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Bimeromorphic Geometry of Locally Conformally Kähler manifolds

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Abstract

We study compact complex manifolds bimeromorphic to locally conformally Kähler (LCK) manifolds. This is an analogy of studying a compact complex manifold bimeromorphic to a Kähler manifold. We give a negative answer for a question of Ornea, Verbitsky, Vuletescu by showing that there exists no LCK current on blow ups along a submanifold (dim ≥ 1) of Vaisman manifolds. We show that a compact complex manifold with LCK currents satisfying a certain condition can be modified to an LCK manifold. Based on this fact, we define a compact complex manifold with a modification from an LCK manifold as a locally conformally class C (LC class C) manifold. We give examples of LC class C manifolds that are not LCK manifolds. Finally, we show that all LC class C manifolds are locally conformally balanced manifolds.

1 Introduction

In this paper, we will study compact complex manifolds bimeromorphic to locally conformally Kähler manifolds.

A locally conformally Kähler manifold (LCK-manifold) is a complex manifold X, dim $X > 1$, admitting a Kähler covering $(\tilde{X}, \tilde{\omega})$, with the covering transformation group acting on $(\tilde{X}, \tilde{\omega})$ by homotheties. Compact complex manifolds bimeromorphic to LCK manifolds are not necessarily LCK manifolds. For example, it is known that a blow up of a Vaisman manifold along a submanifold of dimension ≥ 1 is not an LCK manifold [11].

Recall that X is said to be a class C manifold if X is bimeromorphic to a compact Kähler manifold, or equivalently, if there exists a proper holomorphic bimeromorphic map (i.e. a modification)

$$\mu \colon \tilde{X} \to X$$

from a compact Kähler manifold $\tilde{X}$. In general, class C manifolds are not Kähler manifolds. However it is known that there exists a Kähler current, which is, roughly speaking, a singular Kähler metric [7]. In the case of compact complex manifolds bimeromorphic to LCK manifolds, it is natural to consider an LCK current as an LCK metric with singularities.

Definition 1.1. [11] *Let X be a compact complex manifold, θ a closed real 1-form on X, Ξ a positive, real (1,1)-current satisfying $d\Xi = \theta \wedge \Xi$ and $\Xi > \omega$ for some Hermitian form ω on X. Then Ξ is called an LCK current.*

There exist Kähler currents on any class C manifolds, but there exist a lot of examples of compact complex manifolds with no LCK currents that are bimeromorphic to LCK manifolds. Theorem 1.2 gives a negative answer for the question of Ornea-Verbitsky-Vuletescu [11].

Theorem 1.2. *Let X be a compact complex manifold (*dim $X \geq 3$*), and $D \subset X$ be a submanifold of X (*codim $D \geq 2$*). Let $f \colon \tilde{X} \to X$ be a blow up of X along D, and E be the exceptional set. Suppose that there exists an LCK currrent $\tilde{T}$ with θ as the Lee form on $\tilde{X}$. Then $[\theta_{|E}] = 0$ in $H^1(E, \mathbb{R})$.*

Proof . See Theorem 5.1.

Corollary 1.3. *Let X be a Vaisman manifold. Suppose that* dim $D \geq 1$ *and* codim $D \geq 2$*. Then a blow up of X along D does not admit any LCK current.*

Proof . See Corollary 5.2.

By Demailly's regularization theorem [6], the Kähler current can be chosen to have analytic singularities, which in particular implies that it is a smooth Kähler metric on a Zariski open subset. Similarly to a Kähler current with analytic singularities, an LCK current with analytic singularities can also be defined, but by Theorem 1.2, in general, compact complex manifolds bimeromorphic to LCK manifolds do not admit any LCK current with analytic singularities.

Therefore, properties of compact complex manifolds bimeromorphic to LCK manifolds are different from that of class C manifolds. This is because a blow up of a Kähler manifold is always a Kähler manifold, but a blow up of an LCK manifold is not necessarily LCK. As a result, for class C manifolds there exists a modification from a compact Kähler manifold, but for a compact complex manifold bimeromorphic to an LCK manifold, a modification from an LCK manifold does not necessarily exists. So we define locally conformally class C (LC class C) manifolds as an analogue of class C manifolds as follows.

Definition 1.4. *Let X be a compact complex manifold. It is called a locally conformally class C manifold (LC class C manifold) if there exists a modification*

$$\mu\colon \tilde{X} \to X$$

from an LCK manifold $\tilde{X}$.

There are many LC class C manifolds which are not LCK manifolds.

Proposition 1.5. *Let X be an LCK manifold. Then there exist LC class C manifolds bimeromorphic to X which are not LCK manifolds.*

Proof . See Proposition 8.1.

This result is based on Hironaka'example [8], which shows that non Kähler Moishezon manifolds exist. As a corollary, it follows that a blow down of a LCK-manifold is, in general, not an LCK-manifold. This was conjectured in [11].

It is easy to see that LC class C manifolds admit LCK currents wiht analytic singularities. Conversely, we will consider the question whether a compact complex manifold with an LCK current with analytic singularities admits a modification from an LCK manifold.

Theorem 1.6. *Let X be a compact complex manifold with an LCK current T with analytic singularities supported on B, and θ the Lee form. We assume that there exists an open neighborhood V of B such that $j^*[\theta] = 0$ in $H^1(V, \mathbb{R})$, where j is a holomorphic embedding of V into X. Then there exists a modification*

$$\mu\colon \tilde{X} \to X$$

from an LCK manifold $\tilde{X}$.

Proof . See Theorem 7.4.

In Theorem 1.6, the condition $j^*[\theta] = 0$ can not be omitted. This is because a blow up of X along a submanifold such that $j^*[\theta] \neq 0$, in general, does not admit any LCK currents with analytic singularities by Theorem 1.2, but we believe that there does not exist any LCK current with singularities supported on submanifolds such that $j^*[\theta] \neq 0$ on X, in Theorem 1.6. So we will propose a conjecture as follows.

Conjecture 1.7. *A compact complex manifold X is of LC class C if and only if there exists an LCK current T on X.*

Recall the definition of LCK manifolds. Similar to LCK manifolds, we can define locally conformally balanced (*LC-balanced*) manifolds, by asking them to admit an LC-balanced metric. By proving that LC-balanced manifolds are stable under some proper modification, we will show that every LC class C manifolds admit LC-balanced metric.

