results mentioned above to perfect rings. Following Harada [4], we call a module L_A M-simple-injective if for any submodule N of M_A every $\theta : N_A \to L_A$ with $\mathrm{Im}\,\theta$ simple can be extended to some $\phi : M_A \to L_A$. For a local idempotent e in a left perfect ring A, we will show that eA_A is A-simple-injective if and only if there exists a local idempotent f in A such that (eA, Af) is an i-pair in A and $r_{Af}(\ell_{eA}(M)) = M$ for every submodule M of Af_{fAf}, and that eA_A is injective if it is A-simple-injective and has finite Loewy length. We will show also that for an i-pair (eA, Af) in a left perfect ring A the following are equivalent: (1) ${}_{eAe}eA$ is artinian; (2) Af_{fAf} is artinian; and (3) both eA_A and ${}_AAf$ are injective.

1. Localization and injective objects

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ covariant functors, and $\varepsilon : \mathbf{1}_\mathcal{A} \to GF$ and $\delta : FG \to \mathbf{1}_\mathcal{B}$ homomorphisms of functors, where $\mathbf{1}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ and $\mathbf{1}_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ are identity functors. We assume the conditions: (a) $\delta_F \circ F\varepsilon = \mathrm{id}_F$; (b) $G\delta \circ \varepsilon_G = \mathrm{id}_G$; (c) F is exact; and (d) δ is an isomorphism.

REMARK 1. (1) By the conditions (a) and (b), for each pair of $X \in \mathrm{Ob}(\mathcal{A})$ and

$M \in \mathrm{Ob}(\mathcal{B})$ we have a natural isomorphism

$$\theta_{X,M} : \mathrm{Hom}_{\mathcal{B}}(FX, M) \to \mathrm{Hom}_{\mathcal{A}}(X, GM), \beta \mapsto G\beta \circ \varepsilon_X$$

with ${\theta_{X,M}}^{-1}(\alpha) = \delta_M \circ F\alpha$ for $\alpha \in \mathrm{Hom}_{\mathcal{A}}(X, GM)$. Namely, G is a right adjoint of F. In particular, G is left exact.

(2) By the conditions (a), (b) and (d), $G : \mathcal{B} \to \mathcal{A}$ is fully faithful.

(3) By the conditions (a) and (d), $F\varepsilon : F \to FGF$ is an isomorphism with $F\varepsilon^{-1} = \delta_F$.

(4) By the conditions (b) and (d), $\varepsilon_G : G \to GFG$ is an isomorphism with ${\varepsilon_G}^{-1} = G\delta$.

Though the following lemmas are well known and more or less obvious, we include proofs for completeness.

Lemma 1.1. *Let $X \in \mathrm{Ob}(\mathcal{A})$ be simple with $FX \neq 0$. Then $FX \in \mathrm{Ob}(\mathcal{B})$ is simple.*

Proof. Let $\beta : FX \to M$ be a nonzero morphism in $\mathcal{B}$. We claim β monic. Note that $\beta = \delta_M \circ F(G\beta \circ \varepsilon_X)$. Thus $G\beta \circ \varepsilon_X : X \to GM$ is nonzero and monic, so is $\beta = \delta_M \circ F(G\beta \circ \varepsilon_X)$. □

Lemma 1.2. *Let $\mu : Y \to X$ be an essential monomorphism in $\mathcal{A}$ with ε_Y monic. Then $F\mu : FY \to FX$ is an essential monomorphism in $\mathcal{B}$.*

Proof. Let $\beta : FX \to M$ be a morphism in $\mathcal{B}$ with $\beta \circ F\mu$ monic. We claim β monic. Since $(G\beta \circ \varepsilon_X) \circ \mu = G\beta \circ GF\mu \circ \varepsilon_Y = G(\beta \circ F\mu) \circ \varepsilon_Y$ is monic, $G\beta \circ \varepsilon_X$ is monic and so is $\beta = \delta_M \circ F(G\beta \circ \varepsilon_X)$. □

