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EXISTENCE OF PENCILS WITH PRESCRIBED SCROLLAR INVARIANTS OF SOME GENERAL TYPE

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0. Introduction

Let C be an irreducible smooth projective non-hyperelliptic curve of genus g defined over the field \mathbb{C} of complex numbers. Let g_k^1 be a complete base-point free special linear system on C . The scrollar invariants of g_k^1 are defined as follows. Let C be canonically embedded in \mathbb{P}^{g-1} and let X be the union of the linear spans $\langle D \rangle$ with $D \in g_k^1$. This defines a set of integers $e_1 \geq \dots \geq e_{k-1} \geq 0$ such that X is the image of the projective bundle $\mathbf{P}(e_1; \dots; e_{k-1}) = \mathbf{P}(O_{\mathbb{P}^1}(e_1) \oplus \dots \oplus O_{\mathbb{P}^1}(e_{k-1}))$ using the tautological bundle (see e.g. [2]; [7]). Those integers $e_1; e_2; \dots; e_{k-1}$ are called the scrollar invariants of g_k^1 .

Those scrollar invariants determine (and are determined by) the complete linear systems associated to multiples of the linear system g_k^1 . For $1 \leq i \leq k-1$ the invariant e_i is one less than the number of non-negative integers j satisfying $\dim(|K_C - jg_k^1|) - \dim(|K_C - (j+1)g_k^1|) \geq i$. Here K_C denotes a canonical divisor on C . Let $m = e_{k-1} + 2$. Then m is defined by the following conditions: $\dim(|(m-1)g_k^1|) = m-1$ and $\dim(|mg_k^1|) > m$. In case $|mg_k^1|$ is birationally very ample then the scrollar invariants satisfy the inequalities $e_i \leq e_{i+1} + m$ for $1 \leq i \leq k-2$ (see [3]). In case $k=3$ this number $m = e_2$ determines also the other scrollar invariant e_1 . It is the starting point for so-called Maroni-theory for linear systems on trigonal curves (see [4]; [5]). Scrollar invariants for 4-gonal curves are intensively studied in [1]; [3] and for 5-gonal curves in [6].

For $(j-1)m - 1 < x \leq jm - 1$ with $j \leq k-1$ the inequalities between the scrollar invariants imply $\dim(|xg_k^1|) \geq \frac{j(j-1)}{2}m - 1 + (x - (j-1)m + 1)j$. Equality (if not in conflict with the Riemann-Roch Theorem) can be expected being the most general case for a fixed value of m . The inequalities also imply $\dim(|(k-1)mg_k^1|) = \dim(|((k-1)m - 1)g_k^1|) + k$. This implies that $|((k-1)m - 1)g_k^1|$ is not special. Using the dimension bound one obtains $g \leq [(k^2 - k)m - 2k + 2]/2$. (This easy but interesting consequence from the inequalities is not mentioned by Kato and Ohbuchi.) In this paper we prove the following theorem.

Theorem. *For all nonnegative integers k ; m and g satisfying $k \geq 3$; $m \geq 2$ and $k-1 \leq g \leq [(k^2 - k)m - 2k + 2]/2$ there exists a smooth curve C of genus g possessing*

a complete base point free linear system g_k^1 satisfying the following property. For each nonnegative integer x with $x \leq (k - 1)m - 1$ define the nonnegative integer j such that $(j - 1)m - 1 < x \leq jm - 1$. Then $\dim(|xg_k^1|) = \max\left(\left\{\frac{j(j-1)}{2}m - 1 + (x - (j - 1)m + 1)j; kx - g\right\}\right)$. Also $|mg_k^1|$ is birationally very ample.

The curves C are obtained using special plane curves degenerating to special types of rational curves. First we construct those rational curves Γ_0 using some linear system g_k^1 on P^1 . In order to prove the theorem we study canonical adjoint curves of Γ_0 containing all points belonging to a given number of divisors from g_k^1 .

SOME NOTATIONS. On a smooth surface X ; if Γ_1 and Γ_2 are two effective divisors intersecting at $x \in X$ (no common component containing x) then we write $i(\Gamma_1, \Gamma_2; x)$ for the intersection multiplicity of Γ_1 and Γ_2 at x . We write (Γ_1, Γ_2) for the intersection number of Γ_1 and Γ_2 . We also write K_X for a canonical divisor on X .