Theorem 1.8. *Let $\mu\colon \tilde{X} \to X$ be a proper modification of compact complex manifolds. Then if $\tilde{X}$ is an LC-balanced manifold, X is an LC-balanced manifold.*

Proof . See Theorem 9.9.

Corollary 1.9. *Every LC class C manifolds admit an LC-balanced metric.*

The organization of this paper is as follows. In Section 2, we recall some basic terms. In Section 3, we show that a blow up along a submanifold of a Vaisman manifold does not admit LCK currents (not necessarily with analytic singularities). In Section 4, we show that a compact complex manifolds with some kind of LCK current can be modified to an LCK manifold by a proper holomorphic bimeromorphic map. In Section 5, we give exampes of LC class C manifolds. In Section 6, we show that LC class C manifolds are LC-balanced.

2 Bimeromorphic maps and Positive currents

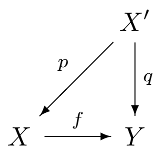
In this section, we review definitions and basic facts on bimeromorphic maps and positive currents.

First we will introduce the notion of a proper modification.

Definition 2.1. Let X and Y be complex manifolds. A proper surjective holomorphic map $f\colon X \to Y$ is called a proper modification, if there eixst nowhere dense analytic subsets M of X and N of Y such that f incuces a biholomorphic mapping of $X - M$ onto $Y - N$.

The following theorem is called Hironaka Chow lemma.

Theorem 2.2. *Let X and Y be compact complex manifolds. Let $f\colon X \to Y$ be a proper modification. Then there is a commutative diagram*



where p and q are obtained by finite sequences of blowing up along smooth centers.

Definition 2.3. Let X and Y be complex manifolds. A mapping ϕ of X into the power set of Y is called a meromorphic mapping of X into Y, if X satisfies the following conditions.
1. The graph $G_\phi = \{(x, y) \in X \times Y | y \in \phi(x)\}$ is an irreducible analytic subset in $X \times Y$.
2. The projection map $p_X : G_\phi \to X$ is a proper modification.

Definition 2.4. A meromorphic mapping $\phi\colon X \to Y$ of complex manifolds is called a bimeromorphic mapping if $p_Y : G \to Y$ is also a proper modification.

The following theorem is the standard fact giving the vanishing of the higher direct image sheaves of structual sheaf under modification.

Theorem 2.5. *Let X and Y be complex manifolds. If X and Y are bimeromorphically equivalent, then*
1. $f_*\mathcal{O}_X = \mathcal{O}_Y$,
2. $R^i f_*\mathcal{O}_X = 0$ for all $i \geq 1$.

Next we will review about Positive current.

Definition 2.6. Let X be a complex manifold (n = dim X). We call linear forms T on $A_c^{(n-p,n-q)}(X)$ as (p, q) currents. We denote by $D^{(p,q)}(X)$ the space of (p, q) current on X. We define derivative of current

$$\partial\colon D^{(p,q)}(X) \to D^{(p+1,q)}(X),$$
$$\overline{\partial}\colon D^{(p,q)}(X) \to D^{(p,q+1)}(X)$$

by

$$(\partial T)(\phi) = (-1)^{p+q+1} T(\partial\phi),$$
$$(\overline{\partial} T)(\phi) = (-1)^{p+q+1} T(\overline{\partial}\phi).$$

Moreover we can define $d = \partial + \overline{\partial}$. A real (p, p) current T is said to be positive if

$$(\sqrt{-1})^{p(n-p)} T(\eta \wedge \overline{\eta}) \geq 0 \ \ (\eta \in A_c^{(n-p,0)}(X)).$$

Example 2.7. For $\eta \in A^{(p,q)}(X)$, η is (p, q) current by

$$\eta(\phi) := \int_X \eta \wedge \phi \ \ (\phi \in A_c^{(n-p,n-q)}(X)).$$

Example 2.8. Let X be a complex manifold (dim $X = n$) and $V \subset X$ a submanifold (codim $V = k$). For $\phi \in A_c^{(n-k,n-k)}(X)$, define

$$V(\phi) := \int_V \phi.$$

Then we can regard V as closed positive (k, k) current.

3 LCK manifolds and LCK currents

Definition 3.1. Let M be a complex manifold. Suppose that there exists a non zero closed real 1-form θ and a Hermitian form ω satisfy the following condition.

$$d\omega = \theta \wedge \omega. \tag{3.1}$$

Then M is called a *locally conformally Kähler* manifold (LCK manifold). We call θ the Lee form of ω.

Let M be LCK. By definition, M admits a Kähler covering $\pi : \tilde{M} \to M$ ($\pi^*[\theta] = 0$ in $H^1(\tilde{M}, \mathbb{R})$). If θ is exact, then M is called *globally conformally Kähler* (GCK).

Let $\pi : \tilde{M} \to M$ be the universal covering and Γ the covering transformation group. Then $\pi^*\omega$ is globally conformal to a Kähler metric $\tilde{\omega}$. Moreover, Γ acts by holomorphic homotheties with respect to $\tilde{\omega}$. This defines a character

$$\chi : \Gamma \to \mathbb{R}_{>0}, \quad \gamma^*\tilde{\omega} = \chi(\gamma)\tilde{\omega}. \tag{3.2}$$

If $\chi(\gamma) \equiv 1$ for any $\gamma \in \Gamma$, then M is GCK. Moreover, there exists an open cover $(U_i)_{i \in I}$ of M endowed with local Kähler metrics g_i on U_i, conformal to g_j on overlaps $U_i \cap U_j : g_i = c_{ij} g_j$. By difinition, c_{ij} are positive constants.

A particular class of LCK manifolds are the Vaisman manifolds.

Definition 3.2. Let M be LCK and θ the Lee form. Then M is called Vaisman if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection.

In the same way, we define locally conformally balanced manifolds.

Definition 3.3. Let X be a complex manifold of $\dim_{\mathbb{C}} X = n > 1$. Suppose that there exists a non zero closed real 1-form θ and a Hermitian form ω which satisfy the following condition

$$d\omega^{n-1} = \theta \wedge \omega^{n-1}. \tag{3.3}$$

Then X is called a *locally conformally balanced* manifold (LC-balanced). We call θ the Lee form of ω.

Definition 3.4. Let M be an LCK manifold, and $j : Y \hookrightarrow M$ a complex subvariety. We say that Y is an *induced globally conformally Kähler* (IGCK) subvariety if the cohomology class $j^*[\theta] \in H^1(Y)$ vanishes, where θ is the Lee form. Similarly, for an LC-balanced manifold, we say that Y is an *induced globally conformally balanced* (IGC-balanced) if the cohomology class $j^*[\theta]$ vanishes.