Lemma 1.3. *Let $X \in \mathrm{Ob}(\mathcal{A})$ be injective with ε_X monic. Then $\varepsilon_X : X \to GFX$ is an isomorphism and $FX \in \mathrm{Ob}(\mathcal{B})$ is injective.*

Proof. Since $F\varepsilon_X$ is an isomorphism, $F(\mathrm{Cok}\,\varepsilon_X) \cong \mathrm{Cok}\,F\varepsilon_X = 0$ and $\mathrm{Hom}_{\mathcal{A}}(\mathrm{Cok}\,\varepsilon_X, GFX) \cong \mathrm{Hom}_{\mathcal{B}}(F(\mathrm{Cok}\,\varepsilon_X), FX) = 0$. Thus, since $\varepsilon_X : X \to GFX$ is a split monomorphism, $\mathrm{Cok}\,\varepsilon_X = 0$. Hence for each $M \in \mathrm{Ob}(\mathcal{B})$ we have a natural isomorphism

$$\eta_M : \mathrm{Hom}_{\mathcal{B}}(M, FX) \to \mathrm{Hom}_{\mathcal{A}}(GM, X), \beta \mapsto {\varepsilon_X}^{-1} \circ G\beta.$$

Let $\nu : N \to M$ be a monomorphism in $\mathcal{B}$. Since $G\nu$ is monic, $\mathrm{Hom}_{\mathcal{A}}(G\nu, X)$ is epic and so is $\mathrm{Hom}_{\mathcal{B}}(\nu, FX) = {\eta_N}^{-1} \circ \mathrm{Hom}_{\mathcal{A}}(G\nu, X) \circ \eta_M$. □

REMARK 2. (1) An object $M \in \mathrm{Ob}(\mathcal{B})$ is injective if and only if so is $GM \in$

$\mathrm{Ob}(\mathcal{A})$.

(2) The canonical monomorphism $\mathrm{Im}\,\varepsilon_X \to GFX$ is an essential monomorphism for every $X \in \mathrm{Ob}(\mathcal{A})$ with $FX \neq 0$.

(3) If $\nu : N \to M$ is an essential monomorphism in $\mathcal{B}$, so is $G\nu : GN \to GM$.

(4) For $X \in \mathrm{Ob}(\mathcal{A})$ with ε_X monic, a monomorphism $\mu : Y \to X$ in $\mathcal{A}$ is an essential monomorphism if and only if so is $F\mu : FY \to FX$.

2. Injective pairs

Throughout the rest of this note, A stands for a ring with Jacobson radical J. For an i-pair (eA, Af) in A, we denote by $\mathcal{A}_\ell(eA, Af)$ the lattice of submodules X of ${}_{eAe}eA$ with $\ell_{eA}(r_{Af}(X)) = X$ and by $\mathcal{A}_r(eA, Af)$ the lattice of submodules M of Af_{fAf} with $r_{Af}(\ell_{eA}(M)) = M$.

REMARK 3. Let (eA, Af) be an i-pair in A. Let X be a submodule of ${}_{eAe}eA$. Then $Xr_{Af}(X) = 0$ implies $X \subset \ell_{eA}(r_{Af}(X))$ and thus $r_{Af}(\ell_{eA}(r_{Af}(X))) \subset r_{Af}(X)$. Also, $\ell_{eA}(r_{Af}(X))r_{Af}(X) = 0$ implies $r_{Af}(X) \subset r_{Af}(\ell_{eA}(r_{Af}(X)))$. Thus $r_{Af}(X) \in \mathcal{A}_r(eA, Af)$. Similarly, $\ell_{eA}(M) \in \mathcal{A}_\ell(eA, Af)$ for every submodule M of Af_{fAf}. It follows that $\mathcal{A}_\ell(eA, Af)$ is anti-isomorphic to $\mathcal{A}_r(eA, Af)$.

