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<thead>
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<th>Title</th>
<th>On the Euler characteristic of the orbit space of a proper $\Gamma$-complex</th>
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</thead>
<tbody>
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Osaka University
ON THE EULER CHARACTERISTIC OF
THE ORBIT SPACE OF A PROPER $\Gamma$-COMPLEX

Dedicated to Professor Fuichi Uchida on his 60th birthday

TOSHIYUKI AKITA *

(Received November 17, 1997)

1. Introduction

Let $\Gamma$ be a discrete groups. A $\Gamma$-CW-complex $X$ is said to be $\Gamma$-finite if it satisfies the following two conditions:

1. For each cell $\sigma$ of $X$, the isotropy subgroup $\Gamma_\sigma$ of $\sigma$ is of finite order.
2. The orbit space $X/\Gamma$ is a finite complex.

In other words, $X$ is $\Gamma$-finite if the action of $\Gamma$ on $X$ is proper and cocompact. For a $\Gamma$-finite $\Gamma$-CW-complex $X$, define the equivariant Euler characteristic $e(\Gamma, X)$ by

$$e(\Gamma, X) = \sum_{\sigma \in \mathcal{E}} (-1)^{\dim \sigma} \frac{1}{|\Gamma_\sigma|} \in \mathbb{Q},$$

where $\mathcal{E}$ is a set of representatives of $\Gamma$-orbits of cells of $X$ and $|\Gamma_\sigma|$ is the order of $\Gamma_\sigma$. We agree $e(\Gamma, X) = 0$ when $X = \emptyset$. The equivariant Euler characteristic and its variants appear in various contexts of mathematics. See [3, 4, 9, 11] for instance. In particular, when $X$ is a manifold, the orbit space $X/\Gamma$ can be regarded as an orbifold and $e(\Gamma, X)$ is the orbifold Euler characteristic of $X/\Gamma$ in the sense of [11].

In this paper, we prove the formula expressing the Euler characteristic of the orbit space of a $\Gamma$-finite $\Gamma$-CW-complex in terms of equivariant Euler characteristics. More precisely, let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. For each $\gamma \in \Gamma$, the centralizer $C_\Gamma(\gamma)$ acts on the fixed point set $X^\gamma$. In this way $X^\gamma$ is naturally a $C_\Gamma(\gamma)$-finite $C_\Gamma(\gamma)$-CW-complex and hence $e(C_\Gamma(\gamma), X^\gamma)$ is defined for each $\gamma \in \Gamma$. Our result is:

Theorem 1. Let $\Gamma$ be a discrete group, and $X$ a $\Gamma$-finite $\Gamma$-CW-complex. Then

$$\chi(X/\Gamma) = \sum_{\gamma \in \mathcal{F}(\Gamma)} e(C_\Gamma(\gamma), X^\gamma),$$

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where $\mathcal{F}(\Gamma)$ is a set of representatives of conjugacy classes of elements of finite order in $\Gamma$.

Note that there are finitely many $\gamma$'s in $\mathcal{F}(\Gamma)$ with $X^\gamma \neq \emptyset$ and hence the summation in (1) makes sense.

In case a $\Gamma$-finite $\Gamma$-CW-complex $X$ is a smooth $\Gamma$-manifold, then $X^\gamma$ is a submanifold of $X$ for each $\gamma \in \Gamma$ of finite order, so that the terms $e(C_\Gamma(\gamma), X^\gamma)$ in (1) can be regarded as orbifold Euler characteristics of orbifolds $X^\gamma/C_\Gamma(\gamma)$. Thus Theorem 1 gives the expression of the Euler characteristic of $X/\Gamma$ in terms of orbifold Euler characteristics of $X^\gamma/C_\Gamma(\gamma)$.

When $\Gamma$ is a finite group, a $\Gamma$-finite $\Gamma$-CW-complex is simply a finite $\Gamma$-CW-complex, and Theorem 1 implies

\begin{equation}
\chi(X/\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(X^\gamma).
\end{equation}

Thus Theorem 1 is a generalization of the well-known equation (2) for finite group actions. For the direct proof of the equation (2), see [6, p. 225].

