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Classifications and Constructions of Lattice Polytopes

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List of Papers

The following papers are included in this thesis.

1. T. Hibi, K. Matsuda, and A. Tsuchiya. Quadratic Gröbner bases arising from partially ordered sets. *Math. Scand.*, 121:19–25, 2017.

2. T. Hibi, M. Olsen, and A. Tsuchiya. Self dual reflexive simplices with Eulerian polynomials. *Graphs Combin.*, 33:1401–1404, 2017.

3. T. Hibi and A. Tsuchiya. Facets and volume of Gorenstein Fano polytopes. *Math. Nachr.*, 290:2619–2628, 2017.

4. T. Hibi and A. Tsuchiya. Reflexive polytopes arising from perfect graphs. *J. Combin. Theory Ser. A*, 157:233–246, 2018.

5. T. Hibi and A. Tsuchiya. Reflexive polytopes arising from partially ordered sets and perfect graphs. *J. Algebraic Combin.*, to appear.

6. A. Tsuchiya. The δ-vectors of reflexive polytopes and of the dual polytopes. *Discrete Math.*, 339:2450–2456, 2016.

7. A. Tsuchiya. Gorenstein simplices and the associated finite abelian groups. *European J. Combin.*, 67:145–157, 2018.

8. A. Tsuchiya. Volume, facets and dual polytopes of twinned chain polytopes. *Ann. Comb.*, to appear.

9. A. Tsuchiya. Ehrhart polynomials of lattice polytopes with normalized volumes 5. *J. Comb.*, to appear.

Contents

Chapter 1

Introduction

A *convex polytope* is the convex hull of finitely many points in a Euclidean space (see the books [28] and [98]). Convex polytopes are fundamental objects in mathematics, and they are currently in an extremely active area of research. A *lattice polytope* or an *integral polytope* is a convex polytope all of whose vertices have integer coordinates. Lattice polytopes are interesting objects arising in many different branches of pure mathematics and applied mathematics such as number theory, commutative algebra, combinatorics, toric geometry, optimization and mirror symmetry. They possess rich structures arising from the interaction of algebraic, geometric, analytic and combinatorial properties, and lattice polytopes are useful combinatorial objects to understand these fields. On the other hand, we can realize properties of lattice polytopes from the theories of closely related areas. In fact, there are several results on lattice polytopes which are proven by using methods of commutative algebra and algebraic geometry. Moreover, no combinatorial proofs of some of these results are known.

In this thesis, we pick up properties of lattice polytopes which are closely related with combinatorics, algebraic geometry and commutative algebra. The keywords are Ehrhart theory and reflexive polytopes. In particular, we focus on the following two topics: classification problems on lattice polytopes and constrictions of new classes of reflexive polytopes.

Classification problems on lattice polytopes

One of the final, but, unreachable goals of the study on lattice polytopes is to classify all of the lattice polytopes, up to unimodular equivalence. Recently, many authors have studied classifications of lattice polytopes from several viewpoints. We recall some results on classifications of lattice polytopes. From a viewpoint of counting lattice points, the following classes of lattice polytopes are classified:

- 3-dimensional lattice polytopes with at most 6 lattice points ([14, 15]);
- 3-dimensional lattice polytopes with one interior lattice point ([58]);
- 3-dimensional lattice polytopes with two interior lattice points ([3]).

Meanwhile, from a viewpoint of toric geometry, there are several classifications of lattice polytopes. In particular, the following classes of lattice polytopes correspond to an important class of toric varieties, which are called toric Fano varieties:

- Centrally symmetric smooth Fano polytopes [95];
- Pseudo-symmetric smooth Fano polytopes [26, 95];
- Smooth Fano polytopes up to dimension 9 [61, 68].
- Pseudo-symmetric simplicial reflexive polytopes [65];
- Reflexive polytopes up to dimension 4 [59, 60].

Additionally, there exist other classifications of smooth Fano polytopes and reflexive polytopes (e.g. [1, 19, 69]).

To work towards classification problems of lattice polytopes, we focus on the theory of Ehrhart polynomials and δ-polynomials, which is called Ehrhart theory. Ehrhart discovered that the function which counts the number of lattice points in dilations of a lattice polytope is a polynomial. The polynomial is called the Ehrhart polynomial of a lattice polytope, and some of its coefficients have combinatorial interpretations (see the books [13] and [37]). In particular, we can know the volume of a lattice polytope from its Ehrhart polynomial, and Ehrhart theory can be seen as a higher-dimensional generalization of Pick's theorem. On the other hand, the δ-polynomial of a lattice polytope is an equivalent invariant to the Ehrhart polynomial. It is known that the coefficients of the δ-polynomial are nonnegative integers and they have combinatorial interpretations (see [30]). One of the most important problems of Ehrhart theory is to characterize δ-polynomials. However, this is also a hard problem. Thus, we focus on δ-polynomials that have especially simple forms. The restrictions enable us to characterize them and to classify all lattice polytopes with these δ-polynomials. Moreover, such results help to learn what to expect in more general situations. In this thesis, we will characterize δ-polynomials which satisfy some reasonable conditions and classify all lattice polytopes with these δ-polynomials.

Constructions of new classes of reflexive polytopes

A reflexive polytope is one of the keywords belonging to the current trends in the research of lattice polytopes. Many authors have studied reflexive polytopes from the viewpoints of combinatorics, commutative algebra and algebraic geometry. In fact, it is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related to mirror symmetry (see, e.g., [4, 21]). Since a Gorenstein toric Fano variety is defined from a reflexive polytope completely, it has a lot of information of a Gorenstein toric Fano variety. Thus, from a viewpoint of algebraic geometry, reflexive polytopes are useful combinatorial objects to understand Gorenstein toric Fano varieties. Please refer to [64] for the details of Gorenstein toric Fano varieties. On the other hand, reflexive polytopes give crucial examples in commutative algebra. In fact, the semigroup algebra associated to the cone over a reflexive polytope is a Gorenstein algebra of a-invariant -1. Equivalently, a reflexive polytope is a lattice polytope having a unique interior lattice point and a palindromic δ-polynomial. As above, reflexive polytopes have many important properties in combinatorics, commutative algebra, toric geometry and other areas.

One of the most precious properties of reflexive polytopes is that in each dimension, there exist only finitely many reflexive polytopes up to unimodular equivalence ([62]). This fact motivates us to classify reflexive polytopes up to unimodular equivalence. So far, all of them are known up to dimension 4 ([59, 60]). However, it is too hard to classify higher-dimensional reflexive polytopes. Therefore, in order to understand reflexive polytopes, finding new classes of reflexive polytopes is an important problem. In this thesis, we will give several new classes of reflexive polytopes arising from combinatorial objects.

Structure of this thesis

The organization of this thesis is as follows. We divide this thesis into two parts. Each part includes the author's results on each topic.

- Part I is devoted to the studies on classification problems of lattice polytopes. There are six chapters in Part I. In Chapter 2, we will recall the notations of lattice polytopes and introduce Ehrhart theory. In Chapters 3 and 4, we will try to classify the lattice polytopes with small volumes. In Chapters 5, 6 and 7, we will discuss a classification problem of a special class of lattice polytopes, which are called Gorenstein polytopes. In particular, we focus on Gorenstein simplices. Part I contains the results of [48, 52, 54, 91, 93].
- Part II is devoted to the studies on constructions of new classes of reflexive polytopes. There are six chapters in Part II. In Chapter 8, we will recall the notations of toric ideals and introduce Gröbner bases. By using the theory of Gröbner bases, we will construct several classes of reflexive polytopes. In Chapters 9, 10, 11 and 12, we will give new large classes of reflexive polytopes arising from other combinatorial objects and investigate the combinatorial properties of these reflexive polytopes. Finally, in Chapter 13, we will discuss constructions of self

dual reflexive polytopes. This property is an extremely rare property in reflexive polytopes. Part II contains the results of [45, 46, 47, 49, 50, 51, 63, 90, 92].

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Part I

Classification problems on lattice polytopes

Chapter 2

Introduction to Ehrhart theory

For a lattice polytopes $\mathscr{P}$, let $\text{aff}(\mathscr{P})$ be the affine span of $\mathscr{P}$ and $\dim(\mathscr{P})$ the dimension of $\mathscr{P}$. We say that two lattice polytopes $\mathscr{P} \subset \mathbb{R}^d$ and $\mathscr{Q} \subset \mathbb{R}^e$ are *unimodularly equivalent* if there exists an affine map $\text{aff}(\mathscr{P}) \to \text{aff}(\mathscr{Q})$ that maps $\mathbb{Z}^d \cap \text{aff}(\mathscr{P})$ bijectively onto $\mathbb{Z}^e \cap \text{aff}(\mathscr{Q})$, and which maps $\mathscr{P}$ to $\mathscr{Q}$.

One of the most important problems on lattice polytopes is to classify all of the lattice polytopes, up to unimodular equivalence. In this part, we will classify some classes of lattice polytopes from a viewpoint of Ehrhart theory. Ehrhart theory is the theory of Ehrhart polynomials and δ-polynomials. These polynomials often appear in the area of enumerative combinatorics and they are important combinatorial invariants of lattice polytopes. In this chapter, we will recall the definitions of Ehrhart polynomials and δ-polynomials, and some related facts.

2.1 Ehrhart polynomials and δ-polynomials

Let $\mathscr{P} \subset \mathbb{R}^N$ be a lattice polytope of dimension d. Given a positive integer k, we define

$$L_{\mathscr{P}}(k) = |k\mathscr{P} \cap \mathbb{Z}^N|,$$

where $k\mathscr{P} = \{k\mathbf{x} : \mathbf{x} \in \mathscr{P}\}$ and $|X|$ is the cardinality of a finite set X. The study on $L_{\mathscr{P}}(k)$ originated in Ehrhart [24] who proved that $L_{\mathscr{P}}(k)$ is a polynomial in k of degree d with the constant term 1. We say that $L_{\mathscr{P}}(k)$ is the *Ehrhart polynomial* of $\mathscr{P}$.

The generating function of the lattice point enumerator, i.e., the formal power series

$$\text{Ehr}_{\mathscr{P}}(t) = 1 + \sum_{k=1}^{\infty} L_{\mathscr{P}}(k) t^k$$

is called the *Ehrhart series* of $\mathscr{P}$. It is well known that it can be expressed as a rational function of the form

$$\mathrm{Ehr}_{\mathscr{P}}(t) = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}}.$$

The sequence of the coefficients of the polynomial in the numerator

$$\delta(\mathscr{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$$

is called the δ*-vector* (or the h^**-vector)* of $\mathscr{P}$ and the polynomial

$$\delta(\mathscr{P}, t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$$

is called the δ*-polynomial* (or the h^**-polynomial*) of $\mathscr{P}$. The degree of the polynomial $\delta(\mathscr{P}, t)$ is called the *degree* of $\mathscr{P}$, denoted by $\deg(\mathscr{P})$.

For a general lattice polytope $\mathscr{P}$, we can extract the Ehrhart polynomial of $\mathscr{P}$ from its δ-vector:

Proposition 2.1. *Let $\mathscr{P} \subset \mathbb{R}^N$ be a lattice polytope of dimension d with its δ-vector $\delta(\mathscr{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$. Then one has*

$$L_{\mathscr{P}}(k) = \sum_{i=0}^{d} \delta_i \binom{k+d-i}{d}.$$

The following properties of $\delta(\mathscr{P})$ are known:

- $\delta_0 = 1$, $\delta_1 = |\mathscr{P} \cap \mathbb{Z}^N| - (d+1)$ and $\delta_d = |\mathrm{int}(\mathscr{P}) \cap \mathbb{Z}^N|$, where $\mathrm{int}(\mathscr{P})$ is the relative interior of $\mathscr{P}$. Hence one has $\delta_1 \geq \delta_d$;
- $\delta_i \geq 0$ for each i ([80]);
- When $\delta_d \neq 0$, one has $\delta_i \geq \delta_1$ for $1 \leq i \leq d-1$ ([38]);
- When $\mathscr{P}$ is *full-dimensional*, namely, $N = d$. then $\sum_{i=0}^{d} \delta_i / d!$ coincides with the usual volume of $\mathscr{P}$ ([84, Proposition 4.6.30]). In general, the positive integer $\sum_{i=0}^{d} \delta_i$ is said to be the *normalized volume* of $\mathscr{P}$, denoted by $\mathrm{Vol}(\mathscr{P})$.

Moreover, there are two well-known inequalities on δ-vectors. Set $s = \deg(\mathscr{P})$. In [82], Stanley proved that

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq \lfloor s/2 \rfloor, \tag{2.1}$$

while in [38], Hibi proved that

$$\delta_{d-1} + \delta_{d-2} + \cdots + \delta_{d-i} \leq \delta_2 + \delta_3 + \cdots + \delta_{i+1}, \quad 1 \leq i \leq \lfloor (d-1)/2 \rfloor. \tag{2.2}$$

Recently, more general results of inequalities on δ-vectors are in [85, 86].

Finally, we recall that the δ-vector of a lattice polytope has a monotonicity property:

Lemma 2.2 ([83, Theorem 3.3]). *Let $\mathscr{P} \subset \mathbb{R}^N$ be a lattice polytope of dimension d with the δ-vector $(\delta_0(\mathscr{P}), \delta_1(\mathscr{P}), \dots, \delta_d(\mathscr{P}))$ and $\mathscr{Q} \subset \mathbb{R}^n$ a lattice subpolytope of $\mathscr{P}$ with the δ-vector $(\delta_0(\mathscr{Q}), \delta_1(\mathscr{Q}), \dots, \delta_d(\mathscr{Q}))$. Then one has $\delta_i(\mathscr{P}) \geq \delta_i(\mathscr{Q})$ for any $0 \leq i \leq d$.*

Refer the reader to [13] and [37, Part II] for the detailed information about Ehrhart polynomials and δ-polynomials.

2.2 Lattice pyramids and Cayley polytopes

In this section, we recall well-known constructions of lattice polytopes. For a positive integer N, let $\mathbf{e}_1^{(N)}, \dots, \mathbf{e}_N^{(N)}$ denote the canonical unit vectors of $\mathbb{R}^N$ and $\mathbf{0}_N$ denote the origin of $\mathbb{R}^N$. If $\mathbb{R}^N$ is clear from the context, we will write $\mathbf{e}_1, \dots, \mathbf{e}_N$ and $\mathbf{0}$.

Let $\mathrm{conv}(S)$ be the convex hull of a subset $S \subset \mathbb{R}^N$. For a lattice polytope $\mathscr{P} \subset \mathbb{R}^N$ of dimension d, the *lattice pyramid* over $\mathscr{P}$ is defined by

$$\mathrm{conv}(\{\mathscr{P} \times \{0\}, \mathbf{e}_{N+1}\}) \subset \mathbb{R}^{N+1}.$$

Let $\mathrm{Pyr}(\mathscr{P})$ denote this polytope. We often use lattice pyramid shortly for a lattice polytope that has been obtained by successively taking lattice pyramids. Note that the δ-polynomial does not change under lattice pyramids:

Lemma 2.3 ([13, Theorem 2.4.]). *Let $\mathscr{P} \subset \mathbb{R}^N$ be a lattice polytope. Then one has*

$$\delta(\mathscr{P}, t) = \delta(\mathrm{Pyr}(\mathscr{P}), t).$$

Moreover, it is known that there are only finitely many lattice polytopes of fixed degree s and fixed volume V up to unimodular equivalence and lattice pyramid constructions ([67, Corollary 1.4]). Therefore, it is essential that we classify lattice polytopes which are not lattice pyramids over any lower-dimensional lattice polytope. Now, we recall a sufficient condition that a lattice polytope is a lattice pyramid:

Lemma 2.4 ([67, Theorem 7]). *Let $\mathscr{P} \subset \mathbb{R}^N$ be a lattice polytope of dimension d with $d + c + 1$ vertices and s the degree of the δ-polynomial of $\mathscr{P}$. If $d \geq c(2s+1) + 4s - 1$, then $\mathscr{P}$ is a lattice pyramid.*

We also define a lattice polytope $\mathscr{P}$ to be a *Cayley polytope* of $\mathscr{P}_1, \dots, \mathscr{P}_n \subset \mathbb{R}^N$ if $\mathscr{P}$ is unimodularly equivalent to the lattice polytope

$$\mathrm{conv}(\{\{\mathbf{e}_1^{(n)}\} \times \mathscr{P}_1, \dots, \{\mathbf{e}_n^{(n)}\} \times \mathscr{P}_n\}) \subset \mathbb{R}^n \times \mathbb{R}^N = \mathbb{R}^{n+N},$$

see e.g. [9]. We denote this by $\mathscr{P}_1 * \mathscr{P}_2 * \cdots * \mathscr{P}_n$. Note that a lattice pyramid of a lattice polytope is unimodularly equivalent to a Cayley polytope.

2.3 Spanning Polytopes

In this section, we introduce an important class of lattice polytopes. A full-dimensional lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is called *spanning* if any lattice point in $\mathbb{Z}^d$ is an affine integer combination of the lattice points in $\mathscr{P}$. This is equivalent to that any lattice point in $\mathbb{Z}^{d+1}$ is a linear integer combination of the lattice points in $\mathscr{P} \times \{1\}$. A spanning polytope is also called a *primitive* polytope.

For the δ-polynomial of a spanning polytope, the following result is known:

Lemma 2.5 ([56, Theorem 1.3])**.** *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d whose δ-polynomial equals $\delta_0 + \delta_1 t + \cdots + \delta_s t^s$, where $\delta_s \neq 0$. If $\mathscr{P}$ is spanning, then one has $\delta_i \geq 1$ for any $0 \leq i \leq s$.*

Recently, there is a more general result of the δ-polynomials of spanning polytopes in [55].

2.4 Reflexive polytopes and Gorenstein polytopes

In this section, we introduce reflexive polytopes and Gorenstein polytopes. A full-dimensional lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is called *reflexive* if the origin of $\mathbb{R}^d$ is a unique lattice point belonging to the interior of $\mathscr{P}$ and its dual polytope

$$\mathscr{P}^\vee := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{x} \in \mathscr{P}\}$$

is also a lattice polytope, where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the usual inner product of $\mathbb{R}^d$. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related with mirror symmetry (see, e.g., [4, 21]). It is known from the work of Lagarias and Ziegler ([62]) that there are only finitely many reflexive polytopes (up to unimodular equivalence) in each dimension, with one reflexive polytope in dimension one, 16 in dimension two, 4319 in dimension three, and 473800776 in dimension four according to computations by Kreuzer and Skarke ([59, 60]). Moreover, every lattice polytope is unimodularly equivalent to a face of some reflexive polytope ([29]). We say that a lattice polytope $\mathscr{P}$ is *Gorenstein of index r* where $r \in \mathbb{Z}_{>0}$ if $r\mathscr{P}$ is unimodularly equivalent to a reflexive polytope ([23]). Equivalently, the semigroup algebra associated to the cone over $\mathscr{P}$ is a Gorenstein algebra of a-invariant $-r$. Gorenstein polytopes are of interest in combinatorial commutative algebra, mirror symmetry, and tropical geometry (we refer to [8, 10, 57]).

A polynomial $\sum_{i=0}^{s} a_i t^i$ (with $a_s \neq 0$) is *palindromic*, if $a_i = a_{s-i}$ for $i = 0, \dots, s$. From a viewpoint of Ehrhart theory, we can characterize reflexive polytopes and Gorenstein polytopes by the following:

Lemma 2.6. *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d with the δ-polynomial $\delta(\mathscr{P},t) = \delta_0 + \delta_1 t + \cdots + \delta_s t^s$, where $\delta_s \neq 0$. Then the following conditions are equivalent:*

- *$\mathscr{P}$ is Gorenstein of index r;*
- *$r = d - s + 1$ and $\delta(\mathscr{P},t)$ is palindromic.*

In particular, $\mathscr{P}$ is unimodularly equivalent to a reflexive polytope if and only if $s = d$ and $\delta(\mathscr{P},t)$ is palindromic.

By Lemmas 2.3 and 2.6, it follows that $\mathscr{P}$ is Gorenstein of index r if and only if $\mathrm{Pyr}(\mathscr{P})$ is Gorenstein of index $r+1$. Hence, if we construct all Gorenstein polytopes which are not lattice pyramids, we can obtain all Gorenstein polytopes.

Finally, we give a characterization of reflexive polytopes in terms of their facets:

Lemma 2.7 ([37, Corollary 35.6])**.** *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d containing the origin in its interior. Then a point $\mathbf{a} \in \mathbb{R}^d$ is a vertex of $\mathscr{P}^\vee$ if and only if $\mathscr{H} \cap \mathscr{P}$ is a facet of $\mathscr{P}$, where $\mathscr{H}$ is the hyperplane*

$$\left\{ \mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle = 1 \right\}$$

in $\mathbb{R}^d$.

2.5 The associated abelian groups of lattice simplices

In this section, we introduce the associated finite abelian groups of lattice simplices, which are very useful tools for a classification of lattice simplices. For a lattice simplex $\Delta \subset \mathbb{R}^N$ of dimension d whose vertices are $\mathbf{v}_0, \dots, \mathbf{v}_d \in \mathbb{Z}^N$, set

$$\Lambda_\Delta = \{(\lambda_0, \dots, \lambda_d) \in (\mathbb{R}/\mathbb{Z})^{d+1} : \sum_{i=0}^{d} \lambda_i (\mathbf{v}_i, 1) \in \mathbb{Z}^{N+1}\}.$$

The collection Λ_Δ forms a finite abelian group with addition defined as follows: For $(\lambda_0, \dots, \lambda_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$ and $(\lambda'_0, \dots, \lambda'_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$, $(\lambda_0, \dots, \lambda_d) + (\lambda'_0, \dots, \lambda'_d) = (\lambda_0 + \lambda'_0, \dots, \lambda_d + \lambda'_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$. We denote the unit of Λ_Δ by $\mathbf{0}$, and the inverse of λ by $-\lambda$, and the order of λ by $\mathrm{ord}(\lambda)$, and also denote $\underbrace{\lambda + \cdots + \lambda}_{j}$ by $j\lambda$ for an integer

$j > 0$ and $\lambda \in \Lambda_\Delta$. Note that e.g. $-(1/3,2/3) = (2/3,1/3)$. For $\lambda = (\lambda_0,\ldots,\lambda_d) \in \Lambda_\Delta$, where each λ_i is taken with $0 \le \lambda_i < 1$, we set $\mathrm{ht}(\lambda) = \sum_{i=0}^{d} \lambda_i \in \mathbb{Z}$.

In [7], it is shown that there is a bijection between unimodular equivalence classes of d-dimensional lattice simplices with a chosen ordering of their vertices and finite subgroups of $(\mathbb{R}/\mathbb{Z})^{d+1}$ such that the sum of all entries of each element is an integer. In particular, two lattice simplices Δ and Δ' are unimodularly equivalent if and only if there exists an ordering of their vertices such that $\Lambda_\Delta = \Lambda_{\Delta'}$. Moreover, we can characterize lattice pyramids in terms of the associated finite abelian groups by using the following lemma:

Lemma 2.8 ([67, Lemma 12])**.** *Let $\Delta \subset \mathbb{R}^N$ be a lattice simplex of dimension d. Then Δ is a lattice pyramid if and only if there is $i \in \{0,\ldots,d\}$ such that $\lambda_i = 0$ for all $(\lambda_0,\ldots,\lambda_d) \in \Lambda_\Delta$.*

It is well known that the δ-polynomial of the lattice simplex Δ can be computed as follows:

Lemma 2.9 ([9, Proposition 2.6])**.** *Let Δ be a lattice simplex of dimension d whose δ-polynomial equals $\delta_0 + \delta_1 t + \cdots + \delta_d t^d$. Then for each i, we have $\delta_i = |\{\lambda \in \Lambda_\Delta : \mathrm{ht}(\lambda) = i\}|$. In particular, one has $\mathrm{Vol}(\Delta) = |\Lambda_\Delta|$.*

2.6 Characterizations of δ-polynomials

In this section, we recall some results on characterizations of the δ-polynomials of lattice polytopes.

Small dimension

Let us describe what is known about δ-polynomials of small-dimensional lattice polytopes. In dimension $d = 1$, for a given lattice segment of length $a+1$, we know that the δ-polynomial is $1 + at$. In dimension 2, the δ-polynomials of lattice polygons have been classified by Scott [78]. It holds that $1 + at + bt^2$ with $a, b \in \mathbb{Z}_{\ge 0}$ is the δ-polynomial of a lattice polygon $\mathscr{P}$ if and only if

- $b = 0$ (i.e., $\mathscr{P}$ has no interior lattice points), or
- $b = 1$ and $a = 7$ (here, $\mathscr{P}$ is unimodularly equivalent to $\mathrm{conv}(\{(0,0),(3,0),(0,3)\})$), or
- $b \ge 1$ and $b \le a \le 3b+3$.

We refer to [32] for a thorough discussion.

In dimension 3 there are currently only partial results. The arguably most significant one is White's theorem [97]: a three-dimensional lattice simplex is empty (i.e., $\delta_1 = 0$) if and only if it is the Cayley polytope of two empty line segments in $\mathbb{R}^2$. Recently, all three-dimensional lattice polytopes with at most 6 lattice points (i.e., $\delta_1 \leq 2$) have been classified [14, 15].

Small degree

It is natural to take the degree of the δ-polynomial as a measure of complexity. We recall what is known about δ-polynomial of small degrees. Any degree zero lattice polytope is a unimodular simplex (i.e., the convex hull of affine lattice basis). Lattice polytopes of degree one are completely classified [9]:

- Lattice pyramids over $\operatorname{conv}(\{(0,0),(2,0),(0,2)\})$, or
- Cayley polytopes of line segments in $\mathbb{R}^1$.

Lattice polytopes of degree two are not yet classified. However, their δ-polynomials are completely known [34, 89]. A polynomial $1+at+bt^2$ with $a,b \in \mathbb{Z}_{\geq 0}$ is the δ-polynomial of a lattice polytope $\mathscr{P}$ (in some dimension) if and only if

- $b = 0$, or
- $b = 1$ and $a = 7$ (here, $\mathscr{P}$ is unimodularly equivalent to a lattice pyramid over $\operatorname{conv}(\{(0,0),(3,0),(0,3)\})$), or
- $b \geq 1$ and $a \leq 3b+3$.

Note how close this is to the characterization in dimension two above. It follows from the proof in [34] that any such polynomial can be given by the δ-polynomial of a lattice polytope in dimension three. Most recently, Balletti and Higashitani [2] improved the result further to any lattice polytope whose δ-polynomial satisfies $\delta_3 = 0$.

Small number of monomials

An even more general problem is to consider the number of terms in the δ-polynomial. Batyrev and Hofscheier [6, 7] have recently classified all lattice polytopes whose δ-polynomials are binomials, i.e., of the form $1+at^k$. Let $\mathscr{P}$ be a d-dimensional lattice polytopes with such a binomial δ-polynomial. Since the degree one case $k=1$ is known, let $k \geq 2$. Hence, $\delta_1 = 0$ implies that $\mathscr{P}$ is an empty simplex. It can be observed [7, Prop.1.5] that $d \geq 2k-1$. Let $d = 2k-1$. In this case, it is proven in [6] that $\mathscr{P}$ has δ-polynomial $1+at^k$ (with $a \geq 1$) if and only if $\mathscr{P}$ is a Cayley polytope of k empty line segments in $\mathbb{R}^k$. Note that for $d = 3$ and $k = 2$ this recovers White's theorem. In particular, one sees from [7, Example 2.2] that any $a \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}_{\geq 1}$ is possible for an δ-polynomial of the form $1+at^k$. The reader might notice the analogy with the degree one case above.

For $d \geq 2k$, we are in an exceptional situation. Let us consider only δ-polynomials of lattice polytopes that are not lattice pyramids (otherwise, by what we've just seen, any $1+at^k$ can appear). Note that since Δ is a simplex, it follows from Lemma 2.4 that $d \leq 4k-2$. Now, the following characterization can be deduced from the results in [7]: $1+at^k$ (with $a \in \mathbb{Z}_{\geq 1}$) is the δ-polynomial of a d-dimensional lattice polytope $\mathscr{P}$ with $d \geq 2k$ where $\mathscr{P}$ is not a lattice pyramid if and only if $a = \frac{2kp}{d+1-p(d+1-2k)} - 1$ and $\frac{2k}{d+1-p(d+1-2k)}$ is a power of a prime p. It is not hard to see that this implies $p \leq k$, in particular, $\mathrm{Vol}(\mathscr{P}) = a+1 < 2k^2$. Hence, there are only finitely many non-lattice-pyramid lattice polytopes with binomial δ-polynomials for given k and arbitrary $d \geq 2k$. They are completely classified by Batyrev and Hofscheier [7]. It turns out that they are uniquely determined by their δ-polynomial.

Palindromic δ-polynomials

By Lemma 2.6, palindromic δ-polynomials correspond to Gorenstein polytopes. From a complete classification of reflexive polytopes up to dimension 4, we can characterize the palindromic δ-polynomials of lattice polytopes whose dimensions are at most 4. Moreover, for fixed degree, there exist only finitely many Gorenstein polytopes that are not lattice pyramids [31]. They have been completely classified by Batyrev and Juny up to degree two [8]. In particular, their results imply that a polynomial $1+(m-2)t+t^2$ with $m \in \mathbb{Z}_{\geq 2}$ is the δ-polynomial of a d-dimensional lattice polytope that is not a lattice pyramid if and only if

- $d = 2$ and $3 \leq m \leq 9$, or
- $d = 3$ and $2 \leq m \leq 8$, or
- $d = 4$ and $3 \leq m \leq 6$, or
- $d = 5$ and $m = 4$.

Small volumes

We recall a complete characterization of the δ-polynomials of lattice polytopes whose normalized volumes are at most 4. In [41, Theorem 0.1], the possible δ-polynomials with $\delta_0 + \cdots + \delta_d \leq 3$ are completely classified by the inequalities (2.1) and (2.2). However, this is not true for $\delta_0 + \cdots + \delta_d = 4$. In [40, Theorem 5.1], the complete classification of the possible δ-polynomials with $\delta_0 + \cdots + \delta_d = 4$ is given.

Theorem 2.10 ([40, 41] **).** *Let $2 \leq V \leq 4$ be a positive integer and $1 + t^{i_1} + \cdots + t^{i_{V-1}}$ a polynomial with $1 \leq i_1 \leq \cdots \leq i_{V-1} \leq d$. Then there exists a lattice polytope of dimension d whose δ-polynomial equals $1 + t^{i_1} + \cdots + t^{i_{V-1}}$ if and only if one of the followings is satisfies:*

(1) *$V = 2$ and $i_1 \leq \lfloor (d+1)/2 \rfloor$;*

(2) *$V = 3$, $2i_1 \geq i_2$ and $i_2 \leq \lfloor (d+1)/2 \rfloor$;*

(3) *$V = 4$, $i_3 \leq i_1 + i_2$, $i_1 + i_3 \leq d + 1$ and $i_2 \leq \lfloor (d+1)/2 \rfloor$, and the additional condition*

$$2i_2 \leq i_1 + i_3 \text{ or } i_2 + i_3 \leq d + 1.$$

We remark that when $\delta_0 + \cdots + \delta_d \leq 4$, all the possible δ-polynomials can be obtained by simplices. However, when $\delta_0 + \cdots + \delta_d = 5$, this is not true ([40, Remark 5.3]).

The structure of the rest of Part I

The organization of the rest of this part is as follows. In Chapter 3, we will classify the lattice polytopes whose normalized volumes are at most 4. In Chapter 4, we will characterize the δ-polynomials of lattice polytopes whose normalized volumes equal 5. In Chapter 5, to work towards a classification of the Gorenstein simplices, we will discuss a characterization of Gorenstein simplices in terms of their associated finite abelian groups. In Chapter 6, we consider Gorenstein simplices with a given δ-polynomial. Finally, in Chapter 7, we will finish the compete classification of the lattice polytopes that are not lattice pyramids and whose δ-polynomials are palindromic and have precisely three terms.

Chapter 3

Classification of lattice polytopes with small volumes

Taking into consideration the fact that a complete characterization of the δ-polynomials of lattice polytopes whose normalized volumes are at most 4 is known ([40, 41]), it is reasonable to classify the lattice polytopes whose normalized volumes are at most 4. In fact, in this chapter, this job will be done.

In the frame of a classification of general square systems of polynomial equations solvable by radicals, Esterov and Gusev [25] succeeded in classifying all lattice spanning polytopes $\mathscr{P}$ whose normalized volumes are at most 4. However, the condition of spanning lattice polytopes is rather strong for achieving a classification of lattice polytopes. For example, no empty simplex does enjoy the property and, in addition, there exists a lattice non-simplex whose normalized volume is 4 and which lacks the property. Combining our work with Esterov and Gusev [25] will establish a complete classification of lattice polytopes whose normalized volumes are at most 4. Moreover, our classification work will be making steady progress by means of δ-polynomials.

In this chapter, we will classify, up to unimodular equivalence and lattice pyramid constructions, the lattice polytopes whose normalized volumes are at most 4. The complete classification of the lattice polytopes whose normalized volumes are at most 4 up to unimodular equivalence consists of these polytopes and lattice pyramids over them. Note that every lattice simplex of dimension d with $\mathrm{Vol}(\mathscr{P}) = 1$ is unimodularly equivalent to the standard simplex of dimension d. In order to do this job, we divide into the following three cases:

(1) Lattice simplices $\Delta \subset \mathbb{R}^d$ with $\mathrm{Vol}(\Delta) \leq 4$;

(2) Spanning lattice non-simplices $\mathscr{P} \subset \mathbb{R}^d$ with $\mathrm{Vol}(\mathscr{P}) \leq 4$;

(3) Non-spanning lattice non-simplices $\mathscr{P} \subset \mathbb{R}^d$ with $\mathrm{Vol}(\mathscr{P}) \leq 4$.

The complete classification of the case (2) can be obtained from [25]. Therefore, we will show the cases (1) and (3).

First, the complete classification of the lattice simplices $\Delta \subset \mathbb{R}^d$ with $\mathrm{Vol}(\Delta) \leq 4$ can be obtained from the following:

Theorem 3.1 ([48, Theorem 1.2]). *Let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-polynomial equals $1+t^{i_1}+\cdots+t^{i_{V-1}}$ with $2 \leq V \leq 4$, where $(i_1,\ldots,i_{V-1})$ satisfies the conditions in* Theorem 2.10*. Assume that Δ is not a lattice pyramid. Then there exist, up to unimodular equivalence, exactly the following 5 possibilities for Δ:*

(1) $V=2:\Delta^{(2)}$;

(2) $V=3:\Delta^{(3)}$;

(3) $V=4:\Delta_i^{(4)}$, $1 \leq i \leq 3$.

The conditions and vertices of Δ are presented in TABLE 3.1.

	conditions	vertices
$\Delta^{(2)}$	$d=2i_1-1$	$\mathbf{0},\mathbf{e}_1,\ldots,\mathbf{e}_{d-1},\mathbf{e}_1+\cdots+\mathbf{e}_{d-1}+2\mathbf{e}_d$
$\Delta^{(3)}$	$d=i_1+i_2-1$	$\mathbf{0},\mathbf{e}_1,\ldots,\mathbf{e}_{d-1}$, $2\sum_{i=1,i\neq d}^{-i_1+2i_2-1}\mathbf{e}_i+\sum_{i=-i_1+2i_2}^{d-1}\mathbf{e}_i+3\mathbf{e}_d$
$\Delta_1^{(4)}$	$i_1<i_2<i_3$, $i_1+i_3 \leq 2i_2$, $d=i_1+i_3-1$	$\mathbf{0},\mathbf{e}_1,\ldots,\mathbf{e}_{d-1}$, $2\sum_{i=1}^{i_1-2i_2+i_3}\mathbf{e}_i+\sum_{i=i_1-2i_2+i_3+1,i\neq d}^{2i_1-i_2}\mathbf{e}_i+3\sum_{i=2i_1-i_2+1}^{d-1}\mathbf{e}_i+4\mathbf{e}_d$
$\Delta_2^{(4)}$	$d=i_2+i_3-1$	$\mathbf{0},\mathbf{e}_1,\ldots,\mathbf{e}_{d-1}$, $2\sum_{i=1}^{-2i_1+i_2+i_3}\mathbf{e}_i+\sum_{i=-2i_1+i_2+i_3+1,i\neq d}^{-i_1+2i_2}\mathbf{e}_i+3\sum_{i=-i_1+2i_2+1}^{d-1}\mathbf{e}_i+4\mathbf{e}_d$
$\Delta_3^{(4)}$	$d=i_1+i_2+i_3-1$	$\mathbf{0},\mathbf{e}_1,\ldots,\mathbf{e}_{d-2}$, $\sum_{i=-i_1+i_2+i_3}^{d-2}\mathbf{e}_i+2\mathbf{e}_{d-1}$, $\sum_{i=1}^{-i_1+i_2+i_3-1}\mathbf{e}_i+\sum_{i=2i_3-1}^{d-2}\mathbf{e}_i+2\mathbf{e}_d$

TABLE 3.1: The lattice simplices $\Delta \subset \mathbb{R}^d$ with $\mathrm{Vol}(\Delta) \leq 4$.

Second, the complete classification of the spanning lattice non-simplices $\mathscr{P} \subset \mathbb{R}^d$ with $\mathrm{Vol}(\mathscr{P}) \leq 4$ can be obtained from the following:

Theorem 3.2 ([25]). *Let $2 \leq V \leq 4$ be a positive integer and $\mathscr{P} \subset \mathbb{R}^d$ a lattice spanning non-simplex with $\mathrm{Vol}(\mathscr{P})=V$. Assume that $\mathscr{P}$ is not a lattice pyramid. Then there exist up to unimodular equivalence exactly the following 24 possibilities for $\mathscr{P}$:*

(1) $\delta(\mathscr{P},t) = 1+t : \mathscr{P}^{(2)}$;

(2) $\delta(\mathscr{P},t) = 1+2t : \mathscr{P}_i^{(3)}, 1 \leq i \leq 2$;

(3) $\delta(\mathscr{P},t) = 1+t+t^2 : \mathscr{Q}_i^{(3)}$, $1 \leq i \leq 2$;

(4) $\delta(\mathscr{P},t) = 1+3t : \mathscr{P}_i^{(4)}, 1 \leq i \leq 4$;

(5) $\delta(\mathscr{P},t) = 1+2t+t^2 : \mathscr{Q}_i^{(4)}$, $1 \leq i \leq 9$;

(6) $\delta(\mathscr{P},t) = 1+t+t^2 : \mathscr{R}_i^{(4)}, 1 \leq i \leq 2$;

(7) $\delta(\mathscr{P},t) = 1+t+t^2+t^3 : \mathscr{S}_i^{(4)}$, $1 \leq i \leq 4$.

The dimension and vertices of $\mathscr{P}$ are presented in TABLE 3.2.

Finally, the complete classification of the non-spanning lattice non-simplices $\mathscr{P} \subset \mathbb{R}^d$ with $\mathrm{Vol}(\mathscr{P}) \leq 4$ can be obtained the following:

Theorem 3.3 ([48, Theorem 1.4])**.** *Let $2 \leq V \leq 4$ be a positive integer and $\mathscr{P} \subset \mathbb{R}^d$ a non-spanning lattice non-simplex with* $\mathrm{Vol}(\mathscr{P}) = V$*. Assume that $\mathscr{P}$ is not a lattice pyramid. Then there exist, up to unimodular equivalence, exactly the following* 4 *possibilities for $\mathscr{P}$:*

(1) $\delta(\mathscr{P},t) = 1+t+t^k+t^{k+1}$ *with* $k \geq 2 : \mathscr{A}_i^{(4)}, 1 \leq i \leq 3$;

(2) $\delta(\mathscr{P},t) = 1+t+2t^k$ *with* $k \geq 2 : \mathscr{B}^{(4)}$.

The dimension and vertices of $\mathscr{P}$ are presented in TABLE 3.3.

The this chapter is organized as follows: We, in Section 3.1, prove Theorem 3.1. Finally, in Section 3.2, we prove Theorem 3.3.

3.1 Proof of Theorem 3.1

In this section, we classify the lattice simplices $\Delta \subset \mathbb{R}^d$ of dimension d with $\mathrm{Vol}(\Delta) \leq 4$ up to unimodular equivalence and lattice pyramid constructions. Namely, we prove Theorem 3.1. In order to do this job, we divide into the following three cases:

(1) $\mathrm{Vol}(\Delta) = 2$ (Subsection 3.1.1);

(2) $\mathrm{Vol}(\Delta) = 3$ (Subsection 3.1.2);

(3) $\mathrm{Vol}(\Delta) = 4$ (Subsection 3.1.3).

	d	vertices
$\mathscr{P}^{(2)}$	2	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2$
$\mathscr{P}_1^{(3)}$	2	$\mathbf{0},2\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2$
$\mathscr{P}_2^{(3)}$	3	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2+\mathbf{e}_3$
$\mathscr{Q}_1^{(3)}$	3	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2-2\mathbf{e}_3$
$\mathscr{Q}_2^{(3)}$	4	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,-\mathbf{e}_1-\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4$
$\mathscr{P}_1^{(4)}$	2	$\mathbf{0},2\mathbf{e}_1,\mathbf{e}_2,2\mathbf{e}_1+\mathbf{e}_2$
$\mathscr{P}_2^{(4)}$	2	$\mathbf{0},3\mathbf{e}_1,\mathbf{e}_1+\mathbf{e}_2,2\mathbf{e}_1+\mathbf{e}_2$
$\mathscr{P}_3^{(4)}$	3	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2+\mathbf{e}_3,2\mathbf{e}_3$
$\mathscr{P}_4^{(4)}$	4	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_4$
$\mathscr{Q}_1^{(4)}$	2	$\mathbf{e}_1,-\mathbf{e}_2,\mathbf{e}_1-\mathbf{e}_2,-\mathbf{e}_1+\mathbf{e}_2$
$\mathscr{Q}_2^{(4)}$	2	$\mathbf{e}_1,\mathbf{e}_2,-\mathbf{e}_1,-\mathbf{e}_2$
$\mathscr{Q}_3^{(4)}$	3	$\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2,-\mathbf{e}_3$
$\mathscr{Q}_4^{(4)}$	3	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2,2\mathbf{e}_3$
$\mathscr{Q}_5^{(4)}$	3	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$
$\mathscr{Q}_6^{(4)}$	3	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3$
$\mathscr{Q}_7^{(4)}$	4	$\mathbf{0},2\mathbf{e}_1,\mathbf{e}_4,\mathbf{e}_2+\mathbf{e}_4,\mathbf{e}_3+\mathbf{e}_4,\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4$
$\mathscr{Q}_8^{(4)}$	4	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_3+\mathbf{e}_4$
$\mathscr{Q}_9^{(4)}$	5	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_5,\mathbf{e}_3+\mathbf{e}_5,\mathbf{e}_4+\mathbf{e}_5,\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5$
$\mathscr{R}_1^{(4)}$	3	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2-3\mathbf{e}_3$
$\mathscr{R}_2^{(4)}$	4	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,-2\mathbf{e}_1-\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4$
$\mathscr{S}_1^{(4)}$	4	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+\mathbf{e}_4$
$\mathscr{S}_2^{(4)}$	4	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+2\mathbf{e}_4$
$\mathscr{S}_3^{(4)}$	5	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5,-2\mathbf{e}_1-\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5$
$\mathscr{S}_4^{(4)}$	6	$\mathbf{0},\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6,-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5+\mathbf{e}_6$

TABLE 3.2: The spanning lattice non-simplices $\mathscr{P}$ with $\mathrm{Vol}(\mathscr{P}) \leq 4$.

3.1.1 The case $\mathrm{Vol}(\Delta) = 2$

In this subsection, we consider the case where $\mathrm{Vol}(\Delta) = 2$. Since $|\Lambda_\Delta| = 2$, for any $\lambda \in \Lambda_\Delta \setminus \{\mathbf{0}\}$, $\mathrm{ord}(\lambda) = 2$. Hence since Δ is not a lattice pyramid, by using Lemma 2.3, it follows that Λ_Δ is generated by one element $(1/2,\dots,1/2)$ and $d+1$ is an even number. Set $d = 2k-1$ with some positive integer k. By using Lemma 2.9, one has $\delta(\Delta,t) = 1+t^k$. Moreover it is easy to see that $\Lambda_{\Delta^{(2)}} = \Lambda_\Delta$ with any ordering of the vertices of $\Delta^{(2)}$. Hence this completes the proof of the case where $\mathrm{Vol}(\Delta) = 2$.

	d	vertices
$\mathscr{A}_1^{(4)}$	$2k$	$\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d, \mathbf{e}_1 - \mathbf{e}_{d-1}$
$\mathscr{A}_2^{(4)}$	$2k+1$	$\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d, \mathbf{e}_{d-2} + \mathbf{e}_{d-1}$
$\mathscr{A}_3^{(4)}$	$2k+2$	$\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \sum_{j=1}^{d-4} \mathbf{e}_j + 2\mathbf{e}_d, -\mathbf{e}_{d-3} + \mathbf{e}_{d-2} + \mathbf{e}_{d-1}$
$\mathscr{B}^{(4)}$	$2k$	$\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d, -\mathbf{e}_1 + \mathbf{e}_{d-1}$

TABLE 3.3: The lattice non-spanning non-simplices $\mathscr{P}$ with $\mathrm{Vol}(\mathscr{P}) \leq 4$.

3.1.2 The case $\mathrm{Vol}(\Delta) = 3$

In this subsection, we consider the case where $\mathrm{Vol}(\Delta) = 3$. For nonnegative integers a and b, we let $\Lambda(a,b)$ be the finite abelian subgroups of $(\mathbb{R}/\mathbb{Z})^{a+b}$ defined as follows:

$$\Lambda(a,b) = \left\langle \left(\underbrace{\frac{1}{3}, \dots, \frac{1}{3}}_{a}, \underbrace{\frac{2}{3}, \dots, \frac{2}{3}}_{b} \right) \right\rangle.$$

Since $\mathrm{Vol}(\Delta) = |\Lambda_\Delta| = 3$, for any $\lambda \in \Lambda_\Delta \setminus \{\mathbf{0}\}$, $\mathrm{ord}(\lambda) = 3$. Hence since Δ is not a lattice pyramid, by Lemma 2.3, there exist nonnegative integers a, b with $a+b = d+1$ such that $\Lambda_\Delta = \Lambda(a,b)$ with some ordering of the vertices of Δ. Since $\Lambda(a,b)$ coincides with $\Lambda(b,a)$ by reordering of the coordinates, we can assume that $a \geq b$. Then by using Lemma 2.9, one has $i_1 = (a+2b)/3$ and $i_2 = (2a+b)/3$. Hence we obtain $a = -i_1 + 2i_2, b = 2i_1 - i_2$ and $d+1 = a+b = i_1 + i_2$. Moreover, it is easy to see that $\Lambda_{\Delta^{(3)}} = \Lambda(a,b)$ with some ordering of the vertices of $\Delta^{(3)}$. Hence this completes the proof of the case where $\mathrm{Vol}(\Delta) = 3$.

3.1.3 The case $\mathrm{Vol}(\Delta) = 4$

In this subsection, we consider the case where $\mathrm{Vol}(\Delta) = 4$. For nonnegative integers a, b, c, we let $\Lambda_1(a,b,c)$ and $\Lambda_2(a,b,c)$ be the finite abelian subgroups of $(\mathbb{R}/\mathbb{Z})^{a+b+c}$ defined as follows:

$$\Lambda_1(a,b,c) = \left\langle \left(\underbrace{\frac{1}{4}, \dots, \frac{1}{4}}_{a}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{b}, \underbrace{\frac{3}{4}, \dots, \frac{3}{4}}_{c} \right) \right\rangle;$$

$$\Lambda_2(a,b,c) = \left\langle \left(\underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{a}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{b}, \underbrace{0, \dots, 0}_{c} \right), \left(\underbrace{0, \dots, 0}_{a}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{b}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{c} \right) \right\rangle.$$

Since $\mathrm{Vol}(\Delta) = |\Lambda_\Delta| = 4$, for any $\lambda \in \Lambda_\Delta \setminus \{\mathbf{0}\}$, $\mathrm{ord}(\lambda) \in \{2,4\}$. Hence since Δ is not a lattice pyramid, by Lemma 2.3, there exist nonnegative integers a,b,c with $d+1 = a+b+c$ such that Λ_Δ coincides with $\Lambda_1(a,b,c)$ or $\Lambda_2(a,b,c)$ with some ordering of the vertices of Δ.

At first, suppose that $\Lambda_\Delta = \Lambda_1(a,b,c)$ with some ordering of the vertices of Δ. Then since $\Lambda_1(a,b,c)$ coincides with $\Lambda_1(c,b,a)$ by reordering of the coordinates, we may assume that $a \geq c$. Moreover, by using Lemma 2.9, one has $\{i_1,i_2,i_3\} = \{(a+2b+3c)/4, (a+c)/2, (3a+2b+c)/4\}$. Set $(h_1,h_2,h_3) = ((a+2b+3c)/4, (a+c)/2, (3a+2b+c)/4)$. Then we obtain $a = -h_1+h_2+h_3, b = h_1-2h_2+h_3$ and $c = h_1+h_2-h_3$. Since $a \geq c$, (h_1,h_3) is (i_1,i_2), (i_1,i_3) or (i_2,i_3). Hence it follows from $a,b,c \geq 0$ that one of the following conditions is satisfied:

(1) $i_1+i_3 \geq 2i_2$ and $d+1 = i_1+i_3$, and $\Lambda_\Delta = \Lambda_1(-i_1+i_2+i_3, i_1-2i_2+i_3, i_1+i_2-i_3)$;

(2) $i_2+i_3 \geq 2i_1$ and $d+1 = i_2+i_3$, and $\Lambda_\Delta = \Lambda_1(i_1-i_2+i_3, -2i_1+i_2+i_3, i_1+i_2-i_3)$;

(3) $i_1+i_2 \geq 2i_3$ and $d+1 = i_1+i_2$, and $\Lambda_\Delta = \Lambda_1(-i_1+i_2+i_3, i_1+i_2-2i_3, i_1-i_2+i_3)$;

If $i_1 = i_2$ or $i_2 = i_3$, then the condition (1) is equivalent to one of the conditions (2) and (3). Since $i_1+i_2 \geq 2i_3$ implies that $i_1 = i_2 = i_3$, if the condition (3) is satisfied, then condition (2) is satisfied. Moreover, it always follows that $i_2+i_3 \geq 2i_1$. Hence we know that one of the following conditions is satisfied:

(1') $i_1 < i_2 < i_3$, $i_1+i_3 \geq 2i_2$ and $d+1 = i_1+i_3$, and $\Lambda_\Delta = \Lambda_1(-i_1+i_2+i_3, i_1-2i_2+i_3, i_1+i_2-i_3)$;

(2') $d+1 = i_2+i_3$ and $\Lambda_\Delta = \Lambda_1(i_1-i_2+i_3, -2i_1+i_2+i_3, i_1+i_2-i_3)$.

In particular, it is easy to see that if the condition (1') is satisfied, then $\Lambda_{\Delta_1^{(4)}} = \Lambda_1(a,b,c)$ with some ordering of the vertices of $\Delta_1^{(4)}$, and if the condition (2') is satisfied, then $\Lambda_{\Delta_2^{(4)}} = \Lambda_1(a,b,c)$ with some ordering of the vertices of $\Delta_2^{(4)}$.