The following lemmas have been established in [5], [3], [1], [8], [6] and so on. However, for the benefit of the reader, we provide direct proofs.

Lemma 2.1. *Let e, $f \in A$ be idempotents and assume $\ell_{eA}(Af) = 0 = r_{Af}(eA)$. Then the following hold.*
(1) *For a two-sided ideal I of A, $eI = 0$ if and only if $If = 0$.*
(2) *$\ell_{eA}(I) = \ell_{eA}(If)$ for every right ideal I of A.*
(3) *$r_{Af}(I) = r_{Af}(eI)$ for every left ideal I of A.*

Proof. (1) Assume $eI = 0$. Then $eAIf = eIf = 0$ and $If \subset r_{Af}(eA) = 0$. By symmetry, $If = 0$ implies $eI = 0$.

(2) Since $If \subset I$, $\ell_{eA}(I) \subset \ell_{eA}(If)$. For any $x \in \ell_{eA}(If)$, since $xIAf = xIf = 0$, $xI \subset \ell_{eA}(Af) = 0$ and $x \in \ell_{eA}(I)$. Thus $\ell_{eA}(If) \subset \ell_{eA}(I)$.

(3) Similar to (2). □

Lemma 2.2. *Let (eA, Af) be an i-pair in A. Then the following hold.*
(1) *$\ell_{eA}(Af) = 0 = r_{Af}(eA)$.*
(2) *eAf_{fAf} and ${}_{eAe}eAf$ have simple essential socles and $\mathrm{soc}(eA_A)f = \mathrm{soc}(eAf_{fAf}) = \mathrm{soc}({}_{eAe}eAf) = e(\mathrm{soc}({}_AAf))$.*

Proof. (1) For any $0 \neq x \in eA$, since $\mathrm{soc}(eA_A) \subset xA$, $0 \neq \mathrm{soc}(eA_A)f \subset xAf$ and $x \notin \ell_{eA}(Af)$. Thus $\ell_{eA}(Af) = 0$. Similarly $r_{Af}(eA) = 0$.

(2) Since by Lemma 1.1 $\mathrm{soc}(eA_A)f_{fAf}$ and ${}_{eAe}e(\mathrm{soc}({}_AAf))$ are simple, and since by Lemma 1.2 $\mathrm{soc}(eA_A)f_{fAf} \subset eAf_{fAf}$ and ${}_{eAe}e(\mathrm{soc}({}_AAf)) \subset {}_{eAe}eAf$ are essential extensions, the assertion follows. $\square$

Lemma 2.3. *Let (eA, Af) be an i-pair in A. Then for any $n \geq 1$ $eJ^n = 0$ if and only if $J^n f = 0$, so that eA_A and ${}_AAf$ have the same Loewy length.*

Proof. By Lemmas 2.2(1) and 2.1(1). $\square$

Lemma 2.4. *Let (eA, Af) be an i-pair in A. Let N, M be submodules of Af_{fAf} with $N \subset M$ and M/N simple. Assume $N \in \mathcal{A}_r(eA, Af)$. Then the following hold.*

(1) *${}_{eAe}\ell_{eA}(N)/\ell_{eA}(M)$ is simple.*

(2) *$M \in \mathcal{A}_r(eA, Af)$.*

Proof. (1) Let $a \in M$ with $a \notin N$. Then $M = N + afAf$. Also, since $M \neq N = r_{Af}(\ell_{eA}(N))$, $\ell_{eA}(M) \subset \ell_{eA}(N)$ with $\ell_{eA}(N)/\ell_{eA}(M) \neq 0$. Since $0 \neq \ell_{eA}(N)M = \ell_{eA}(N)afAf$ and $\ell_{eA}(N)afJf = 0$, $\ell_{eA}(N)afAf = \mathrm{soc}(eAf_{fAf})$. Thus by Lemma 2.2(2) $\ell_{eA}(N)a = \mathrm{soc}({}_{eAe}eAf)$ and, since $\ell_{eA}(M)a = 0$, ${}_{eAe}\ell_{eA}(N)/\ell_{eA}(M) \cong \mathrm{soc}({}_{eAe}eAf)$.

