<table>
<thead>
<tr>
<th>Title</th>
<th>Existence of pencils with prescribed scrollar invariants of some general type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Coppens, Marc</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 36(4) P.1049-P.1057</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1999</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/7071">https://doi.org/10.18910/7071</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/7071</td>
</tr>
<tr>
<td>Note</td>
<td></td>
</tr>
</tbody>
</table>
EXISTENCE OF PENCILS WITH PRESCRIBED SCROLLAR INVARIANTS OF SOME GENERAL TYPE

MARC COPPENS

(Received January 12, 1998)

0. Introduction

Let $C$ be an irreducible smooth projective non-hyperelliptic curve of genus $g$ defined over the field $C$ of complex numbers. Let $g_k^1$ be a complete base-point free special linear system on $C$. The scrollar invariants of $g_k^1$ are defined as follows. Let $C$ be canonically embedded in $P^{g-1}$ and let $X$ be the union of the linear spans $\langle D \rangle$ with $D \in g_k^1$. This defines a set of integers $e_1 \geq \ldots \geq e_{k-1} \geq 0$ such that $X$ is the image of the projective bundle $P(e_1; \ldots; e_{k-1}) = P(\mathcal{O}_P(1) \oplus \ldots \oplus \mathcal{O}_P(e_{k-1}))$ using the tautological bundle (see e.g. [2]; [7]). Those integers $e_1; e_2; \ldots; e_{k-1}$ are called the scrollar invariants of $g_k^1$.

Those scrollar invariants determine (and are determined by) the complete linear systems associated to multiples of the linear system $g_k^1$. For $1 \leq i \leq k-1$ the invariant $e_i$ is one less than the number of non-negative integers $j$ satisfying $\dim(\langle K_C - jg_k^1 \rangle) - \dim(\langle K_C - (j+1)g_k^1 \rangle) \geq i$. Here $K_C$ denotes a canonical divisor on $C$. Let $m = e_{k-1} + 2$. Then $m$ is defined by the following conditions: $\dim(\langle (m-1)g_k^1 \rangle) = m-1$ and $\dim(\langle mg_k^1 \rangle) > m$. In case $mg_k^1$ is birationally very ample then the scrollar invariants satisfy the inequalities $e_i \leq e_{i+1} + m$ for $1 \leq i \leq k-2$ (see [3]). In case $k = 3$ this number $m = e_2$ determines also the other scrollar invariant $e_1$. It is the starting point for so-called Maroni-theory for linear systems on trigonal curves (see [4]; [5]). Scrollar invariants for 4-gonal curves are intensively studied in [1]; [3] and for 5-gonal curves in [6].

For $(j-1)m - 1 < x \leq jm - 1$ with $j$ satisfying the scrollar invariants imply $\dim(\langle xg_k^1 \rangle) \geq \frac{j(j-1)}{2}m - 1 + (x - (j-1)m + 1)j$. Equality (if not in conflict with the Riemann-Roch Theorem) can be expected being the most general case for a fixed value of $m$. The inequalities also imply $\dim(\langle (k-1)mg_k^1 \rangle) = \dim(\langle ((k-1)m-1)g_k^1 \rangle) + k$. This implies that $\langle ((k-1)m-1)g_k^1 \rangle$ is not special. Using the dimension bound one obtains $g \leq [(k^2 - k)m - 2k + 2]/2$. (This easy but interesting consequence from the inequalities is not mentioned by Kato and Ohbuchi.) In this paper we prove the following theorem.

**Theorem.** For all nonnegative integers $k$; $m$ and $g$ satisfying $k \geq 3$; $m \geq 2$ and $k-1 \leq g \leq [(k^2 - k)m - 2k + 2]/2$ there exists a smooth curve $C$ of genus $g$ possessing
a complete base point free linear system \( g_k \) satisfying the following property. For each nonnegative integer \( x \) with \( x \leq (k-1)m - 1 \) define the nonnegative integer \( j \) such that 
\[(j - 1)m - 1 < x \leq jm - 1.\]
Then \( \dim([xg_k]) = \max\left\{ \left\lfloor \frac{j(j-1)}{2} m - 1 + (x - (j - 1)m + 1)j; kx - g \right\rfloor \right\} \). Also \( |mg_k| \) is birationally very ample.

The curves \( C \) are obtained using special plane curves degenerating to special types of rational curves. First we construct those rational curves \( \Gamma_0 \) using some linear system \( g_k \) on \( P^1 \). In order to prove the theorem we study canonical adjoint curves of \( \Gamma_0 \) containing all points belonging to a given number of divisors from \( g_k \).