Next, we suppose that $\Lambda_\Delta = \Lambda_2(a,b,c)$ with some ordering of the vertices of Δ. It follows that $\Lambda_2(a,b,c)$ coincides with $\Lambda_2(b,a,c)$ (resp. $\Lambda_2(c,b,a)$) by reordering of the coordinates. Hence we may assume that $a \geq b \geq c$. Then by using Lemma 2.9, one has $(i_1,i_2,i_3) = ((b+c)/2, (a+c)/2, (a+b)/2)$. Therefore, we obtain $d+1 = i_1+i_2+i_3$ and $\Lambda_\Delta = \Lambda_2(-i_1+i_2+i_3, i_1-i_2+i_3, i_1+i_2-i_3)$. In particular, it is easy to see that $\Lambda_{\Delta_2^{(4)}} = \Lambda_2(a,b,c)$ with some ordering of the vertices of $\Delta_3^{(4)}$. Hence this completes the proof of the case where $\mathrm{Vol}(\Delta) = 4$.

Therefore, Theorem 3.1 follows.

3.2 Proof of Theorem 3.3

In this section, we classify the non-spanning lattice non-simplices $\mathscr{P} \subset \mathbb{R}^d$ of dimension d with $\mathrm{Vol}(\mathscr{P}) \leq 4$ up to unimodular equivalence and lattice pyramid constructions. Namely, we prove Theorem 3.3.

We recall that a matrix $A \in \mathbb{Z}^{d\times d}$ is *unimodular* if $\det(A) = \pm 1$. Then lattice polytopes $\mathscr{P} \subset \mathbb{R}^d$ and $\mathscr{Q} \subset \mathbb{R}^d$ of dimension d are unimodularly equivalent if and only if there exist a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and a lattice point $\mathbf{w} \in \mathbb{Z}^d$ such that $\mathscr{Q} = f_U(\mathscr{P}) + \mathbf{w}$, where f_U is the linear transformation in $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$.

A *lattice triangulation* of a lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is a finite collection of lattice simplices T such that

(1) every face of a member of T is in T,

(2) any two elements of T intersect in a common (possibly empty) face, and

(3) the union of simplices in T is $\mathscr{P}$.

First, we show the following lemma:

Lemma 3.4. *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice non-simplex. Assume that $\{\Delta_1, \Delta_2\}$ is a lattice triangulation of $\mathscr{P}$ and $\Delta = \Delta_1 \cap \Delta_2$. Then $L_{\mathscr{P}}(n) = L_{\Delta_1}(n) + L_{\Delta_2}(n) - L_{\Delta}(n)$.*

Proof. Since $\{\Delta_1, \Delta_2\}$ is a triangulation of $\mathscr{P}$, it follows that $\{n\Delta_1, n\Delta_2\}$ is a triangulation of $n\mathscr{P}$ for any positive integer n. Hence since

$$n\mathscr{P} \cap \mathbb{Z}^d = (n\Delta_1 \cap \mathbb{Z}^d) \cup (n\Delta_2 \cap \mathbb{Z}^d),$$

one has

$$|n\mathscr{P} \cap \mathbb{Z}^d| = |n\Delta_1 \cap \mathbb{Z}^d| + |n\Delta_2 \cap \mathbb{Z}^d| - |(n\Delta_1 \cap n\Delta_2) \cap \mathbb{Z}^d|.$$

This implies that $L_{\mathscr{P}}(n) = L_{\Delta_1}(n) + L_{\Delta_2}(n) - L_{\Delta}(n)$. □

Now, we consider the proof of Theorem 3.3. Let $\delta(\mathscr{P}, t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$ be its δ-polynomial of $\mathscr{P}$. Since $\mathscr{P}$ is a lattice non-simplex, one has $\delta_1 \geq 1$. Hence by the inequalities (2.1) and (2.2), it is known that $\delta(\mathscr{P}, t)$ forms one of the followings:

- $1 + t + t^k + t^{k+1}$;
- $1 + t + 2t^k$;
- $1 + 2t + t^2$;

- $1+3t$,

where $k \geq 2$ is some integer. In [8, 9], the lattice polytopes whose δ-polynomials equal $1+2t+t^2$ or $1+3t$ are completely classified. Then they appear in the lists of Theorems 3.1 and 3.2. Hence we divide into the following two cases:

(1) $\delta(\mathscr{P},t) = 1+t+t^k+t^{k+1}$ (Subsection 3.2.1);

(2) $\delta(\mathscr{P},t) = 1+t+2t^k$ (Subsection 3.2.2).

In these cases, since $\mathscr{P}$ is a non-simplex and $\delta_1 = 1$, the number of vertices of $\mathscr{P}$ equals $d+2$ and $\mathscr{P}$ has no lattice points expect for its vertices. Let $\mathbf{v}_0,\dots,\mathbf{v}_{d+1}$ be vertices of $\mathscr{P}$ and for $0 \leq i \leq d+1$, let Δ_i be the lattice polytope which is the convex hull of $\mathbf{v}_0,\dots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\dots,\mathbf{v}_{d+1}$. Remark that each Δ_i is not necessarily a lattice simplex of dimension d. Let $\{T_1,\dots,T_m\}$ be a triangulation of $\mathscr{P}$. Then since $\mathrm{Vol}(\mathscr{P}) = 4$, it follows that $2 \leq m \leq 4$ and we may assume that $T_1 = \Delta_{d+1}$ and $\mathrm{Vol}(T_1) \geq \cdots \geq \mathrm{Vol}(T_m)$. If for some i, $\mathrm{Vol}(T_i) = 1$, namely, T_i is unimodularly equivalent to the standard simplex of dimension d, it then follows that $\mathscr{P}$ is spanning. Hence one has $m = 2$ and $(\mathrm{Vol}(T_1),\mathrm{Vol}(T_2)) = (2,2)$.

3.2.1 The case $\delta(\mathscr{P},t) = 1+t+t^k+t^{k+1}$

In this subsection, we consider the case where $\delta(\mathscr{P},t) = 1+t+t^k+t^{k+1}$. Then by Lemma 2.2, it follows that $\delta(T_1,t) = 1+t, 1+t^k$ or $1+t^{k+1}$. From Theorem 3.1, for every lattice simplex $\Delta \subset \mathbb{R}^d$ with $\delta(\Delta,t) = 1+t$, it follows that Δ is spanning. Moreover, if $\delta(T_1,t) = \delta(T_2,t) = 1+t^{k+1}$, then by using Lemma 3.4, for some n,

$$L_{T_1\cap T_2}(n) = \binom{n+d-1}{d-1} - \binom{n+d-k-1}{d-1} < \binom{n+d-1}{d-1},$$

a contradiction. Hence we may suppose that $\delta(T_1,t) = 1+t^k$. Moreover, by Theorem 3.1, we can assume that

$$\mathbf{v}_i = \begin{cases} \mathbf{0}, & (i=0), \\ \mathbf{e}_i, & (i=1,\dots,d-1), \\ \sum_{j=1}^{2k-2}\mathbf{e}_j + 2\mathbf{e}_d, & (i=d). \end{cases}$$

and set $c = d-2k+1$. Let $\mathbf{v}_{d+1} = (a_1,\dots,a_d) \in \mathbb{Z}^d$. Since $\mathscr{P}$ is not a lattice pyramid, for $2k-1 \leq i \leq d-1$, one has $a_i \neq 0$. For a lattice polytope $\mathscr{Q} \subset \mathbb{R}^d$ of dimension e, we set

$$\mathrm{Vol}_d(\mathscr{Q}) = \begin{cases} \mathrm{Vol}(\mathscr{Q}), & (e=d), \\ 0, & (e<d). \end{cases}$$

Then, one has

$$\mathrm{Vol}_d(\Delta_i) = \begin{cases} |2a_i - a_d|, & (1 \le i \le 2k-2), \\ |2a_i|, & (2k-1 \le i \le d-1), \\ |a_d|, & (i = d). \end{cases}$$

Since $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1} \in \mathscr{P}$ and $\mathscr{P}$ is not spanning, it then follows that $a_d \in 2\mathbb{Z}$. Moreover, since for $0 \le i \le d+1$, $\mathrm{Vol}(\Delta_i) < 4$, one has $a_d \in \{-2, 0, 2\}$ and $a_i \in \{-1, 1\}$ for $2k-1 \le i \le d-1$. Now, we may assume that $T_2 \in \{\Delta_0, \Delta_1, \Delta_{2k-1}, \Delta_d\}$.

3.2.1.1 The case $T_2 = \Delta_d$

Suppose that $T_2 = \Delta_d$. Then since $T_1 \cap T_2$ is a unimodular simplex of dimension $d-1$, by Lemma 3.4, one has $\delta(T_2, t) = 1 + t^{k+1}$. Hence since $(d+1)/2 \ge k+1$ from the inequality (2.2), we obtain $c \ge 2$. Moreover, since $T_1 \cap T_2$ belongs to the hyperplane defined by the equation $x_d = 0$ and $\mathrm{Vol}(T_2) = 2$, one has $a_d = -2$. Hence $(\mathbf{v}_d + \mathbf{v}_{d+1})/2 \in T_1 \cap T_2$. Then it follows that $a_i + 1 \ge 0$ for $1 \le i \le 2k-2$ and $a_i \ge 0$ for $2k-1 \le i \le d-1$. Therefore, for $1 \le i \le 2k-2$, one has $a_i \in \{-1, 0\}$ and for $2k-2 \le i \le d-1$, one has $a_i = 1$. Thus we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{-1, \dots, -1}_{a}, \underbrace{0, \dots, 0}_{2k-2-a}, \underbrace{1, \dots, 1}_{c}, -2).$$

Then one has

$$(\mathbf{v}_d + \mathbf{v}_{d+1})/2 = (\underbrace{0, \dots, 0}_{a}, \underbrace{1/2, \dots, 1/2}_{2k-2-a+c}, 0).$$

Hence it follows from $(\mathbf{v}_d + \mathbf{v}_{d+1})/2 \in T_1 \cap T_2$ that $2k-2-a+c \le 2$. Therefore, since $c \ge 2$, one has $(a, c) = (2k-2, 2)$. Then we have $\delta(\mathscr{P}, t) = 1 + t + t^k + t^{k+1}$.

3.2.1.2 The case $T_2 = \Delta_0$

Suppose that $T_2 = \Delta_0$. Then $\Lambda_{T_1 \cap T_2} = \{(0, \dots, 0)\} \subset (\mathbb{R}/\mathbb{Z})^d$. Hence $T_1 \cap T_2$ is a unimodular simplex of dimension $d-1$. Then it follows from Subsubsection 3.2.1.1 that $\delta(T_2, t) = 1 + t^{k+1}$ and $c \ge 2$. If for some i, $a_i < 0$, then $\mathbf{v}_{d+1}/2 \in \Delta_0$. This implies that $\mathbf{0} \in T_2$. Hence we obtain $a_i \ge 0$ for $1 \le i \le d$. Moreover, one has $a_d \in \{0, 2\}$ and for $2k-1 \le i \le d-1$, one has $a_i = 1$.

Assume that $a_d = 0$. Then for $1 \le i \le 2k-2$, we obtain $a_i \in \{0, 1\}$. Hence we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{1, \dots, 1}_{a}, \underbrace{0, \dots, 0}_{2k-2-a}, \underbrace{1, \dots, 1}_{c}, 0).$$

Then since $a+c-1 \geq 1$,

$$\lambda = \left(\underbrace{\frac{a+c-2}{a+c-1}, \ldots, \frac{a+c-2}{a+c-1}}_{a}, \underbrace{0, \ldots, 0}_{2k-2-a}, \underbrace{\frac{a+c-2}{a+c-1}, \ldots, \frac{a+c-2}{a+c-1}}_{c}, 0, \frac{1}{a+c-1} \right)$$

is an element of Λ_{T_2} with the ordering $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$. Since $\mathrm{Vol}(T_2) = 2$, one has $a+c \leq 3$. If $a+c = 2$, namely, $(a,c) = (0,2)$, then we obtain $\delta(\mathscr{P},t) = 1+t+t^k+t^{k+1}$. Suppose that $a+c = 3$. Then $\mathrm{ht}(\lambda) = a+c-1 = 2 < k+1$, a contradiction.

Assume that $a_d = 2$. Then for $1 \leq i \leq 2k-2$, one has $a_i \in \{0,1,2\}$. If for some $1 \leq i \leq 2k-2$, $a_i = 0$, then $\mathbf{v}_{d+1}/2 \in T_2$ since the ith coordinate and the dth coordinate are 0 and positive. Set $\mathbf{v}_{d+1}/2 = c_1\mathbf{v}_1 + \cdots + c_{d+1}\mathbf{v}_{d+1}$, where $0 \leq c_1, \ldots, c_{d+1}$ and $c_1 + \cdots + c_{d+1} = 1$. Then it follows that $c_d = 0$ and $c_{d+1} = 1/2$. This implies that $\mathbf{0} \in T_2$, a contradiction. Hence $\{a_1, \ldots, a_{2k-2}\}$ is in $\{1,2\}^{2k-2}$. Therefore, we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{2, \ldots, 2}_{a}, \underbrace{1, \ldots, 1}_{2k-2-a}, \underbrace{1, \ldots, 1}_{c}, 2).$$

Then

$$\lambda = \left(\underbrace{\frac{1}{a+c}, \ldots, \frac{1}{a+c}}_{a}, \underbrace{0, \ldots, 0}_{2k-2-a}, \underbrace{\frac{1}{a+c}, \ldots, \frac{1}{a+c}}_{c}, \frac{1}{a+c}, \frac{a+c-1}{a+c} \right) \in (\mathbb{R}/\mathbb{Z})^{d+1}$$

is an element of Λ_{T_2} with the ordering $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$. Hence since $a+c \geq 2$, it follows that $\mathrm{Vol}(T_2) \geq a+c$. Therefore, one has $a+c = 2$ and $\mathrm{ht}(\lambda) = 2$. However, this contradicts that $\delta(T_2,t) = 1+t^{k+1}$ and $k+1 \geq 3$.

3.2.1.3 The case $T_2 = \Delta_1$

Suppose that $T_2 = \Delta_1$. Then $\Lambda_{T_1 \cap T_2} = \{(0, \ldots, 0)\} \subset (\mathbb{R}/\mathbb{Z})^d$. Hence $T_1 \cap T_2$ is a unimodular simplex of dimension $d-1$. Then it follows from Subsubsection 3.2.1.1 that $\delta(T_2,t) = 1+t^{k+1}$ and $c \geq 2$. Since $T_1 \cap T_2$ belongs to the hyperplane defined by the equation $2x_1 - x_d = 0$, one has $(a_1, a_d) \in \{(-2,-2), (-1,0), (0,2)\}$. Then it follows that $(\mathbf{v}_1 + \mathbf{v}_{d+1})/2 \in T_1 \cap T_2$. If $(a_1, a_d) = (-2,-2)$, then since the dth coordinate is negative, the point does not belong to $T_1 \cap T_2$. Set $(\mathbf{v}_1 + \mathbf{v}_{d+1})/2 = c_0\mathbf{v}_0 + c_2\mathbf{v}_2 + \cdots + c_d\mathbf{v}_d$, where $0 \leq c_0, c_2, \ldots, c_d$ and $c_0 + c_2 + \cdots + c_d = 1$. Assume that $(a_1, a_d) = (-1,0)$. Then we obtain $c_d = 0$. Hence it follows that for $2 \leq i \leq d-1$, $c_i = a_i/2 \geq 0$. Thus since $c_0 + c_2 + \cdots + c_d = 1$ and $c \geq 2$, one has $c = 2$ and $a_i = 0$ for $2 \leq i \leq 2k-2$ and $a_i = 1$ for $i = 2k-1, 2k$. Then we obtain $\delta(\mathscr{P},t) = 1+t+t^k+t^{k+1}$. Assume that $(a_1, a_d) = (0,2)$. Then we obtain $c_d = 1/2$. Moreover, since $c_i = a_i/2$ for $2k-1 \leq i \leq d-1$, it follows that $a_i = 1$ for $2 \leq i \leq d-1$. However, from $c \geq 2$, one has $c_0 + c_2 + \cdots + c_d > 1$, a contradiction.

3.2.1.4 The case $T_2 = \Delta_{2k-1}$

Suppose that $c \geq 1$ and $T_2 = \Delta_{2k-1}$. Since $T_1 \cap T_2$ belongs to the hyperplane defined by the equation $x_{2k-1} = 0$ and since $\mathrm{Vol}(T_2) = 2$, one has $a_{2k-1} = -1$. Moreover, we obtain $(\mathbf{v}_{2k-1} + \mathbf{v}_{d+1})/2 \in T_1 \cap T_2$. Hence it follows that for $1 \leq i \leq 2k-2$, $a_i \geq 0$ and $2k \leq i \leq d-1$, $a_i = 1$ and $a_d \in \{0, 2\}$. In particular, if $a_d = 2$, then $a_i > 0$ for $1 \leq i \leq 2k-2$.

Assume that $a_d = 0$. Then for $1 \leq i \leq 2k-2$, one has $a_i \in \{0, 1\}$. Hence we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{1, \ldots, 1}_{a}, \underbrace{0, \ldots, 0}_{2k-2-a}, -1, \underbrace{1, \ldots, 1}_{c-1}, 0).$$

Set $(\mathbf{v}_{2k-1} + \mathbf{v}_{d+1})/2 = \sum_{i=0, i \neq 2k-1}^{d} c_i \mathbf{v}_i$, where $0 \leq c_0, \ldots, c_{2k-2}, c_{2k}, \ldots, c_d$ and $c_0 + \cdots + c_{2k-2} + c_{2k} + \cdots + c_d = 1$. Then one has $c_d = 0$ and $c_i = 1/2$ for $1 \leq i \leq a$ or for $2k \leq i \leq d-1$. Hence we obtain $a + (c-1) \leq 2$, in particular, $1 \leq a + c \leq 3$. If $a + c = 1$, namely, $(a, c) = (0, 1)$, then $(\mathbf{v}_{2k-1} + \mathbf{v}_{d+1})/2 = \mathbf{0}$. This implies that $\mathbf{v}_0$ is not a vertex of $\mathscr{P}$, a contradiction. Hence one has $2 \leq a + c \leq 3$. In each case, we obtain $\delta(\mathscr{P}, t) = 1 + t + t^k + t^{k+1}$.

Assume that $a_d = 2$. Then for $1 \leq i \leq 2k-2$, one has $a_i \in \{1, 2\}$. Hence we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{2, \ldots, 2}_{a}, \underbrace{1, \ldots, 1}_{2k-2-a}, -1, \underbrace{1, \ldots, 1}_{c-1}, 2).$$

Set $(\mathbf{v}_{2k-1} + \mathbf{v}_{d+1})/2 = \sum_{i=0, i \neq 2k-1}^{d} c_i \mathbf{v}_i$, where $0 \leq c_0, \ldots, c_{2k-2}, c_{2k}, \ldots, c_d$ and $c_0 + \cdots + c_{2k-2} + c_{2k} + \cdots + c_d = 1$. Then one has $c_d = 1/2$ and $c_i = 1/2$ for $1 \leq i \leq a$ and $c_i = 0$ for $a + 1 \leq i \leq 2k-2$, and $c_i = 1/2$ for $2k \leq i \leq d-1$. Hence since $c_0 + \cdots + c_{2k-2} + c_{2k} + \cdots + c_d = 1$, we obtain $a + (c-1) \leq 1$, in particular, $1 \leq a + c \leq 2$. In each case, we obtain $\delta(\mathscr{P}, t) = 1 + t + t^k + t^{k+1}$.

Hence we know that $\mathscr{P}$ is unimodularly equivalent to the lattice polytope which is the convex hull of $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{d-1}, \mathbf{v}$ and $\mathbf{v}'$, where $\mathbf{v}$ and $\mathbf{v}'$ satisfy one of the following conditions:

(1) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \mathbf{e}_1 - \mathbf{e}_{d-1}$;

(2) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{d-1}$;

(3) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \sum_{j=1}^{d-2} \mathbf{e}_j - \mathbf{e}_{d-1} + 2\mathbf{e}_d$;

(4) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = 2\mathbf{e}_1 + \sum_{j=2}^{d-2} \mathbf{e}_j - \mathbf{e}_{d-1} + 2\mathbf{e}_d$;

(5) $d = 2k+1$, $\mathbf{v} = \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\sum_{j=1}^{d-3} \mathbf{e}_j + \mathbf{e}_{d-2} + \mathbf{e}_{d-1} - 2\mathbf{e}_d$;

(6) $d = 2k+1$, $\mathbf{v} = \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \mathbf{e}_{d-2} + \mathbf{e}_{d-1}$;

(7) $d = 2k+1$, $\mathbf{v} = \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\mathbf{e}_1 + \mathbf{e}_{d-2} + \mathbf{e}_{d-1}$;

(8) $d = 2k+1$, $\mathbf{v} = \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\mathbf{e}_{d-2} + \mathbf{e}_{d-1}$;

(9) $d = 2k+1$, $\mathbf{v} = \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \mathbf{e}_1 - \mathbf{e}_{d-2} + \mathbf{e}_{d-1}$;

(10) $d = 2k+1$, $\mathbf{v} = \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \sum_{j=1}^{d-3} \mathbf{e}_j - \mathbf{e}_{d-2} + \mathbf{e}_{d-1} + 2\mathbf{e}_d$;

(11) $d = 2k+2$, $\mathbf{v} = \sum_{j=1}^{d-4} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\mathbf{e}_{d-3} + \mathbf{e}_{d-2} + \mathbf{e}_{d-1}$.

For $i = 1, \ldots, 11$, let $\mathscr{P}_i$ be the lattice polytope which satisfies the condition (i). Set

$$U_{1,2} = \begin{pmatrix} 1 & -1 & & & & \\ & -1 & & & & \\ & -1 & 1 & & & \\ & \vdots & & \ddots & & \\ & -1 & & & 1 & \\ & k-1 & & & & 1 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

$$U_{1,3} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 2 \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & -1 & \cdots & -1 & 0 & -1 \end{pmatrix} \in \mathbb{Z}^{d \times d}$$

and

$$U_{1,5} = \begin{pmatrix} 0 & -1 & \cdots & \cdots & -1 & 0 & -2 \\ -1 & 0 & -1 & \cdots & -1 & 0 & -2 \\ \vdots & -1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 & -2 \\ -1 & -1 & \ldots & -1 & 0 & 0 & -2 \\ -1 & -1 & \ldots & -1 & -1 & 1 & -2 \\ k-2 & k-2 & \ldots & k-2 & k-2 & 0 & 2k-3 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

where other entries equal zero. Then these matrices are unimodular. Moreover, one has $\mathscr{P}_1 = f_{U_{1,2}}(\mathscr{P}_2) + \mathbf{e}_2 = f_{U_{1,3}}(\mathscr{P}_3) = f_{U_{1,4}} + (\sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d)$. Hence $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ and

$\mathscr{P}_4$ are unimodularly equivalent each other. Set

$$U_{5,6} = \begin{pmatrix} 0 & -1 & \cdots & -1 & 0 & 0 & -2 \\ -1 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 0 & -1 & 0 & 0 & -2 \\ -1 & \cdots & -1 & 0 & 0 & 0 & -2 \\ -1 & \cdots & -1 & -1 & 1 & 0 & -2 \\ -1 & \cdots & -1 & -1 & 0 & 1 & -2 \\ k-2 & \cdots & k-2 & k-2 & 0 & 0 & 2k-3 \end{pmatrix} \in \mathbb{Z}^{d\times d},$$

$$U_{5,7} = \begin{pmatrix} 1 & 1 & \cdots & 1 & & & 2 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & -1 & \cdots & -1 & & & -1 \end{pmatrix} \in \mathbb{Z}^{d\times d},$$

$$U_{5,8} = \begin{pmatrix} 0 & -1 & \cdots & -1 & 0 & 0 & -2 \\ -1 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 0 & -1 & 0 & 0 & -2 \\ -1 & \cdots & -1 & 0 & 0 & 0 & -2 \\ -1 & \cdots & -1 & -1 & 1 & 0 & -2 \\ -2 & \cdots & -2 & -2 & 1 & 1 & -4 \\ k-2 & \cdots & k-2 & k-2 & 0 & 0 & 2k-3 \end{pmatrix} \in \mathbb{Z}^{d\times d},$$

$$U_{5,9} = \begin{pmatrix} 1 & 1 & \cdots & 1 & & & 2 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ -1 & -1 & \cdots & -1 & 1 & 1 & -2 \\ & -1 & \cdots & -1 & & & -1 \end{pmatrix} \in \mathbb{Z}^{d\times d},$$

and

$$U_{5,10} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ -1 & \cdots & -1 & 1 & 1 & -2 \\ & & & & & 1 \end{pmatrix} \in \mathbb{Z}^{d\times d},$$

where other entries equal zero. Then these matrices are unimodular. Furthermore, one has $\mathscr{P}_5 = f_{U_{5,6}}(\mathscr{P}_6) + (\sum_{j=1}^{d-3}\mathbf{e}_j + 2\mathbf{e}_d) = f_{U_{5,7}}(\mathscr{P}_7) = f_{U_{5,8}}(\mathscr{P}_8) + (\sum_{j=1}^{d-3}\mathbf{e}_j + 2\mathbf{e}_d) =$

$f_{U_{5,9}}(\mathscr{P}_9) = f_{U_{5,10}}(\mathscr{P}_{10})$. Hence $\mathscr{P}_5, \dots, \mathscr{P}_9$ and $\mathscr{P}_{10}$ are unimodularly equivalent each other.

Finally, we will show that $\mathscr{P}_5$ and $\mathscr{P}_{11}$ are not lattice pyramids. For $i \in \{1,5,11\}$, let $\mathbf{v}_1^{(i)}, \dots, \mathbf{v}_{d+2}^{(i)}$ be the vertices of $\mathscr{P}_i$ and for $i \in \{1,5,11\}$, set

$$L_i = \max\{|V| : V \subset \{1,\dots,d+2,\}, |V|/2 \in \mathbb{Z}, \sum_{j \in V} \mathbf{v}_j^{(i)}/2 \in \mathbb{Z}^d\}.$$

Then one has $L_1 = 2k, L_5 = 2k+2$ and $L_{11} = 2k+4$. This implies that $\text{Pyr}(\mathscr{P}_1)$ and $\mathscr{P}_5$ are not unimodularly equivalent each other, and $\text{Pyr}(\text{Pyr}(\mathscr{P}_1)), \text{Pyr}(\mathscr{P}_5)$ and $\mathscr{P}_{11}$ are not unimodularly equivalent each other. Hence we know that $\mathscr{P}_5$ and $\mathscr{P}_{11}$ are not lattice pyramids.

Therefore, this completes the proof of the case $\delta(\mathscr{P},t) = 1+t+t^k+t^{k+1}$.

3.2.2 The case $\delta(\mathscr{P},t) = 1+t+2t^k$

In this subsection, we consider the case where $\delta(\mathscr{P},t) = 1+t+2t^k$. Recall that if $\delta(T_1,t) = 1+t$ or $\delta(T_2,t) = 1+t$, then $\mathscr{P}$ is spanning. Hence by Lemma 2.2, one has $\delta(T_1,t) = \delta(T_2,t) = 1+t^k$. Moreover, by Lemma 3.4, it follows that $T_1 \cap T_2$ is a unimodular simplex of dimension $d-1$. By Theorem 3.1, we can assume that

$$\mathbf{v}_i = \begin{cases} \mathbf{0}, & (i=0), \\ \mathbf{e}_i, & (i=1,\dots,d-1), \\ \sum_{j=1}^{2k-2} \mathbf{e}_j + 2\mathbf{e}_d, & (i=d), \end{cases}$$

and set $c = d-2k+1$. Let $\mathbf{v}_{d+1} = (a_1,\dots,a_d) \in \mathbb{Z}^d$. By the same way of Subsection 3.2.1, it follows that for $2k-1 \le i \le d-1$, one has $a_i \neq 0$ and $a_d \in \{-2,0,2\}$.

Now, we may assume that $T_2 \in \{\Delta_0, \Delta_1, \Delta_{2k-1}, \Delta_d\}$.

3.2.2.1 The case $T_2 = \Delta_d$

Suppose that $T_2 = \Delta_d$. Then since $T_1 \cap T_2$ belongs to the hyperplane defined by the equation $x_d = 0$ and $\text{Vol}(T_2) = 2$, we obtain $a_d = -2$. Hence $(\mathbf{v}_d + \mathbf{v}_{d+1})/2 \in T_1 \cap T_2$. Therefore, for $1 \le i \le 2k-2$, one has $a_i \in \{-1,0\}$ and for $2k-2 \le i \le d-1$, one has $a_i = 1$. Thus we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{-1,\dots,-1}_{a}, \underbrace{0,\dots,0}_{2k-2-a}, \underbrace{1,\dots,1}_{c}, -2).$$

By the same way of Subsubsection 3.2.1.1, It follows that $2k-2-a+c \le 2$. Moreover from $\delta(T_2,t) = 1+t^k$, we obtain $a+c \in \{2k-2, 2k-1\}$. Hence since $a \le 2k-2$, (a,c) equals $(2k-2,0)$, $(2k-2,1)$ or $(2k-3,1)$. If $(a,c) = (2k-2,0)$, then $(\mathbf{v}_d + \mathbf{v}_{d+1})/2 = \mathbf{v}_0$, a contradiction. Therefore, one has $(a,c) = (2k-3,1)$ or $(2k-2,1)$. In each case, we obtain $\delta(\mathscr{P},t) = 1+t+2t^k$.

3.2.2.2 The case $T_2 = \Delta_0$

Suppose that $T_2 = \Delta_0$. If for some i, $a_i < 0$, then $\mathbf{v}_{d+1}/2 \in T_2$. This implies that $\mathbf{0} \in T_2$, a contradiction. Hence we obtain $a_i \ge 0$ for $1 \le i \le d$. Moreover, one has $a_d \in \{0,2\}$ and for $2k-1 \le i \le d-1$, one has $a_i = 1$.

Assume that $a_d = 0$. Then for $1 \le i \le 2k-2$, we obtain $a_i \in \{0,1\}$. Hence we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{1,\dots,1}_{a},\underbrace{0,\dots,0}_{2k-2-a},\underbrace{1,\dots,1}_{c},0).$$

Since $T_1 \cap T_2$ belongs to the hyperplane defined by the equation $x_1 + \cdots + x_{d-1} - (k-3/2)x_d = 1$, one has $a+c > 1$. Then

$$\lambda = \left(\underbrace{\frac{a+c-2}{a+c-1},\dots,\frac{a+c-2}{a+c-1}}_{a},\underbrace{0,\dots,0}_{2k-2-a},\underbrace{\frac{a+c-2}{a+c-1},\dots,\frac{a+c-2}{a+c-1}}_{c},0,\frac{1}{a+c-1}\right)$$

is an element of Λ_{T_2} with the ordering $\mathbf{v}_1,\dots,\mathbf{v}_{d+1}$. Since $\mathrm{Vol}(T_2) = 2$, one has $a+c \le 3$. If $(a,c) = (0,2)$, then $\delta(T_2,t) = 1+t^{k+1}$, a contradiction, and if $(a,c) = (2,0)$, then $\delta(T_2,t) = 1+t^{k-1}$, a contradiction. Suppose that $(a,c) = (1,1)$. Then $\delta(T_2,t) = 1+t^k$. Therefore, one has $\delta(\mathscr{P},t) = 1+t+2t^k$. Next, suppose that $a+c = 3$. Then since $\mathrm{ht}(\lambda) = a+c-1 = 2$, one has $k = 2$. In each case, it is easy to show that $\mathrm{Vol}(T_2) \ge 3$, a contradiction.

Assume that $a_d = 2$. By the proof of Subsubsection 3.2.1.2, it follows that $\{a_1,\dots,a_{2k-2}\}$ is in $\{1,2\}^{2k-2}$. Therefore, we can assume that

$$\mathbf{v}_{d+1} = (\underbrace{2,\dots,2}_{a},\underbrace{1,\dots,1}_{2k-2-a},\underbrace{1,\dots,1}_{c},2).$$

Then

$$\lambda = \left(\underbrace{\frac{1}{a+c},\dots,\frac{1}{a+c}}_{a},\underbrace{0,\dots,0}_{2k-2-a},\underbrace{\frac{1}{a+c},\dots,\frac{1}{a+c}}_{c},\frac{1}{a+c},\frac{a+c-1}{a+c}\right) \in (\mathbb{R}/\mathbb{Z})^{d+1}$$

is an element of Λ_{T_2} with the ordering $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$. Since $\mathrm{Vol}(T_2) \geq a+c$, one has $a+c \leq 2$. Suppose that $a+c=2$. Then we obtain $\mathrm{ht}(\lambda)=2$. This implies $k=2$. In each case, it is easy to see that $\delta(\mathscr{P},t) \neq 1+t+2t^2$, a contradiction. If $(a,c)=(0,0)$, then $\mathbf{v}_d = \mathbf{v}_{d+1}$, a contradiction. If $(a,c)=(1,0)$, then

$$\lambda' = (0, \underbrace{1/2, \ldots, 1/2}_{2k-3}, 0, 1/2) \in (\mathbb{R}/\mathbb{Z})^{d+1}$$

is an element of Λ_{Δ_0} with the ordering $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$. However, $\mathrm{ht}(\lambda') = k-1$, a contradiction. Hence $(a,c)=(0,1)$. Then one has $\delta(\mathscr{P},t) = 1+t+2t^k$.

3.2.2.3 The case $T_2 = \Delta_1$

Suppose that $T_2 = \Delta_1$. Since $T_1 \cap T_2$ belongs to the hyperplane defined by the equation $2x_1 - x_d = 0$ and since $\mathrm{Vol}(T_1)=2$ and $|a_d| \leq 3$, one has $(a_1, a_d) \in \{(-2,-2), (-1,0), (0,2)\}$. Then it follows that $(\mathbf{v}_1 + \mathbf{v}_{d+1})/2 \in T_1 \cap T_2$. If $(a_1,a_d) = (-2,-2)$, then since the dth coordinate is negative, the point does not belong to $T_1 \cap T_2$. Set $(\mathbf{v}_1 + \mathbf{v}_{d+1})/2 = c_0\mathbf{v}_0 + c_2\mathbf{v}_2 + \cdots + c_d\mathbf{v}_d$, where $0 \leq c_0, c_2, \ldots, c_d$ and $c_0 + c_2 + \cdots + c_d = 1$. Assume that $(a_1,a_d) = (-1,0)$. Then we obtain $c_d = 0$. Hence it follows that for $2 \leq i \leq d-1$, $c_i = a_i/2 \geq 0$. Thus we may assume that

$$\mathbf{v}_{d+1} = (-1, \underbrace{1, \ldots, 1}_{a}, \underbrace{0, \ldots, 0}_{2k-3-a}, \underbrace{1, \ldots, 1}_{c}, 0).$$

and $0 \leq a+c \leq 2$. Then since there exists an element λ of Λ_{T_2} with $\mathrm{ht}(\lambda)=k$ and $\mathrm{ord}(\lambda)=2$, one has $c=a+1$ or $c=a$. Hence (a,c) equals $(0,0)$, $(0,1)$ or $(1,1)$. If $(a,c)=(0,0)$, then it follows that $(\mathbf{v}_1+\mathbf{v}_{d+1})/2 = \mathbf{v}_0$, a contradiction. If $(a,c)=(0,1)$ or $(a,c)=(1,1)$, then we obtain $\delta(\mathscr{P},t) = 1+t+2t^k$. Assume that $(a_1,a_d)=(0,2)$. Then we obtain $c_d = 1/2$. Moreover, for $2 \leq i \leq 2k-2$, $a_i \in \{1,2\}$ and for $2k-1 \leq i \leq d-1$, $a_i = 1$. Hence we may assume that

$$\mathbf{v}_{d+1} = (0, \underbrace{2, \ldots, 2}_{a}, \underbrace{1, \ldots, 1}_{2k-3-a}, \underbrace{1, \ldots, 1}_{c}, 2).$$

Then one has $0 \leq a+c \leq 1$ since $c_2 = \cdots = c_{a+1} = 1/2$, $c_{a+2} = \cdots = c_{2k-2} = 0$ and $c_{2k-1} = \cdots = c_{d-1} = 1/2$. Moreover, since there exists an element λ of Λ_{T_2} with $\mathrm{ht}(\lambda) = k$ and $\mathrm{ord}(\lambda)=2$, one has $c=a+2$ or $c=a+1$. Hence it follows that (a,c) equals $(0,1)$. Then we obtain $\delta(\mathscr{P},t) = 1+t+2t^k$.

3.2.2.4 The case $T_2 = \Delta_{2k-1}$

Suppose that $c \geq 1$ and $T_2 = \Delta_{2k-1}$. Then it is easy to see that $\delta(T_1,t) = \delta(T_1 \cap T_2, t) = 1+t^k$, a contradiction.

Therefore, $\mathscr{P}$ is unimodularly equivalent to the lattice polytope which is the convex hull of $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_{d-1}, \mathbf{v}$ and $\mathbf{v}'$, where $\mathbf{v}$ and $\mathbf{v}'$ satisfy one of the following conditions:

(1) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\sum_{j=1}^{d-3} \mathbf{e}_i + \mathbf{e}_{d-1} - 2\mathbf{e}_d$;

(2) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\sum_{j=1}^{d-2} \mathbf{e}_i + \mathbf{e}_{d-1} - 2\mathbf{e}_d$;

(3) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \mathbf{e}_1 + \mathbf{e}_{d-1}$;

(4) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \sum_{j=1}^{d-1} \mathbf{e}_j + 2\mathbf{e}_d$;

(5) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\mathbf{e}_1 + \mathbf{e}_{d-1}$;

(6) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-2} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{d-1}$;

(7) $d = 2k$, $\mathbf{v} = \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$ and $\mathbf{v}' = \sum_{j=2}^{d-1} \mathbf{e}_j + 2\mathbf{e}_d$.

For $i = 1, \dots, 7$, let $\mathscr{P}_i$ be the lattice polytope of (i). Set

$$U_{5,1} = \begin{pmatrix} 1 & 1 & \cdots & 1 & & & 2 \\ & 1 & & & -1 & & \\ & & \ddots & & \vdots & & \\ & & & 1 & -1 & & \\ & & & & -1 & & \\ & & & & -1 & 1 & \\ & -1 & \cdots & -1 & k-2 & & -1 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

$$U_{5,2} = \begin{pmatrix} 1 & 1 & \cdots & 1 & & 2 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & \\ & -1 & \cdots & -1 & & -1 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

$$U_{5,3} = \begin{pmatrix} -1 & & & & \\ -1 & 1 & & & \\ \vdots & & \ddots & & \\ -1 & & & 1 & \\ k-1 & & & & 1 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

$$U_{5,4} = \begin{pmatrix} & 1 & \cdots & 1 & & 2 \\ -1 & 1 & & & & \\ \vdots & & \ddots & & & \\ -1 & & & 1 & & \\ -1 & & & & 1 & \\ k-2 & -1 & \cdots & -1 & & -1 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

$$U_{5,6} = \begin{pmatrix} 1 & -1 & & & & \\ & -1 & & & & \\ & -1 & 1 & & & \\ & \vdots & & \ddots & & \\ & -1 & & & 1 & \\ & k-1 & & & & 1 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

$$U_{5,7} = \begin{pmatrix} 0 & -1 & \cdots & -1 & 0 & -2 \\ -1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 0 & -1 & 0 & -2 \\ -1 & \cdots & -1 & 0 & 0 & -2 \\ -1 & \cdots & -1 & -1 & 1 & -2 \\ k-2 & \cdots & k-2 & k-2 & 0 & 2k-3 \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

where other entries equal zero. Then it follows that these matrices are unimodular and one has $\mathscr{P}_5 = f_{U_{5,1}}(\mathscr{P}_1) + \mathbf{e}_{2k-2} = f_{U_{5,2}}(\mathscr{P}_2) = f_{U_{5,3}}(\mathscr{P}_3) + \mathbf{e}_1 = f_{U_{5,4}}(\mathscr{P}_4) + \mathbf{e}_1 = f_{U_{5,6}(\mathscr{P}_6)} + \mathbf{e}_2 = f_{U_{5,7}}(\mathscr{P}_7) + \sum_{j=1}^{d-3} \mathbf{e}_j + 2\mathbf{e}_d$. Hence $\mathscr{P}_1, \ldots, \mathscr{P}_7$ are unimodularly equivalent each other.

Therefore, this completes the proof of the case $\delta(\mathscr{P}, t) = 1 + t + 2t^k$.

Thus, Theorem 3.3 follows.

Chapter 4

Ehrhart polynomials of lattice polytopes with normalized volume 5

To work towards a classification of the lattice polytopes whose normalized volumes are 5, we should classify all the possible δ-polynomials of lattice polytopes whose normalized volumes are 5. Higashitani classified all the possible δ-polynomials of lattice simplices whose normalized volumes are 5.

Lemma 4.1 ([53, Theorem 1.2])**.** *Let $1+t^{i_1}+t^{i_2}+t^{i_3}+t^{i_4}$ be a polynomial with some positive integers $i_1 \leq \cdots \leq i_4$. Then there exists a lattice simplex of dimension d whose δ-polynomial equals $1+t^{i_1}+t^{i_2}+t^{i_3}+t^{i_4}$ if and only if the following conditions are satisfied:*

- $i_1+i_4 = i_2+i_3 \leq d+1$*;*
- $i_k+i_\ell \geq i_{k+\ell}$ *for* $1 \leq k \leq \ell \leq 4$ *with* $k+\ell \leq 4$.

In this chapter, we classify all the possible δ-polynomials of lattice non-simplices whose normalized volumes are 5. Thus, we obtain a classification of the possible δ-polynomials of lattice polytopes whose normalized volumes are 5. In fact, we will show the following theorem.

Theorem 4.2 ([93, Theorem 0.4])**.** *Let $1+t^{i_1}+t^{i_2}+t^{i_3}+t^{i_4}$ be a polynomial with some positive integers $i_1 \leq \cdots \leq i_4$. Then there exists a lattice polytope of dimension d whose δ-polynomial equals $1+t^{i_1}+t^{i_2}+t^{i_3}+t^{i_4}$ if and only if (i_1,i_2,i_3,i_4) satisfies the condition of* Theorem 4.1 *or one of the following conditions:*

(1) $(i_1,i_2,i_3,i_4) = (1,1,1,2)$ *and* $d \geq 2$*;*

(2) $(i_1,i_2,i_3,i_4) = (1,2,2,2)$ *and* $d \geq 3$*;*

(3) $(i_1, i_2, i_3, i_4) = (1, 2, 3, 3)$ *and* $d \geq 5$.

In particular, we cannot obtain the δ*-polynomials of* (1)*,* (2) *and* (3) *by lattice simplices.*

This chapter is organized as follows: First, in Section 4.1, we will discuss some properties of lattice polytopes whose normalized volumes are prime integers. In particular, we will show that every full-dimensional lattice polytope which is not an empty simplex and whose normalized volume equals a prime integer is always a spanning polytope (Theorem 4.3). Finally, in Section 4.2, by using this result we will prove Theorem 4.2.

4.1 Lattice polytopes with prime volumes

In this section, we will discuss some properties of lattice polytopes whose normalized volumes are prime integers.

Let $\mathscr{P} \subset \mathbb{Z}^d$ be a lattice polytope of dimension d and $\langle \mathscr{P} \cap \mathbb{Z}^d \rangle_{\mathbb{Z}}$ the affine sublattice generated by $\mathscr{P} \cap \mathbb{Z}^d$. We call the *index* of $\mathscr{P}$ the index of $\langle \mathscr{P} \cap \mathbb{Z}^d \rangle_{\mathbb{Z}}$ as a sublattice of $\mathbb{Z}^d$. Then $\mathscr{P}$ is spanning if and only if its index equals 1. Now, we prove the following theorem.

Theorem 4.3 ([93, Theorem 1.1])**.** *Let* p *be a prime integer and* $\mathscr{P} \subset \mathbb{R}^d$ *be a lattice polytope of dimension* d *whose normalized volume equals* p*. Suppose that* $\mathscr{P}$ *is not an empty simplex. Then* $\mathscr{P}$ *is spanning.*

Proof. Since $\mathscr{P}$ is not an empty simplex, there exists a lattice triangulation $\{\Delta_1, \ldots, \Delta_k\}$ of $\mathscr{P}$ with some positive integer $k \geq 2$. Since the index of P must divide the normalized volume of every Δ_i, and since the sum of those normalized volumes is the prime p, the index must be one. Hence $\mathscr{P}$ is spanning. □

Next, we consider an application of this result to classifying lattice polytopes whose normalized volumes are prime integers. Thanks to Theorem 4.3, every full-dimensional lattice polytope whose normalized volumes equals 5 is either an empty simplex or a spanning polytope. See e.g., [33] for how to classify empty simplices. Now, we focus on spanning polytopes. We recall that there are only finitely many spanning lattice polytopes of given normalized volume (and arbitrary dimension) up to unimodular equivalence and lattice pyramid constructions ([56, Corollary 2.4]). Hence we obtain the following corollary.

Corollary 4.4. *Let* p *be a prime integer and* $\mathscr{P}$ *a lattice polytope of dimension* d *whose normalized volume equals* p*. Suppose that* $\mathscr{P}$ *is not an empty simplex. Then there are only finitely many possibilities for* $\mathscr{P}$ *up to unimodular equivalence and lattice pyramid constructions.*

4.2 Proof of Theorem 4.2

In this section, we will prove Theorem 4.2. From Lemma 2.5 and Theorem 4.3, we obtain the following corollary.

Corollary 4.5. *Let p be a prime integer and $\mathscr{P} \subset \mathbb{R}^d$ a lattice polytope of dimension d whose normalized volume equals p and whose δ-polynomial equals $\delta_0 + \delta_1 t + \cdots + \delta_s t^s$, where $\delta_s \neq 0$. Suppose that $\mathscr{P}$ is not an empty simplex. Then one has $\delta_i \geq 1$ for any $0 \leq i \leq s$.*

Next, we give indispensable examples for our proof of Theorem 4.2.

Example 4.6. (a) Let $\mathscr{P}_1 \subset \mathbb{R}^2$ be the lattice polytope which is the convex hull of the following lattice points:

$$\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, 2\mathbf{e}_1 + 3\mathbf{e}_2 \in \mathbb{R}^2.$$

Then one has $\delta(\mathscr{P}_1, t) = 1 + 3t + t^2$.

(b) Let $\mathscr{P}_2 \subset \mathbb{R}^3$ be the lattice polytope which is the convex hull of the following lattice points:

$$\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + 3\mathbf{e}_3 \in \mathbb{R}^3.$$

Then one has $\delta(\mathscr{P}_2, t) = 1 + t + 3t^2$.

(c) Let $\mathscr{P}_3 \subset \mathbb{R}^5$ be the lattice polytope which is the convex hull of the following lattice points:

$$\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + 2\mathbf{e}_5 \in \mathbb{R}^5.$$

Then one has $\delta(\mathscr{P}_3, t) = 1 + t + t^2 + 2t^3$.

Finally, we prove Theorem 4.2.

Proof of Theorem 4.2. First, we can prove the "If" part of Theorem 4.2 from Lemma 2.3, Theorem 4.1 and Example 4.6. Hence we should prove the "Only if" part of Theorem 4.2. Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice non-simplex of dimension d whose normalized volume equals 5 and $\delta(\mathscr{P}, t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$ the δ-polynomial of $\mathscr{P}$. By Corollary 4.5 and the inequalities (2.1) and (2.2), and the fact $\delta_1 \geq \delta_d$, one of the followings is satisfied:

(1) $\delta(\mathscr{P}, t) = 1 + 4t$ and $d \geq 1$;

(2) $\delta(\mathscr{P}, t) = 1 + 3t + t^2$ and $d \geq 2$;

(3) $\delta(\mathscr{P}, t) = 1 + 2t + 2t^2$ and $d \geq 2$;

(4) $\delta(\mathscr{P}, t) = 1 + t + 3t^2$ and $d \geq 3$;

(5) $\delta(\mathscr{P},t) = 1+t+2t^2+t^3$ and $d \geq 3$;

(6) $\delta(\mathscr{P},t) = 1+t+t^2+2t^3$ and $d \geq 5$;

(7) $\delta(\mathscr{P},t) = 1+t+t^2+t^3+t^4$ and $d \geq 4$.

Then we know that the conditions $(1),(3),(5)$ and (7) satisfy the condition of Theorem 4.1. This completes the proof. $\square$

Chapter 5

Gorenstein simplices and the associated finite abelian groups

In this chapter, to work towards a classification of the Gorenstein simplices, we discuss a characterization of Gorenstein simplices in terms of their associated finite abelian groups. In Section 5.1, we recall the Hermite normal form matrices and some of their properties. In Section 5.2, we prove that a class of simplices arising from Hermite normal form matrices are Gorenstein (Theorem 5.6). Using this result, we characterize Gorenstein simplices whose normalized volume is a prime number. In fact, we will prove the following.

Theorem 5.1 ([91, Theorem 0.1])**.** *Let p be a prime number and $\Delta \subset \mathbb{R}^d$ a d-dimensional lattice simplex with normalized volume p. Suppose that Δ is not a lattice pyramid over any lower-dimensional simplex. Then Δ is Gorenstein of index r if and only if $d = rp-1$ and Λ_Δ is generated by $\left(\frac{1}{p},\dots,\frac{1}{p}\right)$. In this case, one has $\delta(\Delta,t) = 1+t^r+t^{2r}+\cdots+t^{(p-1)r}$.*

In Section 5.3, we extend these results by characterizing Gorenstein simplices whose normalized volume equals p^2 and pq, where p and q are prime numbers with $p \neq q$. In fact, we will prove the following theorems.

Theorem 5.2 ([91, Theorem 0.2])**.** *Let p be a prime number and $\Delta \subset \mathbb{R}^d$ a d-dimensional lattice simplex with normalized volume p^2. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is Gorenstein of index r if and only if one of the followings is satisfied:*

1. *There exists an integer s with $0 \le s \le d-1$ such that $rp^2 - 1 = (d-s) + ps$ and Λ_Δ is generated by* $\left(\underbrace{\frac{1}{p},\dots,\frac{1}{p}}_{s},\underbrace{\frac{1}{p^2},\dots,\frac{1}{p^2}}_{d-s+1}\right)$ *for some ordering of the vertices of Δ.*

2. *$d = rp-1$ and there exist an integer s with $1 \le s \le d-1$ and integers $1 \le a_1,\dots,a_{s-1} \le p-1$ such that Λ_Δ is generated by*

$$\left(\frac{2 - \sum_{1\le i\le s-1} a_i}{p}, \frac{a_1+1}{p}, \dots, \frac{a_{s-1}+1}{p}, 0, \frac{1}{p}, \dots, \frac{1}{p}\right)$$

and

$$\left(\frac{\left(\sum_{1\le i\le s-1} a_i\right) - 1}{p}, \frac{p-a_1}{p}, \dots, \frac{p-a_{s-1}}{p}, \frac{1}{p}, 0, \dots, 0\right)$$

for some ordering of the vertices of Δ.