(2) Since $\ell_{eA}(M) \subset \ell_{eA}(N) \subset {}_{eAe}eA$ with $\ell_{eA}(M) \in \mathcal{A}_\ell(eA, Af)$ and $\ell_{eA}(N)/\ell_{eA}(M)$ simple, we can apply the part (1) to conclude that $r_{Af}(\ell_{eA}(M))/r_{Af}(\ell_{eA}(N))$ is simple. Thus $r_{Af}(\ell_{eA}(N)) = N \subset M \subset r_{Af}(\ell_{eA}(M))$ with both $r_{Af}(\ell_{eA}(M))/r_{Af}(\ell_{eA}(N))$ and M/N simple, so that $M = r_{Af}(\ell_{eA}(M))$. $\square$

Lemma 2.5. *Let (eA, Af) be an i-pair in A. Then $M \in \mathcal{A}_r(eA, Af)$ for every submodule M of Af_{fAf} of finite composition length.*

Proof. Lemma 2.4(2) together with Lemma 2.2(1) enables us to make use of induction on the composition length. $\square$

Lemma 2.6. *Let (eA, Af) be an i-pair in A. Then ${}_{eAe}eA$ and Af_{fAf} have the same composition length.*

Proof. By symmetry, we may assume Af_{fAf} has finite composition length. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_n = Af$ be a composition series of Af_{fAf}. Put $X_i = \ell_{eA}(M_i)$ for $0 \leq i \leq n$. Since by Lemma 2.5 $M_i \in \mathcal{A}_r(eA, Af)$ for all $0 \leq i \leq n$, by Lemmas 2.4(1) and 2.2(1) we have a composition series $0 = X_n \subset \cdots \subset X_1 \subset X_0 = eA$ of ${}_{eAe}eA$. $\square$

Lemma 2.7. *Let (eA, Af) be an i-pair in A. Then the following are equivalent.*

(1) *eA_A is A-simple-injective.*

(2) *$\ell_{eA}(M) = \ell_{eA}(N)$ implies $N = M$ for submodules N, M of Af_{fAf} with $N \subset M$.*

(3) *$M \in \mathcal{A}_r(eA, Af)$ for every submodule M of Af_{fAf}.*

Proof. (1) $\Rightarrow$ (2). Let N, M be submodules of Af_{fAf} with $N \subset M$ and $M/N \neq 0$. Since $(MA/NA)f \cong M/N \neq 0$, there exist submodules K, I of MA_A such that $NA \subset K \subset I$ and $I/K \cong fA/fJ$. Let $\mu : I_A \to A_A$ denote the inclusion. Since we have $\theta : I_A \to eA_A$ with $\operatorname{Im}\theta = \operatorname{soc}(eA_A)$ and $\operatorname{Ker}\theta = K$, there exists $\phi : A_A \to eA_A$ with $\phi \circ \mu = \theta$. Then $\phi(1)I = \phi(I) = \theta(I) \neq 0$ and $\phi(1)K = \phi(K) = \theta(K) = 0$. Thus $\phi(1) \in \ell_{eA}(K)$ and $\phi(1) \notin \ell_{eA}(I)$. Since $\ell_{eA}(M) = \ell_{eA}(MA) \subset \ell_{eA}(I) \subset \ell_{eA}(K) \subset \ell_{eA}(NA) = \ell_{eA}(N)$, $\ell_{eA}(I) \neq \ell_{eA}(K)$ implies $\ell_{eA}(M) \neq \ell_{eA}(N)$.