If $\Gamma$ is virtually torsion-free and $X$ is a $\Gamma$-finite $\Gamma$-CW-complex such that $X^\gamma$ is nonempty and $\mathbb{Q}$-acyclic for every element $\gamma \in \Gamma$ of finite order, then $e(C_\Gamma(\gamma), X^\gamma)$ coincides with the Euler characteristic $\chi(C_\Gamma(\gamma))$ of the group $C_\Gamma(\gamma)$ for every $\gamma \in \Gamma$ of finite order, and Theorem 1 reduces to the following formula due to K. S. Brown (cf. [5, p. 261]):

\[ \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, \mathbb{Q}) = \sum_{\gamma \in \mathcal{F}(\Gamma)} \chi(C_\Gamma(\gamma)). \]

The rest of this paper is organized as follows. In §2, we introduce Hattori-Stallings ranks of finitely generated projective $\mathbb{Q}\Gamma$-modules, where $\mathbb{Q}\Gamma$ denotes the rational group algebra of $\Gamma$.

When $X$ is a $\Gamma$-finite $\Gamma$-CW-complex, its cellular chain groups $C_i(X, \mathbb{Q})$ are finitely generated projective $\mathbb{Q}\Gamma$-modules. In §3, we will see that $e(C_\Gamma(\gamma), X^\gamma)$ can be expressed in terms of Hattori-Stallings ranks of cellular chain groups.

In §4, we will prove Theorem 1. The proof is done by the spectral sequence which converges to the homology of $\Gamma$ with coefficients in the cellular chain complex $C_\ast(X, \mathbb{Q})$, together with properties of Hattori-Stallings ranks which will be discussed in §2 and §3.

In the final section §5, we will consider the two special cases where (i) $\Gamma$ is a finite group or (ii) $\Gamma$ is virtually torsion-free, and $X^\gamma$ is nonempty and $\mathbb{Q}$-acyclic for every $\gamma \in \Gamma$ of finite order, both of which are mentioned above.

Throughout this paper, we employ the following conventions unless otherwise stated: $\Gamma$ is a discrete group and $\mathbb{Q}\Gamma$ is its rational group algebra. A module over $\mathbb{Q}\Gamma$ is understood to be a left $\mathbb{Q}\Gamma$-module.
2. The Hattori-Stallings rank

The Hattori-Stallings rank was introduced by A. Hattori [8] and J. Stallings [10] and was studied by H. Bass in detail [1]. The reader should refer to [2] and [5, Chapter IX] for further detail of the Hattori-Stallings rank.

Let $Q\Gamma$ be the rational group algebra of $\Gamma$. Define $[Q\Gamma, Q\Gamma]$ be the additive subgroup of $Q\Gamma$ generated by $\alpha\beta - \beta\alpha$ $(\alpha, \beta \in Q\Gamma)$. Set $T(Q\Gamma) = Q\Gamma/[Q\Gamma, Q\Gamma]$. It is easy to see that $T(Q\Gamma)$ is isomorphic to the $Q$-linear space spanned by the set of conjugacy classes of elements of $\Gamma$, i.e.,

$$T(Q\Gamma) \cong \bigoplus_{\gamma \in C(\Gamma)} Q \cdot (\gamma),$$

where $C(\Gamma)$ is a set of representatives of conjugacy classes of elements of $\Gamma$ and $(\gamma)$ is the conjugacy class of $\gamma$.

Let $\pi : Q\Gamma \rightarrow T(Q\Gamma)$ be the natural projection. Under the identification (3), $\pi$ assigns $1 \cdot (\gamma) \in T(Q\Gamma)$ to $\gamma \in Q\Gamma$.

Let $P$ be a finitely generated projective $Q\Gamma$-module. Then $P$ is a direct summand of a finitely generated free $Q\Gamma$-module $F$. Choose such $F$ and let $p : F \rightarrow P$ be the natural projection and $i : P \hookrightarrow F$ the inclusion. By fixing a basis of $F$, the composite $i \circ p$ can be identified with a square matrix $M$ over $Q\Gamma$.

**DEFINITION.** Under these assumptions, the **Hattori-Stallings rank** $r_\Gamma(P)$ of a finitely generated projective $Q\Gamma$-module $P$ is defined by

$$r_\Gamma(P) = \pi(trM) \in T(Q\Gamma),$$

where $trM$ is the trace of $M$.

Note that $r_\Gamma(P)$ is well-defined, i.e., it is independent of various choices made. Under the identification (3), denote by $r_\Gamma(P)(\gamma)$ the coefficient of $(\gamma)$ in $r_\Gamma(P)$.

We recall some properties of the Hattori-Stallings rank, which will be used later.