1. Construction of the plane rational curve

Choose a general linear system g_k^1 on P^1 and a general divisor $F \in g_k^1$. Choose a general effective divisor E of degree mk on P^1 . Consider the linear system g_{mk}^2 containing $(m - 1)F + g_k^1$ and E .

Claim 1.1. g_{mk}^2 is a simple base point free linear system on P^1 .

Proof. The linear system g_{mk}^2 has no base points: $mF \in (m - 1)F + g_k^1 \subset g_{mk}^2$ and $E \cap F = \emptyset$. For $P \in E$ and $D_P \in g_k^1$ containing P one has $E \cap D_P = \{P\}$ (the intersection as schemes is reduced), therefore also $E \cap ((m - 1)F + D_P) = \{P\}$. Since $(m - 1)F + D_P \in g_{mk}^2$ this implies that g_{mk}^2 is simple. □

Claim 1.2. The space parametrizing such linear systems g_{mk}^2 on P^1 is irreducible of dimension $mk + 2k - 3$.

Proof. Effective divisors of degree d on P^1 are parametrized by a projective space P^d . Linear systems g_k^1 (resp. g_{mk}^2) on P^1 are parametrized by a grassmannian $G(1; k)$ of lines in P^k (resp. $G(2; mk)$ of planes in P^{mk}). On $G(1; k) \times P^k$ we have the incidence subvariety I defined as $(g_k^1; F) \in I$ if and only if $F \in g_k^1$. Clearly I is irreducible of dimension $\dim(G(1; k)) + 1 = 2k - 1$. The linear systems g_{mk}^2 on P^1 constructed above belong to the image of the rational map $\tau : I \times P^{mk} \rightarrow G(2; mk)$ defined by $\tau((g_k^1; F); E) = \langle (m - 1)F + g_k^1; E \rangle$.

Suppose for $((g_k^1; F); E) \in I \times P^{mk}$ general, there exists another element $((h_k^1; G); E') \in I \times P^{mk}$ with $\tau((g_k^1; F); E) = \tau((h_k^1; G); E')$ but $(g_k^1; F) \neq (h_k^1; G)$. Because

$(m - 1)F + g_k^1$ and $(m - 1)G + h_k^1$ are both lines in g_{mk}^2 , one has $[(m - 1)F + g_k^1] \cap [(m - 1)G + h_k^1] \neq \emptyset$. Assume $(m - 1)F + g_k^1 = (m - 1)G + h_k^1$. Because g_k^1 has no fixed points, this implies $F = G$ and so $g_k^1 = h_k^1$; a contradiction. Choose $D \in h_k^1$ with $(m - 1)G + D \notin (m - 1)F + g_k^1$. Then $g_{mk}^2 = \langle (m - 1)F + g_k^1; (m - 1)G + D \rangle$. Because g_{mk}^2 is base point free (Claim 1.1) we find $F \cap G = \emptyset$. But then $[(m - 1)F + g_k^1] \cap [(m - 1)G + h_k^1] \neq \emptyset$ implies $m = 2$; $F \in h_k^1$; $G \in g_k^1$ and so $g_k^1 = h_k^1 = \langle F; G \rangle$. For $D \in g_k^1$ one finds $F + D; G + D \in g_{2k}^2$ so $g_k^1 + D \subset g_{2k}^2$. It follows that $g_{2k}^2 = \{D_1 + D_2 : D_1; D_2 \in g_k^1\}$. This contradicts g_{2k}^2 being simple (Claim 1.1). So we find for a general $((g_k^1; F); E) \in I \times \mathbf{P}^{mk}$ one has $\tau((g_k^1; F); E) = \tau((h_k^1; G); E')$ if and only if $(g_k^1; F) = (h_k^1; G)$ and $E' \in \langle (m - 1)F + g_k^1; E \rangle$. Therefore the general non-empty fiber of τ has dimension 2. So, the image of τ has dimension $mk + 2k - 3$.

Associated to g_{mk}^2 there exist morphisms $\phi : \mathbf{P}^1 \rightarrow \mathbf{P}^2$. Fix such a morphism and let Γ be the image.