(2) $\Rightarrow$ (3). Let M be a submodule of Af_{fAf} and put $L = r_{Af}(\ell_{eA}(M))$. Then $M \subset L$ and $\ell_{eA}(L) = \ell_{eA}(r_{Af}(\ell_{eA}(M))) = \ell_{eA}(M)$. Thus $M = L$.

(3) $\Rightarrow$ (1). Let I be a nonzero right ideal and $\mu : I_A \to A_A$ the inclusion. Let $\theta : I_A \to eA_A$ with $\operatorname{Im}\theta = \operatorname{soc}(eA_A)$ and put $K = \operatorname{Ker}\theta$. Then by Lemma 1.1 $If/Kf_{fAf} \cong (I/K)f_{fAf}$ is simple, so is ${}_{eAe}\ell_{eA}(Kf)/\ell_{eA}(If)$ by Lemma 2.4(1). Let $a \in If$ with $a \notin Kf$. Then, since $\ell_{eA}(Kf)a \neq 0$ and $\ell_{eA}(If)a = 0$, ${}_{eAe}\ell_{eA}(Kf)a$ is simple. Thus by Lemma 2.2(2) $\ell_{eA}(Kf)a = \operatorname{soc}(eA_A)f$, so that $\theta(a) = \theta(af) = \theta(a)f = ba$ with $b \in \ell_{eA}(Kf)$. Define $\phi : A_A \to eA_A$ by $1 \mapsto b$. Then, since by Lemmas 2.2(1) and 2.1(2) $b \in \ell_{eA}(K)$, and since $I = K + aA$, we have $\phi \circ \mu = \theta$. □

Lemma 2.8. *Let (eA, Af) be an i-pair in A. Assume eA_A is injective. Then the canonical homomorphism ${}_{eAe}eA_A \to {}_{eAe}\operatorname{Hom}_{fAf}(Af, eAf)_A$, $a \mapsto (b \mapsto ab)$, is an isomorphism and eAf_{fAf} is injective.*

Proof. By Lemmas 2.2(1) and 1.3. □

3. Injective pairs in perfect rings

In this section, we extend results of Baba and Oshiro [1] to left perfect rings. We refer to [2] for perfect rings. We abbreviate the ascending (resp. descending) chain condition as the ACC (resp. DCC).

REMARK 4. (1) Let (eA, Af) be an i-pair in A. Then, since $\mathcal{A}_\ell(eA, Af)$ is anti-isomorphic to $\mathcal{A}_r(eA, Af)$, $\mathcal{A}_\ell(eA, Af)$ satisfies the ACC (resp. DCC) if and only if $\mathcal{A}_r(eA, Af)$ satisfies the DCC (resp. ACC).

(2) Let $e \in A$ be an idempotent. Then, since ${}_{eAe}eAe$ appears as a direct sum-

mand in ${}_{eAe}eA$, ${}_{eAe}eA$ is artinian if and only if it has finite composition length.

(3) Every module L_A with $\mathrm{soc}(L_A) = 0$ is A-simple-injective.

Lemma 3.1 (cf. [1, Proposition 5]). *Let (eA, Af) be an i-pair in A. Assume $\mathcal{A}_r(eA, Af)$ satisfies the ACC and fAf is a left perfect ring. Then Af_{fAf} is artinian and $M \in \mathcal{A}_r(eA, Af)$ for every submodule M of Af_{fAf}.*

Proof. It follows by Lemma 2.5 that there exists a maximal element M in the set of submodules of Af_{fAf} of finite composition length. We claim $M = Af_{fAf}$. Otherwise, there exists a submodule L of Af_{fAf} with $M \subset L$ and L/M simple, a contradiction. Thus Af_{fAf} has finite composition length and again by Lemma 2.5 the last assertion follows. □

Proposition 3.2. *Let (eA, Af) be an i-pair in a left perfect ring A. Then the following are equivalent.*