**SOME NOTATIONS.** On a smooth surface \( X \); if \( \Gamma_1 \) and \( \Gamma_2 \) are two effective divisors intersecting at \( x \in X \) (no common component containing \( x \)) then we write \( i(\Gamma_1, \Gamma_2; x) \) for the intersection multiplicity of \( \Gamma_1 \) and \( \Gamma_2 \) at \( x \). We write \( (\Gamma_1, \Gamma_2) \) for the intersection number of \( \Gamma_1 \) and \( \Gamma_2 \). We also write \( K_X \) for a canonical divisor on \( X \).

1. **Construction of the plane rational curve**

Choose a general linear system \( g_k \) on \( P^1 \) and a general divisor \( F \in g_k \). Choose a general effective divisor \( E \) of degree \( mk \) on \( P^1 \). Consider the linear system \( g^2_{mk} \) containing \((m-1)F + g_k \) and \( E \).

**Claim 1.1.** \( g^2_{mk} \) is a simple base point free linear system on \( P^1 \).

**Proof.** The linear system \( g^2_{mk} \) has no base points: \( mF \in (m-1)F + g_k \subset g^2_{mk} \) and \( E \cap F = \emptyset \). For \( P \in E \) and \( D_P \in g_k \) containing \( P \) one has \( E \cap D_P = \{P\} \) (the intersection as schemes is reduced), therefore also \( E \cap ((m-1)F + D_P) = \{P\} \). Since \((m-1)F + D_P \in g^2_{mk} \) this implies that \( g^2_{mk} \) is simple.

**Claim 1.2.** The space parametrizing such linear systems \( g^2_{mk} \) on \( P^1 \) is irreducible of dimension \( mk + 2k - 3 \).

**Proof.** Effective divisors of degree \( d \) on \( P^1 \) are parametrized by a projective space \( P^d \). Linear systems \( g^1_k \) (resp. \( g^2_{mk} \)) on \( P^1 \) are parametrized by a grassmannian \( G(1;k) \) (resp. \( G(2; mk) \)) of lines in \( P^k \) (resp. planes in \( P^{mk} \)). On \( G(1;k) \times P^k \) we have the incidence subvariety \( \mathcal{I} \) defined as \((g^1_k; F) \in \mathcal{I} \) if and only if \( F \in g^1_k \). Clearly \( \mathcal{I} \) is irreducible of dimension \( \dim(G(1;k)) + 1 = 2k - 1 \). The linear systems \( g^2_{mk} \) constructed above belong to the image of the rational map \( \tau : \mathcal{I} \times P^{mk} \to G(2; mk) \) defined by \( \tau((g^1_k; F); E) = ((m-1)F + g^1_k; E) \).

Suppose for \((g^1_k; F); E) \in I \times P^{mk} \) general, there exists another element \((h^1_k; G); E') \in I \times P^{mk} \) with \( \tau((g^1_k; F); E) = \tau((h^1_k; G); E') \) but \((g^1_k; F) \neq (h^1_k; G) \). Because
(m - 1)F + g^1_k and (m - 1)G + h^1_k are both lines in g^2_{mk}, one has [(m - 1)F + g^1_k] \cap [(m - 1)G + h^1_k] \neq \emptyset. Assume (m - 1)F + g^1_k = (m - 1)G + h^1_k. Because g^1_k has no fixed points, this implies F = G and so g^1_k = h^1_k; a contradiction. Choose D \in h^1_k with (m - 1)G + D \notin (m - 1)F + g^1_k. Then g^2_{mk} = \langle(m - 1)F + g^1_k; (m - 1)G + D\rangle. Because g^2_{mk} is base point free (Claim 1.1) we find F \cap G = \emptyset. But then [(m - 1)F + g^1_k] \cap [(m - 1)G + h^1_k] \neq \emptyset implies m = 2; F \in h^1_k; G \in g^1_k and so g^1_k = h^1_k = (F; G). For D \in g^1_k one finds F + D; G + D \in g^2_{2k} so g^1_k + D \subset g^2_{2k}. It follows that g^2_{2k} = \{D_1 + D_2 : D_1; D_2 \in g^1_k\}. This contradicts g^2_{2k} being simple (Claim 1.1). So we find for a general \langle(g^1_k; F); E\rangle \in I \times P^{mk} one has \tau((g^1_k; F); E) = \tau((h^1_k; G); E') if and only if \langle(g^1_k; F) = (h^1_k; G) and E' \in \langle(m - 1)F + g^1_k; E\rangle. Therefore the general non-empty fiber of \tau has dimension 2. So, the image of \tau has dimension mk + 2k - 3.

Associated to g^2_{mk} there exist morphisms \phi : P^1 \to P^2. Fix such a morphism and let \Gamma be the image.