We define the analytic singularites of an LCK current (cf. [5]). Let X be a complex n-dimensional manifold. Let T be an LCK current (Definition 1.1) on X. Suppose that, by definition $dT = \theta \wedge T$, and T can locally be written as $T|_{U_k} = e^{f_k}\partial\bar{\partial}\phi_k, \theta = df_k$ on U_k, where ϕ_k is a plurisubharmonic (psh) function on U_k. When $[\theta] = 0$, T is said to be a *GCK current.* If M admits a GCK current, we call M a *globally conformally class C* manifold.

Definition 3.5. Let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf on X and $c > 0$, where $\mathcal{O}_X$ is the structure sheaf of X. We say that an LCK current T has *singularities of type* $(\mathcal{I},\ c)$ if it can locally be written as $T = e^{g_k}(\theta_k + i\partial\bar{\partial}\phi_k)$ on U_k ; $\phi_k = \frac{c}{2}\log(\sum |f_j|^2)$ for some local generators (f_j) of $\mathcal{I}$ and some smooth form θ. In this case, we also say that T admits *analytic singularities*.

Remark 3.6. Let M, $\tilde{M}$ complex manifolds, and $\pi : \tilde{M} \to M$ be a proper modification. Suppose that $\tilde{M}$ admits an LCK metric. Then the push-forward $\pi_*\omega$ is an LCK current with analytic singularities.

4 Blow up and LCK form

In this section, the following arguments in [11], we give a proof of the form version of Theorem 1.2 (Lemma 3.1 of [11]).

Theorem 4.1. *Let X be an LCK manifold, and $D \subset X$ be a submanifold of X* (codim $D \geq 2$)*. Let $f : \tilde{X} \to X$ be a blow up of X along D, and E be the exceptional set. Suppose that there exists an LCK metric ω with θ as the Lee form on $\tilde{X}$. Then $[\theta_{|E}] = 0$ in $H^1(E, \mathbb{R})$.*

Proof . E is LCK, and $\theta_{|E}$ is the Lee form of E. Let $E^{'}$ be the minimal GCK covering of E, that is, the minimal covering $E^{'} \to E$ such that the pullback of $\theta_{|E}$ is exact. Since $H^1(D) \cong H^1(E)$, there exists a covering $D^{'} \to D$ such that the following diagram is commutative:

$$\begin{array}{ccc} E^{'} & \longrightarrow & E \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow \\ D^{'} & \longrightarrow & D \end{array}$$

Let $F_b := \tilde{\pi}^{-1}(b)$ be the fibers of $\tilde{\pi}$, dim $F_b = k$. All F_b represent tha same homology class in $H_{2k}(E^{'})$.
Denote the Kähler form of $E^{'}$ by $\omega^{'}$.
Since all F_b represent the same homology class, the Rimannian volume

$$Vol_{\omega'}(F_b) := \int_{F_b} {\omega'}^k$$

is independent form $b \in D^{'}$. This gives

$$Vol_{\omega'}(F_b) = \int_{F_b} {\omega'}^k = \int_{F_{\gamma^{-1}(b)}} \gamma^* {\omega'}^k = \int_{F_{\gamma^{-1}(b)}} \chi(\gamma)^k {\omega'}^k = \chi(\gamma)^k Vol_{\omega'}(F_b),$$

hence the constant χ_γ is equal to 1 for all $\gamma \in \Gamma$. Therefore $[\theta_{|E}] = 0$ in $H^1(E, \mathbb{R})$. □

Corollary 4.2. The blow up of a compact Vaisman manifold along a compact complex manifold Y of dimension at least 1 cannot have an LCK metric.

Proof . See Corollary 2.13 of [11].

5 Blow up and LCK currents

There exist a lot of examples of compact complex manifolds bimeromorphic to LCK manifolds which are bimeromorphic to LCK manifolds but admit no LCK currents.

Theorem 5.1. *Let X be a compact complex manifold ($\dim X \geq 3$), and $D \subset X$ a submanifold of X ($\operatorname{codim} D \geq 2$). Let $f \colon \tilde{X} \to X$ be a blow up of X along D, and E the exceptional set. Suppose that there exists an LCK currrent $\tilde{T}$ on $\tilde{M}$. Let θ be the Lee form of $\tilde{T}$. Then $[\theta_{|E}] = 0$ in $H^1(E, \mathbb{R})$.*

Proof . Since $\tilde{T}$ is a d_θ-closed positive current on X (i.e. $d_\theta \tilde{T} = d\tilde{T} - \theta \wedge \tilde{T} = 0$), we can write

$$\tilde{T} = \alpha + \sqrt{-1}\partial_\theta \overline{\partial}_\theta \phi,$$

where α is a smooth d_θ-closed form on X and $\partial_\theta = \partial - \theta^{1,0} \wedge \cdot$. To see this, take an open covering of X by open coordinate balls U_j and plurisubharmonic potentials ϕ_j such that $\tilde{T} = e^{f_j}\sqrt{-1}\partial\overline{\partial}\phi_j$ on U_j. Then, if (η_j) is a partition of unity subordinate to (U_j), it is easy to see that, for $\phi = \Sigma \eta_j e^{f_j}\phi_j$, $\alpha := \tilde{T} - \sqrt{-1}\partial_\theta\overline{\partial_\theta}\phi$ is smooth.

