<table>
<thead>
<tr>
<th>Title</th>
<th>On the Euler characteristic of the orbit space of a proper $\Gamma$-complex</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Akita, Toshiyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 36(4) P.783-P.791</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/7063">https://doi.org/10.18910/7063</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/7063</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
ON THE EULER CHARACTERISTIC OF THE ORBIT SPACE OF A PROPER $\Gamma$-COMPLEX

Dedicated to Professor Fuichi Uchida on his 60th birthday

TOSHIYUKI AKITA *

(Received November 17, 1997)

1. Introduction

Let $\Gamma$ be a discrete groups. A $\Gamma$-CW-complex $X$ is said to be $\Gamma$-finite if it satisfies the following two conditions:

1. For each cell $\sigma$ of $X$, the isotropy subgroup $\Gamma_\sigma$ of $\sigma$ is of finite order.
2. The orbit space $X/\Gamma$ is a finite complex.

In other words, $X$ is $\Gamma$-finite if the action of $\Gamma$ on $X$ is proper and cocompact. For a $\Gamma$-finite $\Gamma$-CW-complex $X$, define the equivariant Euler characteristic $e(\Gamma, X)$ by

$$e(\Gamma, X) = \sum_{\sigma \in \mathcal{E}} (-1)^{\dim \sigma} \frac{1}{|\Gamma_{\sigma}|} \in \mathbb{Q},$$

where $\mathcal{E}$ is a set of representatives of $\Gamma$-orbits of cells of $X$ and $|\Gamma_{\sigma}|$ is the order of $\Gamma_{\sigma}$. We agree $e(\Gamma, X) = 0$ when $X = \emptyset$. The equivariant Euler characteristic and its variants appear in various contexts of mathematics. See [3, 4, 9, 11] for instance. In particular, when $X$ is a manifold, the orbit space $X/\Gamma$ can be regarded as an orbifold and $e(\Gamma, X)$ is the orbifold Euler characteristic of $X/\Gamma$ in the sense of [11].

In this paper, we prove the formula expressing the Euler characteristic of the orbit space of a $\Gamma$-finite $\Gamma$-CW-complex in terms of equivariant Euler characteristics. More precisely, let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. For each $\gamma \in \Gamma$, the centralizer $C_\Gamma(\gamma)$ acts on the fixed point set $X^\gamma$. In this way $X^\gamma$ is naturally a $C_\Gamma(\gamma)$-finite $C_\Gamma(\gamma)$-CW-complex and hence $e(C_\Gamma(\gamma), X^\gamma)$ is defined for each $\gamma \in \Gamma$. Our result is:

**Theorem 1.** Let $\Gamma$ be a discrete group, and $X$ a $\Gamma$-finite $\Gamma$-CW-complex. Then

$$\chi(X/\Gamma) = \sum_{\gamma \in \mathcal{F}(\Gamma)} e(C_\Gamma(\gamma), X^\gamma),$$

*The author is supported by Grant-in-Aid for Encouragement of Young Scientists (No. 09740072), the Ministry of Education, Science, Sports and Culture.*
where \( \mathcal{F}(\Gamma) \) is a set of representatives of conjugacy classes of elements of finite order in \( \Gamma \).

Note that there are finitely many \( \gamma \)'s in \( \mathcal{F}(\Gamma) \) with \( X^\gamma \neq \emptyset \) and hence the summation in (1) makes sense.

In case a \( \Gamma \)-finite \( \Gamma \)-CW-complex \( X \) is a smooth \( \Gamma \)-manifold, then \( X^\gamma \) is a submanifold of \( X \) for each \( \gamma \in \Gamma \) of finite order, so that the terms \( e(C_T(\gamma), X^\gamma) \) in (1) can be regarded as orbifold Euler characteristics of orbifolds \( X^\gamma/C_T(\gamma) \). Thus Theorem 1 gives the expression of the Euler characteristic of \( X/\Gamma \) in terms of orbifold Euler characteristics of \( X^\gamma/C_T(\gamma) \).

When \( \Gamma \) is a finite group, a \( \Gamma \)-finite \( \Gamma \)-CW-complex is simply a finite \( \Gamma \)-CW-complex, and Theorem 1 implies

\[
\chi(X/\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(X^\gamma).
\]

Thus Theorem 1 is a generalization of the well-known equation (2) for finite group actions. For the direct proof of the equation (2), see [6, p. 225].