(1) *${}_{eAe}eA$ is artinian.*

(2) *$\mathcal{A}_\ell(eA, Af)$ satisfies both the ACC and the DCC.*

(3) *$\mathcal{A}_\ell(eA, Af)$ satisfies the ACC.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Since the ascending chain $\ell_{eA}(Af) \subset \ell_{eA}(Jf) \subset \ell_{eA}(J^2 f) \subset \cdots$ in $\mathcal{A}_\ell(eA, Af)$ terminates, $\ell_{eA}(J^n f) = \ell_{eA}(J^{n+1} f)$ for some $n \geq 0$. We claim $\ell_{eA}(J^n f) = eA$. Suppose otherwise. Then there exists a submodule M of eA_A with $\ell_{eA}(J^n f) \subset M$ and $M/\ell_{eA}(J^n f)$ simple. Since $MJ \subset \ell_{eA}(J^n f)$, $MJ^{n+1} f \subset \ell_{eA}(J^n f) J^n f = 0$ and $M \subset \ell_{eA}(J^{n+1} f) = \ell_{eA}(J^n f)$, a contradiction. Thus $\ell_{eA}(J^n f) = eA$ and by Lemma 2.2(1) $J^n f \subset r_{Af}(\ell_{eA}(J^n f)) = 0$. Then by Lemma 2.3 $eJ^n = 0$ and eAe is a semiprimary ring. Thus by Lemma 3.1 ${}_{eAe}eA$ is artinian. □

Lemma 3.3. *Let $e \in A$ be a local idempotent. Assume eA_A is A-simple-injective and has nonzero socle. Then $\mathrm{soc}(eA_A)$ is simple.*

Proof. Let S be a simple submodule of $\mathrm{soc}(eA_A)_A$. We claim $S = \mathrm{soc}(eA_A)$. Suppose otherwise. Let $\pi : \mathrm{soc}(eA_A) \to S_A$ be a projection and $\mu : \mathrm{soc}(eA_A) \to eA_A$, $\nu : S_A \to eA_A$ inclusions. There exists $\phi : eA_A \to eA_A$ with $\phi \circ \mu = \nu \circ \pi$. Since π is not monic, ϕ is not an isomorphism. Thus $\phi(e) \in eJe$ and $(e - \phi(e))$ is a unit in eAe. For any $x \in S$, since $\phi(e)x = \phi(x) = \pi(x) = x$, $(e - \phi(e))x = 0$ and thus $x = 0$, a contradiction. □

Lemma 3.4 (cf. [1, Proposition 2]). *Let A be a semiperfect ring and $e \in A$ a local idempotent. Assume eA_A is A-simple-injective and has finite Loewy length. Then*

eA_A is injective.

Proof. Let I be a nonzero right ideal and $\mu : I_A \to A_A$ the inclusion. Let $\theta : I_A \to eA_A$. We make use of induction on the Loewy length of $\theta(I)$ to show the existence of $\phi : A_A \to eA_A$ with $\theta = \phi \circ \mu$. Let $n = \min\{k \geq 0 | \theta(I)J^k = 0\}$. We may assume $n > 0$. Since eA_A has nonzero socle, by Lemma 3.3 $\mathrm{soc}(eA_A)$ is simple and $\mathrm{soc}(eA_A) = \theta(I)J^{n-1} = \theta(IJ^{n-1})$. Let μ_1 and θ_1 denote the restrictions of μ and θ to IJ^{n-1}, respectively. Then $\mathrm{Im}\,\theta_1 = \mathrm{soc}(eA_A)$ and there exists $\phi_1 : A_A \to eA_A$ with $\phi_1 \circ \mu_1 = \theta_1$. Since $(\theta - \phi_1 \circ \mu)(I)J^{n-1} = 0$, by induction hypothesis there exists $\phi_2 : A_A \to eA_A$ with $\phi_2 \circ \mu = \theta - \phi_1 \circ \mu$. Then $\theta = (\phi_1 + \phi_2) \circ \mu$. □