Claim 1.3. Γ is a plane curve of degree mk . The divisor F induces a singular point s on Γ of multiplicity $(m - 1)k$. The other singular points of Γ are ordinary nodes.

Proof. Since g_{mk}^2 is simple and base point free (Claim 1.1) the plane curve Γ has degree mk . There is a 1-dimensional subsystem of g_{mk}^2 containing $(m - 1)F$. This 1-dimensional subsystem corresponds to a pencil of lines on \mathbf{P}^2 containing some fixed point s . For a general line L containing s there are k intersections each one of multiplicity 1 with Γ outside s . This implies $i(\Gamma; L; s) = (m - 1)k$, hence Γ has multiplicity $(m - 1)k$ at s . Assume s' is another singular point of Γ .

First assume s' has multiplicity $\mu \geq 3$. The pencil of lines on \mathbf{P}^2 containing s' induces a linear subsystem $F' + g_{mk-\mu}^1 \subset g_{mk}^2$. From $s \neq s'$ it follows that $F \cap F' \neq \emptyset$. The line $\langle ss' \rangle$ on \mathbf{P}^2 gives rise to $(m - 1)F + D \in g_{mk}^2$ with $D \in g_k^1$. We find $D = F' + D'$ for some effective divisor D' . Let E' be the divisor corresponding to a general line through s' , then $E' = F' + E''$ for some $E'' \in g_{mk-\mu}^1$. Since $g_{mk}^2 = \langle (m - 1)F + g_k^1; E' \rangle$ we find that g_{mk}^2 belongs to the image of the morphism $\tau' : I' \times \mathbf{P}^{mk-\mu} \rightarrow G(2; mk)$ defined by $\tau'((g_k^1; F; F'); E'') = \langle (m - 1)F + g_k^1; F' + E'' \rangle$ with $I' \subset I \times \mathbf{P}^\mu$ defined by $(g_k^1; F; F') \in I'$ if and only if $D \geq F'$ for some $D \in g_k^1$ (here I is as in the proof of Claim 1.2). The choice of E'' implies that τ' has non-empty fibers of dimension at least 1. Since $\dim(I' \times \mathbf{P}^{mk-\mu}) = 2k + mk - \mu$ we obtain $2k - 1 + mk - \mu \leq (m + 2)k - 3$. This contradicts $\mu \geq 3$. It follows that s' has multiplicity 2.

Using the same notations we have $\mu = 2$; $\deg(F') = 2$. Assume $F' = 2P_0$. Then P_0 is a ramification point of g_k^1 .

This implies that g_{mk}^2 belongs to the image of the morphism $\tau'' : I'' \times \mathbf{P}^{mk-2} \rightarrow G(2; mk)$ defined by $\tau''((g_k^1; F; P_0); E'') = \langle (m - 1)F + g_k^1; 2P_0 + E'' \rangle$, with $I'' \subset I \times \mathbf{P}^1$ defined by $(g_k^1; F; P_0) \in I''$ if and only if P_0 is a ramification point of g_k^1 . Again, the non-empty fibers have dimension at least 1. Since $\dim(I'' \times \mathbf{P}^{mk-2}) = 2k - 1 + mk - 2$, we find a contradiction to $\dim(im\tau) = (m + 2)k - 3$.

We obtain $F' = P_0 + Q_0$ with $P_0 \neq Q_0$. Assume L_0 is a line through s' such that L_0 induces $2(P_0 + Q_0) + E'''$ for some effective divisor E''' of degree $mk-4$. Hence, we assume that s' is a tacnode. This implies g_{mk}^2 belongs to the image of the rational map $\tau''' : I''' \times \mathbf{P}^{mk-4} \rightarrow G(2; mk)$ defined by $\tau'''((g_k^1; F; F'); E''') = \langle (m-1)F + g_k^1; 2F' + E''' \rangle$ with $I''' \subset I \times \mathbf{P}^2$ defined by $(g_k^1; F; F') \in I'''$ if and only if $D \geq F'$ for some $D \in g_k^1$. Because $\dim(I''' \times \mathbf{P}^{mk-4}) = (m+2)k-4 < (m+2)k-3$, once more we obtain a contradiction. This implies that s' is an ordinary node.