Theorem 5.3 ([91, Theorem 0.3])**.** *Let p and q be prime numbers with $p \neq q$ and $\Delta \subset \mathbb{R}^d$ a d-dimensional lattice simplex with normalized volume pq. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is Gorenstein of index r if and only if there exist nonnegative integers s_1, s_2, s_3 with $s_1 + s_2 + s_3 = d+1$ such that the following conditions are satisfied:*

1. *$rpq = s_1 q + s_2 p + s_3$;*

2. *Λ_Δ is generated by* $\left(\underbrace{\frac{1}{p},\dots,\frac{1}{p}}_{s_1},\underbrace{\frac{1}{q},\dots,\frac{1}{q}}_{s_2},\underbrace{\frac{1}{pq},\dots,\frac{1}{pq}}_{s_3}\right)$ *for some ordering of the vertices of Δ.*

Moreover, we give a class of Gorenstein simplices whose normalized volumes equal a power of a prime number (Theorem 5.13). Finally, in Section 5.4, we compute the volume of the associated dual reflexive simplices of the Gorenstein simplices described in Sections 5.2 and 5.3.

5.1 Hermite normal form matrices and lattice simplices

In this section, we recall some basic facts about Hermite normal form matrices. For positive integers d and m, we denote by $\mathrm{Herm}(d,m)$ the finite set of lower triangular matrices $H = (h_{ij})_{1\le i,j\le d} \in \mathbb{Z}_{\ge 0}^{d\times d}$ with determinant m satisfying $h_{ij} < h_{ii}$ for all $i > j$. It is well known that for any $M \in \mathbb{Z}^{d\times d}$ with determinant $m \in \mathbb{Z}_{>0}$ there exist a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and a *Hermite normal form matrix* $H \in \mathrm{Herm}(d,m)$ such that $MU = H$. Let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d with normalized volume m and $\mathbf{v}_0,\dots,\mathbf{v}_d$ the vertices of Δ, and let V be the $d\times d$ matrix whose ith row is $\mathbf{v}_i - \mathbf{v}_0$. Then one has $|\det(V)| = m$ and we may assume that $\det(V) = m$. Hence there exist a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and a Hermite normal form matrix $H \in \mathrm{Herm}(d,m)$ such that $VU = H$. In particular, Δ is unimodularly equivalent to the lattice simplex whose vertices are the origin of $\mathbb{R}^d$ and all rows of H.

Let $H = (h_{ij})_{1\le i,j\le d} \in \mathbb{Z}_{\ge 0}^{d\times d}$ be a Hermite normal form matrix and set $\ell(H) = |\{i \mid h_{ii} > 1\}|$. We then say that H has $\ell(H)$ *nonstandard rows*. Let $\Delta(H)$ be the lattice simplex whose vertices are the origin of $\mathbb{R}^d$ and all rows of H, and set $s = \max\{i \mid h_{ii} > 1\}$. If $\Delta(H)$ is not a lattice pyramid over any lower-dimensional lattice simplex, then $s = d$. In [40], lattice simplices arising from Hermite normal form matrices are discussed.

5.2 Hermite normal form matrices with one nonstandard row

For a sequence of integers $A = (a_1,\dots,a_{d-1},a_d)$ with $1 \le a_1,\dots,a_{d-1} \le a_d$, we set $\Delta(A) = \mathrm{conv}(\{\mathbf{v}_0,\dots,\mathbf{v}_d\}) \subset \mathbb{R}^d$, where

$$\mathbf{v}_i = \begin{cases} \mathbf{0}, & \text{if } i = 0,\\ \mathbf{e}_i, & \text{if } 1 \le i \le d-1,\\ \sum_{j=1}^{d-1}(a_d - a_j)\mathbf{e}_j + a_d\mathbf{e}_d, & \text{if } i = d. \end{cases}$$

Namely, $\Delta(A)$ is a lattice simplex arising from a Hermite normal form matrix with one nonstandard row. In particular, the lattice simplices $\Delta(A)$ are exactly the lattice simplices with one unimodular facet.

At first, we give the equations of the supporting hyperplanes of facets of $\Delta(A)$.

Lemma 5.4. *For $0 \le i \le d$, let $\mathscr{F}_i$ be the facet of $\Delta(A)$ whose vertices are $\mathbf{v}_0,\dots,\mathbf{v}_{i-1}$, $\mathbf{v}_{i+1},\dots,\mathbf{v}_d$ and $\mathscr{H}_i$ the supporting hyperplane of $\mathscr{F}_i$. Then one has*

- $\mathscr{H}_0 = \{(x_1,\dots,x_d) \in \mathbb{R}^d : a_d \sum_{j=1}^{d-1} x_j + (1 - \sum_{j=1}^{d-1}(a_d - a_j))x_d = a_d\}$;
- $\mathscr{H}_i = \{(x_1,\dots,x_d) \in \mathbb{R}^d : -a_d x_i + (a_d - a_i)x_d = 0\}$, $1 \le i \le d-1$;
- $\mathscr{H}_d = \{(x_1,\dots,x_d) \in \mathbb{R}^d : -x_d = 0\}$.

It is easy to compute $\Lambda_{\Delta(A)}$ for the simplex $\Delta(A)$, as demonstrated with the following lemma.

Lemma 5.5. *Let a_0 be an integer with $1 \leq a_0 \leq a_d$ such that $a_d \mid (a_0 + \cdots + a_{d-1} + 1)$. Then the finite abelian group $\Lambda_{\Delta(A)}$ is generated by $\left(\frac{a_0}{a_d}, \frac{a_1}{a_d}, \ldots, \frac{a_{d-1}}{a_d}, \frac{1}{a_d}\right)$. In particular, $\Delta(A)$ is not a lattice pyramid over any lower-dimensional lattice simplex if and only if $1 \leq a_0, a_1, \ldots, a_{d-1} < a_d$.*

Proof. Set

$$(\lambda_0, \ldots, \lambda_d) = \left(\frac{a_0}{a_d}, \frac{a_1}{a_d}, \ldots, \frac{a_{d-1}}{a_d}, \frac{1}{a_d}\right) \in (\mathbb{R}/\mathbb{Z})^{d+1}.$$

Then one has

$$\sum_{i=0}^{d} \lambda_i(\mathbf{v}_i, 1) = (b_1, \ldots, b_{d-1}, 1, \frac{a_0 + \cdots + a_{d-1} + 1}{a_d}) \in \mathbb{Z}^{d+1},$$

where for $1 \leq i \leq d-1$, $b_i = \min\{1, a_d - a_i\}$. Hence we know that $(\lambda_0, \ldots, \lambda_d)$ is an element of $\Lambda_{\Delta(A)}$. Since the normalized volume of $\Delta(A)$ is a_d and the order of $(\lambda_0, \ldots, \lambda_d)$ is a_d, $\Lambda_{\Delta(A)}$ is generated by $(\lambda_0, \ldots, \lambda_d)$. Moreover, by Lemma 2.8, it is follows that $\Delta(A)$ is not a lattice pyramid over any lower-dimensional lattice simplex if and only if $1 \leq a_0, a_1, \ldots, a_{d-1} < a_d$. □

The following theorem characterizes exactly when the simplices $\Delta(A)$ are Gorenstein.

Theorem 5.6 ([91, Theorem 2.3])**.** *Suppose that $1 \leq a_0, \ldots, a_{d-1} < a_d$. Then $\Delta(A)$ is Gorenstein of index r if and only if the following conditions are satisfied:*

- *For $0 \leq i \leq d-1$, $a_i \mid a_d$;*
- *$ra_d = a_0 + \cdots + a_{d-1} + 1$.*

In order to prove this theorem, we show the following lemma.

Lemma 5.7. *Suppose that $1 \leq a_0, \ldots, a_{d-1} < a_d$, $ra_d = a_0 + \cdots + a_{d-1} + 1$ and for $0 \leq i \leq d-1$, $a_i \mid a_d$. Then $\Delta(A)$ is Gorenstein of index r. Moreover, the vertices of the associated dual reflexive simplex are the following lattice points:*

- *$-e_d$;*
- *$-\frac{a_d}{a_i}\mathbf{e}_i + \frac{a_d - a_i}{a_i}\mathbf{e}_d$ for $1 \leq i \leq d-1$;*
- *$\frac{a_d}{a_0}\sum_{j=1}^{d-1}\mathbf{e}_j + \frac{(r-d+1)a_d - a_0}{a_0}\mathbf{e}_d$.*

Proof. Since $a_d(d-1)+(1-\sum_{j=1}^{d-1}(a_d-a_j)) = ra_d - a_1 < ra_d$, by Lemma 5.4, we know that $\mathbf{t} = (1,\dots,1)$ is an interior lattice point of $r\Delta(A)$. Set $\Delta = r\Delta(A) - \mathbf{t}$. Then by Lemma 5.4, the equations of supporting hyperplanes of facets of Δ are as follows:

- $-x_d = 1$;
- $-a_d x_i + (a_d - a_i)x_d = a_i$, $1 \le i \le d-1$;
- $a_d \sum\limits_{j=1}^{d-1} x_j + (1 - \sum\limits_{j=1}^{d-1}(a_d - a_j))x_d = a_0$.

Hence by Lemma 2.7, Δ is reflexive and we can obtain the vertices of $\Delta^\vee$. $\square$

Now, we prove Theorem 5.6.

Proof of Theorem 5.6. Let $\mathbf{t} = (t_1,\dots,t_d) \in \mathbb{R}^d$ be the unique interior lattice point of $r\Delta(A)$ and $\Delta' = r\Delta(A) - \mathbf{t}$. Then for each i, one has $t_i \ge 1$. By Lemma 5.4, the equation $-x_d = t_d$ is a supporting hyperplane of a facet of Δ'. Hence by Lemma 2.7, $\mathbf{w}_0 = -\mathbf{e}_d/t_d$ is a vertex of $(\Delta')^\vee$. Therefore, we obtain $t_d = 1$. If for some i, $t_i \ge 2$, then $(t_1,\dots,t_{i-1},t_i-1,t_{i+1},\dots,t_{d-1},1)$ is the interior lattice point of $r\Delta(A)$. Since $(t_1,\dots,t_d)$ is the unique interior lattice point of $r\Delta(A)$, one has $t_1 = \cdots = t_{d-1} = 1$. Therefore, by Lemma 2.7, the following points are the vertices of $(\Delta')^\vee$:

$$w_i = \begin{cases} -\mathbf{e}_d, & \text{if } i = 0,\\ -\dfrac{a_d}{a_i}\mathbf{e}_i + \dfrac{a_d - a_i}{a_i}\mathbf{e}_d, & \text{if } 1 \le i \le d-1,\\ \dfrac{a_d}{a}\sum\limits_{j=1}^{d-1}\mathbf{e}_j + \dfrac{1 - \sum\limits_{j=1}^{d-1}(a_d - a_j)}{a}\mathbf{e}_d, & \text{if } i = d, \end{cases}$$

where $a = ra_d - \sum_{j=1}^{d-1} a_j - 1$. Since the origin of $\mathbb{R}^d$ belongs to the interior of Δ', we obtain $a > 0$. Moreover, Δ' is reflexive, by Lemma 2.7, it is known that a divides a_d. Hence one has $1 \le a < a_d$. Therefore, since $a_d \mid (a + a_1 + \cdots + a_{d-1} + 1)$ and $1 \le a_0 < a_d$, we obtain $a = a_0$. By Lemma 5.7, this completes the proof. $\square$

By Lemmas 2.7 and 5.7, we can prove Theorem 5.6.

Remark 5.8. If $\Delta(A)$ is Gorenstein of index 1, then $\Delta(A)$ is unimodularly equivalent to a lattice polytope $\Delta = \mathrm{conv}(\{\mathbf{e}_1,\dots,\mathbf{e}_d, -\sum_{i=1}^{d} a_{i-1}\mathbf{e}_i\})$. In [17], properties of this polytope Δ are discussed.

We obtain Theorem 5.1 as a special case of Theorem 5.6.

Proof of Theorem 5.1. Since the normalized volume of Δ is a prime number, there exists a sequence of integers $A = (a_1, \ldots, a_{d-1}, p)$ with $1 \le a_1, \ldots, a_{d-1} \le p$ such that Δ is unimodularly equivalent to $\Delta(A)$. Let a_0 be an integer with $1 \le a_0 \le p$ such that $p \mid (a_0 + \cdots + a_{d-1} + 1)$. Since Δ is not a lattice pyramid over any lower-dimensional simplex, by Lemma 5.5, one has $1 \le a_0, \ldots, a_{d-1} < p$. Hence, by Theorem 5.6, $\Delta(A)$ is Gorenstein of index r if and only if $a_0 = \cdots = a_{d-1} = 1$ and $d = rp - 1$. Therefore, Δ is Gorenstein of index r if and only if $d = rp - 1$ and Λ_Δ is generated by $\left(\frac{1}{p}, \ldots, \frac{1}{p}\right)$. $\square$

5.3 The case when $\mathrm{Vol}(\Delta) = p^2$ or $\mathrm{Vol}(\Delta) = pq$

Let s, d be positive integers with $1 \le s < d$, and let $A = (a_1, \ldots, a_s)$ and $B = (b_1, \ldots, b_d)$ be sequences of integers with $0 \le a_1, \ldots, a_{s-1} < a_s$ and $0 \le b_1, \ldots, b_{d-1} < b_d$. Set $\Delta(A, B) = \mathrm{conv}(\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}) \subset \mathbb{R}^d$, where

$$v_i = \begin{cases} \mathbf{0}, & \text{if } i = 0, \\ \mathbf{e}_i, & \text{if } 1 \le i \le s-1, \\ \sum_{j=1}^{s} a_j \mathbf{e}_j, & \text{if } i = s, \\ \mathbf{e}_i, & \text{if } s+1 \le i \le d-1, \\ \sum_{j=1}^{d} b_j \mathbf{e}_j, & \text{if } i = d. \end{cases}$$

Then $\Delta(A, B)$ is a lattice simplex arising from a Hermite normal form matrix with two nonstandard rows.

We give the equations of the supporting hyperplanes of facets of $\Delta(A, B)$.

Lemma 5.9. *Assume that $b_s = 0$. For $0 \le i \le d$, let $\mathscr{F}_i$ be the facet of $\Delta(A, B)$ whose vertices are $\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_d$ and $\mathscr{H}_i$ the supporting hyperplane of $\mathscr{F}_i$. Then one has*

- $\mathscr{H}_0 = \left\{ \begin{array}{l} (x_1, \ldots, x_d) \in \mathbb{R}^d : \\ a_s b_d \sum_{\substack{1 \le j \le d-1 \\ j \ne s}} x_j + b_d (1 - \sum_{1 \le j \le s-1} a_j) x_s + a_s (1 - \sum_{\substack{1 \le j \le d-1 \\ j \ne s}} b_j) x_d = a_s b_d \end{array} \right\}$;
- $\mathscr{H}_i = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : -a_s b_d x_i + a_i b_d x_s + a_s b_i x_d = 0\}$, $1 \le i \le s-1$;
- $\mathscr{H}_s = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : -x_s = 0\}$;
- $\mathscr{H}_i = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : -b_d x_i + b_i x_d = 0\}$, $s+1 \le i \le d-1$;

- $\mathscr{H}_d = \{(x_1,\dots,x_d) \in \mathbb{R}^d : -x_d = 0\}$.

Let p,q be prime numbers with $p \neq q$. In this section, we characterize Gorenstein simplices whose normalized volume equals p^2 and pq. In particular, we prove Theorems 5.2 and 5.3.

We prove the following lemma.

Lemma 5.10. *Let p and q be prime numbers and set $a_s = p$ and $b_d = q$. Suppose that $\Delta(A,B)$ is Gorenstein of index r. Then we have $b_s = 0$ or $b_s = q-1$. Moreover, if $b_s = q-1$, then there exists a sequence of integers $C = (c_1,\dots,c_{d-1},pq)$ with $1 \leq c_1,\dots,c_{d-1} \leq pq$ such that $\Delta(A,B)$ and $\Delta(C)$ are unimodularly equivalent.*

Proof. The following two equations define supporting hyperplanes of two facets of $r\Delta(A,B)$:

- $-x_d = 0$;
- $-qx_s + b_s x_d = 0$.

Let $\mathbf{t} = (t_1,\dots,t_d) \in \mathbb{R}^d$ be the unique interior lattice point of $r\Delta(A,B)$. Then $t_i \geq 1$ for each i. Set $\Delta = r\Delta(A,B) - \mathbf{t}$. Then the followings are equations of supporting hyperplanes of facets of Δ:

- $-x_d = t_d$;
- $-qx_s + b_s x_d = qt_s - b_s t_d$.

By Lemma 2.7, $-\dfrac{\mathbf{e}_d}{t_d}$ and $\dfrac{-q\mathbf{e}_s + b_s\mathbf{e}_d}{qt_s - b_s t_d}$ are vertices of $\Delta^\vee$. Hence since Δ is reflexive, we know that $t_d = 1$ and $\dfrac{q}{qt_s - b_s}$ is an integer. Therefore, we have $t_s = 1$ and $b_s \in \{0, q-1\}$.

Suppose that $b_s = q-1$. Then we know

$$\left(\frac{\lambda_0}{pq}, \frac{pq - a_1 - pb_1}{pq}, \dots, \frac{pq - a_{s-1} - pb_{s-1}}{pq}, \frac{1}{pq}, \frac{q - b_{s+1}}{q}, \dots, \frac{q - b_{d-1}}{q}, \frac{1}{q}\right)$$

is an element of $\Lambda_{\Delta(A,B)}$, where λ_0 is an integer with $0 \leq \lambda_0 \leq pq-1$ such that the sum of all entries of this element is an integer. Hence by Lemma 5.5, there exists a sequence of integers $C = (c_1,\dots,c_{d-1},pq)$ with $1 \leq c_1,\dots,c_{d-1} \leq pq$ such that $\Delta(A,B)$ and $\Delta(C)$ are unimodularly equivalent. □

At first, we characterize Gorenstein simplices with normalized volume p^2. In order to prove Theorem 5.2, we show the following lemma.

Lemma 5.11. *Let p be a prime number and set $a_s = b_d = p$ and $b_s = 0$. Suppose that $d = rp-1$ and for $1 \le i \le s-1$, $a_i + b_i = p-1$ and for $s+1 \le i \le d-1$, $b_i = p-1$. Then $\Delta(A,B)$ is Gorenstein of index r. Moreover, the vertices of the associated dual reflexive simplex are the following lattice points:*

- $-\mathbf{e}_d$;
- $-p\mathbf{e}_i + a_i\mathbf{e}_s + b_i\mathbf{e}_d$ *for* $1 \le i \le s-1$;
- $-\mathbf{e}_s$;
- $-p\mathbf{e}_i + b_i\mathbf{e}_d$ *for* $s+1 \le i \le d-1$;
- $p \sum_{\substack{1\le j\le d-1\\ j\ne s}} \mathbf{e}_j + (1 - \sum_{1\le j\le s-1} a_j)\mathbf{e}_s + (1 - \sum_{\substack{1\le j\le d-1\\ j\ne s}} b_j)\mathbf{e}_d$.

Proof. Since

$$p(d-2) + (1 - \sum_{1\le j\le s-1} a_j) + (1 - \sum_{\substack{1\le j\le d-1\\ j\ne s}} b_j) = d = rp-1 < rp,$$

by Lemma 5.4, we know that $\mathbf{t} = (1,\dots,1)$ is an interior lattice point of $r\Delta(A,B)$. Set $\Delta = r\Delta(A,B) - \mathbf{t}$. Then by Lemma 5.4, the equations of supporting hyperplanes of facets of Δ are as follows:

- $-x_d = 1$;
- $-px_i + a_ix_s + b_ix_d = 1$, $1 \le i \le s-1$;
- $-x_s = 1$;
- $-px_i + b_ix_d = 1$, $s+1 \le i \le d-1$;
- $p \sum_{\substack{1\le j\le d-1\\ j\ne s}} x_j + (1 - \sum_{1\le j\le s-1} a_j)x_s + (1 - \sum_{\substack{1\le j\le d-1\\ j\ne s}} b_j)x_d = 1$.

Hence by Lemma 2.7, Δ is reflexive and we can obtain the vertices of $\Delta^\vee$. □

Now, we prove Theorem 5.2.

Proof of Theorem 5.2. First notice that, by Theorem 5.6, the case of Hermite normal form matrices with one nonstandard row are captured in the statement (1). Hence, we consider the case of Hermite normal form matrices with two nonstandard rows. Let s,d be positive integers with $s < d$, and let $A = (a_1,\dots,a_{s-1},p)$ and $B = (b_1,\dots,b_{d-1},p)$ be sequences of integers with $0 \le a_1,\dots,a_{s-1},b_1,\dots,b_{d-1} < p$. Assume that $\Delta(A,B)$ is not a lattice pyramid over any lower-dimensional lattice simplex and $\Delta(A,B)$ is Gorenstein of index r. Then for $1 \leq i \leq s-1$, we have $(a_i,b_i) \neq (0,0)$ and for $s+1 \leq i \leq d-1$, we have $b_i \neq 0$. By Lemma 5.10, we only need to consider the case where $b_s = 0$. If for some $1 \leq i \leq s-1$, $a_i = 0$, then $\Delta(A,B)$ is unimodularly equivalent to $\Delta(A',B')$, where

$$A' = (a_1,\dots,a_{i-1},a_{i+1},\dots,a_{s-1},p)$$

and

$$B' = (b_1,\dots,b_{i-1},b_{i+1},\dots,b_{s-1},0,b_i,b_{s+1},\dots,b_{d-1},p).$$

Hence we may assume that $a_1,\dots,a_{s-1} \geq 1$. Let $\mathbf{t} = (t_1,\dots,t_d) \in \mathbb{R}^d$ be the unique interior lattice point of $r\Delta(A,B)$, and set $\Delta' = r\Delta(A,B) - \mathbf{t}$. Then by Lemma 5.9, the equations of supporting hyperplanes of facets of Δ' are as follows:

- $-x_d = t_d$;
- $-px_i + a_ix_s + b_ix_d = pt_i - a_it_s - b_it_d$, $1 \leq i \leq s-1$;
- $-x_s = t_s$;
- $-px_i + b_ix_d = pt_i - b_it_d$, $s+1 \leq i \leq d-1$;
- $p \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} x_j + (1 - \sum_{1\leq j\leq s-1} a_j)x_s + (1 - \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} b_j)x_d$
  $= rp - p \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} t_j - (1 - \sum_{1\leq j\leq s-1} a_j)t_s - (1 - \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} b_j)t_d.$

Hence by Lemma 2.7, it is known that $-\mathbf{e}_d/t_d$ and $-\mathbf{e}_s/t_s$ are vertices of $(\Delta')^\vee$. Therefore, since Δ' is reflexive, we obtain $t_s = t_d = 1$. Similarly, since $pt_i - a_i - b_i > 0$ and $pt_i - a_i - b_i$ divides p, a_i and b_i, and since $(a_i,b_i) \neq (0,0)$, we have that $pt_i - a_i - b_i = 1$. Hence, for any $1 \leq i \leq s-1$, we have $t_i = 1$ and $p - a_i - b_i = 1$. Moreover, since $b_i \neq 0$ for any $s+1 \leq i \leq d-1$, we have that $t_i = 1$ and $p - b_i = 1$. We then obtain

$$rp - p \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} t_j - (1 - \sum_{1\leq j\leq s-1} a_j) - (1 - \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} b_j) = rp - d.$$

Since $rp - d > 0$ and $rp - d$ divides p, we have $rp - d = 1$ or $rp - d = p$.

Assume that $rp - d = p$. Then since $p \mid (1 - \sum_{1\leq j\leq s-1} a_j)$, we know that $\Lambda_{\Delta(A,B)}$ is generated by

$$\left(0, \frac{a_1+1}{p}, \dots, \frac{a_{s-1}+1}{p}, 0, \frac{1}{p}, \dots, \frac{1}{p}\right)$$

and

$$\left(0, \frac{p-a_1}{p}, \dots, \frac{p-a_{s-1}}{p}, \frac{1}{p}, 0, \dots, 0\right).$$

Therefore, by Lemma 2.8, $\Delta(A,B)$ is a lattice pyramid over a lower-dimensional lattice simplex. Thus one has $rp-d=1$. Then it follows that $\Lambda_{\Delta(A,B)}$ is generated by

$$\left(\frac{2-\sum\limits_{1\le i\le s-1} a_i}{p}, \frac{a_1+1}{p}, \dots, \frac{a_{s-1}+1}{p}, 0, \frac{1}{p}, \dots, \frac{1}{p}\right)$$

and

$$\left(\frac{\left(\sum\limits_{1\le i\le s-1} a_i\right)-1}{p}, \frac{p-a_1}{p}, \dots, \frac{p-a_{s-1}}{p}, \frac{1}{p}, 0, \dots, 0\right).$$

By Lemma 5.11, this completes the proof. □

Next, we characterize Gorenstein simplices with normalized volume pq. In order to prove Theorem 5.3, we show the following lemma.

Lemma 5.12. *Let p and q be prime numbers with $p \neq q$ and set $a_s = p$ and $b_d = q$. Assume that $k = rpq - p(d-s) - qs \in \{p,q\}$. Then $\Delta(A,B)$ is Gorenstein of index r. Moreover, the vertices of the associated dual reflexive simplex are the following lattice points:*

- $-\mathbf{e}_d$;
- $-p\mathbf{e}_i + (p-1)\mathbf{e}_s$ *for* $1 \le i \le s-1$;
- $-\mathbf{e}_s$;
- $-q\mathbf{e}_i + (q-1)\mathbf{e}_d$ *for* $s+1 \le i \le d-1$;
- $(c_1, \dots, c_d)$,

where

$$c_i = \begin{cases} \dfrac{q(1-(s-1)(p-1))}{k}, & \text{if } i = s, \\ \dfrac{p(1-(d-s-1)(q-1)}{k}, & \text{if } i = d, \\ \dfrac{pq}{k}, & \text{otherwise.} \end{cases}$$

Proof. Since $pq(d-2)+q(1-(p-1)(s-1))+p(1-(q-1)(d-s-1)) = p(d-s)+qs < rpq$, by Lemma 5.9, it follows that $\mathbf{t} = (1,\dots,1) \in \mathbb{Z}^d$ is an interior lattice point of $r\Delta(A,B)$. Hence by Lemma 5.9, the equations of supporting hyperplanes of facets of $\Delta' = r\Delta(A,B) - \mathbf{t}$ are as follows:

- $-x_d = 1$;

- $-pqx_i + (p-1)qx_s = 1$, $1 \leq i \leq s-1$;
- $-x_s = 1$;
- $-qx_i + (q-1)x_d = 1$, $s+1 \leq i \leq d-1$;
- $pq \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} x_j + q(1-(p-1)(s-1))x_s + p(1-(q-1)(d-s-1)x_d$
  $= rpq - p(d-s) - qs.$

If $rpq - p(d-s) - qs = p$, then $p \mid s$. Hence, $p \mid (1-(p-1)(s-1))$. Moreover, if $rpq - p(d-s) - qs = q$, then $q \mid (d-s)$, and so $q \mid (1-(d-s-1)(q-1)$. Thus by Lemma 2.7, Δ' is reflexive and we can obtain the vertices of $(\Delta')^\vee$. □

Now, we prove Theorem 5.3.

Proof of Theorem 5.3. The case when $s_3 \geq 1$ follows from Theorem 5.6 since this case corresponds to the Hermite normal form matrices with one nonstandard row. Hence, we consider the case of Hermite normal form matrices with two nonstandard rows. Let s,d be positive integers with $s < d$ and p,q prime numbers with $p \neq q$, and we let $A = (a_1,\dots,a_{s-1},p)$ and $B = (b_1,\dots,b_{d-1},q)$ be sequences of integers with $0 \leq a_1,\dots,a_{s-1} < p$ and $0 \leq b_1,\dots,b_{d-1} < q$. Assume that $\Delta(A,B)$ is not a lattice pyramid over any lower-dimensional lattice simplex and $\Delta(A,B)$ is Gorenstein of index r. Then for $1 \leq i \leq s-1$, we have $(a_i,b_i) \neq (0,0)$ and for $s+1 \leq i \leq d-1$, we have $b_i \neq 0$. By Lemma 5.10, we need only consider the case where $b_s = 0$.

Let $\mathbf{t} = (t_1,\dots,t_d) \in \mathbb{R}^d$ be the unique interior lattice point of $r\Delta(A,B)$. Analogous to the proof in Theorem 5.2, we have $t_i = 1$ for each i and so we set $\Delta' = r\Delta(A,B) - \mathbf{t}$. Then by Lemma 5.9, the equations of supporting hyperplanes of facets of Δ' are as follows:

- $-x_d = 1$;
- $-pqx_i + a_iqx_s + pb_ix_d = pq - pb_i - a_iq$, $1 \leq i \leq s-1$;
- $-x_s = 1$;
- $-qx_i + b_ix_d = q - b_i$, $s+1 \leq i \leq d-1$;
- $pq \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} x_j + q(1 - \sum_{1\leq j\leq s-1} a_j)x_s + p(1 - \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} b_j)x_d$
  $= rpq - pq(d-2) - q(1 - \sum_{1\leq j\leq s-1} a_j) - p(1 - \sum_{\substack{1\leq j\leq d-1 \\ j\neq s}} b_j).$

Since Δ' is reflexive, by Lemma 2.7, for $1 \le i \le s-1$ we have $pq - pb_i - a_i q \in \{1, p, q\}$ and for $s+1 \le i \le d-1$ we have $b_i = q-1$. If for some $1 \le i \le s-1$, $pq - pb_i - a_i q = 1$, then since

$$a = \left(\frac{\left(\sum_{j=1}^{s-1} b_j\right) - (d-s)}{q}, \frac{q-b_1}{q}, \ldots, \frac{q-b_{s-1}}{q}, 0, \frac{1}{q}, \ldots, \frac{1}{q} \right)$$

and

$$b = \left(\frac{\left(\sum_{j=1}^{s-1} a_j\right) - 1}{p}, \frac{p-a_1}{p}, \ldots, \frac{p-a_{s-1}}{p}, \frac{1}{p}, 0, \ldots, 0 \right)$$

are elements of $\Lambda_{\Delta(A,B)}$, we know that the ith entry of $a+b$ equals $\frac{1}{pq}$. Hence this is the case where $s_3 \ge 1$. If for some $1 \le i \le s-1$, $pq - pb_i - a_i q = p$, then since $(a_i, b_i) = (0, q-1)$, it follows that $\Delta(A,B)$ is unimodularly equivalent to $\Delta(A', B')$, where

$$A' = (a_1, \ldots, a_{i-1}, a_{i+1} \ldots, a_{s-1}, p)$$

and

$$B' = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{s-1}, 0, b_i, b_{s+1}, \ldots, b_{d-1}, q).$$

Hence we may assume that for any $1 \le i \le s-1$, we have that $pq - pb_i - a_i q = q$. In particular, $(a_i, b_i) = (p-1, 0)$. Then we know that an element

$$\left(-\frac{p(d-s)+qs}{pq}, \underbrace{\frac{1}{p}, \ldots, \frac{1}{p}}_{s}, \underbrace{\frac{1}{q}, \ldots, \frac{1}{q}}_{d-s} \right)$$

of $(\mathbb{R}/\mathbb{Z})^{d+1}$ generates Λ_Δ. Moreover, we obtain

$$1 - \sum_{1 \le j \le s-1} a_j = -p(s-1) + s,$$

$$1 - \sum_{\substack{1 \le j \le d-1 \\ j \ne s}} b_j = -q(d-s-1) + (d-s),$$

and

$$rpq - pq(d-2) - q\Big(1 - \sum_{1 \le j \le s-1} a_j\Big) - p\Big(1 - \sum_{\substack{1 \le j \le d-1 \\ j \ne s}} b_j\Big) = rpq - p(d-s) - qs.$$

Since Δ' is reflexive, by Lemma 2.7, it follows that $rpq-p(d-s)-qs \in \{1,p,q,pq\}$. By Lemma 2.8, we know that $rpq-p(d-s)-qs \neq pq$. If $rpq-p(d-s)-qs = 1$, we have $\frac{-s}{p}+\frac{-d+s}{q} = \frac{-rpq+1}{pq}$. Hence, this is again the case where $s_3 \geq 1$. Therefore, we may just consider the case where $rpq-p(d-s)-qs \in \{p,q\}$. However, it is clear that this case satisfies the statement (2). By Lemma 5.12, this completes the proof. $\square$

By Theorem 5.6, we can construct Gorenstein simplices whose normalized volume is equal p^ℓ, where p is a prime number and ℓ is a positive integer. Finally, we give other examples of Gorenstein simplices whose normalized volume equals p^ℓ. These simplices arise from Hermite normal form matrices with ℓ nonstandard rows. In particular, Theorem 5.2 (2) is the motivation for the following theorem.

Theorem 5.13 ([91, Theorem 3.5])**.** *Let p be a prime number, and let d and ℓ be positive integers with $\ell \leq d$, and let $1 \leq s_1 < s_2 < \cdots < s_\ell = d$ be positive integers. For $1 \leq i \leq k$ and $0 \leq j \leq d$, we set*

$$\mathbb{R}/\mathbb{Z} \ni g_{ij} = \begin{cases} -\sum_{k=1}^{d} g_{ik}, & \text{if } j=0, \\ \dfrac{p-a_{ij}}{p}, & \text{if } 1 \leq j \leq s_i - 1 \text{ and } j \neq s_1, \dots, s_{i-1}, \\ \dfrac{1}{p}, & \text{if } j = s_i, \\ 0, & \text{otherwise,} \end{cases}$$

where each a_{ij} is a positive integer with $1 \leq a_{ij} \leq p-1$. Suppose that there exists an integer r with $d = rp-1$, and for $1 \leq j \leq d-1$ with $j \neq s_1, \dots, s_\ell$, there exists a positive integer t_j such that $\sum_i a_{ij} = t_j p - 1$. If $\Delta \subset \mathbb{R}^d$ is a d-dimensional simplex such that Λ_Δ is generated by $(g_{10}, \dots, g_{1d}), \dots, (g_{\ell 0}, \dots, g_{\ell d})$, then Δ is Gorenstein of index r and $\mathrm{Vol}(\Delta) = p^\ell$.

Proof. Set $\Delta = \mathrm{conv}(\{\mathbf{v}_0, \dots, \mathbf{v}_d\}) \subset \mathbb{R}^d$, where

$$v_i = \begin{cases} \mathbf{0}, & \text{if } i = 0, \\ \mathbf{e}_i, & \text{if } i \neq 0, s_1, \dots, s_\ell, \\ \sum\limits_{\substack{1 \leq j < s_k \\ j \neq s_1, \dots, s_{k-1}}} a_{ij}\mathbf{e}_j + p\mathbf{e}_{s_k}, & \text{if } i = s_k. \end{cases}$$

Then $\Delta \subset \mathbb{R}^d$ is a d-dimensional simplex such that $\mathrm{Vol}(\Delta) = p^\ell$ and Λ_Δ is generated by $(g_{10}, \dots, g_{1d}), \dots, (g_{\ell 0}, \dots, g_{\ell d})$. Let $s_0 = 0$. Then the equations of supporting hyperplanes of facets of $r\Delta$ are as follows:

- $-x_{s_k} = 0$, for $k = 1, \dots, \ell$;

- $-px_i + \sum_{j=k+1}^{\ell} a_{ji}x_{s_j} = 0$, for $s_k < i < s_{k+1}$;
- $$p \sum_{\substack{1\le j<s_\ell \\ j\neq s_1,\dots,s_{\ell-1}}} x_j + \sum_{1\le k\le \ell}\left(\left(1-\sum_{\substack{1\le j<s_k \\ j\neq s_1,\dots,s_{k-1}}} a_{kj}\right)x_{s_k}\right) = rp.$$

Let $\mathbf{t}' = (t'_1,\dots,t'_d)$ be a lattice point of $\mathbb{R}^d$, where

$$t'_i = \begin{cases} 1, & \text{if } i = s_1,\dots,s_\ell, \\ t_i, & \text{if } i \neq s_1,\dots,s_\ell. \end{cases}$$

Now, we claim t' is an interior lattice point of $r\Delta$. Indeed, for $s_k < i < s_{k+1}$, we have

$$-pt_i + \sum_{j=k+1}^{\ell} a_{ji} = -1 < 0$$

and

$$p \sum_{\substack{1\le j<s_\ell \\ j\neq s_1,\dots,s_{\ell-1}}} t_j + \sum_{1\le k\le \ell}\left(1-\sum_{\substack{1\le j<s_k \\ j\neq s_1,\dots,s_{k-1}}} a_{kj}\right) = d = rp-1 < rp.$$

Now set $\Delta' = r\Delta - \mathbf{t}'$. Then the equations of supporting hyperplanes of facets of Δ' are as follows:

- $-x_{s_k} = 1$, for $k = 1,\dots,\ell$;
- $-px_i + \sum\limits_{j=k+1}^{\ell} a_{ji}x_{s_j} = 1$, for $s_k < i < s_{k+1}$;
- $$p \sum_{\substack{1\le j<s_\ell \\ j\neq s_1,\dots,s_{\ell-1}}} x_j + \sum_{1\le k\le \ell}\left(\left(1-\sum_{\substack{1\le j<s_k \\ j\neq s_1,\dots,s_{k-1}}} a_{kj}\right)x_{s_k}\right) = 1.$$

Hence by Lemma 2.7, Δ' is reflexive, and so Δ is Gorenstein of index r. □

Remark 5.14. Let Δ be the Gorenstein simplex as in Theorem 5.13. Then the vertices of the associated dual reflexive simplex of Δ are following lattice points:

- $-\mathbf{e}_{s_k}$, for $k = 1,\dots,\ell$;
- $-p\mathbf{e}_i + \sum\limits_{j=k+1}^{\ell} a_{ji}\mathbf{e}_{s_j}$, for $s_k < i < s_{k+1}$;
- $$p \sum_{\substack{1\le j<s_\ell \\ j\neq s_1,\dots,s_{\ell-1}}} \mathbf{e}_j + \sum_{1\le k\le \ell}\left(\left(1-\sum_{\substack{1\le j<s_k \\ j\neq s_1,\dots,s_{k-1}}} a_{kj}\right)\mathbf{e}_{s_k}\right).$$

5.4 Volume of the associated dual reflexive simplex

In this section, we compute the volume of the associated dual reflexive simplices of the Gorenstein simplices we constructed in Sections 5.2 and 5.3. We first consider the case of Gorenstein simplices arising from Hermite normal form matrices with one nonstandard row.

Theorem 5.15 ([91, Theorem 4.1])**.** *Let $\Delta(A) \subset \mathbb{R}^d$ be a d-dimensional Gorenstein simplex of index r as in Theorem 5.6 and set $\Delta = r\Delta(A) - (1,\dots,1)$. For $0 \le i \le d-1$, we set $b_i = a_d/a_i$. Then we have $\mathrm{Vol}(\Delta^\vee) = r\prod_{j=0}^{d-1} b_i$.*

Proof. By Lemma 5.7, we know that $\Delta^\vee = \mathrm{conv}(\{\mathbf{w}_0,\dots,\mathbf{w}_d\})$, where

$$\mathbf{w}_i = \begin{cases} -\mathbf{e}_d, & \text{if } i = 0, \\ -\dfrac{a_d}{a_i}\mathbf{e}_i + \dfrac{a_d - a_i}{a_i}\mathbf{e}_d, & \text{if } 1 \le i \le d-1, \\ \dfrac{a_d}{a_0}\displaystyle\sum_{j=1}^{d-1}\mathbf{e}_j + \dfrac{(r-d+1)a_d - a_0}{a_0}\mathbf{e}_d, & \text{if } i = d. \end{cases}$$

It is easy show $\Delta^\vee$ is unimodularly equivalent to a d-dimensional simplex Δ' whose vertices $\mathbf{v}'_0,\dots,\mathbf{v}'_d$ are the following:

$$\mathbf{v}'_i = \begin{cases} \mathbf{0}, & \text{if } i = 0, \\ -b_i\mathbf{e}_i, & \text{if } 1 \le i \le d-1, \\ b_0\displaystyle\sum_{j=1}^{d-1}\mathbf{e}_j + rb_0\mathbf{e}_d, & \text{if } i = d. \end{cases}$$

Hence we have $\mathrm{Vol}(\Delta^\vee) = r\prod_{j=0}^{d-1} b_i$, as desired. □

From this theorem, we immediately obtain the following corollary.

Corollary 5.16. *Let $\Delta \subset \mathbb{R}^d$ be a d-dimensional Gorenstein simplex of index r whose normalized volume equals a prime number p. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex and the unique interior lattice point of $r\Delta$ is the origin of $\mathbb{R}^d$. Then we have $\mathrm{Vol}((r\Delta)^\vee) = rp^d$.*

Next, we consider the case of Gorenstein simplices with normalized volume p^2, where p is a prime number. By Theorem 5.15, we can compute the volume of the associated dual reflexive simplices of the Gorenstein simplices in Theorem 5.2 (1).

The Gorenstein simplices in Theorem 5.2 (2) are included in the Gorenstein simplices in Theorem 5.13. Hence, we consider the case of the Gorenstein simplices in Theorem 5.13. In fact, we can obtain the following Theorem.

Theorem 5.17 ([91, Theorem 4.3]). *Let $\Delta \subset \mathbb{R}^d$ be a d-dimensional Gorenstein polytope of index r as in Theorem 5.13 such that the unique interior lattice point of $r\Delta$ is the origin in $\mathbb{R}^d$. Then we have* $\mathrm{Vol}((r\Delta)^\vee) = rp^{d-\ell+1}$.

Proof. By Remark 5.14, $(r\Delta)^\vee$ is the convex hull of the following lattice points:

- $-\mathbf{e}_{s_k}$, for $k = 1, \dots, \ell$;
- $-p\mathbf{e}_i + \sum_{j=k+1}^{\ell} a_{ji}\mathbf{e}_{s_j}$, for $s_k < i < s_{k+1}$;
- $p \sum_{\substack{1 \le j < s_\ell \\ j \ne s_1, \dots, s_{\ell-1}}} \mathbf{e}_j + \sum_{1 \le k \le \ell} \left(\left(1 - \sum_{\substack{1 \le j < s_k \\ j \ne s_1, \dots, s_{k-1}}} a_{kj} \right) \mathbf{e}_{s_k} \right)$.

So we set

$$U = \begin{pmatrix} 1 & 0 & \cdots & 0 & t'_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & t'_{d-1} \\ 0 & 0 & \cdots & 0 & t'_d \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

Since $t'_d = t_\ell = 1$, it follows that U is unimodular. Letting $\Delta' = f_U((r\Delta)^\vee + \mathbf{e}_d)$, we know that Δ' is the convex hull of the following lattice points:

- $\mathbf{0}$;
- $-\mathbf{e}_{s_k}$, for $k = 1, \dots, \ell - 1$;
- $-p\mathbf{e}_i + \sum_{j=k+1}^{\ell-1} a_{ji}\mathbf{e}_{s_j}$, for $s_k < i < s_{k+1}$;
- $p \sum_{\substack{1 \le j < s_\ell \\ j \ne s_1, \dots, s_{\ell-1}}} \mathbf{e}_j + \sum_{1 \le k \le \ell-1} \left(\left(1 - \sum_{\substack{1 \le j < s_k \\ j \ne s_1, \dots, s_{k-1}}} a_{kj} \right) \mathbf{e}_{s_k} \right) + rp\mathbf{e}_d$.

Hence we have $\mathrm{Vol}(\Delta') = rp^{d-\ell+1}$, as desired. □

Corollary 5.18. *Let p be a prime number, and let $\Delta \subset \mathbb{R}^d$ be a d-dimensional Gorenstein simplex of index r whose normalized volume equals p^2.*
(1) Suppose that Δ and s satisfy the condition of Theorem 5.2 (1) and the unique interior lattice point of $r\Delta$ is the origin in $\mathbb{R}^d$. Then we have $\mathrm{Vol}((r\Delta)^\vee) = rp^{2d-s}$.
(2) Suppose that Δ satisfies the condition of Theorem 5.2 (2) and the unique interior lattice point of $r\Delta$ is the origin in $\mathbb{R}^d$. Then we have $\mathrm{Vol}((r\Delta)^\vee) = rp^{d-1}$.

Finally, we consider the case of Gorenstein simplices whose normalized volume equals pq, where p and q are prime numbers with $p \neq q$.

Theorem 5.19 ([91, Theorem 4.5]). *Let p and q be prime integers with $p \neq q$ and $\Delta \subset \mathbb{R}^d$ a d-dimensional Gorenstein simplex of index r whose normalized volume equals pq. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex and the unique interior lattice point of $r\Delta$ is the origin in $\mathbb{R}^d$. Then we have* $\mathrm{Vol}((r\Delta)^\vee) = rp^{s_1+s_3-1}q^{s_2+s_3-1}$*, where s_1, s_2, s_3 are nonnegative integers which satisfy the conditions of Theorem 5.3.*

Proof. First, assume that $s_3 \geq 1$. Then by Theorem 5.15, we obtain $\mathrm{Vol}((r\Delta)^\vee) = rp^{s_1+s_3-1}q^{s_2+s_3-1}$.

Next, assume that $s_3 = 0$. Then by the condition (1) of Theorem 5.3, we know that $(s_1, s_2) \neq (1, d)$ and $(s_1, s_2) \neq (d, 1)$. Moreover, by the condition (2) of Theorem 5.3 and the normalized volume of Δ, we have $(s_1, s_2) \neq (d+1, 0)$ and $(s_1, s_2) \neq (0, d+1)$. Hence, we have $s_1, s_2 \geq 2$. Since Λ_Δ is generated by

$$\left(\underbrace{\frac{1}{p}, \dots, \frac{1}{p}}_{s_1}, \underbrace{\frac{1}{q}, \dots, \frac{1}{q}}_{s_2}\right),$$

we may assume that $r\Delta = r\Delta(A, B) - (1, \dots, 1)$, where

$$A = (\underbrace{p-1, \dots, p-1}_{s_1-1}, p)$$

and

$$B = (\underbrace{0, \dots, 0}_{s_1}, \underbrace{q-1, \dots, q-1}_{s_2-2}, q).$$

Then by Lemma 5.12, we know that $(r\Delta)^\vee = \mathrm{conv}(\{\mathbf{w}_0, \dots, \mathbf{w}_d\})$ where

$$\mathbf{w}_i = \begin{cases} -\mathbf{e}_d, & \text{if } i = 0, \\ -p\mathbf{e}_i + (p-1)\mathbf{e}_{s_1}, & \text{if } 1 \leq i \leq s_1 - 1, \\ -\mathbf{e}_i, & \text{if } i = s_1, \\ -q\mathbf{e}_i + (q-1)\mathbf{e}_d, & \text{if } s_1 + 1 \leq i \leq d - 1, \\ (c_1, \dots, c_d), & \text{if } i = d, \end{cases}$$

and

$$c_i = \begin{cases} \dfrac{q(1-(s_1-1)(p-1))}{p}, & \text{if } i = s_1, \\ 1-(d-s_1-1)(q-1), & \text{if } i = d, \\ q, & \text{otherwise.} \end{cases}$$

It is easy show $(r\Delta)^\vee$ is unimodularly equivalent to a d-dimensional simplex Δ' whose vertices $\mathbf{v}'_0, \dots, \mathbf{v}'_d$ are the following:

$$\mathbf{v}'_i = \begin{cases} \mathbf{0}, & \text{if } i = 0, \\ -(p-1)\mathbf{e}_i + p\mathbf{e}_{s_1}, & \text{if } 1 \le i \le s_1 - 1, \\ -\mathbf{e}_i, & \text{if } i = s_1, \\ -q\mathbf{e}_i, & \text{if } s_1 + 1 \le i \le d-1, \\ (c_1, \dots, c_{d-1}, c_1 + \dots + c_d + 1), & \text{if } i = d, \end{cases}$$

Since $c_1 + \dots + c_d + 1 = rq$, we have that $\mathrm{Vol}((r\Delta)^\vee) = rp^{s_1-1}q^{s_2-1}$, as desired. □

Chapter 6

Gorenstein simplices with a given δ-polynomial

In Corollary 5.1, it is shown that if Δ is a Gorenstein simplex whose normalized volume $\text{Vol}(\Delta)$ is a prime number p, then its δ-polynomial is of the form

$$\delta(\Delta,t) = 1 + t^k + \cdots + t^{(p-1)k},$$

where $k \geq 1$ is a positive integer. Once the fact became known, we cannot escape from the temptation to achieve the study on the following problem:

Problem 6.1. *Given positive integers $k \geq 1$ and $v \geq 1$, classify the Gorenstein simplices with the δ-polynomial $1 + t^k + \cdots + t^{(v-1)k}$.*

In particular, in Problem 6.1, when $k \geq 2$, the targets are Gorenstein empty simplices. In this chapter, we focus on Problem 6.1.

This chapter is organized as follows. We devote Section 6.1 to discuss lower bounds on the dimensions of Gorenstein simplices with a given δ-polynomial of Problem 6.1 and, in addition, to classify the Gorenstein simplices when the lower bounds are held (Theorem 6.2). The highlight of this chapter is Section 6.2, where a complete answer of Problem 6.1 when v is either p^2 or pq, where p and q are distinct prime integers (Theorems 6.5 and 6.6). Finally, in Section 6.3, we will discuss the number of Gorenstein simplices, up to unimodular equivalence, with a given δ-polynomial of Problem 6.1.

6.1 Existence

In this section, we prove that for positive integers $k \geq 1$ and $v \geq 1$, there exists a lattice simplex with the δ-polynomial $1 + t^k + t^{2k} + \cdots + t^{(v-1)k}$. Moreover, we give a lower bound and an upper bound on the dimension of such a lattice simplex which is not a lattice pyramid. In fact, we obtain the following theorem.

Theorem 6.2 ([52, Theorem 2.1])**.** *Let k and v be positive integers. Then there exists a lattice simplex $\Delta \subset \mathbb{R}^d$ of dimension d whose δ-polynomial is $1+t^k+t^{2k}+\cdots+t^{(v-1)k}$. Furthermore, if Δ is not a lattice pyramid over any lower-dimensional lattice simplex, then one has $vk-1 \leq d \leq 4(v-1)k-2$. In particular, the lower bound holds if and only if Λ_Δ is generated by $(1/v,\dots,1/v)$.*

Proof. We assume that there exists a lattice simplex $\Delta \subset \mathbb{R}^d$ of dimension d whose δ-polynomial is $1+t^k+t^{2k}+\cdots+t^{(v-1)k}$. Let $\mathbf{x}=(x_0,\dots,x_d) \in \Lambda_\Delta$ be an element such that $\mathrm{ht}(\mathbf{x})=(v-1)k$. Then we have that $\mathrm{ht}(-\mathbf{x}) \geq k$. Hence since $\mathrm{ht}(\mathbf{x})+\mathrm{ht}(-\mathbf{x}) \leq d+1$, we obtain $d \geq vk-1$. From Lemma 2.4, if Δ is not a lattice pyramid over any lower-dimensional lattice simplex, then one has $d \leq 4(v-1)k-2$. Now, we assume that $d = vk-1$. Since for each i, one has $0 \leq x_i \leq (v-1)/v$, we obtain $\mathrm{ht}(\mathbf{x}) \leq (d+1)(v-1)/v = (v-1)k$. Hence for each i, it follows that $x_i=(v-1)/v$. Therefore Λ_Δ is generated by $(1/v,\dots,1/v)$. Then it is easy to show that $\delta(\Delta,t)=1+t^k+t^{2k}+\cdots+t^{(v-1)k}$, as desired. □

6.2 Classification

In this section, we give a complete ansewer of Problem 6.1 for the case that v is the product of two prime integers. First, we consider the case where v is a prime integer. Corollary 5.1 says that for each positive integers k and v, if v is a prime integer, then there exists just one lattice simplex up to unimodular equivalence such that its δ-polynomial equals $1+t^k+t^{2k}+\cdots+t^{(v-1)k}$. By the following proposition, we know that if v is not a prime integer, then there exist at least two such simplices up to unimodular equivalence.