**Claim 1.3.** \(\Gamma\) is a plane curve of degree mk. The divisor F induces a singular point s on \(\Gamma\) of multiplicity \((m - 1)k\). The other singular points of \(\Gamma\) are ordinary nodes.

**Proof.** Since g^2_{mk} is simple and base point free (Claim 1.1) the plane curve \(\Gamma\) has degree mk. There is a 1-dimensional subsystem of g^2_{mk} containing \((m - 1)F\). This 1-dimensional subsystem corresponds to a pencil of lines on P^2 containing some fixed point s. For a general line L containing s there are k intersections each one of multiplicity 1 with \(\Gamma\) outside s. This implies \(i(\Gamma; L; s) = (m - 1)k\), hence \(\Gamma\) has multiplicity \((m - 1)k\) at s. Assume s' is another singular point of \(\Gamma\).

First assume s' has multiplicity \(\mu \geq 3\). The pencil of lines on P^2 containing s' induces a linear subsystem F' \in g^{1}_{mk-\mu} \subset g^2_{mk}. From s \neq s' it follows that \(F \cap F' \neq \emptyset\). The line \langle ss'\rangle on P^2 gives rise to \((m - 1)F + D \in g^2_{mk} \cap D \in g^1_k\). We find D = F' + D' for some effective divisor D'. Let E' be the divisor corresponding to a general line through s', then E' = F' + E'' for some E'' \in g^1_{mk-\mu}. Since g^2_{mk} = \langle(m - 1)F + g^1_k; E'\rangle we find that g^2_{mk} belongs to the image of the morphism \(\tau' : I' \times P^{mk-\mu} \to G(2; mk)\) defined by \(\tau'((g^1_k; F'; E')); E'') = \langle(m - 1)F + g^1_k; E' + E''\rangle\) with \(I' \subset I \times P^\mu\) defined by \(\langle(g^1_k; F; F'); E'\rangle\) if and only if \(D \geq F'\) for some \(D \in g^1_k\) (here I is as in the proof of Claim 1.2). The choice of E'' implies that \(\tau'\) has non-empty fibers of dimension at least 1. Since \(\dim(I' \times P^{mk-\mu} = 2k + mk - \mu \leq (m + 2)k - 3\). This contradicts \(\mu \geq 3\). It follows that s' has multiplicity 2.

Using the same notations we have \(\mu = 2; \deg(F') = 2\). Assume F' = 2P_0. Then P_0 is a ramification point of \(g^1_k\).

This implies that g^2_{mk} belongs to the image of the morphism \(\tau^u : I'' \times P^{mk-2} \to G(2; mk)\) defined by \(\tau^u((g^1_k; F; P_0); E'') = \langle(m - 1)F + g^1_k; 2P_0 + E''\rangle\), with \(I'' \subset I \times P^1\) defined by \(\langle(g^1_k; F; P_0) \in I''\) if and only if \(P_0\) is a ramification point of \(g^1_k\). Again, the non-empty fibers have dimension at least 1. Since \(\dim(I'' \times P^{mk-2}) = 2k - 1 + mk - 2\), we find a contradiction to \(\dim(imr) = (m + 2)k - 3\).
We obtain $F' = P_0 + Q_0$ with $P_0 \neq Q_0$. Assume $L_0$ is a line through $s'$ such that $L_0$ induces $2(P_0 + Q_0) + E'''$ for some effective divisor $E'''$ of degree $mk-4$. Hence, we assume that $s'$ is a tacnode. This implies $g_{mk}^2$ belongs to the image of the rational map $\tau'' : I'' \times P^{mk-4} \to G(2; mk)$ defined by $\tau''((g_1^k, F; F'); E'') = ((m-1)F + g_1^k; 2F' + E'')$ with $I'' \subset I \times P^2$ defined by $(g_1^k; F; F') \in I''$ if and only if $D \geq F'$ for some $D \in g_1^k$. Because $\dim(I'' \times P^{mk-4}) = (m + 2)k - 4 < (m + 2)k - 3$, once more we obtain a contradiction. This implies that $s'$ is an ordinary node.

Because $mF \in g_{mk}^2$, there exists a line $\Gamma$ on $P^2$ through $s$ inducing $mF$. This line $T$ intersects $\Gamma$ only at $s$, hence $i(T, \Gamma; s) = mk$. We can consider the singularity of $\Gamma$ at $s$ as follows. It consists of exactly $k$ locally irreducible branches (we use $F \in P^k$ is general), each one having multiplicity $m - 1$ at $s$ and having $T$ as "tangent line" intersecting the branch with multiplicity $m$ at $s$. From now on we fix $s$ and $T$.