Because $mF \in g_{mk}^2$ there exists a line T on \mathbf{P}^2 through s inducing mF . This line T intersects Γ only at s , hence $i(T, \Gamma; s) = mk$. We can consider the singularity of Γ at s as follows. It consists of exactly k locally irreducible branches (we use $F \in \mathbf{P}^k$ is general), each one having multiplicity $m - 1$ at s and having T as “tangent line” intersecting the branch with multiplicity m at s . From now on we fix s and T . □

Claim 1.4. *We obtain a family of plane curves of dimension $(m + 2)k - 1$.*

Proof. This follows from Claim 1.2 taking into account that $\dim(\text{Aut}(\mathbf{P}^1)) = 3$; $\dim(\text{Aut}(\mathbf{P}^2)) = 8$ and fixing s and T imposes 3 independent conditions on Γ . □

2. Blowing up the projective plane

Let $\pi_1 : X_1 \rightarrow \mathbf{P}^2$ be the blowing-up of \mathbf{P}^2 at s ; let E_1 be the exceptional divisor. Let T_1 (resp. Γ_1) be the strict transform of T (resp. Γ) on X_1 . Let L be the inverse image of a line on \mathbf{P}^2 . Then $T_1 \in |L - E_1|$; $\Gamma_1 \in |kmL - k(m - 1)E_1|$. Let $s_1 = E_1 \cap T_1$.

The linear system $|L - E_1|$ induces g_k^1 on \mathbf{P}^1 and T_1 induces F . Since the images of points of F under the morphism $\mathbf{P}^1 \rightarrow \Gamma_1$ are contained in E_1 , it follows that $i(T_1, \Gamma_1; s_1) = k$. Hence the k different points of F correspond to k different irreducible branches of Γ_1 at s_1 . Hence Γ_1 has a singular point of multiplicity k at s_1 . Also $E_1 \cap \Gamma_1 = \{s_1\}$ and since $(E_1, \Gamma_1) = k(m - 1)$ it follows that $i(E_1, \Gamma_1; s_1) = (m - 1)k$. Because $T_1 + E_1$ induces mF on \mathbf{P}^1 , it follows that E_1 intersects each branch of Γ_1 at s_1 with multiplicity $m - 1$ at s_1 .

Let $\pi_2 : X_2 \rightarrow X_1$ be the blowing-up of X_1 at s_1 . Let E_2 be the exceptional divisor. We continue to write L for the inverse image of a general line on \mathbf{P}^2 . Let E_{12} (resp. $T_2; \Gamma_2$) be the strict transforms of E_1 (resp. $T; \Gamma$) on X_2 . We also write E_1 to denote the inverse image of E_1 on X_2 . Then $E_{12} \in |E_1 - E_2|$; $T_2 \in |L - E_1 - E_2|$; $\Gamma_2 \in |kmL - k(m - 1)E_1 - kE_2|$. Let $s_2 = E_2 \cap E_{12}$. One has $(T_2, \Gamma_2) = 0$ hence $T_2 \cap \Gamma_2 = \emptyset$. In case $m = 2$ we find $(\Gamma_2, E_{12}) = 0$ hence $\Gamma_2 \cap E_{12} = \emptyset$.

Assume $m > 2$. From $(\Gamma_2, E_2) = k$ it follows that each branch of Γ_2 corresponding to a point of F is smooth and intersects E_2 transversally at one point. Because $E_2 + E_{12}$ induces $(m - 1)F$ on \mathbf{P}^1 it follows that those points of F map to s_2 and E_{12} intersects each branch with multiplicity $m - 2$ at s_2 . It follows that Γ_2 has multiplicity k at s_2 .

We continue to make blowings-up. For each $i \leq m$ we obtain the blowing-up $\pi_i : X_i \rightarrow X_{i-1}$ with exceptional divisor E_i . On X_i we continue to write L to denote the inverse image of a general line on \mathbf{P}^2 . We write Γ_i (resp. $E_{i-1,i}; T_i$) to denote the strict transform of Γ (resp. $E_{i-1}; T$) on X_i . Also, for $j \leq i - 2$ we write $E_{j,i}$ for the strict transform of $E_{j,i-1}$. Let $s_i = E_i \cap E_{i-1,i}$. In case $i < m$ the multiplicity of Γ_i at s_i is k . At s_i the curve Γ_i has k smooth locally irreducible branches. Also $E_{1,i}$ intersects each branch with multiplicity $m - i$ at s_i . On X_m we also write E_i for the inverse image of E_i on X_i (for $i < m$). We obtain $\Gamma_m \in \mathbf{P} := |kmL - k(m-1)E_1 - kE_2 - \dots - kE_m|$, $T_m \in |L - E_1 - E_2|$, $E_{1m} \in |E_1 - E_2 - \dots - E_m|$ and $E_{im} \in |E_i - E_{i+1}|$ for $2 \leq i \leq m - 1$.