Proposition 6.3. *Given positive integers k, v and a proper divisor u of v, let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d such that Λ_Δ is generated by*

$$\Bigg(\underbrace{u/v,\dots,u/v}_{(v-1)k},\underbrace{1/v,\dots,1/v}_{uk}\Bigg) \in (\mathbb{R}/\mathbb{Z})^{(v+u-1)k}.$$

Then one has $\delta(\Delta,t)=1+t^k+t^{2k}+\cdots+t^{(v-1)k}$.

Proof. Set $\mathbf{x} = \left(\underbrace{u/v, \dots, u/v}_{(v-1)k}, \underbrace{1/v, \dots, 1/v}_{uk} \right)$ and $\mathbf{y} = (v/u)\mathbf{x} = \left(\underbrace{0, \dots, 0}_{(v-1)k}, \underbrace{1/u, \dots, 1/u}_{uk} \right)$. Then we obtain $\mathrm{ht}(\mathbf{x}) = uk$ and $\mathrm{ht}(\mathbf{y}) = k$. Moreover, it follows that

$$\Lambda_\Delta = \{ i\mathbf{x} + j\mathbf{y} \in (\mathbb{R}/\mathbb{Z})^{d+1} : i = 0, \dots, v/u - 1, \ j = 0, \dots, u-1 \}.$$

For any integers $0 \le i \le v/u - 1$ and $0 \le j \le u - 1$, one has

$$\mathrm{ht}(i\mathbf{x} + j\mathbf{y}) = i\mathrm{ht}(\mathbf{x}) + j\mathrm{ht}(\mathbf{y}) = (iu + j)k.$$

Hence, it follows from Lemma 2.9 that $\delta(\Delta, t) = 1 + t^k + t^{2k} + \cdots + t^{(v-1)k}$, as desired. □

Furthermore, the following proposition can immediately be obtained by Lemma 2.9.

Proposition 6.4. *Given positive integers v_1, v_2 and k, let $\Delta_1 \subset \mathbb{R}^{d_1}$ and $\Delta_2 \subset \mathbb{R}^{d_2}$ be lattice simplices of dimension d_1 and d_2 such that $\delta(\Delta_1, t) = 1 + t^k + t^{2k} + \cdots + t^{(v_1-1)k}$ and $\delta(\Delta_2, t) = 1 + t^{v_1 k} + t^{2v_1 k} + \cdots + t^{v_1(v_2-1)k}$. Let $\Delta \subset \mathbb{R}^{d_1+d_2+1}$ be a lattice simplex of dimension $d_1 + d_2 + 1$ such that*

$$\Lambda_\Delta = \{ (\mathbf{x}, \mathbf{y}) \in (\mathbb{R}/\mathbb{Z})^{d_1+d_2+2} : \mathbf{x} \in \Lambda_{\Delta_1}, \mathbf{y} \in \Lambda_{\Delta_2} \}.$$

Then one has $\delta(\Delta, t) = 1 + t^k + t^{2k} + \cdots + t^{(v_1 v_2 - 1)k}$. In particular, if neither Δ_1 nor Δ_2 is not a lattice pyramid, then Δ is not a lattice pyramid.

Now, we consider Problem 6.1 for the case that v is p^2 or pq, where p and q are prime integers with $p \neq q$. The following theorems are the main results of this chapter.

Theorem 6.5 ([52, Theorem 3.4])**.** *Let p be a prime integer and k a positive integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1 + t^k + t^{2k} + \cdots + t^{(p^2-1)k}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then one of the followings is satisfied:*

(1) $d = p^2 k - 1$;

(2) $d = p^2 k + (p-1)k - 1$;

(3) $d = p^2 k + pk - 1$.

Moreover, in each case, a system of generators of the finite abelian group Λ_Δ is the set of row vectors of the matrix which can be written up to permutation of the columns as follows:

(1) $(1/p^2 \ \cdots \ 1/p^2) \in (\mathbb{R}/\mathbb{Z})^{1 \times p^2 k}$;

(2) $$\left(\underbrace{1/p \ \cdots \ 1/p}_{(p^2-1)k} \ \underbrace{1/p^2 \ \cdots \ 1/p^2}_{pk}\right) \in (\mathbb{R}/\mathbb{Z})^{1\times(p^2+p-1)k};$$

(3) $$\begin{pmatrix} 1/p \ \cdots \ 1/p & 0 \ \cdots \ 0 \\ \underbrace{0 \ \cdots \ 0}_{pk} & \underbrace{1/p \ \cdots \ 1/p}_{p^2k} \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{2\times p(p+1)k}.$$

Theorem 6.6 ([52, Theorem 3.5])**.** *Let p and q be prime integers with $p \neq q$ and k a positive integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1+t^k+t^{2k}+\cdots+t^{(pq-1)k}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then one of the followings is satisfied:*

(1) $d = pqk-1$;

(2) $d = pqk+pk-1$;

(3) $d = pqk+qk-1$;

(4) $d = pqk+(p-1)k-1$;

(5) $d = pqk+(q-1)k-1$.

Moreover, in each case, the finite abelian group Λ_Δ is generated by one element which can be written up to permutation of the coordinates as follows:

(1) $(1/(pq),\dots,1/(pq)) \in (\mathbb{R}/\mathbb{Z})^{pqk}$;

(2) $$\left(\underbrace{1/p,\dots,1/p}_{pk},\underbrace{1/q,\dots,1/q}_{pqk}\right) \in (\mathbb{R}/\mathbb{Z})^{p(q+1)k};$$

(3) $$\left(\underbrace{1/q,\dots,1/q}_{qk},\underbrace{1/p,\dots,1/p}_{pqk}\right) \in (\mathbb{R}/\mathbb{Z})^{(p+1)qk};$$

(4) $$\left(\underbrace{1/q,\dots,1/q}_{(pq-1)(k+1)},\underbrace{1/(pq),\dots,1/(pq)}_{p(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{(pq+p-1)(k+1)};$$

(5) $$\left(\underbrace{1/p,\dots,1/p}_{(pq-1)(k+1)},\underbrace{1/(pq),\dots,1/(pq)}_{q(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{(pq+q-1)(k+1)}.$$

Remark 6.7. *The lattice simplices in Theorems 6.5 and 6.6 can be constructed by Propositions 6.3 and 6.4.*

Before proving these theorems, we give the vertex representations of Gorenstein simplices in Theorems 6.5 and 6.6. Given a sequence $A = (a_1, \dots, a_d)$ of integers, let $\Delta(A) \subset \mathbb{R}^d$ be the lattice simplex described in Section 5.2. Given sequences $A = (a_1, \dots, a_s)$ and $B = (b_1, \dots, b_d)$ of integers with $1 \leq s < d$, let $\Delta(A,B) \subset \mathbb{R}^d$ be the lattice simplex described in Section 5.3.

Corollary 6.8. *Let p be a prime integer and k a positive integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1+t^k+t^{2k}+\cdots+t^{(p^2-1)k}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is unimodularly equivalent to one of $\Delta(A_1)$, $\Delta(A_2)$ and $\Delta(A_3,B_3)$, where*

(1) $A_1 = (\underbrace{1,\dots,1}_{p^2k-2}, p^2)$;

(2) $A_2 = \left(\underbrace{1,\dots,1}_{pk-1}, \underbrace{p,\dots,p}_{(p^2-1)k-1}, p^2\right)$;

(3) $A_3 = \left(\underbrace{p-1,\dots,p-1}_{pk-1}, p\right)$, $B_3 = \left(\underbrace{0,\dots,0}_{pk}, \underbrace{p-1,\dots,p-1}_{p^2k-2}, p\right)$.

Corollary 6.9. *Let p and q be prime integers with $p \neq q$ and k a positive integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1+t^k+t^{2k}+\cdots+t^{(pq-1)k}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is unimodularly equivalent to one of $\Delta(A_1)$, $\Delta(A_2,B_2)$, $\Delta(A_3,B_3)$, $\Delta(A_4)$ and $\Delta(A_5)$, where*

(1) $A_1 = (\underbrace{1,\dots,1}_{pqk-2}, pq)$;

(2) $A_2 = \left(\underbrace{p-1,\dots,p-1}_{pk-1}, p\right)$, $B_2 = \left(\underbrace{0,\dots,0}_{pk}, \underbrace{q-1,\dots,q-1}_{pqk-2}, q\right)$;

(3) $A_3 = \left(\underbrace{q-1,\dots,q-1}_{qk-1}, q\right)$, $B_3 = \left(\underbrace{0,\dots,0}_{qk}, \underbrace{p-1,\dots,p-1}_{pqk-2}, p\right)$;

(4) $A_4 = \left(\underbrace{1,\dots,1}_{pk-1}, \underbrace{p,\dots,p}_{(pq-1)k-1}, pq\right)$;

(5) $A_5 = \left(\underbrace{1,\dots,1}_{qk-1}, \underbrace{q,\dots,q}_{(pq-1)k-1}, pq\right)$.

In order to prove Theorems 6.5 and 6.6, we use the following lemma.

Lemma 6.10. *Let v and k be positive integers, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-polynomial equals $1+t^k+t^{2k}+\cdots+t^{(v-1)k}$. Assume that $\mathbf{x} \in (\mathbb{R}/\mathbb{Z})^{d+1}$ is an element of Λ_Δ such that $\mathrm{ht}(\mathbf{x}) = k$ and set $m = \mathrm{ord}(\mathbf{x})$. Then by reordering the coordinates, we obtain $\mathbf{x} = \left(\underbrace{1/m,\dots,1/m}_{s},\underbrace{0,\dots,0}_{d-s+1}\right)$ for some positive integer s.*

Proof. Since $m = \mathrm{ord}(\mathbf{x})$, $\mathbf{x}$ must be of a form $(k_1/m,\dots,k_s/m,0,\dots,0)$ for a positive integer s and integers $1 \le k_1,\dots,k_s \le m-1$ by reordering the coordinates. If there exists an integer $k_i \ge 2$ for some $1 \le i \le s$, then one has $k_i(m-1)/m \ge 1$. Therefore, we obtain $\mathrm{ht}((m-1)\mathbf{x}) < (m-1)\mathrm{ht}(\mathbf{x}) = (m-1)k$. Since $m = \mathrm{ord}(\mathbf{x})$, $(m-1)\mathbf{x}$ is different from $\mathbf{0},\mathbf{x},\dots,(m-2)\mathbf{x}$. We remark that for any $\mathbf{a}$, $\mathbf{b} \in (\mathbb{R}/\mathbb{Z})^{d+1}$, one has $\mathrm{ht}(\mathbf{a}+\mathbf{b}) \le \mathrm{ht}(\mathbf{a})+\mathrm{ht}(\mathbf{b})$. This fact and the supposed δ-polynomial imply that $\mathrm{ht}(t\mathbf{x}) = t\mathrm{ht}(\mathbf{x}) = tk$ for any $1 \le t \le m-1$. This is a contradiction, as desired. □

Finally, we prove Theorem 6.5 and Theorem 6.6.

Proof of Theorem 6.5. Theorem 5.2 implies that Δ is unimodularly equivalent to either Δ_1 or Δ_2, where Δ_1 and Δ_2 are lattice simplices such that each system of generators of Λ_{Δ_1} and Λ_{Δ_2} is the set of vectors of matrix as follows:

(i) $$\left(\underbrace{1/p \cdots 1/p}_{d-s+1}\ \underbrace{1/p^2 \ \dots\ 1/p^2}_{s}\right) \in (\mathbb{R}/\mathbb{Z})^{1\times(d+1)};$$

(ii) $$\begin{pmatrix} (a_0+1)/p & \cdots & (a_{d-2}+1)/p & 0 & 1/p \\ (p-a_0)/p & \cdots & (p-a_{d-2})/p & 1/p & 0 \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{2\times(d+1)},$$

where s is a positive integer and $0 \le a_0,\dots,a_{d-2} \le p-1$ are integers.

At first, we assume that Δ is unimodularly equivalent to Δ_1. If $s = d+1$, then one has $(d+1)/p^2 = k$, hence, $d = p^2k-1$. This is the case (1). Now, we suppose that $s \ne d+1$. Let $\mathbf{x}$ be an element of Λ_{Δ_1} with $\mathrm{ht}(\mathbf{x}) = k$. Then by Lemma 6.10, one has $\mathbf{x} = \left(\underbrace{0,\dots,0}_{d-s+1},\underbrace{1/p,\dots,1/p}_{s}\right)$, hence $s = pk$. Set $\mathbf{y} = \left(\underbrace{1/p,\dots,1/p}_{d-s+1},\underbrace{1/p^2,\dots,1/p^2}_{s}\right)$. Since for any $1 \le m \le p-1$, $\mathrm{ht}(m\mathbf{x}) = mk$, we have $\mathrm{ht}(\mathbf{y}) = pk$. Hence it follows that $d-s+1 = p^2k-k$, namely, $d = p^2k+(p-1)k-1$. This is the case (2).

Next, we assume that Δ is unimodularly equivalent to Δ_2. By Lemma 6.10, it follows that for any $0 \le i \le d-2$, $a_i \in \{0, p-1\}$. Hence by reordering the coordinates of Λ_{Δ_2}, we can assume that Λ_{Δ_2} is generated by

$$\mathbf{x}_1 = \left(\underbrace{1/p,\dots,1/p}_{s},\underbrace{0,\dots,0}_{d-s+1}\right), \mathbf{x}_2 = \left(\underbrace{0,\dots,0}_{s},\underbrace{1/p,\dots,1/p}_{d-s+1}\right),$$

where $1 \le s \le \lfloor (d+1)/2 \rfloor$. Then since $\mathrm{ht}(\mathbf{x}_1) = k$, one has $s = pk$. Moreover, since $\mathrm{ht}(\mathbf{x}_2) = pk$, we have $d-s+1 = p^2k$, namely, $d = p^2k + pk - 1$. Therefore, This is the case (3).

Conversely, in each case, it is easy to show that $\delta(\Delta,t) = 1 + t^k + t^{2k} + \cdots + t^{(p^2-1)k}$, as desired. □

Proof of Theorem 6.6. By Theorem 5.3, we can suppose that Λ_Δ is generated by

$$\mathbf{x} = \left(\underbrace{1/p,\dots,1/p}_{s_1},\underbrace{1/q,\dots,1/q}_{s_2},\underbrace{1/(pq),\dots,1/(pq)}_{s_3}\right),$$

where $s_1 + s_2 + s_3 = d+1$ with nonnegative integers s_1, s_2, s_3. If $s_1 = s_2 = 0$, since $\mathrm{ht}(\mathbf{x}) = k$, one has $d = pqk - 1$. This is the case (1). If $s_3 = 0$, we can assume that Λ_Δ is generated by

$$\mathbf{x}_1 = \left(\underbrace{1/p,\dots,1/p}_{s_1},\underbrace{0,\dots,0}_{s_2}\right), \mathbf{x}_2 = \left(\underbrace{0,\dots,0}_{s_1},\underbrace{1/q,\dots,1/q}_{s_2}\right),$$

with $s_1, s_2 > 0$. Then it follows that $\mathrm{ht}(\mathbf{x}_1) = k$ and $\mathrm{ht}(\mathbf{x}_2) = pk$, or $\mathrm{ht}(\mathbf{x}_1) = qk$ and $\mathrm{ht}(\mathbf{x}_2) = k$. Assume that $\mathrm{ht}(\mathbf{x}_1) = k$ and $\mathrm{ht}(\mathbf{x}_2) = pk$. Then one has $s_1 = pk$ and $s_2 = pqk$. Hence since $d = pqk + pk - 1$, this is the case (2). Similarly, we can show the case (3).

Next we suppose that $s_1, s_2, s_3 > 0$. Let $\mathbf{a}$ be an element of Λ_Δ such that $\mathrm{ht}(\mathbf{a}) = k$. By Lemma 6.10, we know that $\mathrm{ord}(\mathbf{a}) \neq pq$. Hence, it follows that $\mathrm{ord}(\mathbf{a})$ equals p or q. Now we assume that $\mathrm{ord}(\mathbf{a}) = p$. By Lemma 6.10 again, $\mathbf{a}$ must be of a form $\left(\underbrace{1/p,\dots,1/p}_{s_1},\underbrace{0,\dots,0}_{s_2},\underbrace{1/p,\dots,1/p}_{s_3}\right)$. Let $\mathbf{b} = (b_1,\dots,b_{d+1})$ be an element of Λ_Δ such that $\mathrm{ht}(\mathbf{b}) = pk$. If there exists an index $1 \le i \le s_1$ such that $b_i = n/p$ with an integer $1 \le n \le p-1$, then $\mathrm{ht}(\mathbf{b} + (p-1)\mathbf{a}) < \mathrm{ht}(\mathbf{b}) + (p-1)\mathrm{ht}(\mathbf{a})$. Since $\mathbf{b} + (p-1)\mathbf{a}$ is different from $\mathbf{0}, \mathbf{a}, 2\mathbf{a}, \dots, (p-1)\mathbf{a}, \mathbf{b}, \mathbf{b}+\mathbf{a}, \dots, \mathbf{b} + (p-2)\mathbf{a}$, this contradicts to that $\delta_\Delta(t) = 1 + t^k + t^{2k} + \cdots + t^{(pq-1)k}$. Hence one obtains $b_i = 0$ for any $1 \le i \le s_1$. Therefore, we can assume that $\mathbf{b} = \left(\underbrace{0,\dots,0}_{s_1},\underbrace{\ell/q,\dots,\ell/q}_{s_2},\underbrace{m/q,\dots,m/q}_{s_3}\right)$ for some positive integers ℓ, m. Then whenever $(g_1, h_1) \neq (g_2, h_2)$ with $0 \leq g_1, g_2 \leq p-1$ and $0 \leq h_1, h_2 \leq q-1$, $g_1\mathbf{a} + h_1\mathbf{b}$ and $g_2\mathbf{a} + h_2\mathbf{b}$ are different elements of Λ_Δ. Hence since $\delta_\Delta(t) = 1 + t^k + t^{2k} + \cdots + t^{(pq-1)k}$, one has

$$\mathrm{ht}(g\mathbf{a} + h\mathbf{b}) = g\mathrm{ht}(\mathbf{a}) + h\mathrm{ht}(\mathbf{b})$$

for any $0 \leq g \leq p-1$ and $0 \leq h \leq q-1$. This implies that $\ell = m = 1$. However since $(p-1)/p + (q-1)/q > 1$, we have $\mathrm{ht}((p-1)\mathbf{a} + (q-1)\mathbf{b}) < (p-1)\mathrm{ht}(\mathbf{a}) + (q-1)\mathrm{ht}(\mathbf{b})$, a contradiction. Therefore, it does not follow $s_1, s_2, s_3 > 0$.

Finally, we assume that $s_1 = 0$ and $s_2 > 0$. Then one has $\mathrm{ht}(q\mathbf{x}) = k$, hence, $s_3 = pk$. Moreover, since $\mathrm{ht}(\mathbf{x}) = pk$, we obtain $s_2 = (pq-1)k$. Therefore, this is the case (4). Similarly, we can show the case (5).

Conversely, in each case, it is easy to see that $\delta(\Delta, t) = 1 + t^k + t^{2k} + \cdots + t^{(pq-1)k}$, as desired. $\square$

6.3 The number of Gorenstein simplices

In this section, we consider how many Gorenstein simplices which have a given δ-polynomial of Problem 6.1.

Given positive integers v and k, let $N(v,k)$ denote the number of Gorenstein simplices, up to unimodular equivalence, which are not lattice pyramids over any lower-dimensional lattice simplex and whose δ-polynomials equal $1 + t^k + t^{2k} + \cdots + t^{(v-1)k}$. For example, by Corollary 5.1, $N(p,k) = 1$ for any prime integer p. Moreover, from Theorems 6.5 and 6.6, $N(p^2,k) = 3$ and $N(pq,k) = 5$ for any distinct prime integers p and q. However, in other case, it is hard to determine $N(v,k)$. Therefore, our aim of this section is to construct more examples of Gorenstein simplices of Problem 6.1 and to give a lower bound on $N(v,k)$.

The following theorem gives us more examples of Gorenstein simplces of Problem 6.1.

Theorem 6.11 ([52, Theorem 4.1]). *Given a positive integer v, let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d such that Λ_Δ is generated by*

$$\Big(\underbrace{1/v_1, \ldots, 1/v_1}_{s_1}, \underbrace{1/v_2, \ldots, 1/v_2}_{s_2}, \ldots, \underbrace{1/v_t, \ldots, 1/v_t}_{s_t}\Big) \in (\mathbb{R}/\mathbb{Z})^{d+1},$$

where $1 < v_1 < \cdots < v_t = v$ and for any $1 \leq i \leq t-1$, $v_i \mid v_{i+1}$ and $s_1, \ldots, s_t$ are positive integers. Then $\delta(\Delta, t) = 1 + t^k + t^{2k} + \cdots + t^{(v-1)k}$ with a positive integer k if and only *if*

$$s_i = \begin{cases} \left(\dfrac{v_t}{v_{i-1}} - \dfrac{v_t}{v_{i+1}}\right)k, & 1 \leq i \leq t-1 \\ \dfrac{v_t}{v_{t-1}}k, & i = t, \end{cases}$$

where $v_0 = 1$.

Proof. Let

$$\mathbf{x}_0 = \left(\underbrace{1/v_1,\dots,1/v_1}_{s_1},\underbrace{1/v_2,\dots,1/v_2}_{s_2},\dots,\underbrace{1/v_t,\dots,1/v_t}_{s_t}\right) \in (\mathbb{R}/\mathbb{Z})^{d+1},$$

and for $i = 1,\dots,t-1$, we set $\mathbf{x}_i = v_i\mathbf{x}_0$. Then it follows that

$$\Lambda_\Delta = \left\{\sum_{i=0}^{t-1} c_i\mathbf{x}_i : c_i \in \mathbb{Z}_{\geq 0}, 0 \leq c_i \leq v_{i+1}/v_i - 1 \text{ for } i = 0,\dots,t-1\right\}.$$

Moreover, we obtain $\mathrm{ht}(\mathbf{x}_i) = \sum_{j=1}^{t-i} \dfrac{v_i}{v_{i+j}} s_{i+j}$ for $i = 0,\dots,t-1$. Since

$$\mathrm{ht}(\mathbf{x}_i) = \mathrm{ht}\left(\frac{v_i}{v_{i-1}}\mathbf{x}_{i-1}\right) = \frac{v_i}{v_{i-1}}\mathrm{ht}(\mathbf{x}_{i-1}) - s_i$$

for any $1 \leq i \leq t-1$, it follows that for any $1 \leq i \leq t-1$, $s_i = \left(\dfrac{v_t}{v_{i-1}} - \dfrac{v_t}{v_{i+1}}\right)k$ and $s_t = \dfrac{v_t}{v_{t-1}}k$ if and only if for any $0 \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}k$. Hence we should prove that $\delta(\Delta,t) = 1 + t^k + t^{2k} + \cdots + t^{(v-1)k}$ if and only if for any $0 \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}k$.

At first, we assume that $\delta(\Delta,t) = 1 + t^k + t^{2k} + \cdots + t^{(v-1)k}$. By Lemma 6.10, one has $\mathrm{ht}(\mathbf{x}_{t-1}) = k$. Suppose that for any $n \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}k$ with an integer $1 \leq n \leq t-1$. Then since $\mathrm{ht}(\sum_{i=n}^{t-1}(v_{i+1}/v_i - 1)\mathbf{x}_i) = (v_t/v_n - 1)k$, there exists an integer m with $0 \leq m \leq n-1$ such that $\mathrm{ht}(\mathbf{x}_m) = \dfrac{v_t}{v_n}k$. Now, we assume that $m < n-1$. Set

$$\Lambda' = \left\{c_m\mathbf{x}_m + \sum_{i=n}^{t-1} c_i\mathbf{x}_i : 0 \leq c_i \leq v_{i+1}/v_i - 1 \text{ for } i = m,n,n+1,\dots,t-1\right\}.$$

Then one has $\{\mathrm{ht}(\mathbf{x}) : \mathbf{x} \in \Lambda'\} = \{j(k+1) : j = 0,\dots,(v_{m+1}v_t)/(v_mv_n) - 1\}$. However,

$$\mathrm{ht}(\mathbf{x}_{m+1}) = \mathrm{ht}\left(\frac{v_{m+1}}{v_m}\mathbf{x}_m\right) < \frac{v_{m+1}}{v_m}\mathrm{ht}(\mathbf{x}_m) = \left(\frac{v_{m+1}v_t}{v_mv_n}\right)k.$$

and $\mathbf{x}_{m+1}$ is not in Λ', a contradiction. Hence we obtain $\mathrm{ht}(\mathbf{x}_{i-1}) = \dfrac{v_t}{v_i}k$ for any $0 \leq i \leq t-1$.

Conversely, we assume that for any $0 \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}k$. Since for any c_i with $0 \leq c_i \leq v_{i+1}/v_i - 1$, $\mathrm{ht}(\sum_{i=0}^{t-1} c_i \mathbf{x}_i) = \sum_{i=0}^{t-1} c_i \mathrm{ht}(\mathbf{x}_i)$, one has $\delta(\Delta,t) = 1 + t^k + t^{2k} + \cdots + t^{(v-1)k}$, as desired. □

By Theorems 6.11, we can answer to Problem 6.1 when v is a power of a prime integer and the associated finite abelian group is cyclic, namely, it is generated by one element.

Corollary 6.12. *Let p be a prime integer, ℓ and k positive integers, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d such that Λ_Δ is cyclic and $\delta(\Delta,t) = 1 + t^k + t^{2k} + \cdots + t^{(p^\ell - 1)k}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then there exist positive integers $0 < \ell_1 < \cdots < \ell_t = \ell$ and $s_1, \ldots, s_t$ such that the following conditions are satisfied:*

- *It follows that*
$$s_i = \begin{cases} (p^{\ell-\ell_{i-1}} - p^{\ell-\ell_{i+1}})k, & 1 \leq i \leq t-1 \\ p^{\ell-\ell_{t-1}}k, & i = t, \end{cases}$$
where $\ell_0 = 0$;
- *Λ_Δ is generated by*
$$\Bigl(\underbrace{1/p^{\ell_1}, \ldots, 1/p^{\ell_1}}_{s_1}, \underbrace{1/p^{\ell_2}, \ldots, 1/p^{\ell_2}}_{s_2}, \ldots, \underbrace{1/p^{\ell_t}, \ldots, 1/p^{\ell_t}}_{s_t}\Bigr) \in (\mathbb{R}/\mathbb{Z})^{d+1}$$
for some ordering of the vertices of Δ.

Now, we consider to give a lower bound on $N(v,k)$. Given positive integers v and k, let $M(v,k)$ denote the number of Gorenstein simplices, up to unimodular equivalence, which are appeared in Theorem 6.11. Then one has $N(v,k) \geq M(v,k)$. By Theorem 6.11, we can determine $M(v,k)$ in terms of the divisor lattice of v. Given a positive integer v, let D_v the set of all divisors of v, ordered by divisibility. Then D_v is a partially ordered set, in particular, a lattice, called the *divisor lattice* of v. We call subset $C \subset D_v$ a *chain* of D_v if C is a totally ordered subset with respect to the induced order.

Corollary 6.13. *Let v and k be positive integers. Then $M(v,k)$ equals the number of chains from a non-least element to the greatest element in D_v. In particular, one has $M(v,k) = \sum_{n \in D_v \setminus \{v\}} M(n,k)$.*

This corollary says that $M(v,k)$ depends on only the divisor lattice D_v. In particular, letting $v = p_1^{a_1} \cdots p_t^{a_t}$ with distinct prime integers $p_1, \ldots, p_t$ and positive integers $a_1, \ldots, a_t$, $M(v,k)$ depends on only $(a_1, \ldots, a_t)$.

Finally, we give examples of $M(v,k)$.

Example 6.14. (1) Let $v = p^\ell$ with a prime integer p and a positive integer ℓ. Then from Corollary 6.13, we know that $M(v,k)$ equals the number of subsets of $\{1,\dots,\ell-1\}$. Hence one has $M(v,k) = 2^{\ell-1}$.

(2) Let $v = p_1 \cdots p_t$, where $p_1,\dots,p_t$ are distinct prime integers. From Corollary 6.13, we know that $M(v,k)$ depends on only t. Now, let $a(t) = M(v,k)$, where we define $a(0) = M(1,k) = 1$. Then one has

$$a(t) = M(v,k) = \sum_{n \in D_v \setminus \{v\}} M(n,k) = 1 + \sum_{i=1}^{t-1} \binom{t}{i} M(p_1 \cdots p_i, k) = \sum_{i=0}^{t-1} \binom{t}{i} a(i).$$

We remark that $a(t)$ is the well-known recursive sequence ([79, A000670]) which is called the *ordered Bell numbers* or *Fubini numbers*.

Chapter 7

Gorenstein polytopes with trinomial δ-polynomials

In this chapter, we finish the complete classification of all lattice polytopes that are not lattice pyramids and whose δ-polynomial is palindromic and has precisely three terms (Theorem 7.6). In the case of degree two, this was already done by Batyrev and Juny [8]. Here, we only consider the case when the degree is strictly larger than two. In this situation, the lattice polytope is necessarily an empty simplex, and we can apply methods and results of Batyrev and Hofscheier [6, 7]. Since the precise formulation of Theorem 7.6 needs some more notation, let us describe here only two immediate consequences. First, the complete characterization of palindromic δ-trinomials:

Corollary 7.1. *Let $d \geq 2$, $m \geq 2$ and $k \geq 1$ be integers. The polynomial $1+(m-2)t^k+t^{2k}$ is the δ-polynomial of a lattice polytope of dimension d if and only if the integers k, m, d satisfy one of the following conditions:*

(1) $k = 1$, $3 \leq m \leq 9$ *and* $d = 2$;

(2) $k = 1$, $2 \leq m \leq 9$ *and* $d \geq 3$;

(3) $k \geq 2$, $m \in \{3, 4, 6, 8\}$ *and* $d \geq 3k-1$;

(4) $k = 2^{\ell-3}a$, $m = 2^{\ell}$ *and* $d \geq 4k-1$, *where* $a \geq 1$ *and* $\ell \geq 4$;

(5) $k = 3^{\ell-2}a$, $m = 3^{\ell}$ *and* $d \geq 3k-1$, *where* $a \geq 1$ *and* $\ell \geq 3$.

The case $k = 1$ was already known, as described in Section 2.6.

Secondly, Theorem 7.6 implies the following uniqueness result:

Corollary 7.2. *A lattice simplex Δ that is not a lattice pyramid is uniquely determined by its dimension and its δ-polynomial if it is of the form $1+(m-2)t^k+t^{2k}$ with $k \geq 2$.*

Let us note that for $k \geq 2$ any of these lattice simplices that are not lattice pyramids have dimension $3k-1$ or $4k-1$, see Theorem 7.6.

This chapter is organized as follows. In Section 7.1 we recall the notation and results by Batyrev and Hofscheier. In Section 7.2 we present and prove the main result of this chapter (Theorem 7.6): the classification of Gorenstein polytopes with δ-trinomials of degree ≥ 3. Finally, in Section 7.3, we discuss a future problem.

7.1 The approach by Batyrev and Hofscheier

In this section, we summarize results by Batyrev and Hofscheier from [6] and [7] which play a crucial role in our proof of Theorem 7.6.

First, let us describe their generalization of White's theorem.

Theorem 7.3 ([6]). *Let $k \geq 2$ and let Δ be a lattice simplex of dimension $2k-1$ with $\mathrm{Vol}(\Delta) = m$ which is not a lattice pyramid. Then the following statements are equivalent:*

(a) *the δ-polynomial of Δ is $1+(m-1)t^k$;*

(b) *Δ is isomorphic to the Cayley polytope $\Delta_1 * \cdots * \Delta_k$ of empty simplices $\Delta_i \subset \mathbb{R}^k$ of dimension 1;*

(c) *Λ_Δ is cyclic and generated by $(a_1/m, (m-a_1)/m, \ldots, a_k/m, (m-a_k)/m) \in (\mathbb{R}/\mathbb{Z})^{2k}$ after reordering, where each $0 < a_i \leq m/2$ is an integer which is coprime to m.*

Batyrev and Hofscheier use the language of linear codes to consider the case $d > 2k-1$. A *linear code* over $\mathbb{F}_p$ with *block length* n is a subspace L of the finite vector space $\mathbb{F}_p^n$ (where p is a prime). $A \in \mathbb{F}_p^{r\times n}$ (an $r \times n$ matrix with entries in $\mathbb{F}_p$) is the *generator matrix* of such an r-dimensional linear code L if the rows of A form a basis of L.

Definition 7.4. Fix a natural number r and a prime number p. Let $n = (p^r-1)/(p-1)$ be the number of points in $(r-1)$-dimensional projective space over $\mathbb{F}_p$. Consider the $r \times n$ matrix $A \in \mathbb{F}_p^{r\times n}$ whose columns consist of nonzero vectors from each 1-dimensional subspace of $\mathbb{F}_p^r$. Then A is the generator matrix of the *simplex code* of dimension r over $\mathbb{F}_p$ with block length n.

Theorem 7.5 ([7]). *Let $d \geq 3$ and let Δ be a lattice simplex of dimension d which is not a lattice pyramid. Let the δ-polynomial of Δ be $1+(m-1)t^k$ for some $m \geq 2$ and $1 < k < (d+1)/2$. Then there exists a prime number p such that every non-trivial element of Λ_Δ has order p. In particular, Λ_Δ can be identified with $p\Lambda_\Delta \subseteq \{0,\ldots,p-1\}^{d+1}$, a linear code over $\mathbb{F}_p$ with block length $d+1$. The order m of Λ_Δ is equal to p^r, where* *the positive integer r is the dimension of the linear code $p\Lambda_\Delta$. The numbers p,d,k,r are related by the equation*

$$(p^r - p^{r-1})(d+1) = 2k(p^r - 1). \tag{7.1}$$

A generator matrix of the linear code $p\Lambda_\Delta$ is given (up to permutation of the columns) by the rows in the following $r \times (d+1)$ matrix:

$$(A,\dots,A) \text{ if } p = 2, \text{ or } (A,-A,\dots,A,-A) \text{ if } p > 2,$$

where A is the generator matrix of the r-dimensional simplex code over $\mathbb{F}_p$ and A (resp. the pair $(A,-A)$) is repeated $k/2^{r-2}$ (resp. k/p^{r-1}) times if $p = 2$ (resp. if $p > 2$).

Let us note that also the converse of the theorem holds, so the linear codes defined by the generator matrices given in the theorem correspond to lattice simplices with δ-polynomial $1+(p^r-1)t^k$ if the numerical condition (7.1) holds, see also [7, Proposition 5.2].

7.2 The classification of lattice polytopes with palindromic δ-trinomials

If B is a matrix, we denote by $(B,0)$ the matrix with one additional zero column. The following is the main result in this chapter.

Theorem 7.6 ([54, Theorem 3.1])**.** *Let $m \geq 3$ and $k \geq 2$ be integers and let Δ be a (necessarily empty) lattice simplex of dimension d whose δ-polynomial is $1+(m-2)t^k+t^{2k}$. Assume that Δ is not a lattice pyramid over any lower-dimensional simplex. Then the integers k,m,d satisfy one of the following:*

(a) *$m \in \{3,4,6,8\}$ and $d = 3k-1$ or $m = 4$ and $d = 4k-1$;*

(b) *$k = 2^{\ell-3}a$, $m = 2^\ell$ and $d = 2^{\ell-1}a-1$, where $a \geq 1$ and $\ell \geq 3$ with $(a,\ell) \neq (1,3)$;*

(c) *$k = 3^{\ell-2}a$, $m = 3^\ell$ and $d = 3^{\ell-1}a-1$, where $a \geq 1$ and $\ell \geq 2$ with $(a,\ell) \neq (1,2)$.*

Moreover, in each case, a system of generators of the finite abelian group Λ_Δ is the set of row vectors of the matrix which can be written up to permutation of the columns as follows:

(a)

$$\begin{array}{ll}
(1/3\ 1/3\ \cdots\ 1/3) \in (\mathbb{R}/\mathbb{Z})^{1\times 3k} & \text{in the case } m=3;\\
\left(\underbrace{1/4\ \cdots\ 1/4}_{2k}\ \underbrace{1/2\ \cdots\ 1/2}_{k}\right) \in (\mathbb{R}/\mathbb{Z})^{1\times 3k} & \text{in the case } m=4 \text{ with } d=3k-1;\\
\left(\begin{array}{c} \underbrace{\begin{matrix} 1/2 & \cdots & 1/2 \\ 1/2 & \cdots & 1/2 \end{matrix}}_{2k} \end{array} \begin{array}{c} \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 1/2 & \cdots & 1/2 \end{matrix}}_{2k} \end{array}\right) \in (\mathbb{R}/\mathbb{Z})^{2\times 4k} & \text{in the case } m=4 \text{ with } d=4k-1;\\
\left(\underbrace{1/6\ \cdots\ 1/6}_{k}\ \underbrace{1/3\ \cdots\ 1/3}_{k}\ \underbrace{1/2\ \cdots\ 1/2}_{k}\right) \in (\mathbb{R}/\mathbb{Z})^{1\times 3k} & \text{in the case } m=6;\\
\left(\begin{array}{c} \underbrace{\begin{matrix} 1/2 & \cdots & 1/2 \\ 1/4 & \cdots & 1/4 \end{matrix}}_{k} \end{array} \begin{array}{c} \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 1/4 & \cdots & 1/4 \end{matrix}}_{k} \end{array} \begin{array}{c} \underbrace{\begin{matrix} 1/2 & \cdots & 1/2 \\ 1/2 & \cdots & 1/2 \end{matrix}}_{k} \end{array}\right) \in (\mathbb{R}/\mathbb{Z})^{2\times 3k} & \text{in the case } m=8.
\end{array}$$

(b)

$$\begin{pmatrix} (B^{(2)}_{\ell-1},0) & (B^{(2)}_{\ell-1},0) & \cdots & (B^{(2)}_{\ell-1},0) \\ 1/2 & \cdots & \cdots & 1/2 \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{\ell\times 2^{\ell-1}a},$$

where $A^{(2)}_{\ell-1} \in \{0,1\}^{(\ell-1)\times(2^{\ell-1}-1)}$ *is the generator matrix of the simplex code over* $\mathbb{F}_2$ *of dimension* $(\ell-1)$ *with block length* $(2^{\ell-1}-1)$ *and* $B^{(2)}_{\ell-1} \in \{0,1/2\}^{(\ell-1)\times(2^{\ell-1}-1)}$ *is the matrix all of whose entries are divided by* 2 *from those of* $A^{(2)}_{\ell-1}$*, and where in above matrix* $(B^{(2)}_{\ell-1},0) \in \{0,1/2\}^{(\ell-1)\times 2^{\ell-1}}$ *is repeated a times.*

(c)

$$\begin{pmatrix} (B^{(3)}_{\ell-1},-B^{(3)}_{\ell-1},0) & (B^{(3)}_{\ell-1},-B^{(3)}_{\ell-1},0) & \cdots & (B^{(3)}_{\ell-1},-B^{(3)}_{\ell-1},0) \\ 1/3 & \cdots & \cdots & 1/3 \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{\ell\times 3^{\ell-1}a},$$

where $A^{(3)}_{\ell-1} \in \{0,1,2\}^{(\ell-1)\times(3^{\ell-1}-1)/2}$ *is the generator matrix of the simplex code over* $\mathbb{F}_3$ *of dimension* $(\ell-1)$ *with block length* $(3^{\ell-1}-1)/2$ *and* $B^{(3)}_{\ell-1} \in \{0,1/3,2/3\}^{(\ell-1)\times(3^{\ell-1}-1)/2}$ *(resp.* $-B^{(3)}_{\ell-1} \in \{0,2/3,1/3\}^{(\ell-1)\times(3^{\ell-1}-1)/2}$*) is the matrix all of whose entries are divided by* 3 *from those of* $A^{(3)}_{\ell-1}$ *(resp.* $-A^{(3)}_{\ell-1}$*), and where in above matrix* $(B^{(3)}_{\ell-1},-B^{(3)}_{\ell-1},0) \in \{0,1/3,2/3\}^{(\ell-1)\times 3^{\ell-1}}$ *is repeated a times.*

Example 7.7. In case (b) for $k=2$ and $\ell=3$ the rows of the following matrix generate Λ_Δ of size $m=8$:

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$

In case (c) for $k = 2$ and $\ell = 2$ the rows of the following matrix generate Λ_Δ of size $m = 9$:

$$\begin{pmatrix} 1/3 & 2/3 & 0 & 1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 1/3 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$

In this section, we prove Theorem 7.6.

7.2.1 Preliminary results

For the proof of Theorem 7.6, we prepare some lemmas. Throughout this section, let Δ be a lattice simplex of dimension d whose δ-polynomial equals $1+(m-2)t^k+t^{2k}$ with $k \geq 2$ and $m \geq 3$. Note that Δ is necessarily empty.

For $\mathbf{x} = (x_0,\dots,x_d) \in \Lambda_\Delta$, let $\mathrm{supp}(\mathbf{x}) = \{i : x_i \neq 0\}$. The following equality will be used throughout:

$$|\mathrm{supp}(\mathbf{x})| = \mathrm{ht}(\mathbf{x}) + \mathrm{ht}(-\mathbf{x}).$$

Lemma 7.8. *Let $\mathbf{x} \in \Lambda_\Delta$ be an element whose order is n and let $1 \leq j \leq n-1$ be coprime to n. Then we have $\mathrm{supp}(\mathbf{x}) = \mathrm{supp}(j\mathbf{x})$. Hence,*

$$\mathrm{ht}(\mathbf{x}) + \mathrm{ht}((n-1)\mathbf{x}) = \mathrm{ht}(j\mathbf{x}) + \mathrm{ht}((n-j)\mathbf{x}).$$

Proof. Let $i \in \mathrm{supp}(x)$, $x_i = \frac{a}{b} \neq 0$ with $\gcd(a,b) = 1$. By the definition of n, we observe that b divides an, so also n. Hence, $\gcd(b,j) = 1$. Therefore, b does not divide ja, so $i \in \mathrm{supp}(jx)$. □

Lemma 7.9. *Let $\mathbf{x} \in \Lambda_\Delta$ be the unique element with $\mathrm{ht}(\mathbf{x}) = 2k$. Then,*

(a) *for any $\mathbf{y} = (y_0,\dots,y_d) \in \Lambda_\Delta \setminus \{\mathbf{0}, \pm\mathbf{x}\}$, we have $|\mathrm{supp}(\mathbf{y})| = 2k$;*

(b) *there is no integer j and $\mathbf{y} \in \Lambda_\Delta \setminus \{\mathbf{0}, \pm\mathbf{x}\}$ such that $\mathbf{x} = j\mathbf{y}$.*

Proof. (a) Since $\mathrm{ht}(\mathbf{y}) = k$, $\mathrm{ht}(-\mathbf{y}) = k$, and $\mathbf{y} \neq \mathbf{x} \neq -\mathbf{y}$, we have

$$2k = \mathrm{ht}(\mathbf{y}) + \mathrm{ht}(-\mathbf{y}) = |\mathrm{supp}(\mathbf{y})|.$$

(b) For any integer j and $y \in \Lambda_\Delta \setminus \{\mathbf{0}, \pm\mathbf{x}\}$, since $|\operatorname{supp}(\mathbf{y})| = 2k$ by (a), we have $|\operatorname{supp}(j\mathbf{y})| \leq 2k$. However, by $|\operatorname{supp}(\mathbf{x})| > \operatorname{ht}(\mathbf{x}) = 2k$, $\mathbf{x} = j\mathbf{y}$ never happens. □

The following proposition is crucial for the proof of Theorem 7.6.

Proposition 7.10. *Let Δ be a lattice simplex which is not a lattice pyramid whose δ-polynomial is $1+(m-2)t^k+t^{2k}$ with $m \geq 3$ and $k \geq 2$. Let $\mathbf{x} \in \Lambda_\Delta$ be the unique element with $\operatorname{ht}(\mathbf{x}) = 2k$. Then the order of $\mathbf{x}$ must be 2 or 3 or 4 or 6, and up to permutation of coordinates x is given as follows:*

- $\mathbf{x} = (1/2,\dots,1/2) \in (\mathbb{R}/\mathbb{Z})^{4k}$ *when its order is 2;*
- $\mathbf{x} = (2/3,\dots,2/3) \in (\mathbb{R}/\mathbb{Z})^{3k}$ *when its order is 3;*
- $\mathbf{x} = (\underbrace{3/4,\dots,3/4}_{2k},\underbrace{1/2,\dots,1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ *when its order is 4;*
- $\mathbf{x} = (\underbrace{5/6,\dots,5/6}_{k},\underbrace{2/3,\dots,2/3}_{k},\underbrace{1/2,\dots,1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ *when its order is 6.*

In particular, the dimension of Δ is $4k-1$ if the order of $\mathbf{x}$ is 2 and $3k-1$ otherwise.

Proof. Let $m' \geq 2$ be the order of $\mathbf{x}$. Suppose that $m' = 5$ or $m' \geq 7$. Then $\varphi(m') > 2$, where φ is the Eulerian φ-function. In particular, there exists an integer $2 \leq j \leq m'-2$ which is coprime to m'. By Lemma 7.8 and $-\mathbf{x} \neq \mathbf{x}$, we obtain

$$3k = \operatorname{ht}(\mathbf{x}) + \operatorname{ht}((m'-1)\mathbf{x}) = \operatorname{ht}(j\mathbf{x}) + \operatorname{ht}((m'-j)\mathbf{x}) = 2k,$$

implying that $k = 0$, a contradiction. Thus, $m' \leq 6$ and $m' \neq 5$. Hence, $m' \in \{2,3,4,6\}$.

<u>$m' = 2$</u>: Then each x_i is $1/2$ or 0. From $\operatorname{ht}(\mathbf{x}) = 2k$, we have $\mathbf{x} = (\underbrace{1/2,\dots,1/2}_{4k},\underbrace{0,\dots,0}_{s})$ after reordering. Fix $\mathbf{y} \in \Lambda_\Delta \setminus \{\mathbf{0},\mathbf{x}\}$ and let $q = |(\operatorname{supp}(\mathbf{y})| \setminus \operatorname{supp}(\mathbf{x}))$. Since $|\operatorname{supp}(\mathbf{x}+\mathbf{y})| = 2k$ by Lemma 7.9 (a), we have $2k = |\operatorname{supp}(\mathbf{x}+\mathbf{y})| = 4k-k'+q$, where $k' = |\{i \in \operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y}) : y_i = 1/2\}|$. Hence, $k'-q = 2k$. On the other hand, since $|\operatorname{supp}(\mathbf{y})| = 2k$, we also have $k'+q \leq 2k$. Thus, $q \leq 0$, i.e., $q = 0$. This means that $\operatorname{supp}(\mathbf{y}) \subset \operatorname{supp}(\mathbf{x})$. Hence, if $s > 0$, then Δ is a lattice pyramid by Lemma 2.8, a contradiction. Thus $s = 0$ and we conclude that $\mathbf{x} = (1/2,\dots,1/2) \in (\mathbb{R}/\mathbb{Z})^{4k}$.

<u>$m' = 3$</u>: Then each x_i is $1/3$ or $2/3$ or 0. It follows from $\operatorname{ht}(\mathbf{x}) = 2k$ and $\operatorname{ht}(-\mathbf{x}) = k$ that $\mathbf{x} = (\underbrace{2/3,\dots,2/3}_{3k},\underbrace{0,\dots,0}_{s})$ after reordering. Fix $\mathbf{y} \in \Lambda_\Delta \setminus \{\mathbf{0},\pm\mathbf{x}\}$ and let $q = |(\operatorname{supp}(\mathbf{y})| \setminus \operatorname{supp}(\mathbf{x}))$. Since $|\operatorname{supp}(\mathbf{x}+\mathbf{y})| = 2k$, we have $2k = |\operatorname{supp}(\mathbf{x}+\mathbf{y})| = 3k - k_1 + q$, where $k_1 = |\{i \in \operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y}) : y_i = 1/3\}|$. Hence, $k_1 - q = k$. Similarly, since $|\operatorname{supp}(2\mathbf{x}+\mathbf{y})| = 2k$, we have $2k = |\operatorname{supp}(2\mathbf{x}+\mathbf{y})| = 3k - k_2 + q$, where $k_2 =$

$|\{i \in \mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y}) : y_i = 2/3\}|$. Hence, $k_2 - q = k$. On the other hand, since $|\mathrm{supp}(\mathbf{y})| = 2k$, we also have $k_1 + k_2 + q \leq 2k = k_1 + k_2 - 2q$. Thus, $q \leq 0$, i.e., $q = 0$, implying that $s = 0$. Hence we conclude that $\mathbf{x} = (2/3, \ldots, 2/3) \in (\mathbb{R}/\mathbb{Z})^{3k}$.

$\underline{m' = 4}$: Then each x_i is $1/4$ or $1/2$ or $3/4$ or 0. For $j = 1,2,3$, let $q_j = |\{i : x_i = j/4\}|$. Since $\mathrm{ht}(\mathbf{x}) = (q_1 + 2q_2 + 3q_3)/4 = 2k$, $\mathrm{ht}(2\mathbf{x}) = (q_1 + q_3)/2 = k$ and $\mathrm{ht}(3\mathbf{x}) = (3q_1 + 2q_2 + q_3)/4 = k$, we obtain $q_1 = 0$, $q_2 = k$ and $q_3 = 2k$, that is,

$$\mathbf{x} = (\underbrace{3/4, \ldots, 3/4}_{2k}, \underbrace{1/2, \ldots, 1/2}_{k}, \underbrace{0, \ldots, 0}_{s})$$

after reordering. Fix $\mathbf{y} \in \Lambda_\Delta \setminus \{j\mathbf{x} : j = 0,1,2,3\}$ and let $q = |(\mathrm{supp}(\mathbf{y}) \setminus \mathrm{supp}(\mathbf{x}))|$. Let $k_j = |\{i \in \mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y}) : x_i = 3/4, y_i = j/4\}|$ for $j = 1,2,3$ and let $k' = |\{i \in \mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{y}) : x_i = y_i = 1/2\}|$. Since $|\mathrm{supp}(\mathbf{x}+\mathbf{y})| = |\mathrm{supp}(2\mathbf{x}+\mathbf{y})| = |\mathrm{supp}(3\mathbf{x}+\mathbf{y})| = 2k$, we have the following:

- $2k = |\mathrm{supp}(\mathbf{x}+\mathbf{y})| = 2k - k_1 + k - k' + q$, i.e., $k + q = k_1 + k'$;
- $2k = |\mathrm{supp}(2\mathbf{x}+\mathbf{y})| \geq 2k - k_2 + k' + q$, i.e., $k' + q \leq k_2$;
- $2k = |\mathrm{supp}(3\mathbf{x}+\mathbf{y})| = 2k - k_3 + k - k' + q$, i.e., $k + q = k_3 + k'$.