Let $\pi \colon \tilde{X}^{'} \to \tilde{X}$ be the universal covering. Then the pull back of $\tilde{T}$ by π is defined, i.e. $\pi^*\tilde{T} = \pi^*\alpha + \sqrt{-1}\partial_{\theta'}\overline{\partial_{\theta'}}\phi \circ \pi$, where $\theta^{'} = \pi^*\theta$. Let γ be a real continuous (1,1)-form such that $\sqrt{-1}\partial_{\theta'}\overline{\partial_{\theta'}}\phi > \gamma$. Since $\sqrt{-1}e^h\partial\overline{\partial}e^{-h}\phi \circ \pi > \pi^*\gamma$, $e^{-h}\phi \circ \pi$ is an almost psh function, where h is a function on $\tilde{X}^{'}$ such that $dh = \theta^{'}$. Let S be a fundamental domain of the universal covering π, and U a relatively compact subset of $\tilde{X}^{'}$ such that $S \subset U$. By Demailly's regularization theorem [6], there exists a sequence (ϕ_n) of almost psh functions with analytic singularities on U such that

$$\phi_n \to e^{-h}\phi \circ \pi \ \ as \ \ n \to \infty$$

$$\sqrt{-1}\partial\overline{\partial}\phi_n > e^{-h}\pi^*\gamma - \epsilon(n)\eta, \ \epsilon(n) \to 0 \ \ as \ \ n \to \infty,$$

where η is a Hermitian metric on $\tilde{X}^{'}$. Therefore, $e^h\phi_n \to \phi \circ \pi$ $(n \to \infty)$ on U. Therefore $\pi^*\tilde{T}_n := \pi^*\alpha + \sqrt{-1}\partial_{\theta'}\overline{\partial_{\theta'}}\psi_n$ admits analytic singularities, where $\psi_n := e^h\phi_n$. Since $H^1(D) \cong H^1(E)$ is an isomorphism, there exists a covering $\tilde{D} \to D$ induced by π such that the following diagram is commutative.

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow \\ \tilde{D} & \longrightarrow & D \end{array}$$

Let $F_b := \tilde{\pi}^{-1}(b) \cong \mathbb{P}^k$ be the fibers of $\tilde{\pi}$ and $\mathbb{P}^1_b \subset F_b$ a line. Since D is connected, all $\mathbb{P}^1_b$ represent the same homology class in $H_2(\tilde{E})$. Let $\iota \colon \tilde{E} \to \tilde{X}^{'}$ be a holomorphic embedding and $\tilde{T}_n^{'} := \iota^* e^{-h}(\pi^*\tilde{T}_n)_{ac}$ on U, where $(\pi^*\tilde{T}_n)_{ac}$ is the absolutely continuous part of $\pi^*\tilde{T}_n$. Since $\tilde{T}_n^{'}$ is smooth outside an analytic set of codim ≥ 1, for generic $b \in \tilde{D}$ and generic $\mathbb{P}_b^1 \subset F_b$,

$$\int_{\mathbb{P}_b^1} \tilde{T}_n^{'} < \infty.$$

Let $\tilde{T}_{n,\epsilon}^{'}$ be a smoothing of $\tilde{T}_n^{'}$, i.e.

$$\tilde{T}_{n,\epsilon}^{'} = \iota^* e^{-h}\pi^*\alpha + ((\sqrt{-1}\partial\overline{\partial}\phi_n \circ \pi)_{ac} \circ \iota)_\epsilon.$$

Since $((\sqrt{-1}\partial\overline{\partial}\phi_n \circ \pi)_{ac} \circ \iota)_\epsilon$ is d-exact and $\tilde{T}_{n,\epsilon}^{'}|_{\mathbb{P}_b^1} \to \tilde{T}_n^{'}|_{\mathbb{P}_b^1}$ as a current and for n large $\tilde{T}_n^{'}$ is a Kähler current,

$$\int_{\mathbb{P}_b^1} \iota^* e^{-h}\pi^*\alpha > 0.$$

Since

$$\mathrm{Vol}(\mathbb{P}_b^1) := \int_{\mathbb{P}_b^1} \iota^* e^{-h}\pi^*\alpha$$

is independent from $b \in \tilde{D}$, this gives (recall the definition of the character χ in (3.2))

$$\mathrm{Vol}(\mathbb{P}_b^1) = \int_{\mathbb{P}_b^1} \iota^* e^{-h}\pi^*\alpha = \int_{\mathbb{P}_{\gamma^{-1}(b)}^1} \gamma^*\iota^* e^{-h}\pi^*\alpha = \int_{\mathbb{P}_{\gamma^{-1}(b)}^1} \chi(\gamma)\iota^* e^{-h}\pi^*\alpha = \chi(\gamma)\mathrm{Vol}(\mathbb{P}_b^1),$$

hence tha constant $\chi(\gamma)$ is equal to 1 for all $\gamma \in \Gamma$, where Γ is covering transformation group. Therefore $[\theta_{|E}]= 0$. □

Corollary 5.2. *Let X be a Vaisman manifold. Let D* (dim $D \geq 1$, codim $D \geq 2$) *be a submanifold of X. Then the blow up of X along D does not admit any LCK currents.*

Proof . Let $\pi\colon \tilde{X} \to X$ be a blow up along D and E the exceptional set. Suppose that there exists an LCK current $\tilde{T}$ on $\tilde{X}$. Let $\tilde{\eta}$ be the Lee form of $\tilde{T}$.

Since $[\tilde{\eta}_{|E}] = 0$ by Theorem 5.1, after possibly making a conformal change of the LCK current, we can assume $\tilde{\eta}_{|V} = 0$ where V is a neighborhood of E. In particular, $\tilde{\eta}$ will be the pull-back of a one-form η on X. On the other hand, $\tilde{T}$ gives rise to a current on $\tilde{X}$ and its pushforward defines an LCK positive (1,1) current T on X with associate Lee form η. Clearly, $\eta_{|D} = 0$.

The proof of Corollary 5.2 is now identical to the proof of Corollary 2.13 in [11]. That is, $\tilde{T}$ does not exist. □

6 class C manifolds

In this section, we review definitions and basic facts on class C manifolds.

Definition 6.1. Let X be a compact complex manifold. Suppose that X is a holomolphic imege of a compact Kähler manifold. Then we say that X is a class C manifold.

Let X be a compact complex manifold bimeromorphic to a compact Kähler manifold. Of course, X belongs class C. But by the following theorem, class C manifolds are bimeromorphic to compact Kähler manifolds [13].

Theorem 6.2. *Let X be class C. Then X is bimeromorphic to a compact Kähler manifold.*

Proof . By Hironaka's flatterring theorem [10] (See also Theorem 5 of [13]).

There exists a lot of non Kähler class C manifolds. The following theorem is called Hironaka's example.

Proposition 6.3. *Let X be a compact Kähler manifold. Suppose that* dim $X \geq 3$. *Then there exists a non Kähler class C manifold bimeromorphic to X.*

Proof . Let X' be a blow up at a point of X. Then we can use Hironaka's discussion [8] on X'.

It is natural to think of the following currents for class C manifolds.

Definition 6.4. Let X be a compact complex manifold. Let ω be a Hermitian metric on X. A Kähler current T is a closed (1,1) current such that $T > \epsilon\omega$, a constant $\epsilon > 0$.