**Claim 1.4.** We obtain a family of plane curves of dimension $(m + 2)k - 1$.

**Proof.** This follows from Claim 1.2 taking into account that $\dim(\text{Aut}(P^1)) = 3$; $\dim(\text{Aut}(P^2)) = 8$ and fixing $s$ and $T$ imposes $3$ independent conditions on $\Gamma$. \qed

2. **Blowing up the projective plane**

Let $\pi_1 : X_1 \to P^2$ be the blowing-up of $P^2$ at $s$; let $E_1$ be the exceptional divisor. Let $T_1$ (resp. $\Gamma_1$) be the strict transform of $T$ (resp. $\Gamma$) on $X_1$. Let $L$ be the inverse image of a line on $P^2$. Then $T_1 \subset |L - E_1|; \Gamma_1 \subset |kmL - k(m - 1)E_1|$. Let $s_1 = E_1 \cap T_1$.

The linear system $|L - E_1|$ induces $g_1^k$ on $P^1$ and $T_1$ induces $F$. Since the images of points of $F$ under the morphism $P^1 \to \Gamma_1$ are contained in $E_1$, it follows that $i(T_1, \Gamma_1; s_1) = k$. Hence the $k$ different points of $F$ correspond to $k$ different irreducible branches of $\Gamma_1$ at $s_1$. Hence $\Gamma_1$ has a singular point of multiplicity $k$ at $s_1$. Also $E_1 \cap \Gamma_1 = \{s_1\}$ and since $(E_1, \Gamma_1) = k(m - 1)$ it follows that $i(E_1, \Gamma_1; s_1) = (m - 1)k$. Because $T_1 + E_1$ induces $mF$ on $P^1$, it follows that $E_1$ intersects each branch of $\Gamma_1$ at $s_1$ with multiplicity $m - 1$ at $s_1$.

Let $\pi_2 : X_2 \to X_1$ be the blowing-up of $X_1$ at $s_1$. Let $E_2$ be the exceptional divisor. We continue to write $L$ for the inverse image of a general line on $P^2$. Let $E_{12}$ (resp. $T_2; \Gamma_2$) be the strict transforms of $E_1$ (resp. $T; \Gamma$) on $X_2$. We also write $E_1$ to denote the inverse image of $E_1$ on $X_2$. Then $E_{12} \subset |E_1 - E_2|; T_2 \subset |L - E_1 - E_2|; \Gamma_2 \subset |kmL - k(m - 1)E_1 - kE_2|$. Let $s_2 = E_2 \cap E_{12}$. One has $(T_2, \Gamma_2) = 0$ hence $T_2 \cap \Gamma_2 = \emptyset$. In case $m = 2$ we find $(T_2, E_{12}) = 0$ hence $\Gamma_2 \cap E_{12} = \emptyset$.

Assume $m > 2$. From $(T_2, E_2) = k$ it follows that each branch of $\Gamma_2$ corresponding to a point of $F$ is smooth and intersects $E_2$ transversally at one point. Because $E_2 + E_{12}$ induces $(m - 1)F$ on $P^1$ it follows that those points of $F$ map to $s_2$ and $E_{12}$ intersects each branch with multiplicity $m - 2$ at $s_2$. It follows that $\Gamma_2$ has multiplicity $k$ at $s_2$. 


We continue to make blowings-up. For each \( i \leq m \) we obtain the blowing-up \( \pi_i : X_i \to X_{i-1} \) with exceptional divisor \( E_i \). On \( X_i \) we continue to write \( L \) to denote the inverse image of a general line on \( P^2 \). We write \( \Gamma_i \) (resp. \( E_{i-1,i} ; T_i \)) to denote the strict transform of \( \Gamma \) (resp. \( E_{i-1} ; T \)) on \( X_i \). Also, for \( j \leq i - 2 \) we write \( E_{j,i} \) for the strict transform of \( E_{j,i-1} \). Let \( s_i = E_i \cap E_{i-1,i} \). In case \( i < m \) the multiplicity of \( \Gamma_i \) at \( s_i \) is \( k_i \). At \( s_i \) the curve \( \Gamma_i \) has \( k_i \) smooth locally irreducible branches. Also, for \( j < i - 2 \) we write \( E_{j,i} \) for the strict transform of \( E_{j,i-1} \).

Let \( s_i^* = E_i \cap E_{i-1,i}^* \). In case \( i < m \) the multiplicity of \( \Gamma_i \) at \( s_i^* \) is \( k_i \). At \( s_i^* \) the curve \( \Gamma_i \) has \( k_i \) smooth locally irreducible branches. Also, for \( j < i - 2 \) we write \( E_{j,i} \) for the strict transform of \( E_{j,i-1} \).