In particular, we have $2k + 3q \leq k_1 + k_2 + k_3 + k'$. On the other hand, since $|\mathrm{supp}(\mathbf{y})| = 2k$, we have $k_1 + k_2 + k_3 + k' + q \leq 2k$. Thus we obtain

$$2k + 4q \leq k_1 + k_2 + k_3 + k' + q \leq 2k.$$

This means $q = 0$, and thus, $s = 0$. Hence we conclude that $\mathbf{x} = (\underbrace{3/4, \ldots, 3/4}_{2k}, \underbrace{1/2, \ldots, 1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ after reordering.

$\underline{m' = 6}$: Then each x_i is $1/6, 1/3, 1/2, 2/3, 5/6$ or 0. For $j = 1,2,3,4,5$, let $q_j = |\{i : x_i = j/6\}|$. Then

$$\begin{aligned}
\mathrm{ht}(\mathbf{x}) &= (q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5)/6 = 2k, \\
\mathrm{ht}(2\mathbf{x}) &= (q_1 + 2q_2 + q_4 + 2q_5)/3 = k, \\
\mathrm{ht}(3\mathbf{x}) &= (q_1 + q_3 + q_5)/2 = k, \\
\mathrm{ht}(4\mathbf{x}) &= (2q_1 + q_2 + 2q_4 + q_5)/3 = k \text{ and} \\
\mathrm{ht}(5\mathbf{x}) &= (5q_1 + 4q_2 + 3q_3 + 2q_4 + q_5)/6 = k.
\end{aligned}$$

Thus $q_1 = q_2 = 0$ and $q_3 = q_4 = q_5 = k$, that is,

$$\mathbf{x} = (\underbrace{5/6, \ldots, 5/6}_{k}, \underbrace{2/3, \ldots, 2/3}_{k}, \underbrace{1/2, \ldots, 1/2}_{k}, \underbrace{0, \ldots, 0}_{s})$$

after reordering. Fix $\mathbf{y} \in \Lambda_\Delta \setminus \{j\mathbf{x} : j = 0,1,2,3,4,5\}$ and let $q = |(\text{supp}(\mathbf{y}) \setminus \text{supp}(\mathbf{x}))|$. Let

$$\begin{aligned} k_j &= |\{i \in \text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y}) : x_i = 5/6, y_i = j/6\}| \text{ for } j = 1,2,3,4,5, \\ k'_j &= |\{i \in \text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y}) : x_i = 2/3, y_i = j/3\}| \text{ for } j = 1,2, \\ k'' &= |\{i \in \text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y}) : x_i = y_i = 1/2\}|. \end{aligned}$$

Since $|\text{supp}(j\mathbf{x}+\mathbf{y})| = 2k$ for $j = 1,2,3,4,5$, we have the following:

- $2k = |\text{supp}(\mathbf{x}+\mathbf{y})| = k - k_1 + k - k'_1 + k - k'' + q$, i.e., $k + q = k_1 + k'_1 + k''$;
- $2k = |\text{supp}(2\mathbf{x}+\mathbf{y})| \geq k - k_2 + k - k'_2 + k'' + q$, i.e., $k'' + q \leq k_2 + k'_2$;
- $2k = |\text{supp}(3\mathbf{x}+\mathbf{y})| \geq k - k_3 + k'_1 + k'_2 + k - k'' + q$, i.e., $k'_1 + k'_2 + q \leq k_3 + k''$;
- $2k = |\text{supp}(4\mathbf{x}+\mathbf{y})| \geq k - k_4 + k - k'_1 + k'' + q$, i.e., $k'' + q \leq k_4 + k'_1$;
- $2k = |\text{supp}(5\mathbf{x}+\mathbf{y})| = k - k_5 + k - k'_2 + k - k'' + q$, i.e., $k + q = k_5 + k'_2 + k''$.

By summing up these five inequalities, we have $2k + 5q \leq k_1 + \cdots + k_5 + k'_1 + k'_2 + k''$. On the other hand, since $|\text{supp}(\mathbf{y})| = 2k$, we have $k_1 + \cdots + k_5 + k'_1 + k'_2 + k'' + q \leq 2k$. Thus we obtain

$$2k + 6q \leq k_1 + \cdots + k_5 + k'_1 + k'_2 + k'' + q \leq 2k.$$

This means $q = 0$, and thus, $s = 0$. Hence we conclude that

$$\mathbf{x} = (\underbrace{5/6,\ldots,5/6}_{k}, \underbrace{2/3,\ldots,2/3}_{k}, \underbrace{1/2,\ldots,1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$$

after reordering. □

As a corollary of this proposition, we obtain the following:

Corollary 7.11. *Let $m \geq 3$ and $k \geq 2$ be integers. Let Δ be a lattice polytope with $\delta(\Delta,t) = 1 + (m-2)t^k + t^{2k}$. Assume that Λ_Δ is a cyclic group. Then m must be 3 or 4 or 6. Moreover, the generator of Λ_Δ looks as follows:*

- $(1/3,\ldots,1/3) \in (\mathbb{R}/\mathbb{Z})^{3k}$ *or its inverse when $m = 3$;*
- $(\underbrace{1/4,\ldots,1/4}_{2k}, \underbrace{1/2,\ldots,1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ *or its inverse when $m = 4$;*
- $(\underbrace{1/6,\ldots,1/6}_{k}, \underbrace{1/3,\ldots,1/3}_{k}, \underbrace{1/2,\ldots,1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ *or its inverse when $m = 6$.*

Proof. Let $\mathbf{x} \in \Lambda_\Delta$ be the unique element with $\mathrm{ht}(\mathbf{x}) = 2k$. By Lemma 7.9 (b), $\mathbf{x}$ and its inverse must be a generator of Λ_Δ. On the other hand, by Proposition 7.10 and $m \geq 3$, m is 3 or 4 or 6. The form of $\mathbf{x}$ follows directly from Proposition 7.10. □

7.2.2 Proof of Theorem 7.6

Let Δ be an empty simplex whose δ-polynomial equals $1 + (m-2)t^k + t^{2k}$ for given integers $m \geq 3$ and $k \geq 2$.

By Corollary 7.11 we can assume that Λ_Δ is not cyclic. Namely, we assume that there is a group isomorphism

$$\varphi : \Lambda_\Delta \to \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_\ell\mathbb{Z},$$

where $\ell \geq 2$, $m_i \in \mathbb{Z}_{\geq 2}$ and m_i divides m_{i+1} for each $1 \leq i \leq \ell - 1$.

Let $\mathbf{x} \in \Lambda_\Delta$ be the unique element with $\mathrm{ht}(\mathbf{x}) = 2k$. Then there is $x^{(i)} \in \mathbb{Z}/m_i\mathbb{Z}$ for each $1 \leq i \leq \ell$ such that $\varphi(\mathbf{x}) = (x^{(1)}, \ldots, x^{(\ell)}) \in \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_\ell\mathbb{Z}$. Let $S = \{i \in \{1, \ldots, \ell\} : x^{(i)} \neq 0\}$. Then $S \neq \emptyset$.

We will split the proof into two cases.

7.2.3 The case $\ell \geq 3$

First, we consider the case $\ell \geq 3$.

Assume that $|S| > 1$. Then there are q and q' in S such that $q \neq q'$. Let

$$G = \varphi^{-1}(\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_{q-1}\mathbb{Z} \times \{0\} \times \mathbb{Z}/m_{q+1}\mathbb{Z} \times \cdots \times \mathbb{Z}/m_\ell\mathbb{Z}).$$

Then G is a subgroup of Λ_Δ not containing $\mathbf{x}$. Let $\Delta_G \subset \mathbb{R}^d$ be a lattice simplex such that $\Lambda_{\Delta_G} = G$. Since we have $\mathrm{ht}(y) = k$ for each $y \in G \setminus \{\mathbf{0}\}$, the δ-polynomial of Δ_G equals $1 + (|G| - 1)t^k$. Moreover, since $\ell \geq 3$, G is not cyclic. Although Δ_G might be a lattice pyramid, the structure of Δ_G (equivalently, G) is known by Theorem 7.3 or 7.5. Since G is not cyclic, Δ_G is the case of Theorem 7.5. In particular, there are a prime number p and a positive integer r such that $G \cong (\mathbb{Z}/p\mathbb{Z})^r$. Hence, $m_1 = \cdots = m_{q-1} = m_{q+1} = \cdots = m_\ell = p$ and $r = \ell - 1$. Similarly, let

$$G' = \varphi^{-1}(\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_{q'-1}\mathbb{Z} \times \{0\} \times \mathbb{Z}/m_{q'+1}\mathbb{Z} \times \cdots \times \mathbb{Z}/m_\ell\mathbb{Z}).$$

Then the same discussion as above shows that there is a prime number p' such that $m_1 = \cdots = m_{q'-1} = m_{q'+1} = \cdots = m_\ell = p'$. Since $\ell \geq 3$ and $q \neq q'$, we conclude that

$m_1 = \cdots = m_\ell = p(= p')$, that is, $\Lambda_\Delta \cong (\mathbb{Z}/p\mathbb{Z})^\ell$. Moreover, since the order of $\mathbf{x} \in \Lambda_\Delta$ is 2 or 3 or 4 or 6 by Proposition 7.10, p should be 2 or 3. Therefore,

$$\Lambda_\Delta \cong (\mathbb{Z}/2\mathbb{Z})^\ell \text{ or } \Lambda_\Delta \cong (\mathbb{Z}/3\mathbb{Z})^\ell.$$

In each case, there is another isomorphism $\varphi' : \Lambda_\Delta \to (\mathbb{Z}/b\mathbb{Z})^\ell$, where $b = 2$ or $b = 3$, such that $\varphi'(\mathbf{x}) = (0, \ldots, 0, 1) \in (\mathbb{Z}/b\mathbb{Z})^\ell$.

Hence, we can assume the case $|S| = 1$. By Lemma 7.9 (b) and $|S| = 1$, $\varphi(\mathbf{x})$ generates one direct factor of $\varphi(\Lambda_\Delta)$ and so does $\varphi(-\mathbf{x})$. For the remaining direct factors, the same discussions as above can be applied. Therefore, $\varphi(\Lambda_\Delta)$ must be one of the following (non-cyclic) groups:

(i) $(\mathbb{Z}/2\mathbb{Z})^\ell$;

(ii) $(\mathbb{Z}/2\mathbb{Z})^{\ell-1} \times \mathbb{Z}/4\mathbb{Z}$;

(iii) $(\mathbb{Z}/2\mathbb{Z})^{\ell-1} \times \mathbb{Z}/6\mathbb{Z}$;

(iv) $(\mathbb{Z}/3\mathbb{Z})^\ell$;

(v) $(\mathbb{Z}/3\mathbb{Z})^{\ell-1} \times \mathbb{Z}/6\mathbb{Z}$.

Here, we assume that $\varphi(\mathbf{x})$ belongs to the last direct factor.

By the discussions below, we verify the cases (i) and (iv) can happen but the cases (ii), (iii) and (v) never happen.

The case (i): Let us consider the subgroup $G' = \varphi^{-1}((\mathbb{Z}/2\mathbb{Z})^{\ell-1} \times \{0\})$ of Λ_Δ, where $\mathbf{x} \notin G'$. Then it follows that we have $\mathrm{ht}(\mathbf{y}) = k$ for each $\mathbf{y} \in G' \setminus \{\mathbf{0}\}$. By Theorem 7.5, we know the system of generator of G' as follows: let Δ' be the lattice simplex of dimension $d' \leq d$ which is not a lattice pyramid such that $\Lambda_{\Delta'} = G'$ after taking $(d-d')$-repeated lattice pyramids. Then the system of generators of $\Lambda_{\Delta'}$ is the set of the row vectors of the matrix

$$(B^{(2)}_{\ell-1}, \ldots, B^{(2)}_{\ell-1}),$$

where $A^{(2)}_{\ell-1} \in \{0,1\}^{(\ell-1)\times(2^{\ell-1}-1)}$ is the generator matrix of the simplex code over $\mathbb{F}_2$ of dimension $(\ell-1)$ with block length $(2^{\ell-1}-1)$ and $B^{(2)}_{\ell-1} \in \{0, \frac{1}{2}\}^{(\ell-1)\times(2^{\ell-1}-1)}$ is the matrix all of whose entries are divided by 2 from those of $A^{(2)}_{\ell-1}$, and where $B^{(2)}_{\ell-1} \in \{0, \frac{1}{2}\}^{(\ell-1)\times 2^{\ell-1}}$ is repeated $k/2^{\ell-3}$ times.

Let $a = k/2^{\ell-3}$. Then $k = 2^{\ell-3}a$ and $a \geq 1$. By Theorem 7.5, we know the relation

$$2^{\ell-2}(d'+1) = 2k(2^{\ell-1}-1) = 2^{\ell-2}a(2^{\ell-1}-1).$$

Thus $d'+1 = a(2^{\ell-1}-1)$.

On the other hand, since the order of $\mathbf{x}$ is 2 in this case, we have $|\text{supp}(\mathbf{x})| = 4k = d+1$ by Proposition 7.10. Therefore,

$$d+1-(d'+1) = 2^{\ell-1}a - a(2^{\ell-1}-1) = a.$$

Consequently, in this case, we have $m = 2^\ell$, $k = 2^{\ell-3}a$ and $d = 4k-1 = 2^{\ell-1}a-1$ with $a \geq 1$ and $\ell \geq 3$ and the system of generators of Λ_Δ is the set of row vectors of the matrix

$$\begin{pmatrix} (B^{(2)}_{\ell-1}, 0) & (B^{(2)}_{\ell-1}, 0) & \cdots & (B^{(2)}_{\ell-1}, 0) \\ 1/2 & 1/2 & \cdots & 1/2 \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{\ell \times 4k}$$

up to permutation of the columns. This is the case (b) of Theorem 7.6.

The cases (ii) and (iii): Let G' be the same thing as the case (i) above.

Since the order of $\mathbf{x}$ is 4 or 6, we have $d+1 = 3k$ by Proposition 7.10. Take $\mathbf{y} \in G' \setminus \{\mathbf{0}\}$. Since the order of $\mathbf{y}$ is 2, we have $\mathbf{y} = (\underbrace{1/2, \ldots, 1/2}_{2k}, \underbrace{0, \ldots, 0}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ after reordering. By $|\text{supp}(\mathbf{x}+\mathbf{y})| = 2k$, $|\{i \in \text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y}) : x_i = 1/2\}|$ should be k. Similarly, for $\mathbf{y}' \in G' \setminus \{\mathbf{0}\}$ with $\mathbf{y} \neq \mathbf{y}'$, one has $|\{i \in \text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{y}') : x_i = 1/2\}| = k$. Recall that $|\{i \in \text{supp}(\mathbf{x}) : x_i = 1/2\}| = k$ by Proposition 7.10. Thus, $|\text{supp}(\mathbf{x}+\mathbf{y}+\mathbf{y}')| = 3k$, a contradiction.

The case (iv): Let us consider the subgroup $G' = \varphi^{-1}((\mathbb{Z}/3\mathbb{Z})^{\ell-1} \times \{0\})$ of Λ_Δ, where $\mathbf{x} \notin G'$. Let Δ' be a lattice simplex of dimension $d' \leq d$ which is not a lattice simplex such that $\Lambda_{\Delta'} = G'$ after taking $(d-d')$-repeated lattice pyramids. Then the system of generators of $\Lambda_{\Delta'}$ is the set of the row vectors of the matrix

$$((B^{(3)}_{\ell-1}, -B^{(3)}_{\ell-1}), \ldots, (B^{(3)}_{\ell-1}, -B^{(3)}_{\ell-1})),$$

where $A^{(3)}_{\ell-1} \in \{0,1,2\}^{(\ell-1)\times(3^{\ell-1}-1)/2}$ is the generator matrix of the simplex code over $\mathbb{F}_3$ of dimension $(\ell-1)$ with block length $(3^{\ell-1}-1)/2$ and $B^{(3)}_{\ell-1} \in \{0, \frac{1}{3}, \frac{2}{3}\}^{(\ell-1)\times(3^{\ell-1}-1)/2}$ (resp. $-B^{(3)}_{\ell-1} \in \{0, \frac{2}{3}, \frac{1}{3}\}^{(\ell-1)\times(3^{\ell-1}-1)/2}$) is the matrix all of whose entries are divided by 3 from those of $A^{(3)}_{\ell-1}$ (resp. $-A^{(3)}_{\ell-1}$), and where in above matrix $(B^{(3)}_{\ell-1}, -B^{(3)}_{\ell-1}) \in \{0, \frac{1}{3}, \frac{2}{3}\}^{(\ell-1)\times(3^{\ell-1}-1)}$ is repeated $(k/3^{\ell-2})$ times.

Let $a = k/3^{\ell-2}$. Then $k = 3^{\ell-2}a$ and $a \geq 1$. By Theorem 7.5, we know the relation

$$(3^{\ell-1}-3^{\ell-2})(d'+1) = 2k(3^{\ell-1}-1) = 2 \cdot 3^{\ell-2}a(3^{\ell-1}-1).$$

Thus $d'+1 = a(3^{\ell-1}-1)$.

Since the order of $\mathbf{x}$ is 3, we have $|\operatorname{supp}(\mathbf{x})| = 3k = d+1$ by Proposition 7.10. Therefore,

$$d+1-(d'+1) = 3^{\ell-1}a - a(3^{\ell-1}-1) = a.$$

Consequently, in this case, we have $m = 3^\ell$, $k = 3^{\ell-2}a$ and $d = 3k-1 = 3^{\ell-1}a-1$ with $a \geq 1$ and $\ell \geq 3$ and the system of generators of Λ_Δ is the set of row vectors of the matrix

$$\begin{pmatrix} (B_{\ell-1}^{(3)}, -B_{\ell-1}^{(3)}, 0) & (B_{\ell-1}^{(3)}, -B_{\ell-1}^{(3)}, 0) & \cdots & (B_{\ell-1}^{(3)}, -B_{\ell-1}^{(3)}.0) \\ 1/3 & 1/3 & \cdots & 1/3 \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{\ell \times 3k}$$

up to permutation of the columns. This is the case (c) of Theorem 7.6 with $\ell \geq 3$.

The case (v): Let G' be the same thing as the case (iv).

Take $\mathbf{y} \in G' \setminus \{\mathbf{0}\}$. Then $\mathbf{y} = (\underbrace{1/3, \ldots, 1/3}_{k}, \underbrace{2/3, \ldots, 2/3}_{k}, \underbrace{0, \ldots, 0}_{k})$ after reordering. Since $\sharp\operatorname{supp}(\mathbf{x}+\mathbf{y}) = 2k$, $|\{i \in \operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y}) : x_i = 2/3, y_i = 1/3\}|$ should be k. Thus, we have $\operatorname{supp}(\mathbf{x}+2\mathbf{y}) = 3k$, a contradiction.

7.2.4 The case $\ell = 2$

Next, we consider the case $\ell = 2$.

Let $G_1 = \varphi^{-1}(\mathbb{Z}/m_1\mathbb{Z} \times \{0\})$ and $G_2 = \varphi^{-1}(\{0\} \times \mathbb{Z}/m_2\mathbb{Z})$. Clearly, either G_1 or G_2 does not contain $\mathbf{x}$, say, G_1. Then we have $\operatorname{ht}(\mathbf{y}) = k$ for each $\mathbf{y} \in G_1 \setminus \{\mathbf{0}\}$. By Theorem 7.3, G_1 is generated by $(a_1/m_q, (m_q-a_1)/m_q, \ldots, a_k/m_q, (m_q-a_k)/m_q, 0, \ldots, 0) \in G_1$ after reordering, where $m_q = |G_1|$ and each a_i is an integer with $0 < a_i \leq m_q/2$ which is coprime to m_q. Let $\mathbf{g} = (a_1/m_q, (m_q-a_1)/m_q, \ldots, a_k/m_q, (m_q-a_k)/m_q, 0, \ldots, 0) \in (\mathbb{R}/\mathbb{Z})^{d+1}$.

Let $\varphi(\mathbf{x}) = (x^{(1)}, x^{(2)}) \in \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z}$, where $0 \leq x^{(i)} \leq m_i - 1$ for $i = 1, 2$.

The case where the order of $\mathbf{x}$ is 2: Then $\mathbf{x} = (1/2, \ldots, 1/2) \in (\mathbb{R}/\mathbb{Z})^{4k}$ and $d+1 = 4k$ by Proposition 7.10. Since $|\operatorname{supp}(\mathbf{x}+\mathbf{g})| = 2k$, we obtain that $m_q = 2$ and $a_i = 1$ for each i.

Assume that $x^{(1)} \neq 0$ and $x^{(2)} \neq 0$. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be the generators of G_1 and G_2, respectively, such that $\mathbf{x} = x^{(1)}\mathbf{g}_1 + x^{(2)}\mathbf{g}_2$. Since $|\operatorname{supp}(x^{(1)}\mathbf{g}_1)| = |\operatorname{supp}(x^{(2)}\mathbf{g}_2)| = 2k$ and $\mathbf{x} = (1/2, \ldots, 1/2) \in (\mathbb{R}/\mathbb{Z})^{4k}$, $\mathbf{g}_1$ and $\mathbf{g}_2$ look like $(\underbrace{1/2, \ldots, 1/2}_{2k}, \underbrace{0, \ldots, 0}_{2k})$ after reordering and we also have $x^{(1)} = x^{(2)} = 1$. In particular, $\varphi(\Lambda_\Delta) = (\mathbb{Z}/2\mathbb{Z})^2$. Thus there is another isomorphism $\varphi' : \Lambda_\Delta \to (\mathbb{Z}/2\mathbb{Z})^2$ such that $\varphi'(\mathbf{x}) = (0, 1) \in (\mathbb{Z}/2\mathbb{Z})^2$. Hence we can deduce the case where $x^{(1)} = 0$ or $x^{(2)} = 0$.

Assume that $x^{(1)} = 0$ or $x^{(2)} = 0$. Then $\mathbf{x}$ generates one direct factor of Λ_Δ. Hence the system of generators of Λ_Δ is the set of row vectors of the matrix

$$\left(\begin{array}{cc} \underbrace{\begin{matrix} 1/2 & \cdots & 1/2 \\ 1/2 & \cdots & 1/2 \end{matrix}}_{2k} & \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 1/2 & \cdots & 1/2 \end{matrix}}_{2k} \end{array} \right) \in (\mathbb{R}/\mathbb{Z})^{2\times 4k}$$

after reordering. This is the case (a) with $m = 4$ and $d = 4k-1$ of Theorem 7.6.

The case where the order of $\mathbf{x}$ is 3: Then $\mathbf{x} = (2/3, \dots, 2/3) \in (\mathbb{R}/\mathbb{Z})^{3k}$ by Proposition 7.10. Since $|\operatorname{supp}(\mathbf{x}+\mathbf{g})| = 2k$, we obtain $m_q = 3$ and $a_i = 1$ for each i.

Assume that $x^{(1)} \neq 0$ and $x^{(2)} \neq 0$. By the similar discussions to the above, we see that $\varphi(\Lambda_\Delta) = (\mathbb{Z}/3\mathbb{Z})^2$. Thus there is another isomorphism $\varphi' : \Lambda_\Delta \to (\mathbb{Z}/3\mathbb{Z})^2$ such that $\varphi'(\mathbf{x}) = (0,1) \in (\mathbb{Z}/3\mathbb{Z})^2$. Hence we can deduce the case where $x^{(1)} = 0$ or $x^{(2)} = 0$.

Assume that $x^{(1)} = 0$ or $x^{(2)} = 0$. Then each of $\mathbf{x}$ and $-\mathbf{x}$ generates one direct factor of Λ_Δ. Hence we obtain that the system of generators of Λ_Δ is the set of row vectors of the matrix

$$\left(\begin{array}{ccc} \underbrace{\begin{matrix} 1/3 & \cdots & 1/3 \\ 1/3 & \cdots & 1/3 \end{matrix}}_{k} & \underbrace{\begin{matrix} 2/3 & \cdots & 2/3 \\ 1/3 & \cdots & 1/3 \end{matrix}}_{k} & \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 1/3 & \cdots & 1/3 \end{matrix}}_{k} \end{array} \right) \in (\mathbb{R}/\mathbb{Z})^{2\times 3k}.$$

This is the case (c) with $\ell = 2$ of Theorem 7.6.

The case where the order of $\mathbf{x}$ is 4: Then $\mathbf{x} = (\underbrace{3/4, \dots, 3/4}_{2k}, \underbrace{1/2, \dots, 1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ by Proposition 7.10. Let $k_j = |\{i \in \operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{g}) : x_i = 3/4, g_i = j/4\}|$ for $j = 1,2,3$ and $k' = |\{i \in \operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{g}) : x_i = g_i = 1/2\}|$. Since $|\operatorname{supp}(\mathbf{x}+\mathbf{g})| = |\operatorname{supp}(2\mathbf{x}+\mathbf{g})| = |\operatorname{supp}(3\mathbf{x}+\mathbf{g})| = 2k$, similar to the proof of Proposition 7.10, we obtain $k_1 + k' = k_3 + k' = k$ and $k_2 \geq k'$. Thus we have $k_1 + k_2 + k_3 + k' \geq 2k$. On the other hand, since $|\operatorname{supp}(\mathbf{g})| = 2k$, we also have $k_1 + k_2 + k_3 + k' \leq 2k$. Hence $k_1 + k_2 + k_3 + k' = 2k$. Moreover, since $|\operatorname{supp}(2\mathbf{x}+2\mathbf{g})| = 2k$, one has $|\operatorname{supp}(2\mathbf{x}+2\mathbf{g})| = 2k - k_1 - k_3 = 2k$. Thus $k_1 = k_3 = 0$. Hence it follows from $k_1 + k' = k_3 + k' = k$ that $k_2 = k' = k$. In particular, $\mathbf{g}$ looks like $(\underbrace{1/2, \dots, 1/2}_{2k}, \underbrace{0, \dots, 0}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ after reordering and has order 2.

Assume that $x^{(1)} \neq 0$ and $x^{(2)} \neq 0$. each generator of G_1 and G_2 has order 2, we obtain that $\Lambda_\Delta \cong (\mathbb{Z}/2\mathbb{Z})^2$. However, $(\mathbb{Z}/2\mathbb{Z})^2$ does not contain any element with order 4, a contradiction.

Hence $x^{(1)} = 0$ or $x^{(2)} = 0$. Then each of $\mathbf{x}$ and $-\mathbf{x}$ generates one direct factor of Λ_Δ. Thus we see that the system of generators of Λ_Δ is the set of row vectors of the matrix

$$\left(\begin{array}{ccc} \underbrace{\begin{matrix} 1/2 & \cdots & 1/2 \\ 1/4 & \cdots & 1/4 \end{matrix}}_{k} & \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 1/4 & \cdots & 1/4 \end{matrix}}_{k} & \underbrace{\begin{matrix} 1/2 & \cdots & 1/2 \\ 1/2 & \cdots & 1/2 \end{matrix}}_{k} \end{array} \right) \in (\mathbb{R}/\mathbb{Z})^{2\times 3k}.$$

This is the case (a) with $m = 8$ of Theorem 7.6.

The case where the order of $\mathbf{x}$ is 6: Then $\mathbf{x} = (\underbrace{5/6,\dots,5/6}_{k},\underbrace{2/3,\dots,2/3}_{k},\underbrace{1/2,\dots,1/2}_{k}) \in (\mathbb{R}/\mathbb{Z})^{3k}$ by Proposition 7.10. Let

$$\begin{aligned} k_j &= |\{i \in \mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{g}) : x_i = 5/6, g_i = j/6\}| \text{ for } j = 1,2,3,4,5, \\ k'_j &= |\{i \in \mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{g}) : x_i = 2/3, g_i = j/3\}| \text{ for } j = 1,2, \\ k'' &= |\{i \in \mathrm{supp}(\mathbf{x}) \cap \mathrm{supp}(\mathbf{g}) : x_i = g_i = 1/2\}|. \end{aligned}$$

Since $|\mathrm{supp}(\mathbf{x}+\mathbf{g})| = \cdots = |\mathrm{supp}(5\mathbf{x}+\mathbf{g})| = 2k$, similar to the proof of Proposition 7.10, we see that $k_1 + \cdots + k_5 + k'_1 + k'_2 + k'' \geq 2k$. On the other hand, since $|\mathrm{supp}(\mathbf{g})| = 2k$, we also have $k_1 + \cdots + k_5 + k'_1 + k'_2 + k'' \leq 2k$. Hence, $k_1 + \cdots + k_5 + k'_1 + k'_2 + k'' = 2k$.

Moreover, since $|\mathrm{supp}(\mathbf{x}+2\mathbf{g})| = |\mathrm{supp}(\mathbf{x}+4\mathbf{g})| = 2k$, one also has

$$2k = |\mathrm{supp}(\mathbf{x}+2\mathbf{g})| = k + k - k'_2 + k \text{ and } 2k = |\mathrm{supp}(\mathbf{x}+4\mathbf{g})| = k + k - k'_1 + k.$$

Hence $k = k'_1 = k'_2$. Then it follows that $2k = k'_1 + k'_2 \leq |\{i \in \mathrm{supp}(\mathbf{x}) : x_i = 2/3\}| = k$, a contradiction.

Therefore, we conclude that the order of $\mathbf{x}$ is never 6 when Λ_Δ has exactly two direct factors. This finishes the proof of Theorem 7.6.

7.3 Future work

It is known [31] that there exists a function f in terms of the degree k and the leading coefficient b of an δ-polynomial of a lattice polytope Δ such that $\mathrm{Vol}(\Delta) \leq f(b,k)$. In the situation of Corollary 7.1 (where $b = 1$) one observes that Δ satisfies $m \leq 9k$. In other words,

$$\mathrm{Vol}(\Delta) \leq \frac{9}{2}\deg(\Delta).$$

Moreover, equality implies $k = 1$ and so as described in (2) above Δ is isomorphic to a lattice pyramid over $\mathrm{conv}(\{(0,0),(3,0),(0,3)\})$. Now, having seen how Scott's theorem could be generalized from dimension two to degree two [89], we make the following guess about a more general class of δ-trinomials:

Conjecture 7.12. *Let Δ be a lattice polytope with δ-polynomial $1+at^k+bt^{2k}$ and $b \geq 2$. Then $a+b+1 \leq (4b+4)k$, or equivalently,*

$$\mathrm{Vol}(\Delta) \leq \frac{4b+4}{2} \deg(\Delta).$$

Part II

Constructions of new classes of reflexive polytopes

Chapter 8

Introduction to Gröbner bases

In this part, we will consider constructions of new classes of reflexive polytopes. Reflexive polytopes are interesting objects in commutative algebra, combinatorics, toric geometry and mirror symmetry. Hence, to find large classes of reflexive polytopes is an important problem.

In order to show that a lattice polytope is reflexive, we use the theory of Gröbner bases and toric ideals. In this chapter, we recall basic materials and notation on toric ideals.

Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K with each $\deg x_i = 1$ and let $\mathscr{M}_n$ denote the set of monomials in the variables $x_1, \ldots, x_n$. We say that a monomial $x_1^{a_1} \cdots x_n^{a_n}$ *divides* $x_1^{b_1} \cdots x_n^{b_n}$ if one has $a_i \leq b_i$ for all $1 \leq i \leq n$. Recall that a *partial order* on a set P is a relation $\leq$ on P such that for all $x, y, z \in P$ one has

(i) $x \leq x$ (reflexivity);

(ii) $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry);

(iii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity).

It is custom to write $x < y$ if $x \leq y$ and $x \neq y$. A *partially ordered set* (*poset*, for short) is a set P with a partial order $\leq$ on P. A *total order* on P is a partial order $\leq$ on P such that, for any two elements x and y belonging to P, one has either $x \leq y$ or $y \leq x$. We recall that a *monomial order* on $K[\mathbf{x}]$ is a total order $<$ on $\mathscr{M}_n$ such that

(i) $1 < u$ for all $1 \neq u \in \mathscr{M}_n$;

(ii) if $u, v \in \mathscr{M}_n$ and $u < v$, then $nw < vw$ for all $w \in \mathscr{M}_n$.

We give an example of a monomial order. Let $<_{\text{rev}}$ be the total order on $\mathscr{M}_n$ by setting $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} <_{\text{rev}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if either (i) $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$, or (ii) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the rightmost nonzero component of the vector $(b_1-a_1, b_2-a_2, \dots, b_n-a_n)$ is negative. Then $<_{\text{rev}}$ is a monomial order on $K[\mathbf{x}]$ and is called the *reverse lexicographic order* on $K[\mathbf{x}]$ induced by the ordering $x_n <_{\text{rev}} \cdots <_{\text{rev}} x_1$. A reverse lexicographic order is also called a *graded reverse lexicographic order*.

We will work with a fixed monomial order $<$ on $K[\mathbf{x}]$. Let f be a polynomial in $K[\mathbf{x}]$. The *initial monomial* $\text{in}_<(f)$ of f with respect to $<$ is the biggest monomial with respect to $<$. Let I be a nonzero ideal of $K[\mathbf{x}]$. The monomial ideal generated by $\{\text{in}_<(f) : 0 \neq f \in I\}$ is called the *initial ideal* of I with respect to $<$ and is written as $\text{in}_<(I)$. In general, even if $I = \langle \{f_\lambda\}_{\lambda \in \Lambda} \rangle$, it is not necessarily true that $\text{in}_<(I)$ coincides with $\langle \{\text{in}_<(f_\lambda)\}_{\lambda \in \Lambda} \rangle$.

Definition 8.1. Let I be a nonzero ideal of $K[\mathbf{x}]$. A finite set of nonzero polynomials $\{g_1, \dots, g_s\}$ with each $g_i \in I$ is said to be a *Gröbner* basis of I with respect to $<$ if $\text{in}_<(I)$ is generated by the monomials $\text{in}_<(g_1), \dots, \text{in}_<(g_s)$.

It is known that a Gröbner basis of I with respect to $<$ exists. Moreover, a Gröbner basis of I generates the ideal I.

Lemma 8.2. *Let I be an ideal of $K[\mathbf{x}]$ and $\mathscr{G}$ a Gröbner basis of with respect to some monomial order. Then $\mathscr{G}$ is a generating set of I.*

Next, we introduce the two associated graded ring of lattice polytopes. Let $K[\mathbf{t}^{\pm 1}, s] = K[t_1^{\pm 1}, \dots, t_N^{\pm 1}, s]$ the Laurent polynomial ring in $N+1$ variables over a field K. If $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{Z}^N$, then $\mathbf{t}^{\mathbf{a}} s$ is the Laurent monomial $t_1^{a_1} \cdots t_N^{a_N} s \in K[\mathbf{t}^{\pm 1}, s]$. In particular $\mathbf{t}^{\mathbf{0}} s = s$. Let $\mathscr{P} \subset \mathbb{R}^N$ be a lattice polytope of dimension d and $\mathscr{P} \cap \mathbb{Z}^N = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Then, the *toric ring* $K[\mathscr{P}]$ of $\mathscr{P}$ is defined by

$$K[\mathscr{P}] = K[\{\mathbf{t}^{\mathbf{a}} s : \mathbf{a} \in \mathscr{P} \cap \mathbb{Z}^N\}] \subset K[\mathbf{t}^{\pm 1}, s],$$

and the *Ehrhart ring* $\mathscr{E}_K(\mathscr{P})$ of $\mathscr{P}$ is defined by

$$\mathscr{E}_K(\mathscr{P}) = K[\{\mathbf{t}^{\mathbf{a}} s^m : \mathbf{a} \in m\mathscr{P} \cap \mathbb{Z}^N, m \geq 1\}] \subset K[\mathbf{t}^{\pm 1}, s].$$

We regard $K[\mathscr{P}]$ and $\mathscr{E}_K(\mathscr{P})$ as graded K-algebras by setting each $\deg(\mathbf{t}^{\mathbf{a}} s^m) = m$. Then we know that the Hilbert function of $\mathscr{E}_K(\mathscr{P})$ coincides with the Ehrhart polynomial of $\mathscr{P}$. The *toric ideal* $I_{\mathscr{P}}$ of $\mathscr{P}$ is the kernel of a surjective homomorphism $\pi : K[x_1, \dots, x_n] \to K[\mathscr{P}]$ defined by $\pi(x_i) = \mathbf{t}^{\mathbf{a}_i} s$ for $1 \leq i \leq n$. It is known that $I_{\mathscr{P}}$ is generated by homogeneous binomials. See, e.g., [87]. Now, we give a useful result of Gröbner bases of the toric ideals of lattice polytopes.

Lemma 8.3 ([73, (0.1), p. 1914])**.** *Work with the same situation as above. A finite set $\mathscr{G}$ of $I_{\mathscr{P}}$ is a Gröbner basis of $I_{\mathscr{P}}$ with respect to $<$ if and only if $\pi(u) \neq \pi(v)$ for all $u \notin (\{\text{in}_<(g) : g \in \mathscr{G}\})$ and $v \notin \langle \{\text{in}_<(g) : g \in \mathscr{G}\} \rangle$ with $u \neq v$.*

Please refer [39, Chapters 1 and 5] and [87] for more details on Gröbner bases and toric ideals.

Now, we recall an important class of lattice polytopes. We say that a lattice polytope $\mathscr{P} \subset \mathbb{R}^N$ of dimension d possesses the *integer decomposition property* if, for each integer $k \geq 1$ and for each $\mathbf{a} \in k\mathscr{P} \cap \mathbb{Z}^N$, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_k$ belonging to $\mathscr{P} \cap \mathbb{Z}^N$ with $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_k$. Clearly, $\mathscr{P}$ possesses the integer decomposition property if and only if $K[\mathscr{P}] = \mathscr{E}_K(\mathscr{P})$. In this case, the Hilbert function of $K[\mathscr{P}]$ coincides with the Ehrhart polynomial of $\mathscr{P}$. The integer decomposition property is particularly important in the theory and application of integer programing [77, §22.10]. Moreover, a lattice polytope which possesses the integer decomposition property is normal and very ample. These properties play important roles in algebraic geometry.

Finally, we give indispensable lemmata for this part. Let $<$ be a monomial order on $K[\mathbf{x}]$ and $\mathrm{in}_<(I_\mathscr{P})$ the initial ideal of $I_\mathscr{P}$ with respect to $<$. The initial ideal $\mathrm{in}_<(I_\mathscr{P})$ is called *squarefree* if $\mathrm{in}_<(I_\mathscr{P})$ is generated by squarefree monomials.

Lemma 8.4 ([44, Lemma 1.1])**.** *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d such that the origin of $\mathbb{R}^d$ is contained in its interior and $\mathscr{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Suppose $\mathscr{P}$ is spanning and there exists an ordering of the variables $x_{i_1} < \cdots < x_{i_n}$ for which $\mathbf{a}_{i_1} = \mathbf{0}$ such that the initial ideal $\mathrm{in}_<(I_\mathscr{P})$ of the toric ideal $I_\mathscr{P}$ with respect to the reverse lexicographic order $<$ on the polynomial ring $K[x_1, \ldots, x_n]$ induced by the ordering is squarefree. Then $\mathscr{P}$ is a reflexive polytope which possesses the integer decomposition property.*

Lemma 8.5 ([35, Corollary 6.1.5])**.** *Let S be a polynomial ring and $I \subset S$ a graded ideal of S. Let $<$ be a monomial order on S. Then S/I and $S/\mathrm{in}_<(I)$ have the same Hilbert function.*

The structure of the rest of Part II

The organization of the rest of this part is as follows. In Chapter 9, we will give several new classes of reflexive polytopes with the integer decomposition propery arising from finite posets. In Chapter 10, we will give a class of reflexive polytopes with the integer decomposition propery arising from perfect graphs. In Chapter 11, we will give two classes of reflexive polytopes with the integer decomposition propery arising from finite posets and perfect graphs. In Chapters 9, 10 and 11, we use the technique on Göbner bases. In Chapter 12, we will give a classes of reflexive polytopes arising from finite simple graph by using Matrix Theory. Finally, in Chapter 13, we will construct higher-dimensional self dual reflexive polytopes.

Chapter 9

Reflexive polytopes arising from posets

In this chapter, we give several large classes of reflexive polytope arising from finite posets. In particular, these reflexive polytopes possess the integer decomposition property.

In order to give classes of reflexive polytopes, we introduce two constructions of lattice polytopes. Given two lattice polytopes $\mathscr{P} \subset \mathbb{R}^N$ and $\mathscr{Q} \subset \mathbb{R}^N$ of dimension d, we set the lattice polytopes $\Gamma(\mathscr{P}, \mathscr{Q}) \subset \mathbb{R}^N$ and $\Omega(\mathscr{P}, \mathscr{Q}) \subset \mathbb{R}^{N+1}$ with

$$\Gamma(\mathscr{P}, \mathscr{Q}) = \text{conv}\{\mathscr{P} \cup (-\mathscr{Q})\},$$

$$\Omega(\mathscr{P}, \mathscr{Q}) = \text{conv}\{(\mathscr{P} \times \{1\}) \cup (-\mathscr{Q} \times \{-1\})\}.$$

If $\mathscr{P} = \mathscr{Q}$, then we will write $\Gamma(\mathscr{P}) = \Gamma(\mathscr{P}, \mathscr{P})$ and $\Omega(\mathscr{P}) = \Omega(\mathscr{P}, \mathscr{P})$. We remark that the origin of $\mathbb{R}^n$ is always a relative interior lattice point of $\Gamma(\mathscr{P})$ and the origin of $\mathbb{R}^{n+1}$ is always a relative interior lattice point of $\Omega(\mathscr{P})$. Assume that $\mathscr{P}$ and $\mathscr{Q}$ are full-dimensional, namely, $N = d$. Then $\Gamma(\mathscr{P}, \mathscr{Q})$ and $\Omega(\mathscr{P}, \mathscr{Q})$ are also full-dimensional. In particular, each of $\mathscr{P} \times \{1\}$ and $-\mathscr{Q} \times \{-1\}$ is a facet of $\Omega(\mathscr{P}, \mathscr{Q})$.

In this chapter, the study on $\Gamma(\mathscr{P}, \mathscr{Q})$ and $\Omega(\mathscr{P}, \mathscr{Q})$ will be done when each of $\mathscr{P}$ and $\mathscr{Q}$ is an order polytope and a chain polytope, which are lattice polytopes arising from finite posets. This chapter is organized as follows. In Section 9.1, we recall some terminologies of finite posets and introduce order polytopes and chain polytopes. In Section 9.2, we consider $\Gamma(\mathscr{P}, \mathscr{Q})$ when each of $\mathscr{P}$ and $\mathscr{Q}$ is an order polytope and a chain polytope. Finally, in section 9.3, we consider $\Omega(\mathscr{P}, \mathscr{Q})$ when each of $\mathscr{P}$ and $\mathscr{Q}$ is an order polytope and a chain polytope.

9.1 Two poset polytopes

In this section, we recall some terminologies of finite posets and introduce two lattice polytopes arising from finite posets. Let $P = \{p_1, \ldots, p_d\}$ denote a finite poset. A subset I of P is called a *poset ideal* of P if $p_i \in I$ and $p_j \in P$ together with $p_j \leq p_i$ guarantee $p_j \in I$. Note that the empty set $\emptyset$ and itself P are poset ideals of P. Let $\mathscr{J}(P)$ denote the set of poset ideals of P. A subset A of P is called an *antichain* of P if p_i and p_j belonging to A with $i \neq j$ are incomparable. In particular, the empty set $\emptyset$ and each 1-elemant subsets $\{p_j\}$ are antichains of P. Let $\mathscr{A}(P)$ denote the set of antichains of P. For a poset ideal I of P, we write $\max(I)$ for the set of maximal elements of I. In particular, $\max(I)$ is an antichain. A *linear extension* of P is a permutation $\sigma = i_1 i_2 \cdots i_d$ of $[d] = \{1, 2, \ldots, d\}$ which satisfies $i_a < i_b$ if $p_{i_a} < p_{i_b}$ in P. Let $e(P)$ denote the number of linear extensions of P.

Stanley [81] introduced two classes of lattice polytopes arising from finite posets, order polytopes and chain polytopes. The *order polytope* $\mathscr{O}_P$ of P is defined to be the convex polytope consisting of those $(x_1, \ldots, x_d) \in \mathbb{R}^d$ such that

(1) $0 \leq x_i \leq 1$ for $1 \leq i \leq d$;

(2) $x_i \geq x_j$ if $p_i \leq p_j$ in P.

The *chain polytope* $\mathscr{C}_P$ is defined to be the convex polytope consisting of those $(x_1, \ldots, x_d) \in \mathbb{R}^d$ such that

(1) $x_i \geq 0$ for $1 \leq i \leq d$;

(2) $x_{i_1} + \cdots + x_{i_k} \leq 1$ for every maximal chain $p_{i_1} < \cdots < p_{i_k}$ of P.

It then follows that both order polytopes and chain polytopes are lattice polytopes of dimension d. For each subset $I \subset P$, we define the $(0,1)$-vectors $\rho(I) = \sum_{p_i \in I} \mathbf{e}_i$. In particular $\rho(\emptyset)$ is the origin $\mathbf{0}$ of $\mathbb{R}^d$. In [81, Corollary 1.3 and Theorem 2.2], it is shown that

$$\{\text{the set of vertices of } \mathscr{O}_P\} = \{\rho(I) : I \in \mathscr{J}(P)\},$$

$$\{\text{the set of vertices of } \mathscr{C}_P\} = \{\rho(A) : A \in \mathscr{A}(P)\}.$$

Next, we consider the Ehrhart polynomials of order polytopes and chain polytopes. In fact, $\mathscr{O}_P$ and $\mathscr{C}_P$ have the same Ehrhart polynomial. Moreover, every lattice point in $k\mathscr{O}_P$ corresponds an order preserving map $\varphi : P \to [k+1]$. Counting order preserving maps is classical [84, Sect. 3.15]. The *order polynomial* $\Omega_P(k)$ of P counts the number of order preserving maps into k-chains.

Lemma 9.1 ([81, Theorem 4.1 and Corollary 4.2]). *Let $P = \{p_1, \dots, p_d\}$ be a finite poset. Then for every $n > 0$one has*

$$\Omega_P(k+1) = L_{\mathscr{O}_P}(k) = L_{\mathscr{C}_P}(k).$$

In particular, $e(P) = \text{Vol}(\mathscr{O}_P) = \text{Vol}(\mathscr{C}_P)$.

Finally, we consider the toric ideals of order polytopes and chain polytopes. Recall that a lattice polytope $\mathscr{P}$ is called *compressed* ([88]) if all its "pulling triangulations" are unimodular. If $\mathscr{P}$ is spanning, then $\mathscr{P}$ is compressed if and only if every reverse lexicographic initial ideal of $I_{\mathscr{P}}$ is squarefree ([87]). It follows from [72, Theorem 1.1] that all order polytopes and all chain polytope are compressed and possess the integer decomposition property. Moreover, each of $I_{\mathscr{O}_P}$ and $I_{\mathscr{C}_P}$ possesses a squarefree quadratic initial ideal. Let $K[\mathscr{O}] = K[\{x_I\}_{I \in \mathscr{J}(P)}]$ and $K[\mathscr{C}] = K[\{x_{\max(I)}\}_{I \in \mathscr{J}(P)}]$ denote the polynomial rings over K, and define the surjective ring homomorphisms $\pi_{\mathscr{O}}$ and $\pi_{\mathscr{C}}$ by the following:

- $\pi_{\mathscr{O}} : K[\mathscr{O}] \to K[\mathscr{O}_P]$ by setting $\pi_{\mathscr{O}}(x_I) = \mathbf{t}^{\rho(I)}s$;
- $\pi_{\mathscr{C}} : K[\mathscr{C}] \to K[\mathscr{C}_P]$ by setting $\pi_{\mathscr{C}}(x_{\max(I)}) = \mathbf{t}^{\rho(\max(I))}s$,

where $I \in \mathscr{J}(P)$. Then the toric ideal $I_{\mathscr{O}_P}$ (resp. $I_{\mathscr{C}_P}$) is the kernel of $\pi_{\mathscr{O}}$ (resp. $\pi_{\mathscr{C}}$).

Next, we introduce monomial orders $<_{\mathscr{O}}$ and $<_{\mathscr{C}}$, and $\mathscr{G}_{\mathscr{O}}$ and $\mathscr{G}_{\mathscr{C}}$ which are the sets of binomials. Let $<_{\mathscr{O}}$ denote a reverse lexicographic order on $K[\mathscr{O}]$ satisfying $x_I <_{\mathscr{O}} x_J$ if $I \subset J$, anf let $<_{\mathscr{C}}$ denote a reverse lexicographic order on $K[\mathscr{C}]$ satisfying $x_{\max(I)} <_{\mathscr{C}} x_{\max(J)}$ if $I \subset J$, where I and J are poset ideals of P, Let $\mathscr{G}_{\mathscr{O}}$ be the set of the following binomials:

$$x_I x_J - x_{I \cup J} x_{I \cap J},$$

and $\mathscr{G}_{\mathscr{C}}$ the set of the following binomials:

$$x_{\max(I)} x_{\max(J)} - x_{\max(I \cup J)} x_{\max(I * J)},$$

where I and J are poset ideals of P which are incomparable in $\mathscr{J}(P)$ and $I * J$ is the poset ideal of P generated by $\max(I \cap J) \cap (\max(I) \cup \max(J))$.

Lemma 9.2 ([36]). *Work with the same situation as above. Then $\mathscr{G}_{\mathscr{O}}$ is a Gröbner basis of $I_{\mathscr{O}_P}$ with respect to $<_{\mathscr{O}}$.*

Lemma 9.3 ([42]). *Work with the same situation as above. Then $\mathscr{G}_{\mathscr{C}}$ is a Gröbner basis of $I_{\mathscr{C}_P}$ with respect to $<_{\mathscr{C}}$.*

From these facts, it follows that each of $\mathscr{O}_P$ and $\mathscr{C}_P$ possesses a regular, flag, unimodular triangulation. (Recall that a *flag* complex is a simplicial complex any of its nonface is an edge.) Furthermore, toric ideals of order polytopes naturally appear in algebraic geometry (e.g., [18]) and in representation theory (e.g., [96]).

9.2 Type Γ

In this section, we consider $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ for finite posets $P = \{p_1,\dots,p_d\}$ and $Q = \{q_1,\dots,q_d\}$.

9.2.1 When are they reflexive?

In this subsection, we discuss when $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ are reflexive polytope. In fact, we show the following theorem.