If \( \Gamma \) is virtually torsion-free and \( X \) is a \( \Gamma \)-finite \( \Gamma \)-CW-complex such that \( X^\gamma \) is nonempty and \( \mathbb{Q} \)-acyclic for every element \( \gamma \in \Gamma \) of finite order, then \( e(C_T(\gamma), X^\gamma) \) coincides with the Euler characteristic \( \chi(C_T(\gamma)) \) of the group \( C_T(\gamma) \) for every \( \gamma \in \Gamma \) of finite order, and Theorem 1 reduces to the following formula due to K. S. Brown (cf. [5, p. 261]):

\[
\sum_i (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, \mathbb{Q}) = \sum_{\gamma \in \mathcal{F}(\Gamma)} \chi(C_T(\gamma)).
\]

The rest of this paper is organized as follows. In §2, we introduce Hattori-Stallings ranks of finitely generated projective \( \mathbb{Q}\Gamma \)-modules, where \( \mathbb{Q}\Gamma \) denotes the rational group algebra of \( \Gamma \).

When \( X \) is a \( \Gamma \)-finite \( \Gamma \)-CW-complex, its cellular chain groups \( C_i(X, \mathbb{Q}) \) are finitely generated projective \( \mathbb{Q}\Gamma \)-modules. In §3, we will see that \( e(C_T(\gamma), X^\gamma) \) can be expressed in terms of Hattori-Stallings ranks of cellular chain groups.

In §4, we will prove Theorem 1. The proof is done by the spectral sequence which converges to the homology of \( \Gamma \) with coefficients in the cellular chain complex \( C_*(X, \mathbb{Q}) \), together with properties of Hattori-Stallings ranks which will be discussed in §2 and §3.

In the final section §5, we will consider the two special cases where (i) \( \Gamma \) is a finite group or (ii) \( \Gamma \) is virtually torsion-free, and \( X^\gamma \) is nonempty and \( \mathbb{Q} \)-acyclic for every \( \gamma \in \Gamma \) of finite order, both of which are mentioned above.

Throughout this paper, we employ the following conventions unless otherwise stated: \( \Gamma \) is a discrete group and \( \mathbb{Q}\Gamma \) is its rational group algebra. A module over \( \mathbb{Q}\Gamma \) is understood to be a left \( \mathbb{Q}\Gamma \)-module.
2. The Hattori-Stallings rank

The Hattori-Stallings rank was introduced by A. Hattori [8] and J. Stallings [10] and was studied by H. Bass in detail [1]. The reader should refer to [2] and [5, Chapter IX] for further detail of the Hattori-Stallings rank.

Let $Q\Gamma$ be the rational group algebra of $\Gamma$. Define $[Q\Gamma, Q\Gamma]$ be the additive subgroup of $Q\Gamma$ generated by $\alpha\beta - \beta\alpha$ ($\alpha, \beta \in Q\Gamma$). Set $T(Q\Gamma) = Q\Gamma/[Q\Gamma, Q\Gamma]$. It is easy to see that $T(Q\Gamma)$ is isomorphic to the $Q$-linear space spaned by the set of conjugacy classes of elements of $\Gamma$, i.e.,

\[(3) \quad T(Q\Gamma) \cong \bigoplus_{\gamma \in C(\Gamma)} Q \cdot (\gamma),\]

where $C(\Gamma)$ is a set of representatives of conjugacy classes of elements of $\Gamma$ and $(\gamma)$ is the conjugacy class of $\gamma$.

Let $\pi : Q\Gamma \to T(Q\Gamma)$ be the natural projection. Under the identification (3), $\pi$ assigns $1 \cdot (\gamma) \in T(Q\Gamma)$ to $\gamma \in Q\Gamma$.

Let $P$ be a finitely generated projective $Q\Gamma$-module. Then $P$ is a direct summand of a finitely generated free $Q\Gamma$-module $F$. Choose such $F$ and let $p : F \to P$ be the natural projection and $i : P \hookrightarrow F$ the inclusion. By fixing a basis of $F$, the composite $i \circ p$ can be identified with a square matrix $M$ over $Q\Gamma$.

**DEFINITION.** Under these assumptions, the Hattori-Stallings rank $r_\Gamma(P)$ of a finitely generated projective $Q\Gamma$-module $P$ is defined by

\[r_\Gamma(P) = \pi(\text{tr}M) \in T(Q\Gamma),\]

where $\text{tr}M$ is the trace of $M$.

Note that $r_\Gamma(P)$ is well-defined, i.e., it is independent of various choices made. Under the identification (3), denote by $r_\Gamma(P)(\gamma)$ the coefficient of $(\gamma)$ in $r_\Gamma(P)$.

We recall some properties of the Hattori-Stallings rank, which will be used later.