Lemma 3.5 (cf. [1, Proposition 4]). *Let A be a semiperfect ring and $e \in A$ a local idempotent. Assume eA_A is A-simple-injective and has essential socle. Then there exists a local idempotent $f \in A$ such that (eA, Af) is an i-pair in A.*

Proof. By Lemma 3.3 $S_A = \mathrm{soc}(eA_A)$ is simple. Let $f \in A$ be a local idempotent with $Sf \neq 0$. We claim that (eA, Af) is an i-pair in A. Let $0 \neq a \in Sf$. It suffices to show $a \in Ab$ for all $0 \neq b \in Af$. Let $0 \neq b \in Af$. Define $\alpha : fA_A \to aA_A$ by $x \mapsto ax$ and $\beta : fA_A \to bA_A$ by $x \mapsto bx$. Since $\mathrm{Ker}\,\beta = r_{fA}(b) \subset fJ = r_{fA}(a) = \mathrm{Ker}\,\alpha$, we have $\theta : bA_A \to aA_A = S_A$ with $\alpha = \theta \circ \beta$. Let $\mu : S_A \to eA_A$, $\nu : bA_A \to A_A$ be inclusions. Then there exists $\phi : A_A \to eA_A$ with $\phi \circ \nu = \mu \circ \theta$ and $a = \alpha(f) = \theta(\beta(f)) = \theta(b) = \phi(b) = \phi(1)b \in Ab$. □

Theorem 3.6 (cf. [1, Theorem 1]). *Let A be a left perfect ring and $e \in A$ a local idempotent. Then the following are equivalent.*
(1) *eA_A is A-simple-injective.*
(2) *There exists a local idempotent $f \in A$ such that (eA, Af) is an i-pair in A and $M \in \mathcal{A}_r(eA, Af)$ for every submodule M of Af_{fAf}.*

Proof. By Lemmas 3.5 and 2.7. □

Theorem 3.7 (cf. [1, Theorem 2]). *Let (eA, Af) be an i-pair in a left perfect ring A. Then the following are equivalent.*
(1) *${}_{eAe}eA$ is artinian.*
(2) *Af_{fAf} is artinian.*
(3) *Both eA_A and ${}_AAf$ are injective.*

Proof. (1) ⇔ (2). By Lemma 2.6.
(2) ⇒ (3). By Lemmas 2.6, 2.5 and 2.7 both eA_A and ${}_AAf$ are A-simple-injective. Also, by Lemma 2.3 both eA_A and ${}_AAf$ have finite Loewy length. Thus by Lemma 3.4 both eA_A and ${}_AAf$ are injective.

(3) $\Rightarrow$ (1). By Lemma 2.8 the canonical homomorphism

$$ {}_{eAe}eA_A \to {}_{eAe}\mathrm{Hom}_{fAf}(Af, eAf)_A $$

is an isomorphism and eAf_{fAf} is injective. Similarly, the canonical homomorphism ${}_AAf_{fAf} \to {}_A\mathrm{Hom}_{eAe}(eA, eAf)_{fAf}$ is an isomorphism and ${}_{eAe}eAf$ is injective. It follows that ${}_{eAe}eAf_{fAf}$ defines a Morita duality. Thus by [7, Theorem 3] eAe is left artinian and ${}_{eAe}eA$ has finite Loewy length. Since the canonical homomorphism ${}_{eAe}eA \to {}_{eAe}\mathrm{Hom}_{fAf}(\mathrm{Hom}_{eAe}(eA, eAf), eAf)$ is an isomorphism, it follows by [7, Lemma 13] that ${}_{eAe}eA$ has finite composition length. $\square$

Remark 5. In Theorem 3.7 the assumption that A is left perfect cannot be replaced by a weaker condition that A is semiperfect (see [7, Example 1]).

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