Theorem 9.4 ([43, 45, 46]). *Let $P = \{p_1,\dots,p_d\}$ and $Q = \{q_1,\dots,q_d\}$ be finite posets.*

(1) ([43, Corollary 2.2]) *If P and Q possess a common linear extension, then the lattice polytope $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$ is a reflexive polytope which possesses the integer decomposition property.*

(2) ([46, Corollary 1.2]) *The lattice polytope $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ is a reflexive polytope which possesses the integer decomposition property.*

(3) ([45, Corollary 1.3]) *The lattice polytope $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is a reflexive polytope which possesses the integer decomposition property.*

Let

$$\begin{aligned} K_1[\mathscr{O}\mathscr{O}] &= K[\{x_I\}_{\emptyset\neq I\in\mathscr{J}(P)}\cup\{y_J\}_{\emptyset\neq J\in\mathscr{J}(Q)}\cup\{z\}],\\ K_1[\mathscr{O}\mathscr{C}] &= K[\{x_I\}_{\emptyset\neq I\in\mathscr{J}(P)}\cup\{y_{\max(J)}\}_{\emptyset\neq J\in\mathscr{J}(Q)}\cup\{z\}],\\ K_1[\mathscr{C}\mathscr{C}] &= K[\{x_{\max(I)}\}_{\emptyset\neq I\in\mathscr{J}(P)}\cup\{y_{\max(J)}\}_{\emptyset\neq J\in\mathscr{J}(Q)}\cup\{z\}] \end{aligned}$$

denote the polynomial rings over K, and define the surjective ring homomorphisms $\pi^1_{\mathscr{O}\mathscr{O}}$, $\pi^1_{\mathscr{O}\mathscr{C}}$ and $\pi^1_{\mathscr{C}\mathscr{C}}$ by the following:

- $\pi^1_{\mathscr{O}\mathscr{O}} : K_1[\mathscr{O}\mathscr{O}] \to K[\Gamma(\mathscr{O}_P,\mathscr{O}_Q)]$ by setting
  $\pi^1_{\mathscr{O}\mathscr{O}}(x_I) = \mathbf{t}^{\rho(I)}s$, $\pi^1_{\mathscr{O}\mathscr{O}}(y_J) = \mathbf{t}^{-\rho(J)}s$ and $\pi^1_{\mathscr{O}\mathscr{O}}(z) = s$,

- $\pi^1_{\mathscr{O}\mathscr{C}} : K_1[\mathscr{O}\mathscr{C}] \to K[\Gamma(\mathscr{O}_P,\mathscr{C}_Q)]$ by setting
  $\pi^1_{\mathscr{O}\mathscr{C}}(x_I) = \mathbf{t}^{\rho(I)}s$, $\pi^1_{\mathscr{O}\mathscr{C}}(y_{\max(J)}) = \mathbf{t}^{-\rho(\max(J))}s$ and $\pi^1_{\mathscr{O}\mathscr{C}}(z) = s$,

- $\pi^1_{\mathscr{C}\mathscr{C}} : K_1[\mathscr{C}\mathscr{C}] \to K[\Gamma(\mathscr{C}_P,\mathscr{C}_Q)]$ by setting
  $\pi^1_{\mathscr{C}\mathscr{C}}(x_{\max(I)}) = \mathbf{t}^{\rho(\max(I))}s$, $\pi^1_{\mathscr{C}\mathscr{C}}(y_{\max(J)}) = \mathbf{t}^{-\rho(\max(J))}s$ and $\pi^1_{\mathscr{C}\mathscr{C}}(z) = s$

where $\emptyset \neq I \in \mathscr{J}(P)$ and $\emptyset \neq J \in \mathscr{J}(Q)$. Then the toric ideal $I_{\Gamma(\mathscr{O}_P, \mathscr{O}_Q)}$ of $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ is the kernel of $\pi^1_{\mathscr{O}\mathscr{O}}$. Similarly, the toric ideal $I_{\Gamma(\mathscr{O}_P, -\mathscr{C}_Q)}$ (resp. $I_{\Gamma(\mathscr{C}_P, \mathscr{C}_Q)}$) is the kernel of $\pi^1_{\mathscr{O}\mathscr{C}}$ (resp. $\pi^1_{\mathscr{C}\mathscr{C}}$).

Next, we introduce monomial orders $<^1_{\mathscr{O}\mathscr{O}}$, $<^1_{\mathscr{O}\mathscr{C}}$ and $<^1_{\mathscr{C}\mathscr{C}}$ and $\mathscr{G}^1_{\mathscr{O}\mathscr{O}}$, $\mathscr{G}^1_{\mathscr{O}\mathscr{C}}$ and $\mathscr{G}^1_{\mathscr{C}\mathscr{C}}$ which are the sets of binomials.

Let $<^1_{\mathscr{O}\mathscr{O}}$ denote a reverse lexicographic order on $K_1[\mathscr{O}\mathscr{O}]$ satisfying

- $z <^1_{\mathscr{O}\mathscr{O}} y_J <^1_{\mathscr{O}\mathscr{O}} x_I$;
- $x_{I'} <^1_{\mathscr{O}\mathscr{O}} x_I$ if $I' \subset I$;
- $y_{J'} <^1_{\mathscr{O}\mathscr{O}} y_J$ if $J' \subset J$,

and $\mathscr{G}^1_{\mathscr{O}\mathscr{O}} \subset K[\mathscr{O}\mathscr{O}]$ the set of the following binomials:

(G1) $x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}$;

(G2) $y_J y_{J'} - y_{J \cup J'} y_{J \cap J}$;

(G3) $x_I y_J - x_{I \setminus \{p_i\}} y_{J \setminus \{q_i\}}$,

and let $<^1_{\mathscr{O}\mathscr{C}}$ denote a reverse lexicographic order on $K_1[\mathscr{O}\mathscr{C}]$ satisfying

- $z <^1_{\mathscr{O}\mathscr{C}} y_{\max(J)} <^1_{\mathscr{O}\mathscr{C}} x_I$;
- $x_{I'} <^1_{\mathscr{O}\mathscr{C}} x_I$ if $I' \subset I$;
- $y_{\max(J')} <^1_{\mathscr{O}\mathscr{C}} y_{\max(J)}$ if $J' \subset J$,

and $\mathscr{G}^1_{\mathscr{O}\mathscr{C}} \subset K_1[\mathscr{O}\mathscr{C}]$ the set of the following binomials:

(G4) $x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}$;

(G5) $y_{\max(J)} y_{\max(J')} - y_{\max(J \cup J')} y_{\max(J * J')}$;

(G6) $x_I y_{\max(J)} - x_{I \setminus \{p_i\}} y_{\max(J) \setminus \{q_i\}}$,

and let $<^1_{\mathscr{C}\mathscr{C}}$ denote a reverse lexicographic order on $K_1[\mathscr{C}\mathscr{C}]$ satisfying

- $z <^1_{\mathscr{C}\mathscr{C}} y_{\max(J)} <^1_{\mathscr{C}\mathscr{C}} x_{\max(I)}$;
- $x_{\max(I')} <^1_{\mathscr{C}\mathscr{C}} x_{\max(I)}$ if $I' \subset I$;

- $y_{\max(J')} <^1_{\mathscr{C}\mathscr{C}} y_{\max(J)}$ if $J' \subset J$,

and $\mathscr{G}^1_{\mathscr{C}\mathscr{C}} \subset K_1[\mathscr{C}\mathscr{C}]$ the set of the following binomials:

(G7) $x_{\max(I)}x_{\max(I')} - y_{\max(I\cup I')}y_{\max(I*I')}$;

(G8) $y_{\max(J)}y_{\max(J')} - y_{\max(J\cup J')}y_{\max(J*J')}$;

(G9) $x_{\max(I)}y_{\max(J)} - x_{\max(I)\setminus\{p_i\}}y_{\max(J)\setminus\{q_i\}}$,

where

- $x_\emptyset = y_\emptyset = z$;
- I and I' are poset ideals of P which are incomparable in $\mathscr{J}(P)$;
- J and J' are poset ideals of Q which are incomparable in $\mathscr{J}(Q)$;
- p_i is a maximal element of I and q_i is a maximal element of J.

Proposition 9.5 ([43])**.** *Work with the same situation as above. If P and Q possess a common linear extension, then the origin of $\mathbb{R}^d$ is contained in the interior of $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ and $\mathscr{G}^1_{OO}$ is a Gröbner basis of $I_{\Omega(\mathscr{O}_P,\mathscr{O}_Q)}$ with respect to $<^1_{\mathscr{O}\mathscr{O}}$.*

Proposition 9.6. *Work with the same situation as above. Then $\mathscr{G}^1_{OC}$ is a Gröbner basis of $I_{\Gamma(\mathscr{O}_P,\mathscr{C}_Q)}$ with respect to $<^1_{\mathscr{O}\mathscr{C}}$.*

Proof. It is clear that $\mathscr{G}^1_{\mathscr{O}\mathscr{C}} \subset I_{\Gamma(\mathscr{O}_P,\mathscr{C}_Q)}$. We note that the initial monomial of each of the binomials (G4), (G5) and (G6) with respect to $<^1_{\mathscr{O}\mathscr{C}}$ is its first monomial. Let $\mathrm{in}_{<^1_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^1_{\mathscr{O}\mathscr{C}})$ denote the set of initial monomials of binomials belonging to $\mathscr{G}^1_{\mathscr{O}\mathscr{C}}$. It follows from Lemma 8.3 that, in order to show that $\mathscr{G}^1_{\mathscr{O}\mathscr{C}}$ is a Gröbner basis of $I_{\Gamma(\mathscr{O}_P,\mathscr{C}_Q)}$ with respect to $<^1_{\mathscr{O}\mathscr{C}}$, we must prove the following assertion: If u and v are monomials belonging to $K_1[\mathscr{O}\mathscr{C}]$ with $u \neq v$ such that $u \notin \langle \mathrm{in}_{<^1_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^1_{\mathscr{O}\mathscr{C}})\rangle$ and $v \notin \langle \mathrm{in}_{<^1_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^1_{\mathscr{O}\mathscr{C}})\rangle$, then $\pi^1_{\mathscr{O}\mathscr{C}}(u) \neq \pi^1_{\mathscr{O}\mathscr{C}}(v)$.

Let $u, v \in K_1[\mathscr{O}\mathscr{C}]$ be monomials with $u \neq v$. Write

$$u = z^\alpha x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b}, \quad v = z^{\alpha'} x_{I'_1}^{\xi'_1} \cdots x_{I'_{a'}}^{\xi'_{a'}} y_{\max(J'_1)}^{\nu'_1} \cdots y_{\max(J'_{b'})}^{\nu'_{b'}},$$

where

- $\alpha \geq 0$, $\alpha' \geq 0$;
- $I_1, \ldots, I_a, I'_1, \ldots, I'_{a'} \in \mathscr{J}(P) \setminus \{\emptyset\}$;

- $J_1, \ldots, J_b, J'_1, \ldots, J'_{b'} \in \mathscr{J}(Q) \setminus \{\emptyset\}$;
- $\xi_1, \ldots, \xi_a, \nu_1, \ldots, \nu_b, \xi'_1, \ldots, \xi'_{a'}, \nu'_1, \ldots, \nu'_{b'} > 0$,

and where u and v are relatively prime with $u \notin \langle \mathrm{in}_{<^1_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^1_{\mathscr{O}\mathscr{C}}) \rangle$ and $v \notin \langle \mathrm{in}_{<^1_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^1_{\mathscr{O}\mathscr{C}}) \rangle$. Note that either $\alpha = 0$ or $\alpha' = 0$. Hence we may assume that $\alpha' = 0$. Thus

$$u = z^{\alpha} x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b}, \qquad v = x_{I'_1}^{\xi'_1} \cdots x_{I'_{a'}}^{\xi'_{a'}} y_{\max(J'_1)}^{\nu'_1} \cdots y_{\max(J'_{b'})}^{\nu'_{b'}}.$$

By using (G4) and (G5), it follows that

- $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_a$ and $J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_b$;
- $I'_1 \subsetneq I'_2 \subsetneq \cdots \subsetneq I'_{a'}$ and $J'_1 \subsetneq J'_2 \subsetneq \cdots \subsetneq J'_{b'}$.

Furthermore, by virtue of Lemmas 9.2 and 9.3, it suffices to discuss u and v with $(a, a') \neq (0,0)$ and $(b, b') \neq (0,0)$.

Since $I_a \neq I'_{a'}$, we may assume that $I_a \setminus I'_{a'} \neq \emptyset$. Then there exists a maximal element p_{i^*} of I_a with $p_{i^*} \notin I'_{a'}$.

Now, suppose that $\pi^1_{\mathscr{O}\mathscr{C}}(u) = \pi^1_{\mathscr{O}\mathscr{C}}(v)$. Then we have

$$\sum_{\substack{I \in \{I_1, \ldots, I_a\} \\ p_i \in I}} \xi_I - \sum_{\substack{J \in \{J_1, \ldots, J_b\} \\ q_i \in \max(J)}} \nu_J = \sum_{\substack{I' \in \{I'_1, \ldots, I'_{a'}\} \\ p_i \in I'}} \xi'_{I'} - \sum_{\substack{J' \in \{J'_1, \ldots, J'_{b'}\} \\ q_i \in \max(J')}} \nu'_{J'}.$$

for all $1 \leq i \leq d$ by comparing the degree of t_i. Since $p_{i^*} \notin I'_{a'}$, one has

$$\sum_{\substack{I \in \{I_1, \ldots, I_a\} \\ p_{i^*} \in I}} \xi_I - \sum_{\substack{J \in \{J_1, \ldots, J_b\} \\ q_{i^*} \in \max(J)}} \nu_J = - \sum_{\substack{J' \in \{J'_1, \ldots, J'_{b'}\} \\ q_{i^*} \in \max(J')}} \nu'_{J'} \leq 0.$$

Moreover, since p_{i^*} is belonging to I_a, we also have

$$\sum_{\substack{I \in \{I_1, \ldots, I_a\} \\ p_{i^*} \in I}} \xi_I > 0.$$

Hence there exists an integer c with $1 \leq c \leq b$ such that q_{i^*} is a maximal element of J_c. Therefore we have $x_{I_a} y_{\max(J_c)} \in \langle \mathrm{in}_{<^1_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^1_{\mathscr{O}\mathscr{C}}) \rangle$, but this is a contradiction. □

Proposition 9.7. *Work with the same situation as above. Then $\mathscr{G}^1_{\mathscr{C}\mathscr{C}}$ is a Gröbner basis of $I_{\Gamma(\mathscr{C}_P, \mathscr{C}_Q)}$ with respect to $<^1_{\mathscr{C}\mathscr{C}}$.*

Proof. We can show that the assertion follows by a similar way in the proof of Proposition 9.6. □

Finally, we show Theorem 9.4.

Proof of Theorem 9.4. It is easy to show that $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are spanning. By Lemma 8.4 and Propositions 9.5, 9.6 and 9.7, the assertion follows. $\square$

9.2.2 Their δ-polynomials and volumes

In this subsection, we discuss their δ-polynomials and volumes of $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$.

First, we show the following theorem.

Theorem 9.8 ([45, Theorem 1.1])**.** *Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$ be finite posets. Then we have*

$$\delta(\Gamma(\mathscr{O}_P, \mathscr{C}_Q), t) = \delta(\Gamma(\mathscr{C}_P, \mathscr{C}_Q), t).$$

In particular, the volume of $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is the same as that of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$. Moreover, if P and Q possess a common linear extension, then we have

$$\delta(\Gamma(\mathscr{O}_P, \mathscr{O}_Q), t) = \delta(\Gamma(\mathscr{O}_P, \mathscr{C}_Q), t) = \delta(\Gamma(\mathscr{C}_P, \mathscr{C}_Q), t).$$

In this case, these polytopes have the same volume.

Here, we put

$$R^1_{\mathscr{O}\mathscr{O}} := K_1[\mathscr{O}\mathscr{O}]/\mathrm{in}_{<^1_{\mathscr{O}\mathscr{O}}}(I_{\Gamma(\mathscr{O}_P, \mathscr{O}_Q)}),$$
$$R^1_{\mathscr{O}\mathscr{C}} := K_1[\mathscr{O}\mathscr{C}]/\mathrm{in}_{<^1_{\mathscr{O}\mathscr{C}}}(I_{\Gamma(\mathscr{O}_P, \mathscr{C}_Q)}),$$
$$R^1_{\mathscr{C}\mathscr{C}} := K_1[\mathscr{C}\mathscr{C}]/\mathrm{in}_{<^1_{\mathscr{C}\mathscr{C}}}(I_{\Gamma(\mathscr{C}_P, \mathscr{C}_Q)}).$$

In order to prove Theorem 9.8, we show the following proposition.

Proposition 9.9. *Work with the same situation as above. Then the ring $R^1_{\mathscr{O}\mathscr{C}}$ is isomorphic to the ring $R^1_{\mathscr{C}\mathscr{C}}$. Moreover, if P and Q possess a common linear extension, then these rings $R^1_{\mathscr{O}\mathscr{O}}$, $R^1_{\mathscr{O}\mathscr{C}}$ and $R^1_{\mathscr{C}\mathscr{C}}$ are isomorphic.*

Proof. From Propositions 9.6 and 9.7, we have

$$R^1_{\mathscr{O}\mathscr{C}} \cong \frac{K_1[\mathscr{O}\mathscr{C}]}{\langle \{x_I x_{I'}, y_{\max(J)} y_{\max(J')}, x_I y_{\max(J)} : I, I', J \text{ and } J' \text{ satisfy } (*)\} \rangle},$$

$$R^1_{\mathscr{C}\mathscr{C}} \cong \frac{K_1[\mathscr{C}\mathscr{C}]}{\langle \{x_{\max(I)} x_{\max(I')}, y_{\max(J)} y_{\max(J')}, x_{\max(I)} y_{\max(J)} : I, I', J \text{ and } J' \text{ satisfy } (*)\} \rangle},$$

where the condition $(*)$ is the following:

- I and I' are poset ideals of P which are incomparable in $\mathscr{J}(P)$;
- J and J' are poset ideals of Q which are incomparable in $\mathscr{J}(Q)$;
- There exists $1 \leq i \leq d$ such that p_i is a maximal element of I and q_i is a maximal element of J.

Hence it is easy to see that the ring homomorphism $\varphi : R^1_{\mathscr{O}\mathscr{C}} \to R^1_{\mathscr{C}\mathscr{C}}$ by setting $\varphi(x_I) = x_{\max(I)}$, $\varphi(y_{\max(J)}) = y_{\max(J)}$ and $\varphi(z) = z$ is an isomorphism. Therefore, one has $R^1_{\mathscr{O}\mathscr{C}} \cong R^1_{\mathscr{C}\mathscr{C}}$. Similarly, if P and Q possess a common linear extension, then from Proposition 9.5, we have

$$R^1_{\mathscr{O}\mathscr{O}} \cong \frac{K_1[\mathscr{O}\mathscr{O}]}{\langle \{x_I x_{I'}, y_J y_{J'}, x_I y_J : I, I', J \text{ and } J' \text{ satisfy } (*)\} \rangle}.$$

Moreover, we can see that the ring homomorphism $\varphi' : R^1_{\mathscr{O}\mathscr{O}} \to R^1_{\mathscr{O}\mathscr{C}}$ by setting $\varphi'(x_I) = x_I$, $\varphi'(y_J) = y_{\max(J)}$ and $\varphi'(z) = z$ is an isomorphism. Hence one has $R^1_{\mathscr{O}\mathscr{O}} \cong R^1_{\mathscr{O}\mathscr{C}} \cong R^1_{\mathscr{C}\mathscr{C}}$. □

Now, we can prove Theorem 9.8

Proof of Theorem 9.8. From Theorem 9.4, we have that both $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ possess the integer decomposition property. Hence the Ehrhart polynomial of $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ (resp. $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$) is equal to the Hilbert function of $K[\Gamma(\mathscr{O}_P, \mathscr{C}_Q)]$ (resp. $K[\Gamma(\mathscr{C}_P, \mathscr{C}_Q)]$). By Proposition 9.9, $R^1_{\mathscr{O}\mathscr{C}}$ and $R^1_{\mathscr{C}\mathscr{C}}$ have the same Hilbert function. Hence $K[\Gamma(\mathscr{O}_P, \mathscr{C}_Q)]$ and $K[\Gamma(\mathscr{C}_P, \mathscr{C}_Q)]$ also have the same Hilbert function. Therefore we have the desired conclusion.

If P and Q possess a common linear extension, $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ also possesses the integer decomposition property from Theorem 9.8. Therefore, by the same argument, we have the desired conclusion. □

We immidiately obtain the following corollary.

Corollary 9.10. *Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$ be finite posets. Then we have*

$$\delta(\Gamma(\mathscr{O}_P, \mathscr{C}_Q), t) = \delta(\Gamma(\mathscr{O}_Q, \mathscr{C}_P), t).$$

In particular, these polytopes have the same volume.

As the end of this section, we give an example that P and Q do not have any common linear extension.

Example 9.11. *Let $P = \{p_1 < p_2\}$ and $Q = \{q_2 < q_1\}$ be chains. It is clear that P and Q have no common linear extension. Then*

$$\delta(\Gamma(\mathscr{O}_P, \mathscr{O}_Q), t) = 1 + 2t,$$

$$\delta(\Gamma(\mathscr{O}_P, \mathscr{C}_Q), t) = \delta(\Gamma(\mathscr{C}_P, \mathscr{C}_Q), t) = 1 + 2t + t^2.$$

9.2.3 A volume formula

In this subsection, we give a formula for the volume of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ in terms of the underlying finite posets. For finite posets P and Q with $P \cap Q = \emptyset$, the *ordinal sum* $P \oplus Q$ of P and Q is the finite poset on $P \cup Q$ such that $s \leq t$ in $P \oplus Q$ if (a) $s, t \in P$ and $s \leq t$ in P, or (b) $s, t \in Q$ and $s \leq t$ in Q, or (c) $s \in P$ and $t \in Q$. Then we have $\mathscr{A}(P \oplus Q) = \mathscr{A}(P) \cup \mathscr{A}(Q)$. Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$. Given a subset W of $[d]$, we define the *induced subposet* of P on W to be the finite poset $P_W = \{p_i : i \in W\}$ such that $p_i \leq p_j$ in P_W if and only if $p_i \leq p_j$ in P. For $W \subset [d]$, we let $\Delta_W(P, Q)$ be the ordinal sum of P_W and $Q_{\overline{W}}$, where $\overline{W} = [d] \setminus W$. Note that $|\Delta_W(P, Q)| = d$ and we have $\mathscr{A}(\Delta_W(P, Q)) = \mathscr{A}(P_W) \cup \mathscr{A}(Q_{\overline{W}})$. Let $W = \{i_1, \dots, i_k\} \subset [d]$ and $\overline{W} = \{i_{k+1}, \dots, i_d\} \subset [d]$ with $W \cup \overline{W} = [d]$. Then we have $\Delta_W(P, Q) = \{p_{i_1}, \dots, p_{i_k}, q_{i_{k+1}}, \dots, q_{i_d}\}$. Also, we let $R = \{r_1, \dots, r_d\}$ be the finite poset such that $r_i \leq r_j$ if (a) $i, j \in W$ and $p_i \leq p_j$ in $\Delta_W(P, Q)$, or (b) $i, j \in \overline{W}$ and $q_i \leq q_j$ in $\Delta_W(P, Q)$, or (c) $i \in W, j \in \overline{W}$ and $p_i \leq q_j$ in $\Delta_W(P, Q)$. We call a permutation $\sigma = i_1 i_2 \cdots i_d$ of $[d]$ a linear extension of $\Delta_W(P, Q)$, if σ is a linear extension of R, and we write $e(\Delta_W(P, Q))$ for the number of linear extensions of $\Delta_W(P, Q)$, i.e., $e(\Delta_W(P, Q)) = e(R)$. For $A \subset \Delta_W(P, Q)$, we define the $(-1, 0, 1)$-vector $\rho'(A) = \sum_{p_i \in A} \mathbf{e}_i - \sum_{q_j \in A} \mathbf{e}_j$ and we set

$$\mathscr{C}'_{\Delta_W(P,Q)} = \mathrm{conv}(\{\rho'(A) \mid A \in \mathscr{A}(\Delta_W(P, Q))\}).$$

First, we will show the following lemma.

Lemma 9.12. *Work with the same situation as above. Then $\mathscr{C}'_{\Delta_W(P,Q)}$ and $\mathscr{C}_R$ are unimodularly equivalent. Moreover we have*

$$\mathrm{Vol}(\mathscr{C}'_{\Delta_W(P,Q)}) = e(\Delta_W(P, Q)).$$

Proof. Let $U = (u_{ij})_{1 \leq i,j \leq d} \in \mathbb{Z}^{d \times d}$ be a unimodular matrix such that

$$u_{ij} = \begin{cases} 1, \text{ if } i = j \text{ and } i \in W, \\ -1, \text{ if } i = j \text{ and } i \in \overline{W}, \\ 0, \text{ if } i \neq j. \end{cases}$$

Then $\mathscr{C}'_{\Delta_W(P,Q)} = f_U(\mathscr{C}_R)$, where R is the finite poset defined by the above. This says that $\mathscr{C}'_{\Delta_W(P,Q)}$ and $\mathscr{C}_R$ are unimodularly equivalent. Hence since the normalized volume of $\mathscr{C}_R$ is equal to $e(R)$, We have

$$\mathrm{Vol}(\mathscr{C}'_{\Delta_W(P,Q)}) = \mathrm{Vol}(\mathscr{C}_R) = e(R) = e(\Delta_W(P,Q)),$$

as desired. $\square$

Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope. Then we write $V(\mathscr{P})$ for the vertex set of $\mathscr{P}$, and for $W \subset [d]$, we set

$$\mathscr{P}_W = \{(x_1,\dots,x_d) \in \mathscr{P} : \text{if } i \in W, x_i \geq 0 \text{ and if } j \in \overline{W}, x_j \leq 0\},$$

$$V_W(\mathscr{P}) = \{(x_1,\dots,x_d) \in V(\mathscr{P}) : \text{if } i \in W,\ x_i \geq 0 \text{ and if } j \in \overline{W},\ x_j \leq 0\}.$$

The following is the key proposition in this subsection.

Proposition 9.13. *Work with the same situation as above. Then we have*

$$\Gamma(\mathscr{C}_P, \mathscr{C}_Q) = \bigcup_{W \subset [d]} \mathscr{C}'_{\Delta_W(P,Q)}.$$

In particular, for any subset $W \subset [d]$, we have

$$\Gamma(\mathscr{C}_P, \mathscr{C}_Q)_W = \mathscr{C}'_{\Delta_W(P,Q)}.$$

Proof. For any $W \subset [d]$, we have

$$V_W(\Gamma(\mathscr{C}_P, \mathscr{C}_Q)) = V(\mathscr{C}'_{\Delta_W(P,Q)}) \setminus \{\mathbf{0}\}$$

since $\mathscr{A}(\Delta_W(P,Q)) = \mathscr{A}(P_W) \cup \mathscr{A}(Q_{\overline{W}})$. Hence it follows that

$$\Gamma(\mathscr{C}_P, \mathscr{C}_Q)_W \supset \mathrm{conv}(V_W(\Gamma(\mathscr{C}_P, \mathscr{C}_Q)) \cup \{\mathbf{0}\}) = \mathscr{C}'_{\Delta_W(P,Q)}.$$

Moreover, we obtain

$$\Gamma(\mathscr{C}_P, \mathscr{C}_Q) \supset \bigcup_{W \subset [d]} \mathscr{C}'_{\Delta_W(P,Q)}.$$

We will show that for any $\mathbf{x}, \mathbf{y} \in V(\Gamma(\mathscr{C}_P, \mathscr{C}_Q))$ and $a, b \in \mathbb{R}$ with $a+b=1, a \geq 0$ and $b \geq 0$, there exists $W \subset [d]$ such that $a\mathbf{x} + b\mathbf{y} \in \mathscr{C}'_{\Delta_W(P,Q)}$. This shows that $\bigcup_{W \subset [d]} \mathscr{C}'_{\Delta_W(P,Q)}$ contains any edge of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$, hence, we have

$$\Gamma(\mathscr{C}_P, \mathscr{C}_Q) \subset \bigcup_{W \subset [d]} \mathscr{C}'_{\Delta_W(P,Q)}.$$

When $\mathbf{x}, \mathbf{y} \in \mathscr{C}_P$ or $\mathbf{x}, \mathbf{y} \in (-\mathscr{C}_Q)$, it clearly follows. Let

$$A_1 = \{p_{i_1}, \dots, p_{i_\ell}, p_{i_{\ell+1}}, \dots, p_{i_m}\}$$

and

$$A_2 = \{q_{i_1}, \ldots, q_{i_\ell}, q_{i_{m+1}}, \ldots, q_{i_n}\}$$

be antichains of $\mathscr{A}(P)$ and $\mathscr{A}(Q)$, and we set $\mathbf{x} = \rho(A_1)$ and $\mathbf{y} = -\rho(A_2)$. We should show the case $a \geq b$. Let $W = \{i_1, \ldots, i_m\} \subset [d]$ and $c = a - b$. Then $A'_1 = \{p_{i_1}, \ldots, p_{i_m}\}$, $A'_2 = \{p_{i_{\ell+1}}, \ldots, p_{i_m}\}$ and $A'_3 = \{q_{i_{m+1}}, \ldots, q_{i_n}\}$ are antichains of $\Delta_W(P,Q)$. We set $\mathbf{x}' = \rho'(A'_1), \mathbf{y}' = \rho'(A'_2)$ and $\mathbf{z}' = \rho'(A'_3)$. Then we have $a\mathbf{x} + b\mathbf{y} = c\mathbf{x}' + b\mathbf{y}' + b\mathbf{z}'$ and $c + 2b = 1$. Hence $a\mathbf{x} + b\mathbf{y} \in \mathscr{C}'_{\Delta_W(P,Q)}$.

Therefore, we have

$$\Gamma(\mathscr{C}_P, \mathscr{C}_Q) = \bigcup_{W \subset [d]} \mathscr{C}'_{\Delta_W(P,Q)}.$$

In particular,

$$\Gamma(\mathscr{C}_P, \mathscr{C}_Q)_W = \mathscr{C}'_{\Delta_W(P,Q)}.$$

as desired. □

Now, we give a formula for the volume of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ in terms of the underlying posets. In particular, the following theorem is immediately given Lemma 9.12 and Proposition 9.13.

Theorem 9.14 ([92, Theorem 1.3])**.** *Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite posets. Then we have*

$$\mathrm{Vol}(\Gamma(\mathscr{C}_P, \mathscr{C}_Q)) = \sum_{W \subset [d]} e(\Delta_W(P,Q)).$$

Moreover, by using Theorems 9.8 and 9.14, one obtains the following corollary.

Corollary 9.15. *Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite posets. Then we have*

$$\mathrm{Vol}(\Gamma(\mathscr{O}_P, \mathscr{C}_Q)) = \sum_{W \subset [d]} e(\Delta_W(P,Q)).$$

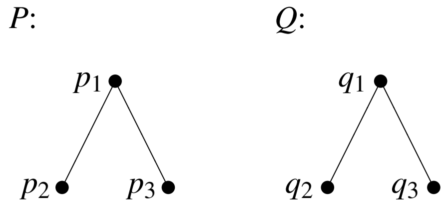
Moreover, if P and Q have a common linear extension, then we have

$$\mathrm{Vol}(\Gamma(\mathscr{O}_P, \mathscr{O}_Q)) = \sum_{W \subset [d]} e(\Delta_W(P,Q)).$$

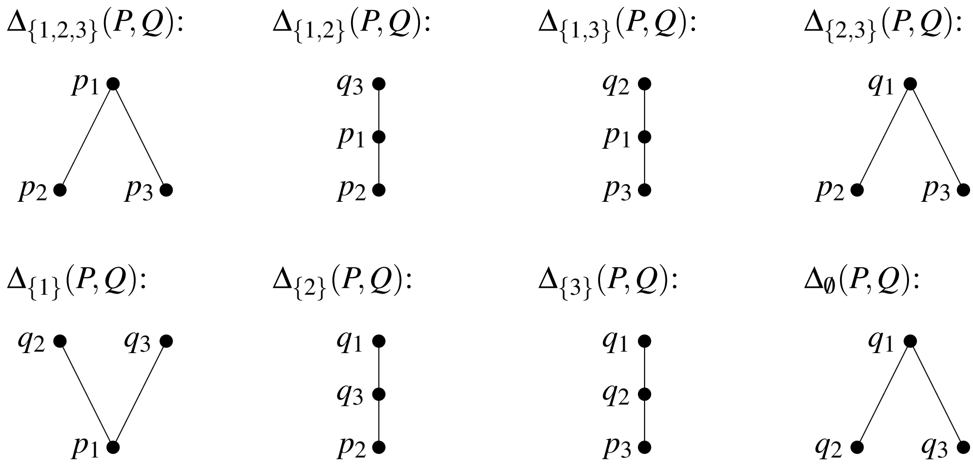
Remark 9.16. By the proof of Proposition 9.13, for any $W \subset [d]$, $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)_W$ is a lattice polytope. However, $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)_W$ and $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)_W$ are not always lattice polytopes. In fact, let $P = \{p_1, p_2\}$ be a 2-element chain with $p_1 \leq p_2$ and $Q = \{q_1, q_2\}$ a 2-element chain with $q_1 \leq q_2$. Then for $W = \{1\}$, we know that $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)_W$ and $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)_W$ are not lattice polytopes. This means that we can not prove Corollary 9.15 by means of a proof similar to that of Theorem 9.14.

We give a few examples.

Example 9.17. Let P and Q be the finite posets with the Haase diagrams shown in the following:



Then $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ is centrally symmetric, i.e., for each facet $\mathscr{F}$ of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$, $-\mathscr{F}$ is a facet of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$. For each subset W of $\{1,2,3\}$, the Haase diagram of $\Delta_W(P,Q)$ is presented in the following:



Hence we have

$$\mathrm{Vol}(\Gamma(\mathscr{C}_P, \mathscr{C}_Q)) = 4 \times 1 + 4 \times 2 = 12.$$

Example 9.18. Let $P = \{p_1, \ldots, p_d\}$ be a d-element antichain and $Q = \{q_1, \ldots, q_d\}$ a d-element chain with $q_1 < \cdots < q_d$. For $W \subset [d]$, we will compute the volume of $\mathscr{C}'_{\Delta_W(P,Q)}$. We set $W = \{1, \ldots, k\}$. Then P_W is a k-element antichain and $Q_{\overline{W}}$ is a $(d-k)$-element chain. Hence we have

$$\mathscr{C}'_{\Delta_W(P,Q)} = \mathrm{conv}(\{[0,1]^k \times \{0\}^{d-k}, -\mathbf{e}_{k+1}, \ldots, -\mathbf{e}_d\})$$

and $\mathrm{Vol}(\mathscr{C}'_{\Delta_W(P,Q)}) = k!$. Therefore, we obtain

$$\begin{aligned}\mathrm{Vol}(\Gamma(\mathscr{C}_P, \mathscr{C}_Q)) &= \sum_{k=0}^{d} \binom{d}{k} k! \\ &= \sum_{k=0}^{d} \frac{d!}{k!}.\end{aligned}$$

For a positive integer d, we write $a(d)$ for the total number of arrangements of a d-element set. Then we have

$$\mathrm{Vol}(\Gamma(\mathscr{C}_P, \mathscr{C}_Q)) = a(d).$$

Next, as an application of Proposition 9.13 we will compute the equations of the supporting hyperplanes of facets and dual polytopes of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$. We begin by recalling these features for the chain polytopes which were originally studied in [81]. Let $P = \{p_1, \dots, p_d\}$ be a finite poset. Then there are two types of the equations of the supporting hyperplanes of facets for the chain polytope $\mathscr{C}_P$:

- for each element p_i of P, $x_i = 0$,
- for each maximal chain C of P, $\sum_{p_i \in C} x_i = 1$.

We write $\mathscr{M}(P)$ for the set of maximal chains of P. Then the number of facets of $\mathscr{C}_P$ equals $|\mathscr{M}(P)| + d$.

The next lemma follows immediately from Lemma 9.12.

Lemma 9.19. *Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$ be finite posets, and let $W \subset [d]$. Then there are three types of the equations of the supporting hyperplane of facets for $\mathscr{C}'_{\Delta_W(P,Q))}$:*

- *for each element p_i of $\Delta_W(P,Q)$, $x_i = 0$,*
- *for each element q_j of $\Delta_W(P,Q)$, $-x_j = 0$,*
- *for each maximal chain C of $\Delta_W(P,Q)$, $\sum_{p_i \in C} x_i - \sum_{q_j \in C} x_j = 1$.*

Now, we characterize the equations of the supporting hyperplanes of facets of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ in terms of the underlying posets. Namely, we prove the following theorem.

Theorem 9.20 ([92, Theorem 2.2])**.** *Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$ be finite posets. The equations of the supporting hyperplanes of facets for $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are given as*

$$\sum_{p_i \in C} x_i - \sum_{q_j \in C} x_j = 1$$

for each $W \subset [d]$ and for each maximal chain C of $\Delta_W(P,Q)$. Moreover, the number of facets of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ equals $|\bigcup_{W \subset [d]} \mathscr{M}(\Delta_W(P,Q))|$.

Proof. We let W be a subset of $[d]$ and let C be a maximal chain of $\Delta_W(P,Q)$. Then by Lemma 9.19, $\mathscr{F}_C = \mathscr{H}_C \cap \mathscr{C}'_{\Delta_W(P,Q)}$ is a facet of $\mathscr{C}'_{\Delta_W(P,Q)}$, where $\mathscr{H}_C$ is the hyperplane

$$\{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{p_i \in C} x_i - \sum_{q_j \in C} x_j = 1\}$$

in $\mathbb{R}^d$. We let $\mathbf{y} = (y_1,\dots,y_d)$ be an interior point of $\mathscr{F}_C$. Then by Lemma 9.19, we know $y_i > 0$ if $i \in W$ and $y_j < 0$ if $j \in \overline{W}$. Hence for any $W' \subset [d]$ with $W \neq W'$, we have $\mathbf{y} \notin \mathscr{C}'_{\Delta_{W'}(P,Q)}$. Therefore, it follows that $\mathbf{y}$ does not belong to the interior of $\Gamma_{\mathscr{C}(P),\mathscr{C}(Q)}$. By Proposition 9.13, $\mathscr{H}_C \cap \Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is a facet of $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$.

Since $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is reflexive, from Lemma 2.7 the equation of the supporting hyperplane of each facet of $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is of the form $a_1x_1+\cdots+a_dx_d = 1$ with each $a_i \in \mathbb{Z}$. Hence the equations of the supporting hyperplanes of facets for $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ are given as

$$\sum_{p_i \in C} x_i - \sum_{q_j \in C} x_j = 1$$

for each $W \subset [d]$ and for each maximal chain C of $\Delta_W(P,Q)$, as desired. □

Remark 9.21. For some finite posets $P = \{p_1,\dots,p_d\}$ and $Q = \{q_1,\dots,q_d\}$, we have

$$\sum_{W \subset [d]} |\mathscr{M}(\Delta_W(P,Q))| \neq |\bigcup_{W \subset [d]} \mathscr{M}(\Delta_W(P,Q))|.$$

For instance, let $P = \{p_1,p_2,p_3\}$ and $Q = \{q_1,q_2,q_3\}$ be 3-element antichains. For $W_1 = \{1\}$, $C_1 = \{p_1,q_3\}$ is a maximal chain of $\Delta_{W_1}(P,Q)$. Then for $W_2 = \{1,2\}$, C_1 is also a maximal chain of $\Delta_{W_2}(P,Q)$. Hence we have

$$\sum_{W \subset [d]} |\mathscr{M}(\Delta_W(P,Q))| > |\bigcup_{W \subset [d]} \mathscr{M}(\Delta_W(P,Q))|.$$

By using Lemma 2.7 and Theorem 9.20, one has the following corollary.

Corollary 9.22. *Let $P = \{p_1,\dots,p_d\}$ and $Q = \{q_1,\dots,q_d\}$ be finite posets. Then we have*

$$V(\Gamma(\mathscr{C}_P,\mathscr{C}_Q)^\vee) = \bigcup_{W \subset [d]} \{\rho'(C) \in \mathbb{R}^d : C \in \mathscr{M}(\Delta_W(P,Q))\}.$$

Namely,

$$\Gamma(\mathscr{C}_P,\mathscr{C}_Q)^\vee = \operatorname{conv}\left(\bigcup_{W \subset [d]} \{\rho'(C) \in \mathbb{R}^d : C \in \mathscr{M}(\Delta_W(P,Q))\}\right).$$

We end this subsection with a pair of examples demonstrating Theorem 9.20 and Corollary 9.22.

Example 9.23. Let P and Q be the finite posets as in Example 9.18. We fix $W = \{i_1,\dots,i_k\} \subset [d]$. Then we have

$$\mathscr{M}(\Delta_W(P,Q)) = \{\{p_{i_s}, q_{i_{k+1}},\dots,q_{i_d}\} : 1 \le s \le k\}$$

and $|\mathscr{M}(\Delta_W(P,Q))| = k$. Hence

$$\begin{aligned} |\bigcup_{W\subset[d]} \mathscr{M}(\Delta_W(P,Q))| &= \sum_{k=1}^{d} \binom{d}{k} k + 1 \\ &= d \cdot 2^{d-1} + 1. \end{aligned}$$

Example 9.24. Let P and Q be the finite posets as in Example 9.17. Then by Corollary 9.22, the vertices of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)^\vee$ are the following:

$$\pm(1,1,0), \pm(1,0,1), \pm(1,-1,0), \pm(1,1,-1), \pm(1,-1,1), \pm(1,0,-1).$$

Moreover, there do not exist finite posets P' and Q' with $|P'| = |Q'| = 3$ such that $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)^\vee$ and $\Gamma(\mathscr{C}_{P'}, \mathscr{C}_{Q'})$ are unimodularly equivalent. Indeed, since $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)^\vee$ is centrally symmetric and the number of its vertices equals 12, each of P' and Q' needs to have just 7 antichains. However, there exists no 3-element partially ordered set which has just 7 antichains.

9.2.4 When are they smooth Fano?

In this subsection, we consider the characterization problem of finite posets yield smooth Fano polytopes. First, we recall several classes of lattice polytopes related to reflexive polytopes. Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d containing the origin of $\mathbb{R}^d$ in its interior.

- $\mathscr{P}$ is called *Fano*, if the vertices are primitive lattice points.
- $\mathscr{P}$ is called *canonical Fano*, if $\text{int}(\mathscr{P}) \cap \mathbb{Z}^d = \{\mathbf{0}\}$.
- $\mathscr{P}$ is called *terminal Fano*, if $\mathscr{P} \cap \mathbb{Z}^d = \{\mathbf{0}\} \cup V(\mathscr{P})$.
- $\mathscr{P}$ is called *smooth Fano*, if the vertices of any facet of $\mathscr{P}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

It is known that Fano polytopes correspond to toric Fano varieties, smooth Fano polytopes to nonsingular toric Fano varieties and canonical (respectively terminal) Fano polytopes to toric Fano varieties with canonical (respectively terminal) singularities. Moreover simplicial Fano polytopes are associated to $\mathbb{Q}$-factorial toric Fano varieties. In particular, each smooth Fano polytope is a simplicial reflexive polytope, and each reflexive polytope is a canonical Fano polytope.

Let $\mathscr{P} \subset \mathbb{R}^d$ be a Fano polytope.

- We call $\mathscr{P}$ *centrally symmetric* if $\mathscr{P} = -\mathscr{P}$.

- We call $\mathscr{P}$ *pseudo-symmetric* if there exists a facet $\mathscr{F}$ of $\mathscr{P}$ such that $-\mathscr{F}$ is also its facet. Note that every centrally symmetric polytope is pseudo-symmetric.

- A *del Pezzo polytope* of dimension $2k$ is a convex polytope

$$V_{2k} = \mathrm{conv}(\{\pm\mathbf{e}_1,\dots\pm\mathbf{e}_{2k},\pm(\mathbf{e}_1+\cdots+\mathbf{e}_{2k})\}).$$

  Note that del Pezzo polytopes are centrally symmetric smooth Fano polytopes.

- A *pseudo del Pezzo polytope* of dimension $2k$ is a convex polytope

$$\tilde{V}_{2k} = \mathrm{conv}(\{\pm\mathbf{e}_1,\dots\pm\mathbf{e}_{2k},\mathbf{e}_1+\cdots+\mathbf{e}_{2k}\}).$$

  Note that pseudo del Pezzo polytopes are pseudo-symmetric smooth Fano polytopes.

- Let us that $\mathscr{P}$ *splits* into $\mathscr{P}_1$ and $\mathscr{P}_2$ if the convex hull of two Fano polytopes $\mathscr{P}_1 \subset \mathbb{R}^{d_1}$ and $\mathscr{P}_2 \subset \mathbb{R}^{d_2}$ with $d = d_1 + d_2$, i.e., by renumbering

$$\mathscr{P} = \mathrm{conv}(\{(\alpha_1,0),(0,\alpha_2) \in \mathbb{R}^d : \alpha_1 \in \mathscr{P}_1, \alpha_2 \in \mathscr{P}_2\}).$$

  Then we write $\mathscr{P} = \mathscr{P}_1 \oplus \mathscr{P}_2$.

There is well-known fact on the characterization of centrally symmetric or pseudo-symmetric smooth Fano polytopes.

- Any centrally symmetric smooth reflexive polytope splits into copies of the closed interval $[-1,1]$ or a del Pezzo polytope [95].

- Any pseudo-symmetric smooth reflexive polytope splits into copies of the closed interval $[-1,1]$ or a del Pezzo polytope or pseudo del Pezzo polytope [26, 95].

Let $P = \{p_1,\dots,p_d\}$ and $Q = \{q_1,\dots,q_d\}$ be finite posets. In this subsection, we consider when each of $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is a smooth Fano polytope.

First, we consider when $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is smooth Fano. For $1 \le i \le d$, we set $\mathscr{A}_i(P) = \{I \in \mathscr{A}(P) : |I| = i\}$.

Theorem 9.25 ([45, Theorem 2.1])**.** *Let $P = \{p_1,\dots,p_d\}$ and $Q = \{q_1,\dots,q_d\}$ be finite posets. Then the following conditions are equivalent:*

(i) *$\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is simplicial;*

(ii) *$\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is smooth Fano;*

(iii) *$\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ splits into copies of the closed interval $[-1,1]$ or a del Pezzo 2-polytope or a pseudo del Pezzo 2-polytope;*

(iv) *For any* $I_1, I_2 \in \mathscr{A}_2(P)$ *with* $I_1 \neq I_2$, $I_1 \cap I_2 = \emptyset$ *and for any* $J_1, J_2 \in \mathscr{A}_2(Q)$ *with* $J_1 \neq J_2$, $J_1 \cap J_2 = \emptyset$, *and for any* $I \in \mathscr{A}_2(P)$ *and for any* $J \in \mathscr{A}_2(Q)$, $|\{i : p_i \in I, q_i \in J\}| \neq 1$.

Proof. **((i) ⇒ (iv))** Let $p_{i_1} < p_{i_2} < \cdots < p_{i_s}$ be a maximal chain of P. Then $x_{i_1} + x_{i_2} + \cdots + x_{i_s} = 1$ is the equation of the supporting hyperplane of a facet of $\mathscr{C}_P$, in particular, this is a facet of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$. Since $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ is simplicial, this facet is a $(d-1)$-simplex. Hence there exist just $d-s$ antichains $I_1, \dots, I_{d-s} \in \mathscr{A}(P) \setminus \mathscr{A}_1(P)$ such that for each I_k, $|\{p_{i_1}, p_{i_2}, \dots, p_{i_s}\} \cap I_k| = 1$. Since for each $j \in P \setminus \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$, there exists $i \in \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$ such that $\{i, j\}$ is an antichain of P, for each $j \in P \setminus \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$, there exists just one $i \in \{p_{i_1}, p_{i_2}, \dots, p_{i_s}\}$ such that $\{i, j\}$ is an antichain of P. Then for $k \geq 3$, $\mathscr{A}_k(P) = \emptyset$.

First, we assume that there exist $I_1, I_2 \in \mathscr{A}_2(P)$ with $I_1 \neq I_2$ such that $I_1 \cap I_2 \neq \emptyset$. Let $I_1 = \{p_{i_1}, p_{i_2}\}$ and $I_2 = \{p_{i_1}, p_{i_3}\}$. Then we know that $\{p_{i_2}, p_{i_3}\}$ is not an antichain of P. Indeed, if $\{p_{i_2}, p_{i_3}\}$ is an antichain of P, then $\{p_{i_1}, p_{i_2}, p_{i_3}\}$ is also an antichain of P. Hence there exists a maximal chain $p_{j_1} < p_{j_2} < \cdots < p_{j_t}$ of P such that $\{p_{i_2}, p_{i_3}\} \subset \{p_{j_1}, p_{j_2} \dots, p_{j_t}\}$. Then since $\{p_{i_1}, p_{i_2}\}$ and $\{p_{i_1}, p_{i_3}\}$ are antichains of P, a facet whose equation is $x_{j_1} + x_{j_2} + \cdots + x_{j_t} = 1$ of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ is not a $(d-1)$-simplex.

Next, we assume that for any $I_1, I_2 \in \mathscr{A}_2(P)$ with $I_1 \neq I_2$, $I_1 \cap I_2 = \emptyset$, and for any $J_1, J_2 \in \mathscr{A}_2(Q)$ with $J_1 \neq J_2$, $J_1 \cap J_2 = \emptyset$, and there exist $I \in \mathscr{A}_2(P)$ and $J \in \mathscr{A}_2(Q)$ such that $|\{i : p_i \in I, q_i \in J\}| = 1$. We let $I = \{p_{i_1}, p_{i_2}\}$ and $J = \{q_{i_1}, q_{i_3}\}$. Then $x_{i_2} - x_{i_3} = 1$ is the equation of the supporting hyperplane of a face of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ and this face is not simplex. Indeed, we set $\mathscr{H} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \colon x_{i_2} - x_{i_3} = 1\}$ and $\mathscr{H}^+ = \{(x_1, \dots, x_d) \in \mathbb{R}^d \colon x_{i_2} - x_{i_3} \leq 1\}$. Then every vertex of $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ belongs to $\mathscr{H}^+$, and

$$\rho(\{p_{i_1}, p_{i_2}\}), \rho(\{p_{i_2}\}), -\rho(\{q_{i_1}, q_{i_3}\}), -\rho(\{q_{i_3}\}) \in \mathscr{H}.$$

Since

$$\begin{aligned}(\rho(\{p_{i_1}, p_{i_2}\}) - (-\rho(\{q_{i_3}\}))) = (&\rho(\{p_{i_2}\}) - (-\rho(\{q_{i_3}\}))) \\ &- (-\rho(\{q_{i_1}, q_{i_3}\}) - (-\rho(\{q_{i_3}\}))),\end{aligned}$$

this face is not a simplex.

((iv) ⇒ (iii)) We assume that

$$\mathscr{A}_2(P) = \{\{p_1, p_2\}, \dots, \{p_{2k-1}, p_{2k}\}, \{p_{2k+1}, p_{2k+2}\}, \dots, \{p_{2k+2l-1}, p_{2k+2l}\}\},$$

$$\mathscr{A}_2(Q) = \{\{q_1, q_2\}, \dots, \{q_{2k-1}, q_{2k}\}, \{q_{2k+2l+1}, q_{2k+2l+2}\}, \dots, \{q_{2k+2l+2m-1}, q_{2k+2l+2m}\}\},$$

where k, l and m are nonnegative integers with $2k + 2l + 2m \leq d$. Then it follows that $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ is the convex full of $\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_d$ and $\pm(\mathbf{e}_1 + \mathbf{e}_2), \dots, \pm(\mathbf{e}_{2k-1} + \mathbf{e}_{2k})$ and $\mathbf{e}_{2k+1} + \mathbf{e}_{2k+2}, \dots, \mathbf{e}_{2k+2l-1} + \mathbf{e}_{2k+2l}$ and $-(\mathbf{e}_{2k+2l-1} + \mathbf{e}_{2k+2l+2}), \dots, -(\mathbf{e}_{2k+2l+2m-1} + \mathbf{e}_{2k+2l+2m})$. Hence $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ splits into copies of the closed interval $[-1, 1]$ or a del Pezzo 2-polytope or a pseudo del Pezzo 2-polytope.