**Proposition 2** (cf. [1, §2 (2.5)]). If $P_1$ and $P_2$ are finitely generated projective $Q\Gamma$-modules, then

\[r_\Gamma(P_1 \oplus P_2) = r_\Gamma(P_1) + r_\Gamma(P_2).\]

**Proposition 3** (cf. [1, §6 (6.3)]). Let $\Gamma'$ be a subgroup of finite index of $\Gamma$, $P$ a finitely generated projective $Q\Gamma'$-module. Regarding $P$ as a finitely generated projective $Q\Gamma'$-modules by the restriction of scalars, one has

\[r_{\Gamma'}(P)(\gamma) = (C_\Gamma(\gamma), C_{\Gamma'}(\gamma)) \cdot r_\Gamma(P)(\gamma)\]
for all $\gamma \in \Gamma'$, where $(C_\Gamma(\gamma), C_{\Gamma'}(\gamma))$ is the index of $C_{\Gamma'}(\gamma)$ in $C_\Gamma(\gamma)$.

Let $f : \Gamma_1 \to \Gamma_2$ be a group homomorphism. Let $P$ be a finitely generated projective $\mathbb{Q}\Gamma_1$-module. Then $\mathbb{Q}\Gamma_2 \otimes_{\mathbb{Q}\Gamma_1} P$ is a finitely generated projective $\mathbb{Q}\Gamma_2$-modules, where $\mathbb{Q}\Gamma_2$ is regarded as a right $\mathbb{Q}\Gamma_1$-module via $f$. Let $T(f) : T(\mathbb{Q}\Gamma_1) \to T(\mathbb{Q}\Gamma_2)$ be the homomorphism induced by $f$.

**Proposition 4** (cf. [1, §2 (2.9)]). Under these assumptions, one has

$$r_{\Gamma_2}(\mathbb{Q}\Gamma_2 \otimes_{\mathbb{Q}\Gamma_1} P) = T(f)(r_{\Gamma_1}(P)).$$

In case $\Gamma$ is a finite group, Hattori-Stallings ranks can be determined by the character:

**Proposition 5** (cf. [1, §5 (5.8)]). Let $\Gamma$ be a finite group. Let $V$ be a $\mathbb{Q}\Gamma$-module which is finite dimensional over $\mathbb{Q}$. Then $V$ is finitely generated and projective, and one has

$$r_{\Gamma}(V)(\gamma) = \frac{\chi(\gamma^{-1})}{|C_{\Gamma}(\gamma)|},$$

where $\chi : \Gamma \to \mathbb{Q}$ is the character of $V$.

3. Hattori-Stallings ranks and equivariant Euler characteristics

Now we consider the equivariant Euler characteristic $e(\Gamma, X)$. First we invoke the following elementary lemma, which may be well-known.

**Lemma 6.** Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Then its cellular chain group $C_i(X, \mathbb{Q})$ is a finitely generated projective $\mathbb{Q}\Gamma$-module.

**Proof.** $C_i(X, \mathbb{Q})$ has a direct sum decomposition as a $\mathbb{Q}\Gamma$-module:

$$C_i(X, \mathbb{Q}) \cong \bigoplus_{\sigma} \mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_\sigma} \mathbb{Q},$$

where $\sigma$ ranges representatives of $\Gamma$-orbits of $i$-cells of $X$, $\Gamma_\sigma$ is the isotropy subgroup of $\sigma$, $\mathbb{Q}$ is regarded as a left $\mathbb{Q}\Gamma_\sigma$-module with the trivial action of $\Gamma_\sigma$, and $\mathbb{Q}\Gamma$ is regarded naturally as a right $\mathbb{Q}\Gamma_\sigma$-module. Since $X$ is $\Gamma$-finite, each $\Gamma_\sigma$ is a finite subgroup of $\Gamma$, which implies that $\mathbb{Q}$ is always finitely generated projective $\mathbb{Q}\Gamma_\sigma$-module (cf. Proposition 5). Thus $\mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_\sigma} \mathbb{Q}$ is a finitely generated projective $\mathbb{Q}\Gamma$-module. As the number of $\Gamma$-orbits of cells of $X$ is finite, so is the number of direct summands in (4), which yields the lemma.

By Lemma 6, the Hattori-Stallings rank of $C_i(X, \mathbb{Q})$ can be defined.
Lemma 7. Let $X$ be a $\Gamma$-CW-complex. Then

$$r_\Gamma(C_i(X, \mathbb{Q}))(1) = \sum_{\sigma} \frac{1}{|\Gamma_\sigma|},$$

where $\sigma$ ranges representatives of $\Gamma$-orbits of $i$-cells of $X$.