Claim 2.1. Γ_m has ordinary nodes as its only singularities. The intersection points of Γ_m and E_m are smooth points on Γ_m .

Proof. Because of Claim 1.3 it is enough to prove that the intersection points of Γ_m and E_m are smooth points on Γ_m . The inverse image on \mathbf{P}^1 of the intersection as schemes of Γ_m and E_m is the divisor F , hence a general divisor of degree k on \mathbf{P}^1 . If that intersection would not be smooth then 2 different points in F would have the same image on Γ_m . Because of monodromy on \mathbf{P}^1 , in that case all k points on F need to have the same image on Γ_m , hence $\Gamma_m \cap E_m$ is a single multiple point s_m of Γ_m . Since $(\Gamma_m \cdot E_{1m}) = 0$ and $(\Gamma_m \cdot E_{m-1,m}) = 0$ it follows that $s_m \notin \{E_{1,m} \cap E_m; E_{m-1,m} \cap E_m\}$. Let $\pi_{m+1} : X_{m+1} \rightarrow X_m$ be the blowing-up of X_m at s_m . Let E_{m+1} be the exceptional divisor of π_{m+1} and let Γ_{m+1} be the strict transform of Γ_m . We find $\Gamma_{m+1} \in \mathbf{P}_{m+1} := |kmL - k(m-1)E_1 - kE_2 - \dots - kE_{m+1}|$. If Γ_{m+1} is not smooth at each point of $\Gamma_{m+1} \cap E_{m+1}$ then as before we find $s_{m+1} \in \Gamma_{m+1}$ such that Γ_{m+1} has multiplicity k at s_{m+1} . In that case we blow-up Γ_{m+1} at s_{m+1} and so on.

For some $m' \geq 1$ we obtain $X_{m+m'}$ and $\Gamma_{m+m'} \in \mathbf{P}_{m+m'} := |kmL - k(m-1)E_1 - kE_2 - \dots - kE_{m+m'}|$ such that $\Gamma_{m+m'}$ has ordinary nodes as its only singularities. The arithmetic genus of $\Gamma_{m+m'}$ is equal to $[(km-1)(km-2) - (k(m-1)-1)k(m-1) - (m+m'-1)(k-1)k]/2$. This has to be at least 0, hence $(m-m')k^2 + (m'-m)k - 2k + 2 \geq 0$. This condition implies $m' \leq m$.

In $\mathbf{P}_{m+m'}$ we find that the locus of irreducible rational nodal curves has a component of dimension at least $mk + 2k - 1 - m'$. (This follows from Claim 1.4 taking into account the choice of s_{m+i} on E_{m+i} for $0 \leq i \leq m'$.) The number of nodes of $\Gamma_{m+m'}$ is equal to the arithmetic genus of $\Gamma_{m+m'}$ being $\delta = [(m-m')(k^2 - k) - 2k + 2]/2$. Because $m' \leq m$ we find $(K_{X_{m+m'}} \cdot \Gamma_{m+m'}) = -3km + k(m-1) + k(m+m'-1) = (m'-m-2)k < 0$. From Lemma 2.2 in [8] it follows that $\dim(\mathbf{P}_{m+m'}) \geq mk + 2k - 1 - m' + \delta$. Also from the end of the proof of Lemma 2.2 in [8] we also obtain $\dim(\mathbf{P}_{m+m'}) = \delta - (K_{X_{m+m'}} \cdot \Gamma_{m+m'}) - 1 = \delta + (m+2-m')k - 1$. This would imply $(m+2-m')k - 1 \geq mk + 2k - 1 - m'$, hence $m' \geq m'k$. Since $m' \geq 1; k \geq 2$ this is a contradiction. This completes the proof of the claim. \square

3. Canonically adjoint curves

In order to study canonically adjoint curves for curves belonging to P we consider the linear system $P'_0 = |(km - 3)L - (k(m - 1) - 1)E_1 - (k - 1)E_2 - \dots - (k - 1)E_m|$.