((iii) $\Rightarrow$ (ii) $\Rightarrow$(i)) Since $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ splits into copies of the closed interval $[-1,1]$ or a del Pezzo 2-polytope or a pseudo del Pezzo 2-polytope, $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$is smooth Fano. Moreover, in general, any smooth Fano polytope is simplicial. $\square$

Next, we consider when $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is smooth Fano.

Theorem 9.26 ([45, Theorem 2.2])**.** *Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite posets. Then the following conditions are equivalent:*

(i) $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ *is simplicial;*

(ii) $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ *is smooth Fano;*

(iii) $\mathscr{J}(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\}$ *or*
$\mathscr{J}(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\}$, *and*
$\mathscr{A}(Q) = \{\{q_{i_1}\}, \{q_{i_2}\}, \ldots, \{q_{i_d}\}\}$ *or*
$\mathscr{A}(Q) = \{\{q_{i_1}\}, \{q_{i_2}\}, \ldots, \{q_{i_d}\}, \{q_{i_1}, q_{i_2}\}\}$.

Proof. **((i) $\Rightarrow$ (iii))** We may assume that p_{i_1} is a minimal element of P and $\mathscr{J}'(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\} \subset \mathscr{J}(P)$. Then $x_{i_1} = 1$ is the equation of the supporting hyperplane of a facet of $\mathscr{O}_P$, in particular, this is a facet of $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$. Since $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is simplicial, this facet is a $(d-1)$-simplex. Hence there is no poset ideal $I \in \mathscr{J}(P)$ such that $p_{i_1} \in I$ and $I \notin \mathscr{J}'(P)$. If there exists $I \in \mathscr{J}(P)$ such that $p_{i_1} \notin I$, there exists a minimal element $p_{i_j} \in I$ of P. Then since $\{\{p_{i_1}, p_{i_j}\}\}$ is a poset ideal of P, we have $j = 2$. Hence we know that $\mathscr{J}(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\}$ or $\mathscr{J}(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\}$. Also, by the proof of Theorem 9.25, we may assume that for any $J_1, J_2 \in \mathscr{A}_2(Q)$ with $J_1 \neq J_2$, $J_1 \cap J_2 = \emptyset$.

We assume that $\mathscr{J}(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\}$. If $\{q_{i_j}, q_{i_k}\}$ is an antichain of Q with $2 \leq j < k$, then $x_{i_1} - x_{i_k} = 1$ is the equation of the supporting hyperplane of a face of $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and this face is not a simplex. Indeed, we set $\mathscr{H}_1 = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_{i_1} - x_{i_k} = 1\}$ and $\mathscr{H}_1^+ = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_{i_1} - x_{i_k} \leq 1\}$. Then every vertex of $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ belongs to $\mathscr{H}_1^+$. Also, one has

$$\rho(\{p_{i_1}\}), \rho(\{p_{i_1}, p_{i_2}\}), \ldots, \rho(\{p_{i_1}, \ldots, p_{i_{k-1}}\}), -\rho(\{q_{i_k}\}), -\rho(\{q_{i_j}, q_{i_k}\}) \in \mathscr{H}_1.$$

Since

$$\begin{aligned}(-\rho(\{q_{i_k}\}) - \rho(\{p_{i_1}\})) = &(-\rho(\{q_{i_j}, q_{i_k}\}) - \rho(\{p_{i_1}\})) \\ &+ (\rho(\{p_{i_1}, \ldots, p_{i_j}\}) - \rho(\{p_{i_1}\})) \\ &- (\rho(\{p_{i_1}, \ldots, p_{i_{j-1}}\}) - \rho(\{p_{i_1}\})),\end{aligned}$$

this face is not a simplex. If $\{q_{i_1}, q_{i_j}\}$ is an antichain of Q with $3 \leq j$, then $-x_{i_1} + 2x_{i_2} = 1$ is the equation of a supporting hyperplane of a face of $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and this face is not a simplex. Indeed, we set $\mathscr{H}_2 = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : -x_{i_1} + 2x_{i_2} = 1\}$ and $\mathscr{H}_2^+ =$

$\{(x_1,\dots,x_d)\in\mathbb{R}^d : -x_{i_1}+2x_{i_2}\le 1\}$. Then each vertex of $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ belongs to $\mathscr{H}_2^+$. Also, one has

$$\rho(\{p_{i_1},p_{i_2}\}),\dots,\rho(\{p_{i_1},\dots,p_{i_d}\}),-\rho(\{q_{i_1}\}),-\rho(\{q_{i_1},q_{i_j}\})\in\mathscr{H}_2.$$

Hence since the face $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)\cap\mathscr{H}_2$ has $d+1$ vertices, this face is not a simplex.

We assume that $\mathscr{J}(P)=\{\{p_{i_1}\},\{p_{i_2}\},\{p_{i_1},p_{i_2}\},\dots,\{p_{i_1},\dots,p_{i_d}\}\}$. If $\{q_{i_j},q_{i_k}\}$ is an antichain of Q with $2\le j<k$, then similarly, $x_{i_1}-x_{i_k}=1$ is the equation of a supporting hyperplane of a face of $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and this face is not a simplex. If $\{q_{i_1},q_{i_j}\}$ is an antichain of Q with $3\le j$, then $x_{i_2}-x_{i_j}=1$ is the equation of a supporting hyperplane of a face of $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and this face is not a simplex. Indeed, we set $\mathscr{H}_3=\{(x_1,\dots,x_d)\in\mathbb{R}^d : x_{i_2}-x_{i_j}=1\}$ and $\mathscr{H}_3^+=\{(x_1,\dots,x_d)\in\mathbb{R}^d : x_{i_2}-x_{i_j}\le 1\}$. Then every vertex of $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ belongs to $\mathscr{H}_3^+$, and one has

$$\rho(\{p_{i_2}\}),\rho(\{p_{i_1},p_{i_2}\}),\dots,\rho(\{p_{i_1},\dots,p_{i_{j-1}}\}),-\rho(\{q_{i_j}\}),-\rho(\{q_{i_1},q_{i_j}\})\in\mathscr{H}_3.$$

Since

$$\begin{aligned}(\rho(\{p_{i_2}\})-\rho(\{p_{i_1},p_{i_2},p_{i_3}\}))=&(\rho(\{p_{i_1},p_{i_2}\})-\rho(\{p_{i_1},p_{i_2},p_{i_3}\}))\\&+(-\rho(\{q_{i_1},q_{i_j}\})-\rho(\{p_{i_1},p_{i_2},p_{i_3}\}))\\&-(-\rho(\{q_{i_j}\})-\rho(\{p_{i_1},p_{i_2},p_{i_3}\})),\end{aligned}$$

this face is not a simplex.

((iii) ⇒ (ii)) If $\mathscr{P}\subset\mathbb{R}^d$ is a smooth Fano polytope of dimension d, the lattice polytope

$$\mathscr{P}'=\mathrm{conv}(\mathscr{P}\cup\{\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_{d+1},-\mathbf{e}_{d+1}\})\subset\mathbb{R}^{d+1}$$

is also smooth Fano. Moreover, if $d=2$, then $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ is smooth Fano. Hence for $d\ge 2$, we know that $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ is smooth.

((ii) ⇒(i)) In general, any smooth Fano polytope is simplicial. □

Finally, we consider when $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$ is smooth Fano.

Theorem 9.27 ([45, Theorem 2.3]). *Let $P=\{p_1,\dots,p_d\}$ and $Q=\{q_1,\dots,q_d\}$ be finite posets. Assume that P and Q have a common linear extention. Then the following conditions are equivalent:*

(i) $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$ *is simplicial;*

(ii) $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$ *is smooth Fano;*

(iii) $\mathscr{J}(P)=\{\{p_{i_1}\},\{p_{i_1},p_{i_2}\},\dots,\{p_{i_1},\dots,p_{i_d}\}\}$ *or*
$\mathscr{J}(P)=\{\{p_{i_1}\},\{p_{i_2}\},\{p_{i_1},p_{i_2}\},\dots,\{p_{i_1},\dots,p_{i_d}\}\}$, *and*

$\mathscr{J}(Q) = \{\{q_{i_1}\}, \{q_{i_1}, q_{i_2}\}, \ldots, \{q_{i_1}, \ldots, q_{i_d}\}\}$ *or*
$\mathscr{J}(Q) = \{\{q_{i_1}\}, \{q_{i_2}\}, \{q_{i_1}, q_{i_2}\}, \ldots, \{q_{i_1}, \ldots, q_{i_d}\}\}$.

Proof. **((i) $\Rightarrow$ (iii))** By the proof of Theorem 9.26, We have $\mathscr{J}(P) = \{\{p_{i_1}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\}$ or $\mathscr{J}(P) = \{\{p_{i_1}\}, \{p_{i_2}\}, \{p_{i_1}, p_{i_2}\}, \ldots, \{p_{i_1}, \ldots, p_{i_d}\}\}$. Also, we have $\mathscr{J}(Q) = \{\{q_{j_1}\}, \{q_{j_1}, q_{j_2}\}, \ldots, \{q_{j_1}, \ldots, q_{j_d}\}\}$ or $\mathscr{J}(Q) = \{\{q_{j_1}\}, \{q_{j_2}\}, \{q_{j_1}, q_{j_2}\}, \ldots, \{q_{j_1}, \ldots, q_{j_d}\}\}$. Since P and Q have a common linear extention, we may assume that for any $1 \leq k \leq d$, $i_k = j_k$.

((iii) $\Rightarrow$ (ii)) If $\mathscr{P} \subset \mathbb{R}^d$ is a smooth Fano polytope of dimension d, the lattice polytope

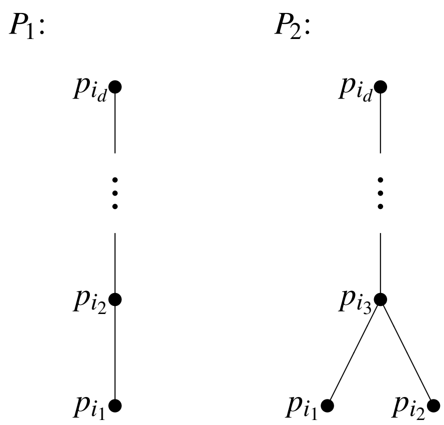
$$\mathscr{P}' = \mathrm{conv}(\mathscr{P} \cup \{\pm(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{d+1})\}) \subset \mathbb{R}^{d+1}$$

is also smooth Fano. Also, if $d = 2$, then $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ is smooth Fano. Hence for $d \geq 2$, we know that $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ is smooth Fano.

((ii) $\Rightarrow$(i)) In general, any smooth Fano polytope is simplicial. $\square$

Theorem 9.28 ([45, Theorem 3.1])**.** *Let $d \geq 3$ and $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite posets. Assume that $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are smooth Fano. Then $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are unimodularly equivalent. However, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is not unimodularly equivalent to these polytopes. Moreover, if $P \neq Q$, then $\Gamma(\mathscr{O}_Q, \mathscr{C}_P)$ is also smooth Fano and is not unimodularly equivalent to $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$.*

Proof. Let P_1, P_2 be the finite posets as follows.



By Theorem 9.27, $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ is smooth Fano if and only if $P, Q \in \{P_1, P_2\}$. Also when $P, Q \in \{P_1, P_2\}$, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are smooth Fano by Theorems 9.25 and 9.26. Hence it follows that $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are smooth Fano if and only if $P, Q \in \{P_1, P_2\}$. We can assume that $i_j = j$ for any $1 \leq j \leq d$.

We should consider the following four cases.

(The case $P = Q = P_1$) $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are unimodularly equivalent, in particular, these polytopes are centrally symmetric. However, since $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is not centrally symmetric, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is not unimodularly equivalent to these polytopes.

(The case $P = Q = P_2$) Similarly, $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are unimodularly equivalent, and $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is not unimodularly equivalent to these polytopes.

(The case $P = P_1$ and $Q = P_2$) $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$ are unimodularly equivalent, in particular, these polytopes are pseudo-symmetric. However, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is not unimodularly equivalent to these polytopes, since it follows that $|\{\mathbf{v} \in V(\Gamma(\mathscr{O}_P, \mathscr{C}_Q)) : -\mathbf{v} \in V(\Gamma(\mathscr{O}_P, \mathscr{C}_Q))\}| \neq |\{\mathbf{v} \in V(\Gamma(\mathscr{O}_P, \mathscr{O}_Q)) : -\mathbf{v} \in V(\Gamma(\mathscr{O}_P, \mathscr{O}_Q))\}|$, where we write $V(\mathscr{P})$ for the vertex set of a polytope $\mathscr{P}$.

(The case $P = P_2$ and $Q = P_1$) Similarly, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is not unimodularly equivalent to $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$. Moreover, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{O}_Q, \mathscr{C}_P)$ are not unimodularly equivalent. Indeed, we assume that these polytopes are unimodularly equivalent. Then there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ such that $\Gamma(\mathscr{O}_P, \mathscr{C}_Q) = f_U(\Gamma(\mathscr{O}_Q, \mathscr{C}_P))$. Also for $\mathbf{v} \in \{\pm\mathbf{e}_1, \pm(\mathbf{e}_1 + \mathbf{e}_2)\}$, there exists $\mathbf{v}' \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$ such that $f_U(\mathbf{v}) = \mathbf{v}'$.

If $f_U(\mathbf{e}_1) = \mathbf{e}_1$ and $f_U(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{e}_2$, we have

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

Then $f_U(-\mathbf{e}_2) = \mathbf{e}_1 - \mathbf{e}_2 \notin V(\Gamma(\mathscr{O}(P), -\mathscr{C}(Q)))$.

If $f_U(\mathbf{e}_1) = \mathbf{e}_1$ and $f_U(\mathbf{e}_1 + \mathbf{e}_2) = -\mathbf{e}_2$, we have

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

Then $f_U(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = (u_{31}, u_{32} - 1, u_{33}, \dots, u_{3d})$ and $f_U(-\mathbf{e}_3) = (-u_{31}, \dots, -u_{3d})$. Since $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is a $(-1,0,1)$-polytope, $u_{32} = 0$ or 1. Then $f_U(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = -\mathbf{e}_2$ or $f_U(-\mathbf{e}_3) = -\mathbf{e}_2$, a contradiction.

If $f_U(\mathbf{e}_1) = -\mathbf{e}_1$ and $f_U(\mathbf{e}_1+\mathbf{e}_2) = \mathbf{e}_2$, we have

$$U = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d\times d}.$$

Then $f_U(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) = (u_{31}, u_{32}+1, u_{33}, \ldots, u_{3d})$ and $f_U(-\mathbf{e}_3) = (-u_{31}, \ldots, -u_{3d})$. Since $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ is a $(-1,0,1)$-polytope, $u_{32} = 0$ or -1. Then $f_U(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3) = \mathbf{e}_2$ or $f(-\mathbf{e}_3) = \mathbf{e}_2$, a contradiction.

If $f_U(\mathbf{e}_1) = -\mathbf{e}_1$ and $f_U(\mathbf{e}_1+\mathbf{e}_2) = -\mathbf{e}_2$, we have

$$U = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & u_{33} & \cdots & u_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & u_{d3} & \cdots & u_{dd} \end{pmatrix} \in \mathbb{Z}^{d\times d}.$$

Then $f_U(-\mathbf{e}_2) = -\mathbf{e}_1+\mathbf{e}_2 \notin V(\Gamma(\mathscr{O}_P, \mathscr{C}_Q))$.

Therefore, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{O}_Q, \mathscr{C}_P)$ are not unimodularly equivalent. □

By Theorems 9.8 and 9.28, the following corollary immidiately follows.

Corollary 9.29. *For any $d \geq 3$, there exist smooth Fano polytopes $\mathscr{P}$ and $\mathscr{Q}$ such that the following conditions satisfied:*

- *$\mathscr{P}$ and $\mathscr{Q}$ have the same Ehrhart polynomial.*
- *$\mathscr{P}$ and $\mathscr{Q}$ are not unimodularly equivalent.*

Now, we recall the following fact.

Lemma 9.30 ([16, Theorem 1])**.** *Let $\mathscr{P}_1$ be a d_1-dimensional reflexive polytope in $\mathbb{R}^{d_1}$ and $\mathscr{P}_2$ a d_2-dimensional lattice polytope in $\mathbb{R}^{d_2}$ with $\mathbf{0} \in int(\mathscr{P}_1)$. Then one has*

$$\delta(\mathscr{P}_1 \oplus \mathscr{P}_2, t) = \delta(\mathscr{P}_1, t)\delta(\mathscr{P}_2, t).$$

In particular,

$$\mathrm{Vol}(\mathscr{P}_1 \oplus \mathscr{P}_2) = \mathrm{Vol}(\mathscr{P}_1)\mathrm{Vol}(\mathscr{P}_2).$$

We let l, m, n be nonnegative integers and

$$\mathscr{P} = (\oplus_l L) \oplus (\oplus_m \tilde{V}_2) \oplus (\oplus_n V_2),$$

where L is the closed interval $[-1,1]$. Since $\mathrm{Vol}(L)=2, \mathrm{Vol}(\tilde{V}_2)=5$ and $\mathrm{Vol}(V_2)=6$, we have $\mathrm{Vol}(\mathscr{P})=2^l\cdot 5^m\cdot 6^n$.

Finally, we consider the volume of each of $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ when these polytopes are smooth Fano. Let P_1 and P_2 be the finite posets as in the proof of Theorem 9.28.

Example 9.31. (i) Let $P=Q=P_1$. Then $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is unimodularly equivalent to $\oplus_d L$. Hence we know the normalized volume of each of $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is equal to 2^d by Theorem 9.8.

(ii) Let $P=Q=P_2$. Then $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is unimodularly equivalent to $(\oplus_{d-2}L)\oplus V_2$. Hence the normalized volume of each of $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is equal to $2^{d-2}\cdot 6$.

(iii) Let $P=P_1$ and $Q=P_2$. Then $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is unimodularly equivalent to $(\oplus_{d-2}L)\oplus\tilde{V}_2$. Hence the normalized volume of each of $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P,\mathscr{C}_Q)$ is equal to $2^{d-2}\cdot 5$. In particular, the normalized volume of $\Gamma(\mathscr{O}_Q,\mathscr{O}_P)$ is also $2^{d-2}\cdot 5$.

9.3 Type Ω

In this section, we consider $\Omega(\mathscr{O}_P,\mathscr{O}_Q)$, $\Omega(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Omega(\mathscr{C}_P,\mathscr{C}_Q)$ for finite posets $P=\{p_1,\dots,p_d\}$ and $Q=\{q_1,\dots,q_d\}$.

9.3.1 When are they reflexive?

In this subsection, we discuss when $\Omega(\mathscr{O}_P,\mathscr{O}_Q)$, $\Omega(\mathscr{O}_P,\mathscr{C}_Q)$ and $\Omega(\mathscr{C}_P,\mathscr{C}_Q)$ are reflexive polytope. In fact, we show the following theorem.

Theorem 9.32 ([49, Theorem 1.3])**.** *Let $P=\{p_1,\dots,p_d\}$ and $Q=\{q_1,\dots,q_d\}$ be finite posets.*

(1) *If P and Q possess a common linear extension, then the lattice polytope $\Omega(\mathscr{O}_P,\mathscr{O}_Q)$ is a reflexive polytope which possesses the integer decomposition property.*

(2) *The lattice polytope $\Omega(\mathscr{O}_P,\mathscr{C}_Q)$ is a reflexive polytope which possesses the integer decomposition property.*

(3) *The lattice polytope $\Omega(\mathscr{C}_P,\mathscr{C}_Q)$ is a reflexive polytope which possesses the integer decomposition property.*

Now, for finite posets $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$, let

$$\begin{aligned} K_2[\mathscr{O}\mathscr{O}] &= K[\{x_I\}_{I \in \mathscr{J}(P)} \cup \{y_J\}_{J \in \mathscr{J}(Q)} \cup \{z\}], \\ K_2[\mathscr{O}\mathscr{C}] &= K[\{x_I\}_{I \in \mathscr{J}(P)} \cup \{y_{\max(J)}\}_{J \in \mathscr{J}(Q)} \cup \{z\}], \\ K_2[\mathscr{C}\mathscr{C}] &= K[\{x_{\max(I)}\}_{I \in \mathscr{J}(P)} \cup \{y_{\max(J)}\}_{J \in \mathscr{J}(Q)} \cup \{z\}] \end{aligned}$$

denote the polynomial rings over K, and define the surjective ring homomorphisms $\pi^2_{\mathscr{O}\mathscr{O}}$, $\pi^2_{\mathscr{O}\mathscr{C}}$ and $\pi^2_{\mathscr{C}\mathscr{C}}$ by the following:

- $\pi^2_{\mathscr{O}\mathscr{O}} : K_2[\mathscr{O}\mathscr{O}] \to K[\Omega(\mathscr{O}_P, \mathscr{O}_Q)]$ by setting $\pi^2_{\mathscr{O}\mathscr{O}}(x_I) = \mathbf{t}^{\rho(I \cup \{d+1\})}s$, $\pi^2_{\mathscr{O}\mathscr{O}}(y_J) = \mathbf{t}^{-\rho(J \cup \{d+1\})}s$ and $\pi^2_{\mathscr{O}\mathscr{O}}(z) = s$,
- $\pi^2_{\mathscr{O}\mathscr{C}} : K_2[\mathscr{O}\mathscr{C}] \to K[\Omega(\mathscr{O}_P, \mathscr{C}_Q)]$ by setting $\pi^2_{\mathscr{O}\mathscr{C}}(x_I) = \mathbf{t}^{\rho(I \cup \{d+1\})}s$, $\pi^2_{\mathscr{O}\mathscr{C}}(y_{\max(J)}) = \mathbf{t}^{-\rho(\max(J) \cup \{d+1\})}s$ and $\pi^2_{\mathscr{O}\mathscr{C}}(z) = s$,
- $\pi^2_{\mathscr{C}\mathscr{C}} : K_2[\mathscr{C}\mathscr{C}] \to K[\Omega(\mathscr{C}_P, \mathscr{C}_Q)]$ by setting $\pi^2_{\mathscr{C}\mathscr{C}}(x_{\max(I)}) = \mathbf{t}^{\rho(\max(I) \cup \{d+1\})}s$, $\pi^2_{\mathscr{C}\mathscr{C}}(y_{\max(J)}) = \mathbf{t}^{-\rho(\max(J) \cup \{d+1\})}s$ and $\pi^2_{\mathscr{C}\mathscr{C}}(z) = s$

where $I \in \mathscr{J}(P)$ and $J \in \mathscr{J}(Q)$. Then the toric ideal $I_{\Omega(\mathscr{O}_P, \mathscr{O}_Q)}$ of $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$ is the kernel of $\pi^2_{\mathscr{O}\mathscr{O}}$. Similarly, the toric ideal $I_{\Omega(\mathscr{O}_P, \mathscr{C}_Q)}$ (resp. $I_{\Omega(\mathscr{C}_P, \mathscr{C}_Q)}$) is the kernel of $\pi^2_{\mathscr{O}\mathscr{C}}$ (resp. $\pi^2_{\mathscr{C}\mathscr{C}}$).

Next, we introduce monomial orders $<^2_{\mathscr{O}\mathscr{O}}$, $<^2_{\mathscr{O}\mathscr{C}}$ and $<^2_{\mathscr{C}\mathscr{C}}$ and $\mathscr{G}^2_{\mathscr{O}\mathscr{O}}$, $\mathscr{G}^2_{\mathscr{O}\mathscr{C}}$ and $\mathscr{G}^2_{\mathscr{C}\mathscr{C}}$ which are the set of binomials. Let $<^2_{\mathscr{O}\mathscr{O}}$ denote a reverse lexicographic order on $K_2[\mathscr{O}\mathscr{O}]$ satisfying

- $z <^2_{\mathscr{O}\mathscr{O}} y_J <^2_{\mathscr{O}\mathscr{O}} x_I$;
- $x_{I'} <^2_{\mathscr{O}\mathscr{O}} x_I$ if $I' \subset I$;
- $y_{J'} <^2_{\mathscr{O}\mathscr{O}} y_J$ if $J' \subset J$,

and $\mathscr{G}^2_{\mathscr{O}\mathscr{O}} \subset K_2[\mathscr{O}\mathscr{O}]$ the set of the following binomials:

(O1) $x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}$;

(O2) $y_J y_{J'} - y_{J \cup J'} y_{J \cap J}$;

(O3) $x_I y_J - x_{I \setminus \{p_i\}} y_{J \setminus \{q_i\}}$;

(O4) $x_\emptyset y_\emptyset - z^2$,

and let $<^2_{\mathscr{O}\mathscr{C}}$ denote a reverse lexicographic order on $K_2[\mathscr{O}\mathscr{C}]$ satisfying

- $z <^2_{\mathscr{O}\mathscr{C}} y_{\max(J)} <^2_{\mathscr{O}\mathscr{C}} x_I$;
- $x_{I'} <^2_{\mathscr{O}\mathscr{C}} x_I$ if $I' \subset I$;
- $y_{\max(J')} <^2_{\mathscr{O}\mathscr{C}} y_{\max(J)}$ if $J' \subset J$,

and $\mathscr{G}^2_{\mathscr{O}\mathscr{C}} \subset K_2[\mathscr{O}\mathscr{C}]$ the set of the following binomials:

(O5) $x_I x_{I'} - x_{I\cup I'} x_{I\cap I'}$;

(O6) $y_{\max(J)} y_{\max(J')} - y_{\max(J\cup J')} y_{\max(J*J')}$;

(O7) $x_I y_{\max(J)} - x_{I\setminus\{p_i\}} y_{\max(J)\setminus\{q_i\}}$;

(O8) $x_\emptyset y_\emptyset - z^2$,

and let $<^2_{\mathscr{C}\mathscr{C}}$ denote a reverse lexicographic order on $K_2[\mathscr{C}\mathscr{C}]$ satisfying

- $z <^2_{\mathscr{C}\mathscr{C}} y_{\max(J)} <^2_{\mathscr{C}\mathscr{C}} x_{\max(I)}$;
- $x_{\max(I')} <^2_{\mathscr{C}\mathscr{C}} x_{\max(I)}$ if $I' \subset I$;
- $y_{\max(J')} <^2_{\mathscr{C}\mathscr{C}} y_{\max(J)}$ if $J' \subset J$,

and $\mathscr{G}^2_{\mathscr{C}\mathscr{C}} \subset K_2[\mathscr{C}\mathscr{C}]$ the set of the following binomials:

(O9) $x_{\max(I)} x_{\max(I')} - y_{\max(I\cup I')} y_{\max(I*I')}$;

(O10) $y_{\max(J)} y_{\max(J')} - y_{\max(J\cup J')} y_{\max(J*J')}$;

(O11) $x_{\max(I)} y_{\max(J)} - x_{\max(I)\setminus\{p_i\}} y_{\max(J)\setminus\{q_i\}}$;

(O12) $x_\emptyset y_\emptyset - z^2$,

where

- I and I' are poset ideals of P which are incomparable in $\mathscr{J}(P)$;
- J and J' are poset ideals of Q which are incomparable in $\mathscr{J}(Q)$;
- p_i is a maximal element of I and q_i is a maximal element of J.

Proposition 9.33. *Work with the same situation as above. If P and Q possess a common linear extension, then the origin of $\mathbb{R}^{d+1}$ is contained in the interior of $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$ and $\mathscr{G}^2_{OO}$ is a Gröbner basis of $I_{\Omega(\mathscr{O}_P,\mathscr{O}_Q)}$ with respect to $<^2_{OO}$.*

Proof. Set $P' = \{p_{d+1}\} \oplus P$ and $Q' = \{q_{d+1}\} \oplus Q$. Then we have

$$\mathscr{J}(P') = \{\emptyset\} \cup \{I \cup \{p_{d+1}\} : I \in \mathscr{J}(P)\},$$

$$\mathscr{J}(Q') = \{\emptyset\} \cup \{J \cup \{q_{d+1}\} : J \in \mathscr{J}(Q)\}.$$

Hence we know that $\Omega(\mathscr{O}_P, \mathscr{C}_Q) = \Gamma(\mathscr{O}_{P'}, \mathscr{O}_{Q'})$. By [43], we can easily show if P and Q possess a common linear extension, then the origin of $\mathbb{R}^{d+1}$ is contained in the interior of $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$ and $\mathscr{G}^2_{OO}$ is a Gröbner basis of $I_{\Omega(\mathscr{O}_P,\mathscr{O}_Q)}$ with respect to $<^2_{OO}$, as desired. □

Proposition 9.34. *Work with the same situation as above. Then $\mathscr{G}^2_{OC}$ is a Gröbner basis of $I_{\Omega(\mathscr{O}_P,\mathscr{C}_Q)}$ with respect to $<^2_{OC}$.*

Proof. It is clear that $\mathscr{G}^2_{\mathscr{O}\mathscr{C}} \subset I_{\Omega(\mathscr{O}_P,\mathscr{C}_Q)}$. We note that the initial monomial of each of the binomials (O5) – (O8) with respect to $<^2_{\mathscr{O}\mathscr{C}}$ is its first monomial. Let $\text{in}_{<^2_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^2_{\mathscr{O}\mathscr{C}})$ denote the set of initial monomials of binomials belonging to $\mathscr{G}^2_{\mathscr{O}\mathscr{C}}$. It follows from Lemma 8.3 that, in order to show that $\mathscr{G}^2_{\mathscr{O}\mathscr{C}}$ is a Gröbner basis of $I_{\Omega(\mathscr{O}_P,\mathscr{C}_Q)}$ with respect to $<^2_{\mathscr{O}\mathscr{C}}$, we must prove the following assertion: If u and v are monomials belonging to $K_2[\mathscr{O}\mathscr{C}]$ with $u \neq v$ such that $u \notin \langle \text{in}_{<^2_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^2_{\mathscr{O}\mathscr{C}})\rangle$ and $v \notin \langle \text{in}_{<^2_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^2_{\mathscr{O}\mathscr{C}})\rangle$, then $\pi^2_{\mathscr{O}\mathscr{C}}(u) \neq \pi^2_{\mathscr{O}\mathscr{C}}(v)$.

Let $u, v \in K[\mathscr{O}\mathscr{C}]$ be monomials with $u \neq v$. Write

$$u = z^{\alpha} x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b}, \quad v = z^{\alpha'} x_{I'_1}^{\xi'_1} \cdots x_{I'_{a'}}^{\xi'_{a'}} y_{\max(J'_1)}^{\nu'_1} \cdots y_{\max(J'_{b'})}^{\nu'_{b'}},$$

where

- $\alpha \geq 0$, $\alpha' \geq 0$;
- $I_1, \ldots, I_a, I'_1, \ldots, I'_{a'} \in \mathscr{J}(P)$;
- $J_1, \ldots, J_b, J'_1, \ldots, J'_{b'} \in \mathscr{J}(Q)$;
- $\xi_1, \ldots, \xi_a, \nu_1, \ldots, \nu_b, \xi'_1, \ldots, \xi'_{a'}, \nu'_1, \ldots, \nu'_{b'} > 0$,

and where u and v are relatively prime with $u \notin \langle \text{in}_{<^2_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^2_{\mathscr{O}\mathscr{C}})\rangle$ and $v \notin \langle \text{in}_{<^2_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^2_{\mathscr{O}\mathscr{C}})\rangle$. Thus By using (O5) and (O6), it follows that

- $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_a$ and $J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_b$;
- $I'_1 \subsetneq I'_2 \subsetneq \cdots \subsetneq I'_{a'}$ and $J'_1 \subsetneq J'_2 \subsetneq \cdots \subsetneq J'_{b'}$.

Now, suppose that $\pi^2_{\mathscr{O}\mathscr{C}}(u) = \pi^2_{\mathscr{O}\mathscr{C}}(v)$. Then we have

$$\sum_{\substack{I\in\{I_1,\dots,I_a\}\\ p_i\in I}} \xi_I - \sum_{\substack{J\in\{J_1,\dots,J_b\}\\ q_i\in\max(J)}} \nu_J = \sum_{\substack{I'\in\{I'_1,\dots,I'_{a'}\}\\ p_i\in I'}} \xi'_{I'} - \sum_{\substack{J'\in\{J'_1,\dots,J'_{b'}\}\\ q_i\in\max(J')}} \nu'_{J'}.$$

for all $1 \le i \le d$ by comparing the degree of t_i.

Assume that $(a,a') \neq (0,0)$ and $I_a \setminus I'_{a'} \neq \emptyset$. Then there exists a maximal element p_{i^*} of I_a with $p_{i^*} \notin I'_{a'}$. Since $p_{i^*} \notin I'_{a'}$, one has

$$\sum_{\substack{I\in\{I_1,\dots,I_a\}\\ p_{i^*}\in I}} \xi_I - \sum_{\substack{J\in\{J_1,\dots,J_b\}\\ q_{i^*}\in\max(J)}} \nu_J = - \sum_{\substack{J'\in\{J'_1,\dots,J'_{b'}\}\\ q_{i^*}\in\max(J')}} \nu'_{J'} \le 0.$$

Moreover, since p_{i^*} is belonging to I_a, we also have

$$\sum_{\substack{I\in\{I_1,\dots,I_a\}\\ p_{i^*}\in I}} \xi_I > 0.$$

Hence there exists an integer c with $1 \le c \le b$ such that q_{i^*} is a maximal element of J_c. Therefore we have $x_{I_a} y_{\max(J_c)} \in \langle \mathrm{in}_{<^2_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^2_{\mathscr{O}\mathscr{C}})\rangle$, but this is a contradiction. By considering the case where $(a,a') \neq (0,0)$ and $I'_{a'} \setminus I_a \neq \emptyset$, it is known that one of the followings is satisfied:

- $(a,a') = (1,0), I_a = \emptyset$;
- $(a,a') = (0,1), I_{a'} = \emptyset$;
- $(a,a') = (0,0)$.

Then we have

$$\sum_{\substack{J\in\{J_1,\dots,J_b\}\\ q_i\in\max(J)}} \nu_J = \sum_{\substack{J'\in\{J'_1,\dots,J'_{b'}\}\\ q_i\in\max(J')}} \nu'_{J'}.$$

for all $1 \le i \le d$. Assume that $(b,b') \neq (0,0)$ and $J_b \setminus J'_{b'} \neq \emptyset$. Then there exists a maximal element $q_{i'}$ of J_b with $q_{i'} \notin J'_{b'}$. Since $q_{i'} \notin J'_{b'}$, one has

$$0 < \sum_{\substack{J\in\{J_1,\dots,J_b\}\\ q_{i'}\in\max(J)}} \nu_J \neq \sum_{\substack{J'\in\{J'_1,\dots,J'_{b'}\}\\ q_{i'}\in\max(J')}} \nu'_{J'} = 0,$$

but this is a contradiction. By considering the case where $(b,b') \neq (0,0)$ and $J'_{b'} \setminus J_b \neq \emptyset$, it is known that one of the followings is satisfied:

- $(b,b') = (1,0), J_b = \emptyset$;

- $(b,b') = (0,1), J'_{b'} = \emptyset$;
- $(b,b') = (0,0)$.

Hence one has $u = z^{\alpha} x_{\emptyset}^{\xi} y_{\emptyset}^{\nu}$ and $v = z^{\alpha'} x_{\emptyset}^{\xi'} y_{\emptyset}^{\nu'}$, where $\xi, \xi', \nu, \nu' \geq 0$. Since $x_{\emptyset} y_{\emptyset} \in \langle \mathrm{in}_{<^2_{\mathscr{O}\mathscr{C}}}(\mathscr{G}^2_{\mathscr{O}\mathscr{C}}) \rangle$ and since u and v are relatively prime, we may assume that $\nu = \xi' = 0$. Thus $u = z^{\alpha} x_{\emptyset}^{\xi}$ and $v = z^{\alpha'} y_{\emptyset}^{\nu'}$. Note that either $\alpha = 0$ or $\alpha' = 0$. Hence by comparing the degree of t^{d+1}, it is known that $\xi = \nu' = \alpha = \alpha' = 0$, contradiction. □

Proposition 9.35. *Work with the same situation as above. Then $\mathscr{G}^2_{CC}$ is a Gröbner basis of $I_{\Omega(\mathscr{C}_P, \mathscr{C}_Q)}$ with respect to $<^2_{CC}$.*

Proof. We can show that the assertion follows by a similar way in the proof of Proposition 9.35. □

Finally, we show Theorem 9.32.

Proof of Theorem 9.32. It is easy to show that $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$, $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Omega(\mathscr{C}_P, \mathscr{C}_Q)$ are spanning. By Lemma 8.4 and Propositions 9.33, 9.34, 9.35, the assertion follows. □

9.3.2 Their δ-polynomials and volumes

In this subsection, we consider combinatorial properties of these polytopes, especially, the δ polynomials and the volume of $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$, $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$, for finite posets $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$. In fact, we show the following theorem.

Theorem 9.36 ([49, Theorem 1.4])**.** *Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite poset. We set $P' = \{p_{d+1}\} \oplus P$ and $Q' = \{q_{d+1}\} \oplus Q$. If P and Q possess a common linear extension, then all of $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$, $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$, $\Omega(\mathscr{C}_P, \mathscr{C}_Q)$, $\Gamma(\mathscr{O}_{P'}, \mathscr{O}_{Q'})$, $\Gamma(\mathscr{O}_{P'}, \mathscr{C}_{Q'})$ and $\Gamma(\mathscr{C}_{P'}, \mathscr{C}_{Q'})$ have the same δ polynomial. In particular, these polytopes have the same volume.*

Here, we put

$$R^2_{\mathscr{O}\mathscr{O}} := K_2[\mathscr{O}\mathscr{O}]/\mathrm{in}_{<^2_{\mathscr{O}\mathscr{O}}}(I_{\Omega(\mathscr{O}_P, \mathscr{O}_Q)}),$$

$$R^2_{\mathscr{O}\mathscr{C}} := K_2[\mathscr{O}\mathscr{C}]/\mathrm{in}_{<^2_{\mathscr{O}\mathscr{C}}}(I_{\Omega(\mathscr{O}_P, \mathscr{C}_Q)}),$$

$$R^2_{\mathscr{C}\mathscr{C}} := K_2[\mathscr{C}\mathscr{C}]/\mathrm{in}_{<^2_{\mathscr{C}\mathscr{C}}}(I_{\Omega(\mathscr{C}_P, \mathscr{C}_Q)}).$$

Proposition 9.37. *Work with the same situation as above. If P and Q possess a common linear extension, then these rings $R^2_{\mathscr{O}\mathscr{O}}$, $R^2_{\mathscr{O}\mathscr{C}}$ and $R^2_{\mathscr{C}\mathscr{C}}$ are isomorphic.*

Proof. By Proposition 9.33, 9.34 and 9.35, we have

$$R^2_{\mathscr{O}\mathscr{O}} \cong \frac{K_2[\mathscr{O}\mathscr{O}]}{\langle\{x_I x_{I'}, y_J y_{J'}, x_I y_J, x_\emptyset y_\emptyset \mid I, I', J \text{ and } J' \text{ satisfy } (*)\}\rangle},$$

$$R^2_{\mathscr{O}\mathscr{C}} \cong \frac{K_2[\mathscr{O}\mathscr{C}]}{\langle\{x_I x_{I'}, y_{\max(J)} y_{\max(J')}, x_I y_{\max(J)}, x_\emptyset y_\emptyset \mid I, I', J \text{ and } J' \text{ satisfy } (*)\}\rangle},$$

$$R^2_{\mathscr{C}\mathscr{C}} \cong \frac{K_2[\mathscr{C}\mathscr{C}]}{\langle\{x_{\max(I)} x_{\max(I')}, y_{\max(J)} y_{\max(J')}, x_{\max(I)} y_{\max(J)}, x_\emptyset y_\emptyset \mid I, I', J \text{ and } J' \text{ satisfy } (*)\}\rangle},$$

where the condition $(*)$ is the following:

- I and I' are poset ideals of P which are incomparable in $\mathscr{J}(P)$;
- J and J' are poset ideals of Q which are incomparable in $\mathscr{J}(Q)$;
- There exists $1 \leq i \leq d$ such that p_i is a maximal element of I and q_i is a maximal element of J.

Hence it is easy to see that the ring homomorphism $\varphi : R^2_{\mathscr{O}\mathscr{C}} \to R^2_{\mathscr{C}\mathscr{C}}$ by setting $\varphi(x_I) = x_{\max(I)}$, $\varphi(y_{\max(J)}) = y_{\max(J)}$ and $\varphi(z) = z$ is an isomorphism. Similarly, if P and Q possess a common linear extension, we can see that the ring homomorphism $\varphi' : R^2_{\mathscr{O}\mathscr{O}} \to R^2_{\mathscr{O}\mathscr{C}}$ by setting $\varphi'(x_I) = x_I$, $\varphi'(y_J) = y_{\max(J)}$ and $\varphi'(z) = z$ is an isomorphism. Hence it is known that $R^2_{\mathscr{O}\mathscr{O}} \cong R^2_{\mathscr{O}\mathscr{C}} \cong R^2_{\mathscr{C}\mathscr{C}}$, as desired. □

Now, we prove Theorem 9.36.

Proof of Theorem 9.36. By Theorem 9.32, it is known that that $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$, $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Omega(\mathscr{C}_P, \mathscr{C}_Q)$ possess the integer decomposition property. Hence the Ehrhart polynomial of $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$ (resp. $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Omega(\mathscr{C}_P, \mathscr{C}_Q)$) is equal to the Hilbert function of $K[\Omega(\mathscr{O}_P, \mathscr{O}_Q)]$ (resp. $K[\Omega(\mathscr{O}_P, \mathscr{C}_Q)]$ and $K[\Omega(\mathscr{C}_P, \mathscr{C}_Q)]$). By Proposition 9.37, $R_{\mathscr{O}\mathscr{O}}$, $R_{\mathscr{O}\mathscr{C}}$ and $R_{\mathscr{C}\mathscr{C}}$ have the same Hilbert function. Hence by Lemma 8.5, $K[\Omega(\mathscr{O}_P, \mathscr{O}_Q)]$, $K[\Omega(\mathscr{O}_P, \mathscr{C}_Q)]$ and $K[\Omega(\mathscr{C}_P, \mathscr{C}_Q)]$ also have the same Hilbert function. On the other hand, in the proof of Proposition 9.33, it is known that $\Omega(\mathscr{O}_P, \mathscr{O}_Q) = \Gamma(\mathscr{O}_{P'}, \mathscr{O}_{Q'})$. Hence by Theorem 9.8, we have the desired conclusion. □

Finally, we give a combinatorial formula to compute the volume of these polytopes in terms of the underlying finite posets P and Q.

By Theorems 9.14 and 9.36, we obtain the following theorem.

Theorem 9.38 ([49, Theorem 3.4]). *Let $P = \{p_1, \dots, p_d\}$ and $Q = \{q_1, \dots, q_d\}$ be finite posets, and set $P' = \{p_{d+1}\} \oplus P$ and $Q' = \{q_{d+1}\} \oplus Q$. If P and Q possess a common linear extension, then we have*

$$\mathrm{Vol}(\Omega(\mathscr{C}_P, \mathscr{C}_Q)) = \sum_{W \subset [d+1]} e(\Delta_W(P', Q')).$$

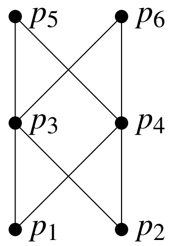
9.3.3 Examples

In this subsection, we give some curious examples of reflexive polytopes.

First, we consider a difference of the class of $\Gamma(\mathscr{P}, \mathscr{Q})$ and the class of $\Omega(\mathscr{P}, \mathscr{Q})$. It is known that the class of $\Omega(\mathscr{O}_P, \mathscr{O}_Q)$ is included in that of $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$.

Example 9.39. Let P be the finite poset as follows,

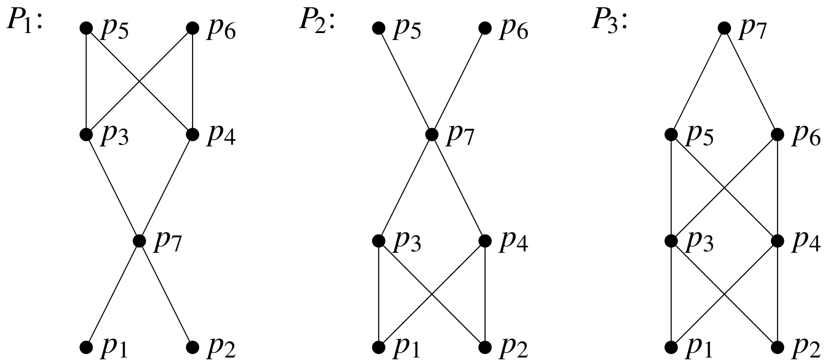
P:



For any finite poset P' with 7 elements, it is known that the f-vector of $\Omega(\mathscr{C}_P, \mathscr{C}_P)$ is not equal to that of $\Gamma(\mathscr{O}_{P'}, \mathscr{O}_{P'})$ and $\Gamma(\mathscr{C}_{P'}, \mathscr{C}_{P'})$. Hence $\Omega(\mathscr{C}_P, \mathscr{C}_P)$ is not unimodularly equivalent to $\Gamma(\mathscr{O}_{P'}, \mathscr{O}_{P'})$ and $\Gamma(\mathscr{C}_{P'}, \mathscr{C}_{P'})$.

By this example, we know that the class of $\Omega(\mathscr{C}_P, \mathscr{C}_Q)$ is not included in that of $\Gamma(\mathscr{O}_P, \mathscr{O}_Q)$, $\Gamma(\mathscr{O}_P, \mathscr{C}_Q)$ and $\Gamma(\mathscr{C}_P, \mathscr{C}_Q)$. Similarly, the class of $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$ is included in none of the above classes. This fact says that the class of $\Omega(\mathscr{O}_P, \mathscr{C}_Q)$ and that of $\Omega(\mathscr{C}_P, \mathscr{C}_Q)$ are new classes of reflexive polytopes which possess the integer decomposition property.

Example 9.40. Let P be the finite poset as in Example 9.39 and $P' = \{p_7\} \oplus P$. Also, we let P_1, P_2 and P_3 be the finite posets as follows:



Then we have

$$\Gamma(\mathscr{C}_{P'},\mathscr{C}_{P'}) = \Gamma(\mathscr{C}_{P_1},\mathscr{C}_{P_1}) = \Gamma(\mathscr{C}_{P_2},\mathscr{C}_{P_2}) = \Gamma(\mathscr{C}_{P_3},\mathscr{C}_{P_3}).$$

Hence it is known that the 11 reflexive polytopes

$$\Omega(\mathscr{O}_P,\mathscr{O}_P),\Omega(\mathscr{O}_P,\mathscr{C}_P),\Omega(\mathscr{C}_P,\mathscr{C}_P),$$

$$\Gamma(\mathscr{O}_{P'},\mathscr{C}_{P'}),\Gamma(\mathscr{C}_{P'},\mathscr{C}_{P'}),$$

$$\Gamma(\mathscr{O}_{P_1},\mathscr{O}_{P_1}),\Gamma(\mathscr{O}_{P_2},\mathscr{O}_{P_2}),\Gamma(\mathscr{O}_{P_3},\mathscr{O}_{P_3}),$$

$$\Gamma(\mathscr{O}_{P_1},\mathscr{C}_{P_1}),\Gamma(\mathscr{O}_{P_2},\mathscr{C}_{P_2}),\Gamma(\mathscr{O}_{P_3},\mathscr{C}_{P_3})$$

possess the integer decomposition property and have the same Ehrhart polynomial. However, these polytopes are not unimodularly equivalent each other.

By these five classes of reflexive polytopes with the integer decomposition property, we can obtain several interesting examples. From this example, one of the future problem is to discuss how many reflexive polytopes which have the same δ-polynomial.

Finally, we give some examples of this problem.

Example 9.41. Let $\mathscr{P} \subset \mathbb{R}^d$ be the reflexive polytope of dimension d whose vertices are followings:

$$\mathbf{e}_1,\dots,\mathbf{e}_d,-\mathbf{e}_1-\cdots-\mathbf{e}_d.$$

Then we have $\delta(\mathscr{P},t) = 1+t+\cdots+t^d$. On the other hand, every reflexive polytope of dimension d whose δ-polynomial is equal to $1+t+\cdots+t^d$ is unimodularly equivalemt to $\mathscr{P}$.

Example 9.42. By checking any reflexive polytopes of dimension 2, we obtain followings:

- The number of reflexive polytopes whose δ-polynomials equal $1+t+t^2$ is 1;

- The number of reflexive polytopes whose δ-polynomials equal $1+2t+t^2$ is 3;
- The number of reflexive polytopes whose δ-polynomials equal $1+3t+t^2$ is 2;
- The number of reflexive polytopes whose δ-polynomials equal $1+4t+t^2$ is 4;
- The number of reflexive polytopes whose δ-polynomials equal $1+5t+t^2$ is 2;
- The number of reflexive polytopes whose δ-polynomials equal $1+6t+t^2$ is 3;
- The number of reflexive polytopes whose δ-polynomials equal $1+7t+t^2$ is 1.

Chapter 10

Reflexive polytopes arising from perfect graphs

In this chapter, the study on $\Gamma(\mathscr{P},\mathscr{Q})$ and $\Omega(\mathscr{P},\mathscr{Q})$ will be done when each of $\mathscr{P}$ and $\mathscr{Q}$ is stable set polytopes, which are lattice polytopes arising from finite simple graphs. This chapter is organized as follows. In Section 10.1, we recall what perfect graphs are and finite posets and introduce the stable set polytopes of finite simple graphs. In Section 10.2, we give a class of reflexive polytope with the integer decomposition property which arise from perfect graphs. In Section 10.3, we consider the δ-polynomials and volumes of these reflexive polytopes. Finally, in Section 10.4, we give some curious examples.

10.1 Perfect graphs and stable set polytopes

In this section, we recall what perfect graphs are and finite posets and introduce the stable set polytopes of finite simple graphs. Let G be a finite simple graph on the vertex set $[d]$ and $E(G)$ the set of edges of G. (A finite graph G is called *simple* if G possesses no loop and no multiple edge.) A subset $W \subset [d]$ is called *stable* if, for all i and j belonging to W with $i \neq j$, one has $\{i,j\} \notin E(G)$. We remark that a stable set is often called an *independent set*. A *clique* of G is a subset $W \subset [d]$ which is a stable set of the complementary graph $\overline{G}$ of G. The *clique number* $\omega(G)$ of G is the maximal cardinality of cliques of G. The *chromatic number* $\chi(G)$ of G is the smallest integer $t \geq 1$ for which there exist stable set $W_1,\dots,W_t$ of G with $[d] = W_1 \cup \cdots \cup W_t$. In general, it follows that $\omega(G) \leq \chi(G)$. A finite simple graph G is said to be *perfect* ([20]) if, for any induced subgraph H of G including G itself, one has $\omega(H) = \chi(G)$. The perfect graphs include many important classes of graphs, for example, chordal graphs and comparability graphs. Moreover, it is known that the complementary graph of a perfect graph is perfect ([20]). This characterization of perfect graphs is called the perfect graph theorem. Recently, a stronger characterization of perfect graphs, which is called the strong perfect graph theorem, is known. An *odd hole* is an induced odd cycle of length ≥ 5 and an *odd antihole* is the complementary graph of an odd hole.