In fact, the definition of Kähler current make senses as the following theorem shows [7].

Theorem 6.5. Let X be a compact complex manifold. Then X belongs class C if and only if X admits a Kähler current.

7 Modification of LC class C manifolds

Let X be a compact complex manifold. A Fujiki's class C manifolds is a compact complex manifold that can be modified by a proper holomorphic bimeromorphic map to a compact Kähler manifold [13]. In analogy with the Fujiki's class C, we say that X is a *locally conformally class C* (LC class C) manifold if X can be modified by a proper holomorphic bimeromorphic map to an LCK manifold. It is trivial that LC class C manifolds admit LCK currents with analytic singularities. Conversely, we will consider the question whether a compact complex manifold with an LCK current with analytic singularities is of LC class C. This discussion is based on [7] where it is proven that a compact complex manifold is of Fujiki's class C if and only if there exists a Kähler current.

The following Proposition 7.1 follows form Hironaka's resolution of singularities theorem [9].

Proposition 7.1. [9] *Let ϕ be a plurisubharmonic function with analytic singularities on a compact complex manifold X. Then, there exists a finite sequence of blow ups $\mu : \tilde{X} \to X$ such that $\mu^*\phi$ has singularities along a divisor of $\tilde{X}$ with normal crossing.*

Let X be a compact complex manifold with an LCK current T which has analytic singularities. By Proposition 7.1, there exists a composition μ of a finite sequence of blow ups such that

$$\mu^*T = \alpha + T_{sg},$$

where α is a smooth positive d_θ-closed form on $\tilde{X}$, T_{sg} is d_θ-closed singular part that the support is included in the exceptional divisor of μ. Decompose μ as a tower of blow ups.

$$\tilde{X} = X_N \overset{\mu_N}{\to} X_{N-1} \overset{\mu_{N-1}}{\to} \cdots \to X_1 \overset{\mu_1}{\to} X_0 = X \tag{7.1}$$

Definition 7.2. Let M be an LC class C manifold and $Y \overset{j}{\hookrightarrow} M$ a complex subvariety. We say that Y is of *induced globally conformally class C* (IGC class C) if the cohomology class $j^*[\theta]$ vanishs, where θ denotes the cohomology class of the Lee form on M.

Proposition 7.3. *Define μ_{k+1}, X_k as (7.1). Suppose that X_k admits an LCK current T_k, $dT_k = \theta_k \wedge T_k, T_k > \omega_k$ and μ_{k+1} is a blow up along a submanifold V. Assume that V is of IGC class C, that is, $j^*[\theta_k] = 0$, where j is a holomorphic embedding of V into X_k. Then there is a $d_{\mu_k^*\theta_k}$-closed $(1,1)$ form u such that $\mu_k^*\omega_k + \epsilon u$ is positice definite ($\epsilon \ll 1$).*

Proof . V is of IGC class C. Let U be an open neighborhood of V such that $V \hookrightarrow U$ induces an isomorphism of the first cohomology, i.e. $\theta_k|U = df$. Let W be a compact neighborhood of V such that $\mu_k^{-1}(W) \subset \tilde{U} = \mu_k^{-1}(U)$, where $\tilde{U}$ is of IGC class C, i.e. $\theta_{k+1}|\tilde{U} = df' = \mu_k^*(df)$.

Let L be a holomorphic line bundle over X_{k+1} such that $L|_{\mu_k^{-1}(V)} \cong \mathcal{O}_{\mathbb{P}(N_{V|X_k})}(1)$ and it is a trivial bundle outside $\mu_k^{-1}(V)$. Let h be a Hermitian structure on $\mathcal{O}_{\mathbb{P}(N_{V|X_k})}(1)$ and ω be the corresponding Chern form. Take an open covering $\{U_i\}_{i\in I}$ of X_{k+1} which trivializes L. By using a partition of unity subordinate to $\{U_i\}_{i\in I}$, it follows that the metric h can be extended to a metric strucure $\tilde{h}$ on L. Let $\tilde{\omega}$ be the corresponding curvature. Then the curvature $\tilde{\omega}$ vanishes on $X_{k+1} \setminus \mu_k^{-1}(W)$ and $\tilde{\omega}|_{\mathbb{P}(N_{V|X_k})} = \omega$. Put $\tilde{u} = 0$ on $\mu_k^{-1}(X_k \setminus W)$ and $u = e^{f'}\tilde{\omega}$ on $\tilde{U}$. Since V is compact, there exists small enough $\epsilon \in \mathbb{R}$, $\epsilon > 0$ such that

$$\mu_k^*\omega_k + \epsilon u$$

is positive definite. □

Theorem 7.4. *Let X be a compact complex manifold with an LCK current T with analytic singularites supported on B and θ the Lee form. We assume that* *there exists an open neighborhood V of B such that $j^*[\theta] = 0$ on U, where j is a holomorphic embedding of V into X. Then X admits a modification $\mu\colon \tilde{X} \to X$ from an LCK manifold $\tilde{X}$.*

Proof . By assumption, we may assume that every center of blow up (7.1) is IGC class C. Therefore, by Proposition 7.3, X_{k+1} admits an LCK current T_{k+1}, i.e,

$$T_{k+1} := \mu_k^* T_k + \epsilon_{k+1} u_{k+1} > \mu_k^* \omega_k + \epsilon_{k+1} u_{k+1} > 0,\ \epsilon_{k+1} \ll 1.$$

Let $\hat{u}_k$ be the pull back of u_k to $\tilde{X}$, then we can conclude inductively that $\mu^* T + \Sigma \epsilon_k \hat{u}_k$ is an LCK current. Therefore the smooth form

$$\hat{\omega} := \alpha + \Sigma \epsilon_k \hat{u}_k = \mu^* T + \Sigma \epsilon_k \hat{u}_k - T_{sg}$$

is an LCK form on $\tilde{X}$, i.e. $\tilde{X}$ is an LCK manifold. □

Remark 7.5. In Theorem 7.4, we can not omit the assumption that the center of every blow up is of IGC class C. For example, any blow up along a submanifold D (dim $D \geq 2$, codim $D \geq 2$) of a regular Vaisman manifold can not admit LCK currents with analytic singularities, that is, such manifolds do not admit any modification from LCK manifolds.

8 Examples of LC class C manifolds

We show that there exists a lot of examples of LC class C manifolds which are not LCK manifolds. It is based on Hironaka's examples of non projective Moishezon manifolds [8].