Let \( s_i^* = E_i \cap E_{i-1,i}^* \). In case \( i < m \) the multiplicity of \( \Gamma_i \) at \( s_i^* \) is \( k_i \). At \( s_i^* \) the curve \( \Gamma_i \) has \( k_i \) smooth locally irreducible branches. Also, for \( j < i - 2 \) we write \( E_{j,i} \) for the strict transform of \( E_{j,i-1} \).

Let \( s_i^* = E_i \cap E_{i-1,i}^* \). In case \( i < m \) the multiplicity of \( \Gamma_i \) at \( s_i^* \) is \( k_i \). At \( s_i^* \) the curve \( \Gamma_i \) has \( k_i \) smooth locally irreducible branches. Also, for \( j < i - 2 \) we write \( E_{j,i} \) for the strict transform of \( E_{j,i-1} \).

Let \( s_i^* = E_i \cap E_{i-1,i}^* \). In case \( i < m \) the multiplicity of \( \Gamma_i \) at \( s_i^* \) is \( k_i \). At \( s_i^* \) the curve \( \Gamma_i \) has \( k_i \) smooth locally irreducible branches. Also, for \( j < i - 2 \) we write \( E_{j,i} \) for the strict transform of \( E_{j,i-1} \).

Claim 2.1. \( \Gamma_m \) has ordinary nodes as its only singularities. The intersection points of \( \Gamma_m \) and \( E_m \) are smooth points on \( \Gamma_m \).

Proof. Because of Claim 1.3 it is enough to prove that the intersection points of \( \Gamma_m \) and \( E_m \) are smooth points on \( \Gamma_m \). The inverse image on \( P^1 \) of the intersection as schemes of \( \Gamma_m \) and \( E_m \) is the divisor \( F \), hence a general divisor of degree \( k \) on \( P^1 \). If that intersection would not be smooth then 2 different points in \( F \) would have the same image on \( \Gamma_m \). Because of monodromy on \( P^1 \), in that case all \( k \) points on \( F \) need to have the same image on \( \Gamma_m \), hence \( \Gamma_m \cap E_m \) is a single multiple point \( s_m \) of \( \Gamma_m \). Since \( (\Gamma_m \cdot E_m) = 0 \) and \( (\Gamma_m \cdot E_{m-1,m}) = 0 \) it follows that \( s_m \notin \{ E_{1,m} \cap E_m; E_{m-1,m} \cap E_m \} \). Let \( \pi_{m+1} : X_{m+1} \to X_m \) be the blowing-up of \( X_m \) at \( s_m \). Let \( E_{m+1} \) be the exceptional divisor of \( \pi_{m+1} \) and let \( \Gamma_{m+1} \) be the strict transform of \( \Gamma_m \).

We find \( \Gamma_{m+1} \in P_{m+1} := [kmL - k(m-1)E_1 - kE_2 - \ldots - kE_{m+1}] \). If \( \Gamma_{m+1} \) is not smooth at each point of \( \Gamma_{m+1} \cap E_{m+1} \) then as before we find \( s_{m+1} \in \Gamma_{m+1} \) such that \( \Gamma_{m+1} \) has multiplicity \( k \) at \( s_{m+1} \). In that case we blow-up \( \Gamma_{m+1} \) at \( s_{m+1} \) and so on.

For some \( m' \geq 1 \) we obtain \( X_{m+m'} \) and \( \Gamma_{m+m'} \in P_{m+m'} := [kmL - k(m-1)E_1 - kE_2 - \ldots - kE_{m+m'}] \) such that \( \Gamma_{m+m'} \) has ordinary nodes as its only singularities. The arithmetic genus of \( \Gamma_{m+m'} \) is equal to \( \frac{(km-1)(km-2) - (km-1)(km-m'-1)(km-m'-1)(k-1)k}{2} \). This has to be at least \( 0 \), hence \( (m-m')k^2 + (m'-m)k-2k+2) \geq 0 \). This condition implies \( m' \leq m \).