Claim 3.1. $P'_0 = P_0 + (\text{fixed components})$ with $P_0 = |(km - 2 - m)L - (k(m - 1) - m)E_1 - (k - 2)E_2 - \dots - (k - 2)E_m|$.

Proof. From $T_m \cdot P'_0 = -1$ it follows that T_m is a fixed component of P'_0 . Deleting T_m from P_0 we obtain $|(km - 4)L - (k(m - 1) - 2)E_1 - (k - 2)E_2 - (k - 1)E_3 - \dots - (k - 1)E_m|$. In case $m = 2$ this finishes the proof of the claim.

Assume $m > 2$. The intersection number with E_{2m} is -1 , hence E_{2m} is a fixed component. Deleting E_{2m} we obtain $|(km - 4)L - (k(m - 1) - 2)E_1 - (k - 1)E_2 - (k - 2)E_3 - (k - 1)E_4 - \dots - (k - 1)E_m|$. Continuing in this way one finds fixed components $E_{3m}, \dots, E_{m-1,m}$. Deleting them, one obtains $|(km - 4)L - (k(m - 1) - 2)E_1 - (k - 1)E_2 - \dots - (k - 1)E_{m-1} - (k - 2)E_m|$. Now T_m is a fixed component. Deleting T_m one obtains $|(km - 5)L - (k(m - 1) - 3)E_1 - (k - 2)E_2 - (k - 1)E_3 - \dots - (k - 1)E_{m-1} - (k - 2)E_m|$. In case $m = 3$ this proves the claim. In case $m > 3$ one has $E_{2m}, \dots, E_{m-2,m}, T_m$ again as fixed components. Deleting them this proves the claim for $m = 4$; in case $m > 4$ one continues.

For curves Γ^c of P we need to investigate canonical adjoint curves containing intersections of Γ^c with elements from $|L - E_1|$ (in terms of linear systems : containing a sum of divisors from g_k^1). For a general element R of $|L - E_1|$ the intersection of R with an element Γ_m of P not containing E_{1m} are k different points. The intersection multiplicity with an element of P_0 is $k - 2 < k$. Therefore an element of P_0 containing this intersection of Γ_m and R contains R as a component. Taking x general elements R_1, \dots, R_x in $|L - E_1|$, the elements of P_0 containing $(R_1 \cup \dots \cup R_x) \cap \Gamma_m$ have R_1, \dots, R_x as components. Deleting R_1, \dots, R_x we obtain $P'_x = |(km - 2 - m - x)L - (k(m - 1) - m - x)E_1 - (k - 2)E_2 - \dots - (k - 2)E_m|$. □

Claim 3.2. Write $x = lm + y$ with $-1 \leq y \leq m - 2$. Then $P'_x = P_x + (\text{fixed components})$ with $P_x = |(km - (l + 1)m - y - 2)L - (k(m - 1) - (l + 1)m + l - y)E_1 - (k - l - 2)E_2 - \dots - (k - l - 2)E_m|$. (We do not claim that P_x has no more fixed components.)

Proof. First, take $0 \leq x \leq m - 2$, hence $x = y$ and $l = 0$. Then $P'_x = P_x$ and there is nothing to prove.

Next, take $x = m - 1$, hence $l = 1; y = -1$. Then $(E_{1m} \cdot P'_{m-1}) = -1$ (the intersection number of E_{1m} with elements of P'_{m-1}), therefore E_{1m} is a fixed component of P'_{m-1} . Deleting E_{1m} we obtain P_{m-1} .

More general, for any $x \geq m$ the curve E_{1m} is a fixed component of P'_x . Deleting E_{1m} we obtain $|(km - 2 - m - x)L - (k(m - 1) - m - x + 1)E_1 - (k - 3)E_2 - \dots - (k - 3)E_m|$. In case $m \leq x < 2m - 1$ this is P_x . For $x = 2m - 1$ (hence $l = 2$; $y = -1$) the intersection number of E_{1m} with that linear system is -1 , hence E_{1m} is a fixed component. Deleting E_{1m} one obtains P_{2m-1} . Continuing in this way one proves the claim. \square

REMARK 3.3. Taking $x = (k - 2)m + m - 2$ (hence $l = k - 2$; $y = m - 2$) one finds $P_{(k-2)m+m-2} = 0$. For $x \geq (k - 2)m + m - 1$ one finds $P_x = \emptyset$.