Proposition 8.1. *Let X be an LCK manifold. Suppose that* $\dim_{\mathbb{C}} X \geq 3$. *Then there exist an LC class C manifold that is bimeromorphic to X but not an LCK manifold.*

Proof . Let $X^{'}$ be a blow up at a point of X. Then $\mathbb{P}^2 \subset X^{'}$. Let $C \subset \mathbb{P}^2 \subset X^{'}$ be a curve that is smooth except for one double point y with normal crossing. There exists an open neighbourhood U of y in $X^{'}$ such that $C \cap U = C_1 \cup C_2$. Let $\pi_1 : U_1 \to U$ be a blow up of C_1. Let $\pi_2 \colon U_2 \to U_1$ be a blow up of the strict transform of C_2. Let $\psi : Z \to X^{'} \setminus y$ be a blow up of $C \setminus y$. There is then a smooth modification $\rho : \tilde{X} \to X^{'}$ that is obtained by glueing ψ and $\pi_2 \circ \pi_1$.

There exists an open neighbourhood V such that $y \in \mathbb{P}^2 \subset \mathbb{P}^{n-1} \subset V$ and V is simply connected. Therefore, since ρ is a proper modification, $\rho^{-1}(V)$ is also simply connected. Notice that any submanifold of an LCK manifold is also an LCK manifold. Therefore if $\tilde{X}$ is an LCK manifold, $\rho^{-1}(V)$ is also an LCK manifold. Since $\rho^{-1}(V)$ is simply connected, $\rho^{-1}(V)$ must be a Kähler manifold. But by construction, $\rho^{-1}(V)$ is non Kähler. Therefore $\tilde{X}$ cannot be an LCK manifold.

Let $\phi\colon X^{'} \to \tilde{X}$ be a bimeromorphic map. By Hironaka's desingularization theorem, there exists a blow up $\pi\colon Y \to X^{'}$(obtained by a sequence of blow ups with smooth centers in V) such that ϕ can be resolved into a modification

$\mu\colon Y \to \tilde{X}$. Since by the argument of [11], Y is an LCK manifold, $\tilde{X}$ must be LC class C. □

Example 8.2. By Theorem 7.4, the compact complex manifolds obtained in Proposition 8.1 are obtained by blowing down LCK manifolds along smooth centers. Therefore, in general, a blow down of an LCK manifold is not an LCK manifold. This fact was conjectured by Ornea-Verbitsky-Vuletescu [11].

Remark 8.3. Let X be an LCK manifold and θ the Lee form. Moreover let $p\colon \tilde{X} \to X$ be a proper modification. Even if an exceptional divisor of p is IGCK, $\tilde{X}$ is not necessarily an LCK manifold. If p is a blow up along a smooth center $B \subset X$ which IGCK, then $\tilde{X}$ is an LCK manifold [11].

9 LC class C manifolds are LC-balanced

Let $\partial_\theta = \partial - \theta^{1,0} \wedge \cdot$, where θ is a closed real 1-form. In this section, let M and $\tilde{M}$ be compact complex manifolds, and let $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} \tilde{M} = n$.

Lemma 9.1. *Let ω be a $\partial_\theta\bar{\partial}_\theta$-closed (1,1)-form, where θ is a closed real 1-form and let $\theta = df$ on a polycylinder D in $\mathbb{C}^n$. Then*

$$\partial\bar{\partial}(e^{-f}\omega) = 0 \text{ on } D,$$

that is,

$$\omega = e^{f}(\partial\bar{\phi} + \bar{\partial}\phi) \text{ on } D,$$

for a certain smooth (1,0)-form ϕ.

Proof . Since $\partial_\theta\bar{\partial}_\theta\omega = 0$, $\partial\bar{\partial}(e^{-f}\omega) = 0$ on D. □

Remark 9.2. Lemma 9.1 also holds for $\partial_\theta\bar{\partial}_\theta$-closed (1,1)-current.

By Lemma 9.1, the following definition makes sense.

Definition 9.3. Let us define the real (1,1)-Aeppli group $V^{1,1}(M)_{\mathbb{R}}$ (cf [1]) and the real (1,1) θ-Aeppli group $V_\theta^{1,1}(M)_{\mathbb{R}}$ as follows.

$$V^{1,1}(M)_{\mathbb{R}} = \frac{\mathrm{Ker}(i\partial\bar{\partial}\colon \mathcal{E}^{1,1}(M)_{\mathbb{R}} \to \mathcal{E}^{2,2}(M)_{\mathbb{R}})}{(\partial\mathcal{E}^{0,1}(M) + \bar{\partial}\mathcal{E}^{1,0})_{\mathbb{R}}},$$

$$V_\theta^{1,1}(M)_{\mathbb{R}} = \frac{\mathrm{Ker}(i\partial_\theta\bar{\partial}_\theta\colon \mathcal{E}^{1,1}(M)_{\mathbb{R}} \to \mathcal{E}^{2,2}(M)_{\mathbb{R}})}{(\partial_\theta\mathcal{E}^{0,1}(M) + \bar{\partial}_\theta\mathcal{E}^{1,0})_{\mathbb{R}}},$$

where θ is a closed real 1-form and $\mathcal{E}^{p,q}(M)$ is the space of smooth (p,q)-forms on M.

Lemma 9.4. *Let T be a $\partial_\theta\bar{\partial}_\theta$-closed (1,1)-current on M. Then we can choose a smooth representative $\phi \in [T]$ in $V_\theta^{1,1}(M)_{\mathbb{R}}$.*

Proof . Take an open covering of M by open coordinate balls U_i such that $T_i = e^{f_i}(\partial\bar{S}_i + \bar{\partial}S_i)$ on U_i for a suitable current S_i. By the definition of T,

$$e^{f_i}(\partial\bar{S}_i + \bar{\partial}S_i) = e^{f_j}(\partial\bar{S}_j + \bar{\partial}S_j) \text{ on } U_i \cap U_j.$$

Hence the following positive constants c_{ij} exist.

$$\partial\bar{S}_i + \bar{\partial}S_i = c_{ij}(\partial\bar{S}_j + \bar{\partial}S_j) \text{ on } U_i \cap U_j. \tag{9.1}$$

Let $\Psi_{ij} = S_i - c_{ij}S_j$. Then Ψ_{ij} is a smooth on $U_i \cap U_j$. In fact, by $\partial\bar{\partial}\Psi_{ij} = 0$, there exist a distribution χ such that $\Psi_{ij} = \partial\chi$. Moreover, since $d(\Psi_{ij} + \overline{\Psi_{ij}}) = 0$ by (9.1), there exists a distribution $\chi^{'}$ such that $\Psi_{ij} + \overline{\Psi_{ij}} = \partial\bar{\partial}\chi^{'}$. Therefore, since Ψ_{ij} is smooth by $\partial\bar{\partial}\chi = 0$, $e^{f_i}S_i - e^{f_j}S_j$ is smooth.