In \( P_{m+m'} \) we find that the locus of irreducible rational nodal curves has a component of dimension at least \( mk + 2k - 1 - m' \). (This follows from Claim 1.4 taking into account the choice of \( s_{m+i} \) on \( E_{m+i} \) for \( 0 \leq i \leq m' \).) The number of nodes of \( \Gamma_{m+m'} \) is equal to the arithmetic genus of \( \Gamma_{m+m'} \) being \( \delta = \frac{(m-m')(k^2 - k) - 2k + 2}{2} \). Because \( m' \leq m \) we find \( (K_{X_{m+m'}}, \Gamma_{m+m'}) = -3km + k(m-1) + k(m+m'-1) = (m'-m-2)k < 0 \). From Lemma 2.2 in [8] it follows that \( \dim(P_{m+m'}) \geq mk + 2k - 1 - m' + \delta \). Also from the end of the proof of Lemma 2.2 in [8] we also obtain \( \dim(P_{m+m'}) = \delta = (K_{X_{m+m'}}, \Gamma_{m+m'}) - 1 = \delta + (m+2-2m')k - 1 \). This would imply \((m+2-m')k - 1 - m'y \geq mk+2k-1-m' \geq m'y \geq k \geq 2 \) this is a contradiction. This completes the proof of the claim.

\( \square \)
3. Canonically adjoint curves

In order to study canonically adjoint curves for curves belonging to \( P \) we consider the linear system \( P'_0 = |(km - 3)L - (k(m - 1) - 1)E_1 - (k - 1)E_2 - \ldots - (k - 1)E_m|. \)

Claim 3.1. \( P'_0 = P_0 + \text{(fixed components)} \) with \( P_0 = |(km - 2 - m)L - (k(m - 1) - m)E_1 - (k - 2)E_2 - \ldots - (k - 2)E_m| \).

Proof. From \( T_m P'_0 = -1 \) it follows that \( T_m \) is a fixed component of \( P'_0 \). Deleting \( T_m \) from \( P_0 \) we obtain \( |(km - 4)L - (k(m - 1) - 2)E_1 - (k - 2)E_2 - (k - 1)E_3 - \ldots - (k - 1)E_m| \). Continuing in this way one finds fixed components \( E_3m, \ldots, E_{m-1,m} \). Deleting them, one obtains \( |(km - 4)L - (k(m - 1) - 2)E_1 - (k - 1)E_2 - \ldots - (k - 1)E_m| \). Now \( T_m \) is a fixed component. Deleting \( T_m \) one obtains \( |(km - 5)L - (k(m - 1) - 3)E_1 - (k - 2)E_2 - (k - 1)E_3 - \ldots - (k - 1)E_{m-1} - (k - 2)E_m| \). In case \( m = 3 \) this proves the claim. In case \( m > 3 \) one has \( E_2m, \ldots, E_{m-2,m}, T_m \) again as fixed components. Deleting them this proves the claim for \( m = 4 \); in case \( m > 4 \) one continues.

For curves \( \Gamma' \) of \( P \) we need to investigate canonical adjoint curves containing intersections of \( \Gamma' \) with elements from \( |L - E_1| \) (in terms of linear systems : containing a sum of divisors from \( q^1 \)). For a general element \( R \) of \( |L - E_1| \) the intersection of \( R \) with an element \( \Gamma_m \) of \( P \) not containing \( E_1m \) are \( k \) different points. The intersection multiplicity with an element of \( P_0 \) is \( k-2<k \). Therefore an element of \( P_0 \) containing this intersection of \( \Gamma_m \) and \( R \) contains \( R \) as a component. Taking one general elements \( R_1, \ldots, R_x \) in \( |L - E_1| \), the elements of \( P_0 \) containing \( (R_1 \cup \ldots \cup R_x) \cap \Gamma_m \) have \( R_1, \ldots, R_x \) as components. Deleting \( R_1, \ldots, R_x \) we obtain \( P'_x = |(km - 2 - m - x)L - (k(m - 1) - m - x)E_1 - (k - 2)E_2 - \ldots - (k - 2)E_m| \).

Claim 3.2. Write \( x = \ell m + y \) with \(-1 \leq y \leq m - 2 \). Then \( P'_x = P_x + \text{(fixed components)} \) with \( P_x = |(km - (\ell + 1)m - y - 2)L - (k(m - 1) - (\ell + 1)m + l - y)E_1 - (k - l - 2)E_2 - \ldots - (k - l - 2)E_m| \). (We do not claim that \( P_x \) has no more fixed components.)

Proof. First, take \( 0 \leq x \leq m - 2 \), hence \( x = y \) and \( l = 0 \). Then \( P'_x = P_x \) and there is nothing to prove.

Next, take \( x = m - 1 \), hence \( l = 1; y = -1 \). Then \( (E_{1m}P'_{m-1}) = -1 \) (the intersection number of \( E_{1m} \) with elements of \( P'_{m-1} \)), therefore \( E_{1m} \) is a fixed component of \( P'_{m-1} \). Deleting \( E_{1m} \) we obtain \( P_{m-1} \).
More general, for any \( x \geq m \) the curve \( E_{1m} \) is a fixed component of \( P'_x \). Deleting \( E_{1m} \) we obtain \(|P - (L - E) - E_{1m}| \). In case \( m \leq x < 2m - 1 \) this is \( P_x \). For \( x = 2m - 1 \) (hence \( l = 2; \ y = -1 \)) the intersection number of \( E_{1m} \) with that linear system is \(-1 \), hence \( E_{1m} \) is a fixed component. Deleting \( E_{1m} \) one obtains \( P_{2m-1} \). Continuing in this way one proves the claim.