Proof. We have

$$r_\Gamma(C_i(X, \mathbb{Q}))(1) = r_\Gamma\left(\bigoplus_{\sigma} \mathbb{Q} \Gamma \otimes_{\mathbb{Q} \Gamma_\sigma} \mathbb{Q}\right)(1) \quad \text{by (4)}$$

$$= \sum_{\sigma} r_\Gamma(\mathbb{Q} \Gamma \otimes_{\mathbb{Q} \Gamma_\sigma} \mathbb{Q})(1) \quad \text{by Proposition 2}$$

$$= \sum_{\sigma} T(i)(r_{\Gamma_\sigma}(\mathbb{Q}))(1) \quad \text{by Proposition 4},$$

where $\sigma$ ranges representatives of $\Gamma$-orbits of $i$-cells of $X$ and $T(i) : T(\mathbb{Q} \Gamma_\sigma) \to T(\mathbb{Q} \Gamma)$ is the map induced by the inclusion $i : \Gamma_\sigma \hookrightarrow \Gamma$. From Proposition 5 we conclude

$$T(i)(r_{\Gamma_\sigma}(\mathbb{Q}))(1) = r_{\Gamma_\sigma}(\mathbb{Q})(1) = \frac{1}{|\Gamma_\sigma|},$$

proving the lemma. $\square$

By virtue of Lemma 7, we have

$$(5) \quad e(\Gamma, X) = \sum_i (-1)^i r_\Gamma(C_i(X, \mathbb{Q}))(1).$$

Together with the result of K. S. Brown [4], we obtain the relation between the Hattori-Stallings rank of $C_i(X, \mathbb{Q})$ and $e(C_\Gamma(\gamma), X^\gamma)$ as follows:

Proposition 8. Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Then

$$(6) \quad e(C_\Gamma(\gamma), X^\gamma) = \sum_i (-1)^i r_\Gamma(C_i(X, \mathbb{Q}))(\gamma)$$

for every $\gamma \in \Gamma$.

Proof. A direct consequence of the equality (5) and [4, Theorem 3.1 (iii)]. $\square$
4. Proof of Theorem 1

Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Let $H_\ast(\Gamma, C_\ast(X, Q))$ be the homology of $\Gamma$ with coefficients in the cellular chain complex $C_\ast(X, Q)$, which is isomorphic to the Borel homology (equivariant homology) $H_\ast^\Gamma(X, Q)$ (cf. [5, Chapter VII]). Since the isotropy subgroup of every cell of $X$ is finite, the Borel homology of $X$ is isomorphic to the rational homology of the orbit space:

$$H_\ast(\Gamma, C_\ast(X, Q)) \cong H_\ast^\Gamma(X, Q) \cong H_\ast(X/\Gamma, Q). \tag{7}$$

**Lemma 9.** Let $X$ be a $\Gamma$-finite $\Gamma$-CW-complex. Then

$$\sum_i (-1)^i \dim_Q H_i(\Gamma, C_\ast(X, Q)) = \sum_i (-1)^i \dim_Q Q \otimes_{Q\Gamma} C_i(X, Q).$$

Proof. Consider the spectral sequence

$$E^{1}_{i,j} = H_j(\Gamma, C_i(X, Q)) \Rightarrow H_{i+j}(\Gamma, C_\ast(X, Q))$$

(cf. [5, §VII.5 and §VII.7]). Since $C_i(X, Q)$ is a projective $Q\Gamma$-module for all $i$, we have

$$E^{1}_{i,j} \cong \begin{cases} Q \otimes_{Q\Gamma} C_i(X, Q) & j = 0 \\ 0 & j \neq 0. \end{cases}$$

As $\dim_Q Q \otimes_{Q\Gamma} C_i(X, Q) < \infty$ for all $i$, we obtain the desired equation. \qed

Now we prove Theorem 1. By Proposition 4 (take $\Gamma_2$ to be the trivial subgroup),

$$\dim_Q(Q \otimes_{Q\Gamma} C_i(X, Q)) = \sum_{\gamma \in C(\Gamma)} r_\Gamma(C_i(X, Q))(\gamma). \tag{8}$$

Hence

$$\chi(X/\Gamma) = \sum_i (-1)^i \dim_Q H_i(\Gamma, C_\ast(X, Q)) \quad \text{by (7)}$$

$$= \sum_i (-1)^i \dim_Q Q \otimes_{Q\Gamma} C_i(X, Q) \quad \text{by Lemma 9}$$

$$= \sum_i (-1)^i \left( \sum_{\gamma \in C(\Gamma)} r_\Gamma(C_i(X, Q))(\gamma) \right) \quad \text{by (8)}$$
Euler Characteristic of Orbit Spaces

\[ = \sum_{\gamma \in C(\Gamma)} \left( \sum_i (-1)^i \tau_i(C_i(X, \mathbb{Q}))(\gamma) \right) \]
\[ = \sum_{\gamma \in C(\Gamma)} e(C_\Gamma(\gamma), X^\gamma) \quad \text{by Proposition 8.} \]

For an element \( \gamma \) of infinite order, we have \( X^\gamma = \emptyset \) and hence \( e(C_\Gamma(\gamma), X^\gamma) = 0 \), which proves Theorem 1.