Given $0 \leq x \leq (k - 2)m + m - 2$ define the integer j by means of the inequalities $(j - 1)m - 1 < x \leq jm - 1$ with $j \leq k - 1$.

Claim 3.4. $\dim(P_x) = \frac{j(j-1)}{2}m - 1 + (x - (j-1)m + 1)j - kx + \frac{(k-1)mk - 2k + 2}{2} - 1$.

Proof. In case $x = (k - 1)m - 2$ we have to prove $\dim(P_{(k-1)m-2}) = 0$. This follows from Remark 3.3.

Now, fix some $x < (k - 1)m - 2$ and assume the claim is proved for $x + 1$ instead of x . Writing $x = lm + y$ with $-1 \leq y \leq m - 2$, from the description in Claim 3.2 we find $\dim(P_x) \geq [(km - (l + 1)m - y + 1)(km - (l + 1)m - y - 2)]/2 - [(k(m - 1) - (l + 1)m + l - y + 1)(k(m - 1) - (l + 1)m + l - y)]/2 - [(m - 1)(k - l - 1)(k - l - 2)]/2$. A computation shows us that we need to prove equality. Assume for x we have strict inequality. For $R \in |L - E_1|$ we have $(R.P_x) = k - l - 2$, hence R imposes at most $k - l - 1$ conditions on P_x . This implies $\dim(|P_x - (L - E_1)|) \geq \dim(P_x) - (k - l - 1)$. In case $y = m - 2$ one finds $((P_x - (L - E_1)).E_{1m}) < 0$, hence $\dim(|P_x - (L - E_1) - E_{1m}|) \geq \dim(P_x) - (k - l - 1)$. But $|P_x - (L - E_1) - E_{1m}| = P_{x+1}$. One more computation shows that $\dim(P_{x+1}) \geq \dim(P_x) - (k - l - 1)$ gives a contradiction to the assumption that the claim holds for P_{x+1} . In case $y < m - 2$ then $|P_x - (L - E_1)| = P_{x+1}$ and again, a computation shows a contradiction.

On X_m we constructed the rational irreducible curve Γ_m belonging to P . From Claim 2.1 we know that Γ_m is a nodal curve, so it has $g_0 := [(k^2 - k)m - 2k + 2]/2$ ordinary nodes. We write s to denote a node of Γ_m . \square

Claim 3.5. We can arrange the nodes $s_1; \dots; s_{g_0}$ in such a way that the following property holds. First we introduce some notation: for $0 \leq \delta \leq g_0$ let $P_x(s_1; \dots; s_\delta) = \{\Gamma \in P_x : s_i \in \Gamma \text{ for } 1 \leq i \leq \delta\}$. Then $P_x(s_1; \dots; s_\delta) = \emptyset$ for $\delta > \dim(P_x)$ and $\dim(P_x(s_1; \dots; s_\delta)) = \dim(P_x) - \delta$ if $\delta \leq \dim(P_x)$.

Proof. For $\delta = 0$ there is nothing to prove.

Fix $\delta > 0$ and assume the claim holds for $\delta - 1$ instead of δ . So, we assume a suited arrangement $s_1; \dots; s_{\delta-1}$ for a suitable part of the set of the nodes. We have to prove that the set of the remaining nodes of Γ_m contains a suited one to be numbered s_δ .