Let (η_i) be a partition of unity subordinate to (U_i). Let $S = \Sigma\eta_i e^{f_i}S_i$ and $\alpha = T - (\partial_\theta\bar{S} + \bar{\partial}_\theta S)$. Since $e^{f_i}S_i - e^{f_j}S_j$ is smooth, $S - e^{f_j}S_j$ is smooth. Therefore α is smooth, and T can be written as $T = \alpha + \partial_\theta S + \overline{\partial_\theta S}$. □

We will prove that compact complex manifolds bimeromorphic to LCK manifolds, in particular LC class C manifolds, admit an LC-balanced metric. To do so, we use Alessandrini and Bassanelli's method [3] [4], i.e, we will show that if there exists a modification

$$\mu \colon \tilde{M} \to M$$

from an LC-balanced manifold $\tilde{M}$, M is an LC-balanced manifold.

We will prove the following theorem.

Theorem 9.5. *Let $f \colon \tilde{M} \to M$ be a proper modification. Let T be a positive $\partial_\theta\overline{\partial}_\theta$-closed $(1,1)$ current on M. Then there exists a positive $\partial_{\theta'}\overline{\partial}_{\theta'}$-closed $(1,1)$ current $\tilde{T}$ on $\tilde{M}$, where $\theta^{'} := f^*\theta$, such that $f_*\tilde{T} = T$ and $\tilde{T} \in f^*[T] \in V^{1,1}_{\theta'}(\tilde{M})_{\mathbb{R}}$.*

To begin with, we quote the following Proposition 9.6 due to Otiman [12].

Proposition 9.6. [12] *Let X be a complex, compact manifold and θ a real closed 1-form. There exists a transverse (p,p) d_θ-closed form if and only if there are no positive currents which are (p,p)-components of d_θ-boundaries.*

In order to prove Theorem 9.5, we prepare some propositions.

Proposition 9.7. *Let $f \colon \tilde{M} \to M$ be a blow up with smooth center Y and $\tilde{T}$ a $\partial_\theta\overline{\partial}_\theta$-closed real $(1,1)$-current on $\tilde{M}$ such that*

(A) For any $x \in M$, there exists an open neighborhood W of x such that $\tilde{T}_{|f^{-1}(W)}$ is a weak limit of currents which are the $(1,1)$-component of d_θ-boundaries.

Then

(B) For any $x \in M$, there exists an open neighborhood W of x such that $\tilde{T}_{|f^{-1}(W)}$ is the $(1,1)$-component of d_θ-boundaries.

Proof . We take a neighborhood W of $x \in M$ and a function h on $f^{-1}(W)$ such that $\partial\overline{\partial}(e^{-h}\tilde{T}_{|f^{-1}(W)}) = 0$ on $f^{-1}(W)$. By assumption (A), there is a sequence (S_m) of currents such that $\partial_{\partial h}\overline{S}_m + \overline{\partial_{\partial h}}S_m \xrightarrow{w} \tilde{T}_{|f^{-1}(W)}$. Therefore $e^{-h}(\partial_{\partial h}\overline{S}_m + \overline{\partial_{\partial h}}S_m) = \partial\overline{S'_m} + \overline{\partial}S'_m \xrightarrow{w} e^{-h}\tilde{T}_{|f^{-1}(W)}$. By Lemma 3.2 of [4], there exists a current S such that $e^{-h}\tilde{T}_{|f^{-1}(W)} = \partial\overline{S} + \overline{\partial}S$. Therefore, we can conclude (B). □

Proposition 9.8. *Let $f\colon \tilde{M} \to M$ be a proper modification and θ a Lee form of M. Let $\tilde{T}$ be a $\partial_{\theta'}\overline{\partial}_{\theta'}$-closed real $(1,1)$-current on $\tilde{M}$ such that $[f_*\tilde{T}]$ in $V^{1,1}_{\theta}(M)_{\mathbb{R}}$, where $\theta' = f^*\theta$, and*

(*) For any $x \in M$, there exists open neighborhood W of x such that $\tilde{T}_{|f^{-1}(W)}$ is the $(1,1)$-component of $d_{\theta'}$-boundaries.

Then $[\tilde{T}] = 0$ in $V^{1,1}_{\theta'}(M)_{\mathbb{R}}$.

Proof . By Lemma 9.4, we may choose a smooth representative ϕ of the class $[\tilde{T}]$ in $V^{1,1}_{\theta}(\tilde{M})_{\mathbb{R}}$. We need only to prove that $[\phi] = 0$ in $V^{1,1}_{\theta'}(\tilde{M})_{\mathbb{R}}$. Let $\{W_j\}$ be a open covering of M such that W_j satisfy (*) for ϕ, in $f^{-1}(W_j)$

$$\phi = e^{g_j \circ f}(\overline{\partial}r_j + \partial\overline{r}_j),\ \theta = dg_j \text{ on } W_j$$

for a suitable $r_j \in E^{1,0}(f^{-1}(W_j))$. Define $\hat{\phi}_j$ as follows,

$$\hat{\phi}_j = \overline{\partial}r_j + \partial\overline{r}_j$$
$$f_*\hat{\phi}_j = \overline{\partial}S_j + \partial\overline{S}_j$$

for a suitable $(1,0)$-current S_j. Therefore

$$f_*r_j - S_j + f_*\overline{r}_j - \overline{S}_j \in H^0(W_j, \mathrm{Ker}(\overline{\partial} + \partial)).$$