5. Remarks

5.1. Finite group actions

Suppose that \( \Gamma \) is a finite group. Let \( X \) be a finite \( \Gamma \)-complex. By Proposition 5, we have

\[ e(C_\Gamma(\gamma), X^\gamma) = \sum_i (-1)^i \tau_i(C_i(X^\gamma, \mathbb{Q}))(1) = \frac{\chi(X^\gamma)}{|C_\Gamma(\gamma)|}. \]

By Theorem 1, we have

\[ \chi(X/\Gamma) = \sum_{\gamma \in C(\Gamma)} \frac{\chi(X^\gamma)}{|C_\Gamma(\gamma)|} = \frac{1}{|\Gamma|} \sum_{\gamma \in C(\Gamma)} \frac{|\Gamma|}{|C_\Gamma(\gamma)|} \chi(X^\gamma). \]

Since \(|\Gamma|/|C_\Gamma(\gamma)|\) is the cardinality of the conjugacy class \((\gamma)\), we obtain

\[ \chi(X/\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(X^\gamma). \]

Hence Theorem 1 implies the well-known equality for finite group actions.

5.2. Euler characteristics of groups

Suppose that \( \Gamma \) is a group of finite homological type, then one can define its Euler characteristic \( \chi(\Gamma) \) in the sense of C. T. C. Wall [12]. See [5, Chapter IX] for relevant definitions. Suppose in addition the centralizer \( C_\Gamma(\gamma) \) is of finite homological type for every \( \gamma \in \Gamma \) of finite order. Under these assumptions, K. S. Brown obtained the following formula:

\[ \tilde{\chi}(\Gamma) = \sum_{\gamma \in \mathcal{C}(\Gamma)} \chi(C_\Gamma(\gamma)), \]

(9)
where $\tilde{\chi}(\Gamma)$ is the naive Euler characteristic of $\Gamma$ defined by $\tilde{\chi}(\Gamma) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, \mathbb{Q})$ (cf. [5, p. 261]). This formula was used by J. Harer and D. Zagier in the computation of the Euler characteristic of the moduli space of curves [7].

We will give a relation between the equation (9) and Theorem 1. Let $\Gamma$ be a discrete group and $X$ a $\Gamma$-finite $\Gamma$-CW complex such that $X^\gamma$ is nonempty and $\mathbb{Q}$-acyclic for every $\gamma \in \Gamma$ of finite order. If $\Gamma$ is virtually torsion-free, then $C_T(\gamma)$ is of finite homological type for every $\gamma \in \Gamma$ of finite order (including $\gamma = 1$), and $\chi(C_T(\gamma))$ coincides with $e(C_T(\gamma), X^\gamma)$ (cf. [4, pp. 111-112]). In this case the equation (1) in Theorem 1 reduces to the equation (9), since

$$H_\ast(\Gamma, \mathbb{Q}) \cong H_\ast^T(X, \mathbb{Q}) \cong H_\ast(X/\Gamma, \mathbb{Q}).$$

However, we claim the equation (9) for this special case can be deduced without the use of the spectral sequence appeared in the proof of Lemma 9. To see this, observe $\varepsilon : C_\ast(X, \mathbb{Q}) \to \mathbb{Q}$ is a projective resolution of $\mathbb{Q}$ over $\mathbb{Q}^\Gamma$, where $\mathbb{Q}$ is regarded as a $\mathbb{Q}^\Gamma$-module with the trivial $\Gamma$-action and $\varepsilon$ is the augmentation. Hence

$$\tilde{\chi}(\Gamma) = \sum_i (-1)^i \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Q}^\Gamma} C_i(X, \mathbb{Q})).$$

Now the claim follows from this together with (8) and Proposition 8.

ACKNOWLEDGEMENT. The author wishes to express his gratitude to Professor Katsuo Kawakubo for his advices.

References


Department of Applied Mathematics
Fukuoka University
Fukuoka 814-0180, Japan

Current address:
Division of Mathematics
Graduate School of Science
Hokkaido University
Sapporo 060-0810, Japan
e-mail:akita@math.sci.hokudai.ac.jp