**Remark 3.3.** Taking \( x = (k - 2)m + m - 2 \) (hence \( l = k - 2; \ y = m - 2 \)) one finds \( P_{(k-2)m+m-2} = 0 \). For \( x \geq (k - 2)m + m - 1 \) one finds \( P_x = \emptyset \).

Given \( 0 \leq x \leq (k - 2)m + m - 2 \) define the integer \( j \) by means of the inequalities \((j - 1)m - 1 < x \leq jm - 1 \) with \( j \leq k - 1 \). **Claim 3.4.** \( \dim(P_x) = \frac{j(j-1)}{2}m-1 + (x-(j-1)m+1)j-kx+\frac{(k-1)mk-2k+2}{2} - 1 \).

**Proof.** In case \( x = (k - 1)m - 2 \) we have to prove \( \dim(P_{(k-1)m-2}) = 0 \). This follows from Remark 3.3.

Now, fix some \( x < (k - 1)m - 2 \) and assume the claim is proved for \( x+1 \) instead of \( x \). Writing \( x = lm+y \) with \(-1 \leq y \leq m-2 \), from the description in Claim 3.2 we find \( \dim(P_x) \geq \frac{1}{2}[(km-(l+1)m-y+1)(km-(l+1)m-y-2)]/2 - [(k(m-1)-(l+1)m+l-y+1)(k(m-1)-(l+1)m+l-y)]/2 - [(m-1)(k-l-1)(k-l-2)]/2 \). A computation shows us that we need to prove equality. Assume for \( x \) we have strict inequality. For \( R \in |L - E_1| \) we have \( (R, P_x) = k-l-2 \), hence \( R \) imposes at most \( k-l-1 \) conditions on \( P_x \). This implies \( \dim(|P_x - (L - E_1)|) \geq \dim(P_x) - (k-l-1) \). In case \( y = m-2 \) one finds \( (|P_x - (L - E_1)|)E_{1m}) < 0 \), hence \( \dim(|P_x - (L - E_1) - E_{1m}|) \geq \dim(P_x) - (k-l-1) \). But \( |P_x - (L - E_1) - E_{1m}| = P_{x+1} \). One more computation shows that \( \dim(P_{x+1}) \geq \dim(P_x) - (k-l-1) \) gives a contradiction to the assumption that the claim holds for \( P_{x+1} \). In case \( y < m-2 \) then \( |P_x - (L - E_1)| = P_{x+1} \) and again, a computation shows a contradiction.

On \( X_m \) we constructed the rational irreducible curve \( \Gamma_m \) belonging to \( P \). From Claim 2.1 we know that \( \Gamma_m \) is a nodal curve, so it has \( g_0 = [(k^2 - k)m - 2k + 2]/2 \) ordinary nodes. We write \( s \) to denote a node of \( \Gamma_m \).

**Claim 3.5.** We can arrange the nodes \( s_1; \ldots; s_{g_0} \) is such a way that the following property holds. First we introduce some notation: for \( 0 \leq \delta \leq g_0 \) let \( P_x(s_1; \ldots; s_\delta) = \{ \Gamma \in P_x : s_i \in \Gamma \ for \ 1 \leq i \leq \delta \} \). Then \( P_x(s_1; \ldots; s_\delta) = \emptyset \) for \( \delta > \dim(P_x) \) and \( \dim(P_x(s_1; \ldots; s_\delta)) = \dim(P_x) - \delta \) if \( \delta \leq \dim(P_x) \).