**Proposition 2** (cf. [1, §2 (2.5)]). If $P_1$ and $P_2$ are finitely generated projective $Q\Gamma$-modules, then

$$r_\Gamma(P_1 \oplus P_2) = r_\Gamma(P_1) + r_\Gamma(P_2).$$

**Proposition 3** (cf. [1, §6 (6.3)]). Let $\Gamma'$ be a subgroup of finite index of $\Gamma$, $P$ a finitely generated projective $Q\Gamma'$-module. Regarding $P$ as a finitely generated projective $Q\Gamma'$-modules by the restriction of scalars, one has

$$r_{\Gamma'}(P)(\gamma) = (C_\Gamma(\gamma), C_{\Gamma'}(\gamma)) \cdot r_\Gamma(P)(\gamma)$$
for all $\gamma \in \Gamma'$, where $(C_{\Gamma}(\gamma), C_{\Gamma'}(\gamma))$ is the index of $C_{\Gamma'}(\gamma)$ in $C_{\Gamma}(\gamma)$.

Let $f : \Gamma_1 \to \Gamma_2$ be a group homomorphism. Let $P$ be a finitely generated projective $\mathbb{Q}\Gamma_1$-module. Then $\mathbb{Q}\Gamma_2 \otimes_{\mathbb{Q}\Gamma_1} P$ is a finitely generated projective $\mathbb{Q}\Gamma_2$-modules, where $\mathbb{Q}\Gamma_2$ is regarded as a right $\mathbb{Q}\Gamma_1$-module via $f$. Let $T(f) : T(\mathbb{Q}\Gamma_1) \to T(\mathbb{Q}\Gamma_2)$ be the homomorphism induced by $f$.

**Proposition 4** (cf. [1, §2 (2.9)]). **Under these assumptions, one has**

$$r_{\Gamma_2}(\mathbb{Q}\Gamma_2 \otimes_{\mathbb{Q}\Gamma_1} P) = T(f)(r_{\Gamma_1}(P)).$$

In case $\Gamma$ is a finite group, Hattori-Stallings ranks can be determined by the character:

**Proposition 5** (cf. [1, §5 (5.8)]). **Let $\Gamma$ be a finite group. Let $V$ be a $\mathbb{Q}\Gamma$-module which is finite dimensional over $\mathbb{Q}$. Then $V$ is finitely generated and projective, and one has**

$$r_{\Gamma}(V)(\gamma) = \frac{\chi(\gamma^{-1})}{|C_{\Gamma}(\gamma)|},$$

where $\chi : \Gamma \to \mathbb{Q}$ is the character of $V$.

### 3. Hattori-Stallings ranks and equivariant Euler characteristics

Now we consider the equivariant Euler characteristic $e(\Gamma, X)$. First we invoke the following elementary lemma, which may be well-known.

**Lemma 6.** **Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Then its cellular chain group $C_i(X, \mathbb{Q})$ is a finitely generated projective $\mathbb{Q}\Gamma$-module.**

**Proof.** $C_i(X, \mathbb{Q})$ has a direct sum decomposition as a $\mathbb{Q}\Gamma$-module:

$$C_i(X, \mathbb{Q}) \cong \bigoplus_{\sigma} \mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_{\sigma}} \mathbb{Q},$$

where $\sigma$ ranges representatives of $\Gamma$-orbits of $i$-cells of $X$, $\Gamma_{\sigma}$ is the isotropy subgroup of $\sigma$, $\mathbb{Q}$ is regarded as a left $\mathbb{Q}\Gamma_{\sigma}$-module with the trivial action of $\Gamma_{\sigma}$, and $\mathbb{Q}\Gamma$ is regarded naturally as a right $\mathbb{Q}\Gamma_{\sigma}$-module. Since $X$ is $\Gamma$-finite, each $\Gamma_{\sigma}$ is a finite subgroup of $\Gamma$, which implies that $\mathbb{Q}$ is always finitely generated projective $\mathbb{Q}\Gamma_{\sigma}$-module (cf. Proposition 5). Thus $\mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_{\sigma}} \mathbb{Q}$ is a finitely generated projective $\mathbb{Q}\Gamma$-module. As the number of $\Gamma$-orbits of cells of $X$ is finite, so is the number of direct summands in (4), which yields the lemma.

By Lemma 6, the Hattori-Stallings rank of $C_i(X, \mathbb{Q})$ can be defined.
Lemma 7. Let $X$ be a $\Gamma$-CW-complex. Then

$$r_\Gamma(C_i(X, \mathbb{Q}))(1) = \sum_{\sigma} \frac{1}{|\Gamma_\sigma|},$$

where $\sigma$ ranges representatives of $\Gamma$-orbits of $i$-cells of $X$.