Moreover, there exist an $\alpha_j \in \Omega^1(W_j)$ and a distribution A_j such that $f_*r_j - S_j = \alpha_j - \partial A_j$ (Remark 1.6 of [4]). Since $\hat{\phi}_j = \overline{\partial}(r_j - f^*\alpha_j) + \partial(\overline{r}_j - f^*\overline{\alpha}_j)$ in $f^{-1}(W_j)$, we can consider $r_j - f^*\alpha_j$ instead of r_j, and write

$$f_*r_j - S_j = -\partial A_j \text{ in } W_j.$$

Therefore

$$f_*(e^{g_j \circ f}r_j) - e^{g_j}S_j = -e^{g_j}\partial A_j,$$

so that

$$f_*(e^{g_j \circ f}r_j - e^{g_k \circ f}r_k + e^{g_j \circ f}\overline{r}_j - e^{g_k \circ f}\overline{r}_k) = d_\theta(A_k - A_j) \text{ in } W_j \cap W_k.$$

Since f is a proper modification, locally, f_* gives an isomorphism

$$H^1_{\theta'}(f^{-1}(W_j \cap W_k)) \cong H^1_\theta(W_j \cap W_k),$$

so that

$$(e^{g_j \circ f}r_j - e^{g_k \circ f}r_k + e^{g_j \circ f}\overline{r}_j - e^{g_k \circ f}\overline{r}_k) = d_{\theta'}f_{jk},\ f_{jk} \in E^0_{\mathbb{R}}(f^{-1}(W_j \cap W_k)).$$

This implies
$$d_\theta(f_* f_{jk}) = d_\theta(A_k - A_j),$$
or
$$f_* f_{jk} = A_k - A_j + e^{g_{jk}} c_{jk} \text{ in } W_j \cap W_k,$$
for suitable real constants c_{jk}. We consider $f_{jk} - e^{g_{jk}\circ f} c_{jk}$ instead of f_{jk}, and write
$$f_* f_{jk} = A_k - A_j \text{ in } W_j \cap W_k.$$
Since f_{jk} is smooth,
$$f_{jk} + f_{kl} + f_{lj} = 0 \text{ in } f^{-1}(W_j \cap W_k \cap W_l).$$
Moreover, locally, there exist functions $f_j \in E^0_{\mathbb{R}}(f^{-1}(W_j))$ such that
$$f_j - f_k = f_{jk} \text{ in } f^{-1}(W_j \cap W_k).$$
Therefore, since
$$e^{g_j\circ f} r_j - \partial_{\theta'} f_j = e^{g_k \circ f} r_k - \partial_{\theta'} f_k,$$
$r|_{f^{-1}(W_j)} = e^{g_j \circ f} r_j - \partial_{\theta'} f_j$ defines a 1-form on $\tilde{M}$, and satisfies $\phi = \overline{\partial}_{\theta'} r + \partial_{\theta'} \overline{r}$. □

Here we give a proof of Theorem 9.5 using Proposition 9.8, which is almost same as Theorem 3 of [3].

Proof of Theorem 9.5. Since f is a proper modification, for any $x \in M$ there exists an open neighborhood V of x in M, a complex manifold Z and holomorphic maps $g\colon Z \to f^{-1}(V), q\colon Z \to V$ such that $q = f \circ g$, where g is a blow up and q is obtained as a finite sequence of blow ups with smooth centers $\{q_1, \dots, q_s\}$, where $q_j\colon V_j \to V_{j-1}, V_0 = V, V_s = Z$.

For every $x \in M$, we assume that V is contained in a chart,
$$T|_V = e^g(\partial \overline{S} + \overline{\partial} S) = e^g \hat{S},$$
and also that T is a weak limit, in V, of a sequence $\{e^g \hat{S}_\epsilon\}$ of smooth positive currents which are $(1,1)$-components of d_θ-boundaries. Define
$$T^j = \lim_{\epsilon \to 0} q_j^* \circ \cdots \circ q_1^* e^g \hat{S}_\epsilon$$
in V_j, and $\tilde{T}_{|f^{-1}(V)} := g_* T^s$.

Applying Proposition 9.7 to T^1, we get (B); moreover $(q_1)_* T^1 = T_{|V}$ is the $(1,1)$-component of a d_θ-boundary, hence by Proposition 9.8, $[T^1] = 0$ in $V^{1,1}_{q_1^*\theta}(V_1)_{\mathbb{R}}$. By induction, we conclude that T^s is the $(1,1)$-component of a $d_{q^*\theta}$-boundary in $V_s = Z$, therefore $[\tilde{T}_{|f^{-1}(V)}] = 0$ in $V^{1,1}_{q^*\theta}(f^{-1}(V))_{\mathbb{R}}$.

Let ϕ be a smooth representative of the class $[T]$ in $V^{1,1}_\theta(M)_{\mathbb{R}}$; $\phi_{|V} = e^g(\overline{\partial} r + \partial \overline{r})$. $f^*\phi_{|V}$ is the $(1,1)$-component of a $d_{\theta'}$-boundary, therefore we can apply Proposition 9.8 to $\tilde{T} - f^*\phi$ and we get $[\tilde{T}] = [f^*\phi] = f^*[T]$ in $V^{1,1}_{\theta'}(\tilde{M})_{\mathbb{R}}$. □

Theorem 9.5 implies:

Theorem 9.9. *Let $\tilde{M}$ be an LC-balanced manifold, and $f\colon \tilde{M} \to M$ be a proper modification with exceptional set E which is mapped onto the subvariety $Y = f(E)$. Then M is an LC-balanced manifold.*

Proof . Let $\theta^{'}$ be a Lee form on $\tilde{M}$ and θ a push-forward of $\theta^{'}$ through f. We may assume that θ is a smooth d-closed 1-form. Suppose that M is not LC-balanced. Then, by Proposition 9.6, there exists a positive $(1,1)$ current $T = \overline{\partial}_\theta S + \partial_\theta \overline{S}$. By Theorem 9.5, there exists a positive $(1,1)$-current $\tilde{T}$ such that $f^*[T] = [\tilde{T}]$. Therefore $\tilde{T}$ is a positive component of a $d_{\theta'}$-boundary on $\tilde{M}$. This implies $\tilde{T} = 0$. Thus supp$T \subset Y$, but the codimension of Y is greater than one; hence by theorem 2.1 of [2], $T = 0$. □

Corollary 9.10. *Any LC class C manifolds admit an LC-balanced metric.*

Since the compact complex manifolds obtained in Proposition 8.1 are LC-class C,

Corollary 9.11. *The compact complex manifolds obtained in Proposition 8.1 are LC-balanced.*

Remark 9.12. It is known that the blow-up of an LC-Balanced manifold along its compact induced globally conformally balanced submanifold also admits a LC-Balanced manifold structure [14].

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