**Proof.** For \( \delta = 0 \) there is nothing to prove.

Fix \( \delta > 0 \) and assume the claim holds for \( \delta - 1 \) instead of \( \delta \). So, we assume a suited arrangement \( s_1; \ldots; s_{\delta - 1} \) for a suitable part of the set of the nodes. We have to prove that the set of the remaining nodes of \( \Gamma_m \) contains a suited one to be numbered \( s_\delta \).
Numbers $x$ satisfying $\delta - 1 > \dim(P_x)$ impose no conditions on $s_\delta$. Let $x_0$ be the minimal number such that $\delta - 1 \leq \dim(P_x)$. We know that $\dim(P_{x_0}(s_1; \ldots ; s_{\delta-1})) = \dim(P_{x_0}) - (\delta - 1) \geq 0$. If each element of $P_{x_0}(s_1; \ldots ; s_{\delta-1})$ would contain all the nodes of $\Gamma_m$ then $\Gamma_m$ possesses a canonically adjoint curve. Since $\Gamma_m$ is a rational curve this is impossible. Hence, there exists a node $s_0$ such that $\dim(P_{x_0}(s_1; \ldots ; s_{\delta-1}; s_0)) = \dim(P_{x_0}) - \delta$ (with $P_{x_0}(s_1; \ldots ; s_{\delta-1}; s_0) = \emptyset$ if $\delta - 1 = \dim(P_{x_0})$). In case for all $x \leq x_0$ we find $\dim(P_x(s_1; \ldots ; s_{\delta-1}; s_0)) = \dim(P_x) - \delta$ then we can take $s_0 = s_\delta$.

Assume $x' < x_0$ such that $\dim(P_{x'}(s_1; \ldots ; s_{\delta-1}; s_0)) = \dim(P_{x'}) - \delta + 1$ while $\dim(P_{x'+1}(s_1; \ldots ; s_{\delta-1}; s_0)) = \dim(P_{x'+1}) - \delta$. Using a general $R \in |L - E_1|$ and using the arguments from the proof of Claim 3.4 one finds a contradiction.

4. Proof of the theorem

Now, we finish the proof of the theorem in the introduction. We start with the rational irreducible nodal curve $\Gamma_m$ on $X_m$. We make an arrangement of the nodes as in Claim 3.5. The main result in Section 2 of [8] implies that there exists a 1-dimensional flat family $Y \rightarrow T$ of curves on $X_m$ belonging to $P$ such that the fiber over a special point $t_0$ of $T$ is the curve $\Gamma_m$ and a general fiber is a nodal curve $\Gamma$ with exactly $g_0 - g$ nodes such that those nodes specialize to the nodes $s_1; \ldots ; s_{g_0 - g}$ on $\Gamma_m$. Define $P_{x, \Gamma} = \{D \in P_x : D$ contains the nodes of $\Gamma\}$. Clearly $\dim(P_{x, \Gamma}) \geq \dim(P_x) - (g_0 - g)$. For the special fiber $\Gamma_m$ we have $P_x(s_1; \ldots ; s_{g_0 - g}) = \emptyset$ if $g_0 - g > \dim(P_x)$ and $\dim(P_x(s_1; \ldots ; s_{g_0 - g})) = \dim(P_x) - (g_0 - g)$ if $g_0 - g \leq \dim(P_x)$. Semicontinuity implies $P_{x, \Gamma} = \emptyset$ if $g_0 - g > \dim(P_x)$ and $\dim(P_{x, \Gamma}) = \dim(P_x) - (g_0 - g)$ if $g_0 - g \leq \dim(P_x)$. Let $C$ be the normalization of $\Gamma$. It is a smooth curve of genus $g$. The linear system $|L - E_1|$ induces a linear system $g^1_k$ on $C$ without base points. Taking $x$ general elements $R_1; \ldots ; R_x$ in $|L - E_1|$ corresponds to taking $x$ general divisors in $g^1_k$. From the description of $P_x$ in 3.2 we find that $\dim(P_{x, \Gamma})$ is equal to the dimension of canonically adjoint curves $\Gamma$ containing the intersection of $\Gamma$ with $R_1 \cup \ldots \cup R_x$, hence it is equal to $\dim(|K_C - xg^1_k|)$. In particular, if $P_{x, \Gamma} = \emptyset$ then $|K_C - xg^1_k| = \emptyset$. So, we find $|K_C - xg^1_k| = \emptyset$ if $g_0 - g > \dim(P_x);$ $\dim(|K_C - xg^1_k|) = \dim(P_x) - (g_0 - g)$ if $g_0 - g \leq \dim(P_x)$. Using 3.4 and the Riemann-Roch Theorem one finds $\dim(|xg^1_k|) = \max(\frac{j(j-1)}{2}m - 1 + (x - (j - 1)m + 1)j; kx - g)$. In particular $\dim(|(m - 1)g^1_k|) = m - 1$ and $\dim(|mg^1_k|) = m + 1$. Since $\Gamma$ is obtained from $C$ using a linear subsystem of $|mg^1_k|$, one also finds $|mg^1_k|$ is birationally very ample. This finishes the proof of the theorem.

References


Katholieke Hogeschool Kempen
Departement Industrieel Ingenieur en Biotechniek
Campus H.I.Kempen
Kleinhoufstraat 4
B 2440 Geel Belgium
e-mail: marc.coppens@khk.be

The author is affiliated with the University of Leuven as a research fellow.