Proof. We have

$$r_\Gamma(C_i(X, \mathbb{Q}))(1) = r_\Gamma \left( \bigoplus_{\sigma} \mathbb{Q}\Gamma \otimes_{\mathbb{Q} \Gamma_{\sigma}} \mathbb{Q} \right)(1) \quad \text{by (4)}$$

$$= \sum_{\sigma} r_\Gamma(\mathbb{Q}\Gamma \otimes_{\mathbb{Q} \Gamma_{\sigma}} \mathbb{Q})(1) \quad \text{by Proposition 2}$$

$$= \sum_{\sigma} T(i)(r_{\Gamma_\sigma}(\mathbb{Q}))(1) \quad \text{by Proposition 4},$$

where $\sigma$ ranges representatives of $\Gamma$-orbits of $i$-cells of $X$ and $T(i) : T(\mathbb{Q}\Gamma_{\sigma}) \to T(\mathbb{Q}\Gamma)$ is the map induced by the inclusion $i : \Gamma_\sigma \hookrightarrow \Gamma$. From Proposition 5 we conclude

$$T(i)(r_{\Gamma_\sigma}(\mathbb{Q}))(1) = r_{\Gamma_\sigma}(\mathbb{Q})(1) = \frac{1}{|\Gamma_\sigma|},$$

proving the lemma. \hfill \Box

By virtue of Lemma 7, we have

$$(5) \quad e(\Gamma, X) = \sum_i (-1)^i r_\Gamma(C_i(X, \mathbb{Q}))(1).$$

Together with the result of K. S. Brown [4], we obtain the relation between the Hattori-Stallings rank of $C_i(X, \mathbb{Q})$ and $e(C_\Gamma(\gamma), X^\gamma)$ as follows:

**Proposition 8.** Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Then

$$(6) \quad e(C_\Gamma(\gamma), X^\gamma) = \sum_i (-1)^i r_\Gamma(C_i(X, \mathbb{Q}))(\gamma)$$

for every $\gamma \in \Gamma$.

Proof. A direct consequence of the equality (5) and [4, Theorem 3.1 (iii)]. \hfill \Box
4. **Proof of Theorem 1**

Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Let $H_\ast(\Gamma, C_\ast(X, \mathbb{Q}))$ be the homology of $\Gamma$ with coefficients in the cellular chain complex $C_\ast(X, \mathbb{Q})$, which is isomorphic to the Borel homology (equivariant homology) $H^\Gamma_\ast(X, \mathbb{Q})$ (cf. [5, Chapter VII]). Since the isotropy subgroup of every cell of $X$ is finite, the Borel homology of $X$ is isomorphic to the rational homology of the orbit space:

$$H_\ast(\Gamma, C_\ast(X, \mathbb{Q})) \cong H^\Gamma_\ast(X, \mathbb{Q}) \cong H_\ast(X/\Gamma, \mathbb{Q}).$$

**Lemma 9.** Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Then

$$\sum_i (-1)^i \dim_\mathbb{Q} H_i(\Gamma, C_\ast(X, \mathbb{Q})) = \sum_i (-1)^i \dim_\mathbb{Q} Q \otimes_{\mathbb{Q} \Gamma} C_i(X, \mathbb{Q}).$$

**Proof.** Consider the spectral sequence

$$E^1_{i,j} = H_j(\Gamma, C_i(X, \mathbb{Q})) \Rightarrow H_{i+j}(\Gamma, C_\ast(X, \mathbb{Q}))$$

(cf. [5, §VII.5 and §VII.7]). Since $C_i(X, \mathbb{Q})$ is a projective $Q\Gamma$-module for all $i$, we have

$$E^1_{i,j} \cong \begin{cases} Q \otimes_{\mathbb{Q} \Gamma} C_i(X, \mathbb{Q}) & j = 0 \\ 0 & j \neq 0. \end{cases}$$

As $\dim_\mathbb{Q} Q \otimes_{\mathbb{Q} \Gamma} C_i(X, \mathbb{Q}) < \infty$ for all $i$, we obtain the desired equation.