Numbers x satisfying $\delta - 1 > \dim(\mathbf{P}_x)$ impose no conditions on s_δ . Let x_0 be the minimal number such that $\delta - 1 \leq \dim(\mathbf{P}_x)$. We know that $\dim(\mathbf{P}_{x_0}(s_1; \dots; s_{\delta-1})) = \dim(\mathbf{P}_{x_0}) - (\delta - 1) \geq 0$. If each element of $\mathbf{P}_{x_0}(s_1; \dots; s_{\delta-1})$ would contain all the nodes of Γ_m then Γ_m possesses a canonically adjoint curve. Since Γ_m is a rational curve this is impossible. Hence, there exists a node s_0 such that $\dim(\mathbf{P}_{x_0}(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\mathbf{P}_{x_0}) - \delta$ (with $\mathbf{P}_{x_0}(s_1; \dots; s_{\delta-1}; s_0) = \emptyset$ if $\delta - 1 = \dim(\mathbf{P}_{x_0})$). In case for all $x \leq x_0$ we find $\dim(\mathbf{P}_x(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\mathbf{P}_x) - \delta$ then we can take $s_0 = s_\delta$.

Assume $x' < x_0$ such that $\dim(\mathbf{P}_{x'}(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\mathbf{P}_{x'}) - \delta + 1$ while $\dim(\mathbf{P}_{x'+1}(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\mathbf{P}_{x'+1}) - \delta$. Using a general $R \in |L - E_1|$ and using the arguments from the proof of Claim 3.4 one finds a contradiction. \square

4. Proof of the theorem

Now, we finish the proof of the theorem in the introduction. We start with the rational irreducible nodal curve Γ_m on X_m . We make an arrangement of the nodes as in Claim 3.5. The main result in Section 2 of [8] implies that there exists a 1-dimensional flat family $Y \rightarrow T$ of curves on X_m belonging to \mathbf{P} such that the fiber over a special point t_0 of T is the curve Γ_m and a general fiber is a nodal curve Γ with exactly $g_0 - g$ nodes such that those nodes specialize to the nodes $s_1; \dots; s_{g_0-g}$ on Γ_m . Define $\mathbf{P}_{x,\Gamma} = \{D \in \mathbf{P}_x : D \text{ contains the nodes of } \Gamma\}$. Clearly $\dim(\mathbf{P}_{x,\Gamma}) \geq \dim(\mathbf{P}_x) - (g_0 - g)$. For the special fiber Γ_m we have $\mathbf{P}_x(s_1; \dots; s_{g_0-g}) = \emptyset$ if $g_0 - g > \dim(\mathbf{P}_x)$ and $\dim(\mathbf{P}_x(s_1; \dots; s_{g_0-g})) = \dim(\mathbf{P}_x) - (g_0 - g)$ if $g_0 - g \leq \dim(\mathbf{P}_x)$. Semicontinuity implies $\mathbf{P}_{x,\Gamma} = \emptyset$ if $g_0 - g > \dim(\mathbf{P}_x)$ and $\dim(\mathbf{P}_{x,\Gamma}) = \dim(\mathbf{P}_x) - (g_0 - g)$ if $g_0 - g \leq \dim(\mathbf{P}_x)$. Let C be the normalization of Γ . It is a smooth curve of genus g . The linear system $|L - E_1|$ induces a linear system g_k^1 on C without base points. Taking x general elements $R_1; \dots; R_x$ in $|L - E_1|$ corresponds to taking x general divisors in g_k^1 . From the description of \mathbf{P}_x in 3.2 we find that $\dim(\mathbf{P}_{x,\Gamma})$ is equal to the dimension of canonically adjoint curves Γ containing the intersection of Γ with $R_1 \cup \dots \cup R_x$, hence it is equal to $\dim(|K_C - xg_k^1|)$. In particular, if $\mathbf{P}_{x,\Gamma} = \emptyset$ then $|K_C - xg_k^1| = \emptyset$. So, we find $|K_C - xg_k^1| = \emptyset$ if $g_0 - g > \dim(\mathbf{P}_x)$; $\dim(|K_C - xg_k^1|) = \dim(\mathbf{P}_x) - (g_0 - g)$ if $g_0 - g \leq \dim(\mathbf{P}_x)$. Using 3.4 and the Riemann-Roch Theorem one finds $\dim(|xg_k^1|) = \max\{\frac{j(j-1)}{2}m - 1 + (x - (j - 1)m + 1)j; kx - g\}$. In particular $\dim(|(m - 1)g_k^1|) = m - 1$ and $\dim(|mg_k^1|) = m + 1$. Since Γ is obtained from C using a linear subsystem of $|mg_k^1|$, one also finds $|mg_k^1|$ is birationally very ample. This finishes the proof of the theorem. \square

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