Now we prove Theorem 1. By Proposition 4 (take $\Gamma_2$ to be the trivial subgroup),

$$\dim_\mathbb{Q}(Q \otimes_{\mathbb{Q} \Gamma} C_i(X, \mathbb{Q})) = \sum_{\gamma \in C(\Gamma)} r_T(C_i(X, \mathbb{Q}))(\gamma).$$

Hence

$$\chi(X/\Gamma) = \sum_i (-1)^i \dim_\mathbb{Q} H_i(\Gamma, C_\ast(X, \mathbb{Q})) \quad \text{by (7)}$$

$$= \sum_i (-1)^i \dim_\mathbb{Q} Q \otimes_{\mathbb{Q} \Gamma} C_i(X, \mathbb{Q}) \quad \text{by Lemma 9}$$

$$= \sum_i (-1)^i \left( \sum_{\gamma \in C(\Gamma)} r_T(C_i(X, \mathbb{Q}))(\gamma) \right) \quad \text{by (8)}$$
For an element $\gamma$ of infinite order, we have $X^\gamma = \emptyset$ and hence $e(C_\Gamma(\gamma), X^\gamma) = 0$, which proves Theorem 1.

5. Remarks

5.1. Finite group actions

Suppose that $\Gamma$ is a finite group. Let $X$ be a finite $\Gamma$-complex. By Proposition 5, we have

$$e(C_\Gamma(\gamma), X^\gamma) = \sum_i (-1)^i r_{C_\Gamma(\gamma)}(C_i(X, \mathbb{Q}))(1) = \frac{\chi(X^\gamma)}{|C_\Gamma(\gamma)|}.$$ 

By Theorem 1, we have

$$\chi(X/\Gamma) = \sum_{\gamma \in C(\Gamma)} \frac{\chi(X^\gamma)}{|C_\Gamma(\gamma)|} = \frac{1}{|\Gamma|} \sum_{\gamma \in C(\Gamma)} \frac{|\Gamma|}{|C_\Gamma(\gamma)|} \chi(X^\gamma).$$

Since $|\Gamma|/|C_\Gamma(\gamma)|$ is the cardinality of the conjugacy class $(\gamma)$, we obtain

$$\chi(X/\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(X^\gamma).$$

Hence Theorem 1 implies the well-known equality for finite group actions.

5.2. Euler characteristics of groups

Suppose that $\Gamma$ is a group of finite homological type, then one can define its Euler characteristic $\chi(\Gamma)$ in the sense of C. T. C. Wall [12]. See [5, Chapter IX] for relevant definitions. Suppose in addition the centralizer $C_\Gamma(\gamma)$ is of finite homological type for every $\gamma \in \Gamma$ of finite order. Under these assumptions, K. S. Brown obtained the following formula:

$$(9) \quad \bar{\chi}(\Gamma) = \sum_{\gamma \in C_\Gamma(\gamma)} \chi(C_\Gamma(\gamma)), $$
where $\chi(\Gamma)$ is the naive Euler characteristic of $\Gamma$ defined by $\chi(\Gamma) = \sum_i (-1)^i \dim_Q H_i(\Gamma, Q)$ (cf. [5, p. 261]). This formula was used by J. Harer and D. Zagier in the computation of the Euler characteristic of the moduli space of curves [7].

We will give a relation between the equation (9) and Theorem 1. Let $\Gamma$ be a discrete group and $X$ a $\Gamma$-finite $\Gamma$-CW complex such that $X^\gamma$ is nonempty and $Q$-acyclic for every $\gamma \in \Gamma$ of finite order. If $\Gamma$ is virtually torsion-free, then $C_\Gamma(\gamma)$ is of finite homological type for every $\gamma \in \Gamma$ of finite order (including $\gamma = 1$), and $\chi(C_\Gamma(\gamma))$ coincides with $\epsilon(C_\Gamma(\gamma), X^\gamma)$ (cf. [4, pp. 111-112]). In this case the equation (1) in Theorem 1 reduces to the equation (9), since

$$H_*(\Gamma, Q) \cong H_*(X, Q) \cong H_*(X/\Gamma, Q).$$

However, we claim the equation (9) for this special case can be deduced without the use of the spectral sequence appeared in the proof of Lemma 9. To see this, observe $\epsilon : C_*(X, Q) \to Q$ is a projective resolution of $Q$ over $Q\Gamma$, where $Q$ is regarded as a $Q\Gamma$-module with the trivial $\Gamma$-action and $\epsilon$ is the augmentation. Hence

$$\chi(\Gamma) = \sum_i (-1)^i \dim_Q (Q \otimes_{Q\Gamma} C_i(X, Q)).$$

Now the claim follows from this together with (8) and Proposition 8.

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