Lemma 10.1 ([20, Strong Perfect Graph Theorem])**.** *A finite simple graph G is perfect if and only if G has no odd hole and no odd antihole as induced subgraph.*

Next, we introduce the stable set polytopes of finite simple graphs. Let $S(G)$ denote the set of stable sets of G. One has $\emptyset \in S(G)$ and $\{i\} \in S(G)$ for each $i \in [d]$. The *stable set polytope* $\mathscr{Q}_G \subset \mathbb{R}^d$ of G is the $(0,1)$-polytope which is the convex hull of $\{\rho(W) : W \in S(G)\}$ in $\mathbb{R}^d$. Then the dimension $\mathscr{Q}_G$ is equal to d. It is known that every chain polytope is a stable set polytope. In fact, let $P = \{p_1, \dots, p_d\}$ be a finite poset. Then its *comparability graph* G_P is the finite simple graph on $[d]$ such that $\{i, j\} \in E(G_P)$ if and only if $p_i < p_j$ or $p_j < p_i$. Then a stable set of G_P corresponds to an antichain of P. Moreover, one has $\mathscr{C}_P = \mathscr{Q}_{G_P}$. Since every comparability graph is perfect, the class of chain polytopes is contained in the class of the stable set polytopes of perfect graphs. Finally, we give a characterization of perfect graphs in terms of the stable set polytopes.

Lemma 10.2 ([72, Example 1.3 (c)])**.** *Let G be a finite simple graph on $[d]$. Then G is perfect if and only if $\mathscr{Q}_G$ is compressed.*

10.2 Squarefree Gröbner basis

In this section, we give a class of reflexive polytope with the integer decomposition property which arise from perfect graphs. In fact, we show the following.

Theorem 10.3 ([50, Theorem 1.1])**.** *Let G_1 and G_2 be finite simple graphs on $[d]$.*

(a) ([74]) *The following conditions are equivalent:*

  (i) *The lattice polytope $\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ is reflexive;*

  (ii) *The lattice polytope $\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ is reflexive and possesses the integer decomposition property;*

  (iii) *Both G_1 and G_2 are perfect.*

(b) *The following conditions are equivalent:*

  (i) *The lattice polytope $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ possesses the integer decomposition property;*

  (ii) *The lattice polytope $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ is reflexive and possesses the integer decomposition property;*

(iii) *Both G_1 and G_2 are perfect.*

In order to prove this theorem, we recall the toric ideals of integer matrices. Let $K[\mathbf{t}^{\pm 1}, s] = K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, s]$ denote the Laurent polynomial ring in $d+1$ variables over a field K. For an integer vector $\mathbf{a} \in \mathbb{Z}^d$, let $\mathbf{a}^\top$ be its transpose vector. Given an integer $d \times n$ matrix $A = (\mathbf{a}_1^\top, \ldots, \mathbf{a}_n^\top)$, where $\mathbf{a}_j^\top = (a_{1j}, \ldots, a_{dj})^\top$ is the jth column of A, then we define the toric ring $K[A]$ of A as follows:

$$K[A] = K[\mathbf{t}^{\mathbf{a}_1} s, \ldots, \mathbf{t}^{\mathbf{a}_n} s] \subset K[\mathbf{t}^{\pm 1}, s].$$

Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K and define the surjective ring homomorphism $\pi : K[\mathbf{x}] \to K[A]$ by setting $\pi(x_j) = \mathbf{t}^{\mathbf{a}_j} s$ for $j = 1, \ldots, n$. The toric ideal of A is the kernel I_A of π.

Let $\mathbb{Z}_{\geq 0}^{d \times n}$ denote the set of $d \times n$ integer matrices $(a_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}}$ with each $a_{ij} \geq 0$. In [74], the concept that $A \in \mathbb{Z}_{\geq 0}^{d \times n}$ and $B \in \mathbb{Z}_{\geq 0}^{d \times m}$ are of *of harmony* is introduced. For an integer vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$, let $\mathbf{a}^{(+)} = (a_1^{(+)}, \ldots, a_d^{(+)}), \mathbf{a}^{(-)} = (a_1^{(-)}, \ldots, a_d^{(-)}) \in \mathbb{Z}_{\geq 0}^d$ where $a_i^{(+)} = \max\{0, a_i\}$ and $a_i^{(-)} = \max\{0, -a_i\}$. Note that $\mathbf{a} = \mathbf{a}^{(+)} - \mathbf{a}^{(-)}$ holds in general. Given $A \in \mathbb{Z}_{\geq 0}^{d \times n}$ and $B \in \mathbb{Z}_{\geq 0}^{d \times m}$ such that the zero vector $\mathbf{0}_d^\top$ is a column in each of A and B, we say that A and B are *of harmony* if the following condition is satisfied: Let $\mathbf{a}^\top$ be a column of A and $\mathbf{b}^\top$ that of B. Let $\mathbf{c} = \mathbf{a} - \mathbf{b} \in \mathbb{Z}^d$. If $\mathbf{c} = \mathbf{c}^{(+)} - \mathbf{c}^{(-)}$, then $(\mathbf{c}^{(+)})^\top$ is a column vector of A and $(\mathbf{c}^{(-)})^\top$ is a column vector of B.

Now we prove the following theorem.

Theorem 10.4 ([50, Theorem 2.1]). *Let $A = (\mathbf{a}_1^\top, \ldots, \mathbf{a}_n^\top) \in \mathbb{Z}_{\geq 0}^{d \times n}$ and $B = (\mathbf{b}_1^\top, \ldots, \mathbf{b}_m^\top) \in \mathbb{Z}_{\geq 0}^{d \times m}$, where $\mathbf{a}_n = \mathbf{b}_m = \mathbf{0}_d \in \mathbb{Z}^d$, be of harmony. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ and $K[\mathbf{y}] = K[y_1, \ldots, y_m]$ be the polynomial rings over a field K. Suppose that $\mathrm{in}_{<_A}(I_A) \subset K[\mathbf{x}]$ and $\mathrm{in}_{<_B}(I_B) \subset K[\mathbf{y}]$ are squarefree with respect to reverse lexicographic orders $<_A$ on $K[\mathbf{x}]$ and $<_B$ on $K[\mathbf{y}]$ respectively satisfying the condition that*

- *$x_i <_A x_j$ if for each $1 \leq k \leq d$ $a_{ki} \leq a_{kj}$.*
- *x_n is the smallest variable with respect to $<_A$.*
- *y_m is the smallest variable with respect to $<_B$.*

Let $(-B, A)^*$ denote the $(d+1) \times (n+m+1)$ integer matrix

$$\begin{pmatrix} -\mathbf{b}_1^\top & \cdots & -\mathbf{b}_m^\top & \mathbf{a}_1^\top & \cdots & \mathbf{a}_n^\top & \mathbf{0}_d^\top \\ -1 & \cdots & -1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

Then the toric ideal $I_{(-B,A)^*}$ of $(-B, A)^*$ possesses a squarefree initial ideal with respect to a reverse lexicographic order whose smallest variable corresponds to the column $\mathbf{0}_{d+1}^\top$ of $(-B, A)^*$.

Proof. Let $I_{(-B,A)^*} \subset K[\mathbf{x},\mathbf{y},z] = K[x_1,\dots,x_n,y_1,\dots,y_m,z]$ be the toric ideal of $(-B,A)^*$ defined by the kernel of

$$\pi^* : K[\mathbf{x},\mathbf{y},z] \to K[(-B,A)^*] \subset K[t_1^{\pm 1},\dots,t_{d+1}^{\pm 1},s]$$

with $\pi^*(z) = s$, $\pi^*(x_i) = \mathbf{t}^{\mathbf{a}_i} t_{d+1} s$ for $i = 1,\dots,n$ and $\pi^*(y_j) = \mathbf{t}^{-\mathbf{b}_j} t_{d+1}^{-1} s$ for $j = 1,\dots,m$. Assume that the reverse lexicographic orders $<_A$ and $<_B$ are induced by the orderings $x_n <_A \cdots <_A x_1$ and $y_m <_B \cdots <_B y_1$. Let $<_{\mathrm{rev}}$ be the reverse lexicographic order on $K[\mathbf{x},\mathbf{y},z]$ induced by the ordering

$$z <_{\mathrm{rev}} x_n <_{\mathrm{rev}} \cdots <_{\mathrm{rev}} x_1 <_{\mathrm{rev}} y_m <_{\mathrm{rev}} \cdots <_{\mathrm{rev}} y_1.$$

In general, for an integer vector $\mathbf{a} = (a_1,\dots,a_d) \in \mathbb{Z}^d$, we let $\mathrm{supp}(\mathbf{a}) = \{i : 1 \le i \le d, a_i \neq 0\}$. Set the following:

$$\mathscr{E} = \{\,(i,j) : 1 \le i \le n,\ 1 \le j \le m,\ \mathrm{supp}(\mathbf{a}_i) \cap \mathrm{supp}(\mathbf{b}_j) \neq \emptyset\,\}.$$

If $\mathbf{c} = \mathbf{a}_i - \mathbf{b}_j$ with $(i,j) \in \mathscr{E}$, then it follows that $\mathbf{c}^{(+)} \neq \mathbf{a}_i$ and $\mathbf{c}^{(-)} \neq \mathbf{b}_j$. Since A and B are of harmony, we know that $(\mathbf{c}^{(+)})^\top$ is a column of A and $(\mathbf{c}^{(-)})^\top$ is a column of B. It follows that $f = x_i y_j - x_k y_\ell$ $(\neq 0)$ belongs to $I_{(-B,A)^*}$, where $\mathbf{c}^{(+)} = \mathbf{a}_k$ and $\mathbf{c}^{(-)} = \mathbf{b}_\ell$. Then since for each $1 \le c \le d$, $a_{ck} \le a_{ci}$, one has $x_k <_A x_i$ and $\mathrm{in}_{<_{\mathrm{rev}}}(f) = x_i y_j$. Hence

$$\{\,x_i y_j : (i,j) \in \mathscr{E}\,\} \subset \mathrm{in}_{<_{\mathrm{rev}}}(I_{(-B,A)^*}).$$

Moreover, it follows that $x_n y_m - z^2 \in I_{(-B,A)^*}$ and $x_n y_m \in \mathrm{in}_{<_{\mathrm{rev}}}(I_{(-B,A)^*})$. We set

$$\mathscr{M} = \{x_n y_m\} \cup \{\,x_i y_j : (i,j) \in \mathscr{E}\,\} \cup \mathscr{M}_A \cup \mathscr{M}_B \ \ (\subset \mathrm{in}_{<_{\mathrm{rev}}}(I_{(-B,A)^*})),$$

where $\mathscr{M}_A$ (resp. $\mathscr{M}_B$) is the minimal set of squarefree monomial generators of $\mathrm{in}_{<_A}(I_A)$ (resp. $\mathrm{in}_{<_B}(I_B)$). Let $\mathscr{G}$ be a finite set of binomials belonging to $I_{(-B,A)^*}$ with $\mathscr{M} = \{\mathrm{in}_{<_{\mathrm{rev}}}(f) : f \in \mathscr{G}\}$.

Now, we prove that $\mathscr{G}$ is a Gröbner base of $I_{(-B,A)^*}$ with respect to $<_{\mathrm{rev}}$. By the following fact ([73, (0.1), p. 1914]) on Gröbner bases, we must prove the following assertion: If u and v are monomials belonging to $K[\mathbf{x},\mathbf{y},z]$ with $u \neq v$ such that $u \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$ and $v \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$, then $\pi^*(u) \neq \pi^*(v)$.

Suppose that there exists a nonzero irreducible binomial $g = u - v$ be belonging to $I_{(-B,A)^*}$ such that $u \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$ and $v \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$. Write

$$u = \left(\prod_{p \in P} x_p^{i_p}\right)\left(\prod_{q \in Q} y_q^{j_q}\right), \quad v = z^\alpha \left(\prod_{p' \in P'} x_{p'}^{i'_{p'}}\right)\left(\prod_{q' \in Q} y_{q'}^{j'_{q'}}\right),$$

where P and P' are subsets of $[n]$, Q and Q' are subsets of $[m]$, α is a nonnegative integer, and each of $i_p, j_q, i'_{p'}, j'_{q'}$ is a positive integer. Since $g = u - v$ is irreducible, one has $P \cap P' = Q \cap Q' = \emptyset$. Furthermore, by the fact that each of $x_i y_j$ with $(i,j) \in \mathscr{E}$ can divide neither u nor v, it follows that

$$\left(\bigcup_{p \in P} \mathrm{supp}(\mathbf{a}_p)\right) \cap \left(\bigcup_{q \in Q} \mathrm{supp}(\mathbf{b}_q)\right) = \left(\bigcup_{p' \in P'} \mathrm{supp}(\mathbf{a}_{p'})\right) \cap \left(\bigcup_{q' \in Q'} \mathrm{supp}(\mathbf{b}_{q'})\right) = \emptyset.$$

Hence, since $\pi^*(u) = \pi^*(v)$, it follows that

$$\sum_{p \in P} i_p \mathbf{a}_p = \sum_{p' \in P'} i'_{p'} \mathbf{a}_{p'}, \quad \sum_{q \in Q} j_q \mathbf{b}_q = \sum_{q' \in Q'} j'_{q'} \mathbf{b}_{q'}.$$

Let $\xi = \sum_{p \in P} i_p$, $\xi' = \sum_{p' \in P'} i'_{p'}$, $\nu = \sum_{q \in Q} j_q$, and $\nu' = \sum_{q' \in Q'} j'_{q'}$. Then $\xi + \nu = \xi' + \nu' + \alpha$. Since $\alpha \geq 0$, it follows that either $\xi \geq \xi'$ or $\nu \geq \nu'$. Assume that $\xi > \xi'$. Then

$$h = \prod_{p \in P} x_p^{i_p} - x_n^{\xi - \xi'} \left(\prod_{p' \in P'} x_{p'}^{i'_{p'}}\right)$$

belongs to I_A and $I_{(-B,A)^*}$. If $h \neq 0$, then $\mathrm{in}_{<_A}(h) = \mathrm{in}_{<_{\mathrm{rev}}}(h) = \prod_{p \in P} x_p^{i_p}$ divides u, a contradiction. Hence $P = \{n\}$ and $Q = \emptyset$. If $\xi = \xi'$, then the binomial

$$h_0 = \prod_{p \in P} x_p^{i_p} - \prod_{p' \in P'} x_{p'}^{i'_{p'}}$$

belongs to I_A and $I_{(-B,A)^*}$. Moreover, if $h_0 \neq 0$, then either $\prod_{p \in P} x_p^{i_p}$ or $\prod_{p' \in P'} x_{p'}^{i'_{p'}}$ must belong to $\mathrm{in}_{<_A}(I_A)$ and $\mathrm{in}_{<_{\mathrm{rev}}}(I_{(-B,A)^*})$. This contradicts the fact that each of u and v can be divided by none of the monomials belonging to $\mathscr{M}$. Hence $h_0 = 0$ and $P = P' = \emptyset$. Similarly, $Q = \{m\}$ and $Q' = \emptyset$, or $Q = Q' = \emptyset$. Hence we know that $g = x_n^k y_m^\ell - z^\alpha$, where k and ℓ are nonnegative integers. Since u cannot be divided by $x_n y_m$, it follows that $g = 0$, a contradiction. Therefore, $\mathscr{G}$ is a Gröbner base of $\mathrm{in}_{<_{\mathrm{rev}}}(I_{(-B,A)^*})$ with respect to $<_{\mathrm{rev}}$. □

Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d with $\mathscr{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Set $A = (\mathbf{a}_1^\top, \ldots, \mathbf{a}_n^\top)$. Then the toric ring $K[\mathscr{P}]$ and the toric ideal $I_{\mathscr{P}}$ of $\mathscr{P}$ coincide with $K[A]$ and I_A. By Lemma 8.4 and Theorem 10.4, we obtain the following corollary.

Corollary 10.5. *Work with the same situation as in Theorem 10.4. Let $\mathscr{P} \subset \mathbb{R}^{d+1}$ be the lattice polytope of dimension $d+1$ with*

$$\mathscr{P} \cap \mathbb{Z}^{d+1} = \{(\mathbf{a}_1, 1), \ldots, (\mathbf{a}_n, 1), (-\mathbf{b}_1, -1), \ldots, (-\mathbf{b}_m, -1), \mathbf{0}_{d+1}\}.$$

Suppose that $\mathbf{0}_{d+1} \in \mathbb{Z}^{d+1}$ belongs to the interior of $\mathscr{P}$ and $\mathscr{P}$ is spanning. Then $\mathscr{P}$ is a reflexive polytope which possesses the integer decomposition property.

Finally, we prove Theorem 10.3.

Proof of Theorem 10.3. For a finite simple graph G on $[d]$, let $A_{S(G)}$ be the matrix whose columns are those $(\rho(W))^\top$ with $W \in S(G)$. If $W \in S(G)$, then each subset of W is also a stable set of G. This means that $S(G)$ is a simplicial complex on $[d]$. Hence it is easy to show that $A_{S(G_1)}$ and $A_{S(G_2)}$ are of harmony. Moreover, by Lemma 10.2, for any perfect graph G, $\mathcal{Q}_G$ is compressed. Let $\mathcal{P} \subset \mathbb{R}^{d+1}$ be the convex hull of $\{\pm(\mathbf{e}_1+\mathbf{e}_{d+1}),\ldots,\pm(\mathbf{e}_d+\mathbf{e}_{d+1}),\pm\mathbf{e}_{d+1}\}$. Then it follows that $\mathbf{0}_{d+1} \in \mathbb{Z}^{d+1}$ belongs to the interior of $\mathcal{P}$ and $\mathcal{P}$ is spanning. Moreover, we have $\mathcal{P} \subset \Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2})$. This implies that $\mathbf{0}_{d+1} \in \mathbb{Z}^{d+1}$ belongs to the interior of $\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2})$ and $\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2})$ is spanning. On the other hand, one has

$$\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2}) \cap \{(a_1,\ldots,a_{d+1}) \in \mathbb{R}^{d+1} : a_{d+1}=0\} = \frac{1}{2}(\mathcal{Q}_{G_1}-\mathcal{Q}_{G_2}) \times \{0\}.$$

Since $\frac{1}{2}(\mathcal{Q}_{G_1}-\mathcal{Q}_{G_2}) \cap \mathbb{Z}^d = \{\mathbf{0}_d\}$, we obtain

$$\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2}) \cap \mathbb{Z}^{d+1} = \left\{(\mathbf{a},1) : \mathbf{a} \in \mathcal{Q}_{G_1} \cap \mathbb{Z}^d\right\} \cup \left\{(-\mathbf{b},-1) : \mathbf{b} \in \mathcal{Q}_{G_2} \cap \mathbb{Z}^d\right\} \cup \{\mathbf{0}_{d+1}\}.$$

Hence, by Corollary 10.5, if G_1 and G_2 are perfect, $\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2})$ is a reflexive polytope which possesses the integer decomposition property.

Next, we prove that if G_1 is not perfect, then $\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2})$ does not possess the integer decomposition property. Assume that G_1 is not perfect and $\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2})$ possesses the integer decomposition property. By Lemma 10.1, G_1 possesses either an odd hole or an odd antihole. Suppose that G_1 possesses an odd hole C of length $2\ell+1$, where $\ell \geq 2$. By renumbering the vertex set, we may assume that the edge set of C is $\{\{i,i+1\} : 1 \leq i \leq 2\ell\} \cup \{1,2\ell+1\}$. Then the maximal stable sets of C in $[2\ell+1]$ are

$$S_1 = \{1,3,\ldots,2\ell-1\}, S_2 = \{2,4,\ldots,2\ell\},\ldots,S_{2\ell+1} = \{2\ell+1,2,4,\ldots,2\ell-2\}$$

and each $i \in [2\ell+1]$ appears ℓ times in the above list. For $1 \leq i \leq 2\ell+1$, we set $\mathbf{v}_i = \sum_{j \in S_i} \mathbf{e}_j + \mathbf{e}_{d+1}$. Then one has

$$\mathbf{a} = \frac{\mathbf{v}_1+\cdots+\mathbf{v}_{2\ell+1}+(-\mathbf{e}_{d+1})}{\ell} = \mathbf{e}_1+\cdots+\mathbf{e}_{2\ell+1}+2\mathbf{e}_{d+1}.$$

Since $2 < (2\ell+2)/\ell \leq 3$, $\mathbf{a} \in 3\Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2}) \cap \mathbb{Z}^{d+1}$. Hence there exist $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3 \in \Omega(\mathcal{Q}_{G_1},\mathcal{Q}_{G_2}) \cap \mathbb{Z}^{d+1}$ such that $\mathbf{a} = \mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3$. Then we may assume that $\mathbf{a}_1,\mathbf{a}_2 \in \mathcal{Q}_C \times \{1\}$ and $\mathbf{a}_3 = \mathbf{0}_{d+1}$. However, since the maximal cardinality of the stable sets of C in $[2\ell+1]$ equals ℓ, a contradiction.

Suppose that G_1 possesses an odd antihole of length $2\ell+1$, where $\ell \geq 2$. Similarly, we may assume that the edge set of $\overline{C}$ is $\{\{i,i+1\} : 1 \leq i \leq 2\ell\} \cup \{1,2\ell+1\}$. Then the maximal stable sets of C are the edges of $\overline{C}$. For $1 \leq i \leq 2\ell$, we set $\mathbf{w}_i = \mathbf{e}_i+\mathbf{e}_{i+1}+\mathbf{e}_{d+1}$

and set $\mathbf{w}_{2\ell+1} = \mathbf{e}_1 + \mathbf{e}_{2\ell+1} + \mathbf{e}_{d+1}$. Then one has

$$\mathbf{b} = \frac{\mathbf{w}_1 + \cdots + \mathbf{w}_{2\ell+1} + (-\mathbf{e}_{d+1})}{2} = \mathbf{e}_1 + \cdots + \mathbf{e}_{2\ell+1} + \ell\mathbf{e}_{d+1}$$

and $\mathbf{b} \in (\ell+1)\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$. Hence there exist $\mathbf{b}_1, \ldots, \mathbf{b}_{\ell+1} \in \Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2}) \cap \mathbb{Z}^{d+1}$ such that $\mathbf{b} = \mathbf{b}_1 + \cdots + \mathbf{b}_{\ell+1}$. Then we may assume that $\mathbf{b}_1, \ldots, \mathbf{b}_\ell \in \mathscr{Q}_C \times \{1\}$ and $\mathbf{b}_{\ell+1} = \mathbf{0}_{d+1}$. However, since the maximal cardinality of the stable sets of C in $[2\ell+1]$ equals 2, a contradiction.

Therefore, if $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ possesses the integer decomposition property, then G_1 and G_2 are perfect, as desired. $\square$

It would, of course, be of interest to find a complete characterization for $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ to be reflexive. For a finite simple graph G on $[d]$, $\Omega(\mathscr{Q}_G)$ is called the *Hansen polytope* of G. This polytope possesses nice properties (e.g., centrally symmetric and 2-level) and is studied in [27, 75]. Especially, in [27], it is shown that if G is perfect, then $\Omega(\mathscr{Q}_G)$ is reflexive. Theorem 10.6 (b) says that G is perfect if and only if the Hansen polytope $\Omega(\mathscr{Q}_G)$ possesses the integer decomposition property.

10.3 δ-polynomials

In this section, we consider the δ-polynomials and the volumes of $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ and $\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ for perfect graphs G_1 and G_2. The *suspension* of a finite simple graph G on $[d]$ is the finite simple graph $\widehat{G}$ on $[d+1]$ with $E(\widehat{G}) = E(G) \cup \{\{i, d+1\} : i \in [d]\}$. Our main theorem of this section is the following.

Theorem 10.6 ([50, Theorem 1.2]). *Let G_1 and G_2 be finite perfect simple graphs on $[d]$. Then one has*

$$\delta(\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2}), t) = \delta(\Gamma(\mathscr{Q}_{\widehat{G_1}}, \mathscr{Q}_{\widehat{G_2}}), t) = (1+t)\delta(\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2}), t).$$

Thus in particular

$$\mathrm{Vol}(\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})) = \mathrm{Vol}(\Gamma(\mathscr{Q}_{\widehat{G_1}}, \mathscr{Q}_{\widehat{G_2}})) = 2 \cdot \mathrm{Vol}(\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})).$$

We obtain this Theorem from the following.

Theorem 10.7 ([50, Theorem 3.1]). *Work with the same situation as in Theorem 10.4. Let $\mathscr{P} \subset \mathbb{R}^d$ be the lattice polytope with $\mathscr{P} \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\mathscr{Q} \subset \mathbb{R}^d$ the lattice polytope with $\mathscr{Q} \cap \mathbb{Z}^d = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$. Suppose that $\Gamma(\mathscr{P}, \mathscr{Q})$ and $\Omega(\mathscr{P}, \mathscr{Q})$ are spanning,*

$$\Gamma(\mathscr{P}, \mathscr{Q}) \cap \mathbb{Z}^d = \{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, -\mathbf{b}_1, \ldots, -\mathbf{b}_{m-1}, \mathbf{0}_d\}$$

and

$$\Omega(\mathscr{P},\mathscr{Q})\cap\mathbb{Z}^{d+1}=\{(\mathbf{a}_1,1),\dots,(\mathbf{a}_n,1),(-\mathbf{b}_1,-1),\dots,(-\mathbf{b}_m,-1),\mathbf{0}_{d+1}\}.$$

Then we obtain

$$\delta(\Omega(\mathscr{P},\mathscr{Q}),\lambda)=(1+\lambda)\delta(\Gamma(\mathscr{P},\mathscr{Q}),\lambda).$$

In particular,

$$\mathrm{Vol}(\Omega(\mathscr{P},\mathscr{Q}))=2\cdot\mathrm{Vol}(\Gamma(\mathscr{P},\mathscr{Q})).$$

Proof. Set $\mathscr{R}=\mathrm{conv}(\{\Gamma(\mathscr{P},\mathscr{Q})\times\{0\},\pm\mathbf{e}_{d+1}\})$. Then it follows from [12, Theorem 1.4] that $\delta(\mathscr{R},t)=(1+t)\delta(\Gamma(\mathscr{P},\mathscr{Q}),t)$. Moreover, by [74, Theorem 1.1] and Theorem 10.4, $\mathscr{R}$ and $\Omega(\mathscr{P},\mathscr{Q})$ possess the integer decomposition property. Hence we should show that $K[\mathscr{R}]$ and $K[\Omega(\mathscr{P},\mathscr{Q})]$ have the same Hilbert function.

Now, use the same notation as in the proof of Theorem 10.4. Then we have

$$\frac{K[\mathbf{x},\mathbf{y},z]}{\mathrm{in}_{<_{\mathrm{ref}}}(I_{\Omega(\mathscr{P},\mathscr{Q})})}=\frac{K[\mathbf{x},\mathbf{y},z]}{\langle\mathscr{M}\rangle}.$$

Set

$$\mathbf{a}'_i=\begin{cases}(\mathbf{a}_i,0), & 1\le i\le n-1,\\ \mathbf{e}_{d+1}, & i=n,\\ \mathbf{0}_{\mathbf{d+1}}, & i=n+1,\end{cases}\quad\text{and } \mathbf{b}'_j=\begin{cases}(\mathbf{b}_i,0), & 1\le j\le m-1,\\ \mathbf{e}_{d+1}, & j=m,\\ \mathbf{0}_{\mathbf{d+1}}, & j=m+1.\end{cases}$$

Then it is easy to show that $A'=(\mathbf{a}_1'^{\top},\dots,\mathbf{a}_{n+1}'^{\top})$ and $B'=(\mathbf{b}_1'^{\top},\dots,\mathbf{b}_{m+1}'^{\top})$ are of harmony. Moreover, $\mathrm{in}_{<_{B'}}(I_{B'})\subset K[y_1,\dots,y_{m+1}]$ and $\mathrm{in}_{<_{A'}}(I_{A'})\subset K[x_1,\dots,x_{n+1}]$ are squarefree with respect to reverse lexicographic orders $<_{A'}$ on $K[x_1,\dots,x_{n+1}]$ and $<_{B'}$ on $K[y_1,\dots,y_{m+1}]$ induced by the orderings $x_{n+1}<_{A'}x_n<_{A'}\cdots<_{A'}x_1$ and $y_{m+1}<_{B'}y_m<_{B'}\cdots<_{B'}y_1$. Now, we introduce the following:

$$\mathscr{E}'=\{\,(i,j)\,:\,1\le i\le n,\,1\le j\le m,\,\mathrm{supp}(\mathbf{a}'_i)\cap\mathrm{supp}(\mathbf{b}'_j)\neq\emptyset\,\}.$$

Then we have $\mathscr{E}'=\mathscr{E}\cup\{(n,m)\}$. Let $\mathscr{M}_{A'}$ (resp. $\mathscr{M}_{B'}$) be the minimal set of squarefree monomial generators of $\mathrm{in}_{<_{A'}}(I_{A'})$ (resp. $\mathrm{in}_{<_{B'}}(I_{B'})$). Then it follows that $\mathscr{M}_{A'}=\mathscr{M}_A$ and $\mathscr{M}_{B'}=\mathscr{M}_B$. This says that $\mathscr{M}=\mathscr{E}'\cup\mathscr{M}_{A'}\cup\mathscr{M}_{B'}$. By the proof of [74, Theorem 1.1], we obtain $\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathscr{R}})=\langle\mathscr{M}\rangle\subset K[\mathbf{x},\mathbf{y},z]$. Hence it follows that

$$\frac{K[\mathbf{x},\mathbf{y},z]}{\mathrm{in}_{<_{\mathrm{rev}}}(I_{\Omega(\mathscr{P},\mathscr{Q})})}=\frac{K[\mathbf{x},\mathbf{y},z]}{\mathrm{in}_{<_{\mathrm{rev}}}(I_{\mathscr{R}})}.$$

Therefore, $K[\mathscr{R}]$ and $K[\Omega(\mathscr{P},\mathscr{Q})]$ have the same Hilbert function, as desired. □

Now, we prove Theorem 10.6.

Proof of Theorem 10.6. For any finite simple graph G on $[d]$, we have $S(\widehat{G}) = S(G) \cup \{d+1\}$. Hence it follows that $\Gamma(\mathscr{Q}_{\widehat{G_1}}, \mathscr{Q}_{\widehat{G_2}}) = \mathrm{conv}(\{\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_1}) \times \{0\}, \pm\mathbf{e}_{d+1}\})$. Therefore, by Theorem 10.7, we obtain

$$\delta(\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2}, t) = \delta(\Gamma(\mathscr{Q}_{\widehat{G_1}}, \mathscr{Q}_{\widehat{G_2}}), t) = (1+t)\delta(\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2}), t),$$

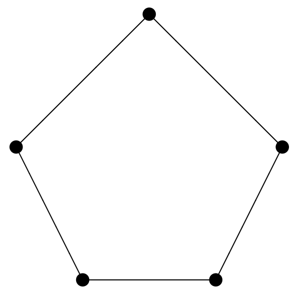
as desired. □

10.4 Examples

In this section, we give some curious examples of $\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ and $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$. First, the following example says that even though G_1 and G_2 are not perfect graphs, $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ may be reflexive.

Example 10.8. Let G be the finite simple graph as follows:

G:



Namely, G is a cycle of length 5. Then G is not perfect. Hence $\Gamma(\mathscr{Q}_G, \mathscr{Q}_G)$ is not reflexive. However, $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ is reflexive. In fact, we have

$$\delta(\Gamma(\mathscr{Q}_G, \mathscr{Q}_G), t) = 1 + 15t + 60t^2 + 62t^3 + 15t^4 + t^5,$$

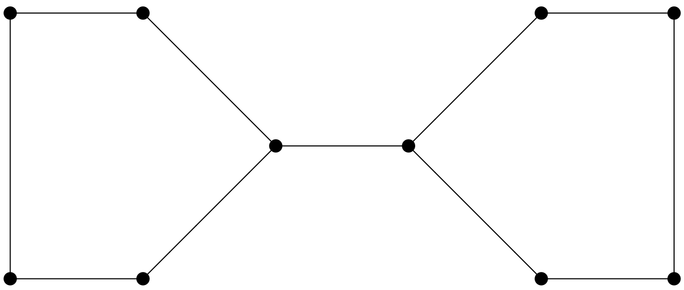
$$\delta(\Omega(\mathscr{Q}_G, \mathscr{Q}_G), t) = 1 + 16t + 75t^2 + 124t^3 + 75t^4 + 16t^5 + t^6.$$

Moreover, $\Gamma(\mathscr{Q}_G, \mathscr{Q}_G)$ possesses the integer decomposition property, but $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ does not possess the integer decomposition property.

For this example, $\Gamma(\mathscr{Q}_G, \mathscr{Q}_G)$ possesses the integer decomposition property. Next example says that if G_1 and G_2 are not perfect, $\Gamma(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ may not possess the integer decomposition property.

Example 10.9. Let G be a finite simple graph whose complementary graph $\overline{G}$ is as follows:

$\overline{G}$:

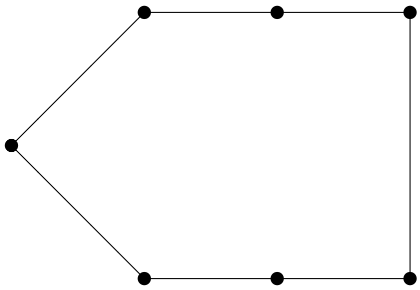


Then G is not perfect. Hence $\Gamma(\mathscr{Q}_G, \mathscr{Q}_G)$ is not reflexive. However, $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ is reflexive. Moreover, in this case, $\Gamma(\mathscr{Q}_G, \mathscr{Q}_G)$ and $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ do not possess the integer decomposition property.

For any finite simple graph G with at most 6 vertices, $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ is always reflexive. However, in the case of finite simple graphs with more than 6 vertices, we obtain a different result.

Example 10.10. Let G be the finite simple graph as follows:

G:



Namely, G is a cycle of length 7. Then G is not perfect. Hence $\Gamma(\mathscr{Q}_G, \mathscr{Q}_G)$ is not reflexive. Moreover, $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ is not reflexive. In fact, we have

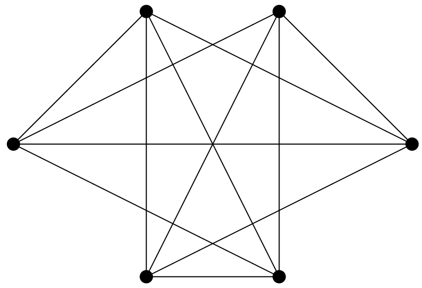
$$\delta(\Gamma(\mathscr{Q}_G, \mathscr{Q}_G), t) = 1 + 49t + 567t^2 + 1801t^3 + 1799t^4 + 569t^5 + 49t^6 + t^7,$$

$$\delta(\Omega(\mathscr{Q}_G, \mathscr{Q}_G), t) = 1 + 50t + 616t^2 + 2370t^3 + 3598t^4 + 2368t^5 + 618t^6 + 50t^7 + t^8.$$

Finally, we show that even though the Ehrhart δ-polynomial of $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ coincides with that of $\Gamma(\mathscr{Q}_{\widehat{G_1}}, \mathscr{Q}_{\widehat{G_2}})$, $\Omega(\mathscr{Q}_{G_1}, \mathscr{Q}_{G_2})$ may not be unimodularly equivalent to $\Gamma(\mathscr{Q}_{\widehat{G_1}}, \mathscr{Q}_{\widehat{G_2}})$.

Example 10.11. Let G be the finite simple graph as follows:

G:



Namely, G is a $(2,2,2)$-complete multipartite graph. Then G is perfect. Hence we know that $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ and $\Gamma(\mathscr{Q}_{\widehat{G}}, \mathscr{Q}_{\widehat{G}})$ have the same Ehrhart δ-polynomial and the same volume. However, $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ has 54 facets and $\Gamma(\mathscr{Q}_{\widehat{G}}, \mathscr{Q}_{\widehat{G}})$ has 432 facets. Hence, $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$ and $\Gamma(\mathscr{Q}_{\widehat{G}}, \mathscr{Q}_{\widehat{G}})$ are not unimodularly equivalent. Moreover, for any finite simple graph G' on $\{1, \ldots, 7\}$ except for $\widehat{G}$, the Ehrhart δ-polynomial of $\Gamma(\mathscr{Q}_{G'}, \mathscr{Q}_{G'})$ is not equal to that of $\Omega(\mathscr{Q}_G, \mathscr{Q}_G)$.

Chapter 11

Reflexive polytopes arising from perfect graphs and posets

In this chapter, the study on $\Gamma(\mathscr{P},\mathscr{Q})$ and $\Omega(\mathscr{P},\mathscr{Q})$ will be done, when $\mathscr{P}$ is the order polytope $\mathscr{O}_P$ of a finite poset $P=\{p_1,\ldots,p_d\}$ and $\mathscr{Q}$ is the stable set polytope $\mathscr{Q}_G$ of a finite simple graph G on $[d]$. In particular, we give two new classes of reflexive polytopes with the integer decomposition property which arise from order polytopes and stable set polytopes. In fact, we show the following.

Theorem 11.1 ([51, Theorem 1.2])**.** *Let G be a finite simple graph on $[d]$. Then the following conditions are equivalent:*

(i) $\Gamma(\mathscr{O}_P,\mathscr{Q}_G)$ *is a reflexive polytope for some finite poset* $P=\{p_1,\ldots,p_d\}$*;*

(ii) $\Gamma(\mathscr{O}_P,\mathscr{Q}_G)$ *is a reflexive polytope for all finite poset* $P=\{p_1,\ldots,p_d\}$*;*

(iii) $\Omega(\mathscr{O}_P,\mathscr{Q}_G)$ *possesses the integer decomposition property for some finite poset* $P=\{p_1,\ldots,p_d\}$*;*

(iv) $\Omega(\mathscr{O}_P,\mathscr{Q}_G)$ *possesses the integer decomposition property for all finite poset* $P=\{p_1,\ldots,p_d\}$*;*

(v) *G is perfect.*

Furthermore, if G is perfect, then each of $\Gamma(\mathscr{O}_P,\mathscr{Q}_G)$ and $\Omega(\mathscr{O}_P,\mathscr{Q}_G)$ is a reflexive polytope with the integer decomposition property for all finite poset $P=\{p_1,\ldots,p_d\}$.

A proof of Theorem 11.1 will be given in Sections 11.1 and 11.2. Furthermore, in Section 11.3, the discussion on δ-polynomials of these reflexive polytopes will be achieved.

11.1 Type Γ

In this section, we prove the equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v) of Theorem 11.1. In fact, we prove the following proposition.

Proposition 11.2. *Let G be a finite simple graph on $[d]$. Then the following conditions are equivalent:*

(i) *$\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ is a reflexive polytope for some finite poset $P = \{p_1, \dots, p_d\}$;*

(ii) *$\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ is a reflexive polytope for all finite poset $P = \{p_1, \dots, p_d\}$;*

(iii) *G is perfect.*

In particular, if G is perfect, then $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ possesses the integer decomposition property for all finite poset $P = \{p_1, \dots, p_d\}$.

Proof. ((iii) $\Rightarrow$ (ii)) Suppose that G is perfect. Let

$$K[\mathscr{O}\mathscr{Q}] = K[\{x_I\}_{\emptyset \neq I \in \mathscr{J}(P)} \cup \{y_C\}_{\emptyset \neq C \in S(G)} \cup \{z\}]$$

denote the polynomial ring over K and define the surjective ring homomorphism $\pi : K[\mathscr{O}\mathscr{Q}] \to K[\Gamma(\mathscr{O}_P, \mathscr{Q}_G)] \subset K[t_1^{\pm 1}, \dots, t_d^{\pm 1}, s]$ by the following:

- $\pi(x_I) = \mathbf{t}^{\rho(I)} s$, where $\emptyset \neq I \in \mathscr{J}(P)$;
- $\pi(y_C) = \mathbf{t}^{-\rho(C)} s$, where $\emptyset \neq C \in S(G)$;
- $\pi(z) = s$.

Then the toric ideal $I_{\Gamma(\mathscr{O}_P, \mathscr{Q}_G)}$ of $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ is the kernel of π.

Let $<_{\mathscr{O}_P}$ and $<_{\mathscr{Q}_G}$ denote reverse lexicographic orders on $K[\mathscr{O}] = K[\{x_I\}_{\emptyset \neq I \in \mathscr{J}(P)} \cup \{z\}]$ and $K[\mathscr{Q}] = K[\{y_C\}_{\emptyset \neq C \in S(G)} \cup \{z\}]$ satisfying

- $z <_{\mathscr{O}_P} x_I$ and $z <_{\mathscr{Q}_G} y_C$;
- $x_{I'} <_{\mathscr{O}_P} x_I$ if $I' \subset I$;
- $y_{C'} <_{\mathscr{Q}_G} y_C$ if $C' \subset C$,

where $I, I' \in \mathscr{J}(P) \setminus \{\emptyset\}$ with $I \neq I'$ and $C, C' \in S(G) \setminus \{\emptyset\}$ with $C \neq C'$. From Lemmas 9.2 and 10.2, we know that $\mathrm{in}_{<_{\mathscr{O}_P}}(I_{\mathscr{O}_P})$ and $\mathrm{in}_{<_{\mathscr{Q}_G}}(I_{\mathscr{Q}_G})$ are squarefree. Let $\mathscr{M}_{\mathscr{O}_P}$ and $\mathscr{M}_{\mathscr{Q}_G}$ be the minimal sets of squarefree monomial generators of $\mathrm{in}_{<_{\mathscr{O}_P}}(I_{\mathscr{O}_P})$ and $\mathrm{in}_{<_{\mathscr{Q}_G}}(I_{\mathscr{Q}_G})$. Then Lemma 9.2, it follows that

$$\mathscr{M}_{\mathscr{O}_P} = \{x_I x_{I'} : I, I' \in \mathscr{J}(P), I \not\subseteq I', I \not\supseteq I'\}. \tag{11.1}$$

Let $<$ be a reverse lexicographic order on $K[\mathscr{O}\mathscr{Q}]$ satisfying

- $z < y_C < x_I$;
- $x_{I'} < x_I$ if $I' \subset I$;
- $y_{C'} < y_C$ if $C' \subset C$,

where $I, I' \in \mathscr{J}(P) \setminus \{\emptyset\}$ with $I \neq I'$ and $C, C' \in S(G) \setminus \{\emptyset\}$ with $C \neq C'$, and set

$$\mathscr{M} = \mathscr{M}_{\mathscr{O}_P} \cup \mathscr{M}_{\mathscr{Q}_G} \cup \{x_I y_C : I \in \mathscr{J}(P), C \in S(G), \max(I) \cap C \neq \emptyset\}.$$

Let $\mathscr{G}$ be a finite set of binomials belonging to $I_{\Gamma(\mathscr{O}_P, \mathscr{Q}_G)}$ with $\mathscr{M} = \{\mathrm{in}_<(g) : g \in \mathscr{G}\}$.

Now, we prove that $\mathscr{G}$ is a Gröbner base of $I_{\Gamma(\mathscr{O}_P, \mathscr{Q}_G)}$ with respect to $<$. Suppose that there exists a nonzero irreducible binomial $f = u - v$ be belonging to $I_{\Gamma(\mathscr{O}_P, \mathscr{Q}_G)}$ such that $u \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$ and $v \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$. Write

$$u = \left(\prod_{1 \leq i \leq a} x_{I_i}^{\mu_{I_i}} \right) \left(\prod_{1 \leq j \leq b} y_{C_j}^{\nu_{C_j}} \right), \quad v = z^{\alpha} \left(\prod_{1 \leq i \leq a'} x_{I'_i}^{\mu'_{I'_i}} \right) \left(\prod_{1 \leq j \leq b'} y_{C'_j}^{\nu'_{C'_j}} \right),$$

where

- $I_1, \ldots, I_a, I'_1, \ldots, I'_{a'} \in \mathscr{J}(P) \setminus \{\emptyset\}$;
- $C_1, \ldots, C_b, C'_1, \ldots, C'_{b'} \in S(G) \setminus \{\emptyset\}$;
- a, a', b, b' and α are nonnegative integers;
- $\mu_I, \mu'_{I'}, \nu_C, \nu'_{C'}$ are positive integers.

By (11.1), we may assume that $I_1 \subsetneq \cdots \subsetneq I_a$ and $I'_1 \subsetneq \cdots \subsetneq I'_{a'}$. If $(a, a') = (0, 0)$, then $\mathrm{in}_{<_{\mathscr{Q}_G}}(f) = \mathrm{in}_<(f)$. Hence we have $(a, a') \neq (0, 0)$. Assume that $I_a \setminus I_{a'} \neq \emptyset$. Then there exists a maximal element i of I_a such that $i \notin I_{a'}$. Hence we have

$$\sum_{\substack{I \in \{I_1, \ldots, I_a\} \\ i \in \max(I)}} \mu_I - \sum_{\substack{C \in \{C_1, \ldots, C_b\} \\ i \in C}} \nu_J = - \sum_{\substack{C' \in \{C'_1, \ldots, C'_{b'}\} \\ i \in C'}} \nu'_{C'} \leq 0.$$

This implies that there exists a stable set $C \in \{C_1, \ldots, C_b\}$ such that $i \in C$. Then $x_{I_a} y_C \in \mathscr{M}$, a contradiction. Similarly, it does not follow that $I_{a'} \setminus I_a \neq \emptyset$. Therefore, $\mathscr{G}$ is a Gröbner base of $I_{\Gamma(\mathscr{O}_P, \mathscr{Q}_G)}$ with respect to $<$.

Thus, by Lemma 8.4, it follows that $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ is reflexive and possesses the integer decomposition property.

((i) $\Rightarrow$ (iii)) Suppose that G is not perfect. By Lemma 10.1, G possesses either an odd hole or an odd antihole. First, suppose that G possesses an odd hole C of length $2\ell+1$, where $\ell \geq 2$. By renumbering the vertex set, we may assume that the edge set of C is $\{\{i, i+1\} : 1 \leq i \leq 2\ell\} \cup \{1, 2\ell+1\}$. Then the hyperplane $\mathscr{H}' \subset \mathbb{R}^d$ defined by the equation $z_1 + \cdots + z_{2\ell+1} = -\ell$ is a supporting hyperplane of $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$. Let $\mathscr{F}$ be a facet of $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ with $\mathscr{H}' \cap \Gamma(\mathscr{O}_P, \mathscr{Q}_G) \subset \mathscr{F}$ and $a_1 z_1 + \cdots + a_d z_d = 1$ with each $a_i \in \mathbb{R}$ the equation of the supporting hyperplane $\mathscr{H} \subset \mathbb{R}^d$ with $\mathscr{F} \subset \mathscr{H}$. The maximal stable sets of C are

$$S_1 = \{1, 3, \ldots, 2\ell-1\}, S_2 = \{2, 4, \ldots, 2\ell\}, \ldots, S_{2\ell+1} = \{2\ell+1, 2, 4, \ldots, 2\ell-2\}$$

and each $i \in [2\ell+1]$ appears ℓ times in the above list. Since for each S_i, we have $-\sum_{j \in S_i} a_j = 1$, it follows that $-\ell(a_1 + \cdots + a_{2\ell+1}) = 2\ell+1$. Hence $a_1 + \cdots + a_{2\ell+1} \notin \mathbb{Z}$. Therefore, $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ is not reflexive.

Finally, we suppose that G possesses an odd antihole C of length $2\ell+1$, where $\ell \geq 2$. Similarly, we may assume that the edge set of $\overline{C}$ is $\{\{i, i+1\} : 1 \leq i \leq 2\ell\} \cup \{1, 2\ell+1\}$. Then the hyperplane $\mathscr{H}' \subset \mathbb{R}^d$ defined by the equation $z_1 + \cdots + z_{2\ell+1} = -2$ is a supporting hyperplane of $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$. Let $\mathscr{F}$ be a facet of $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ with $\mathscr{H}' \cap \Gamma(\mathscr{O}_P, \mathscr{Q}_G) \subset \mathscr{F}$ and $a_1 z_1 + \cdots + a_d z_d = 1$ with each $a_i \in \mathbb{R}$ the equation of the supporting hyperplane $\mathscr{H} \subset \mathbb{R}^d$ with $\mathscr{F} \subset \mathscr{H}$. Then since the maximal stable sets of C is the edges of $\overline{C}$, for each edge $\{i, j\}$ of $\overline{C}$, we have $-(a_i + a_j) = 1$. Hence it follows that $-2(a_1 + \cdots + a_{2\ell+1}) = 2\ell+1$. Thus $a_1 + \cdots + a_{2\ell+1} \notin \mathbb{Z}$. Therefore, $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ is not reflexive, as desired. □

Remark 11.3. Proposition 11.2 is a generalization of Theorem 9.4 (2).

11.2 Type Ω

In this section, we prove the equivalence (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) of Theorem 11.1. In fact, we prove the following proposition.

Proposition 11.4. *Let G be a finite simple graph on $[d]$. Then the following conditions are equivalent:*

(i) *$\Omega(\mathscr{O}_P, \mathscr{Q}_G)$ possesses the integer decomposition property for some finite poset $P = \{p_1, \ldots, p_d\}$;*

(ii) *$\Omega(\mathscr{O}_P, \mathscr{Q}_G)$ possesses the integer decomposition property for all finite poset $P = \{p_1, \ldots, p_d\}$;*

(iii) *G is prefect.*

In particular, if G is perfect, then $\Omega(\mathscr{O}_P, \mathscr{Q}_G)$ is reflexive for all finite poset $P = \{p_1, \ldots, p_d\}$.

Proof. ((iii) $\Rightarrow$ (ii)) Suppose G is perfect. Let

$$K[\mathscr{O}\mathscr{Q}] = K[\{x_I\}_{I \in \mathscr{J}(P)} \cup \{y_C\}_{C \in S(G)} \cup \{z\}]$$

denote the polynomial ring over K and define the surjective ring homomorphism $\pi : K[\mathscr{O}\mathscr{Q}] \to K[\Omega(\mathscr{O}_P, \mathscr{Q}_G)] \subset K[t_1^{\pm 1}, \ldots, t_{d+1}^{\pm 1}, s]$ by the following:

- $\pi(x_I) = \mathbf{t}^{\rho(I)} t_{d+1} s$, where $I \in \mathscr{J}(P)$;
- $\pi(y_C) = \mathbf{t}^{-\rho(C)} t_{d+1}^{-1} s$, where $C \in S(G)$;
- $\pi(z) = s$.

Then the toric ideal $I_{\Omega(\mathscr{O}_P, \mathscr{Q}_G)}$ of $\Omega(\mathscr{O}_P, \mathscr{Q}_G)$ is the kernel of π.

Let $<_{\mathscr{O}_P}$ and $<_{\mathscr{Q}_G}$ denote reverse lexicographic orders on $K[\mathscr{O}] = K[\{x_I\}_{I \in \mathscr{J}(P)}]$ and $K[\mathscr{Q}] = K[\{y_C\}_{C \in S(G)}]$ satisfying

- $x_{I'} <_{\mathscr{O}_P} x_I$ if $I' \subset I$;
- $y_{C'} <_{\mathscr{Q}_G} y_C$ if $C' \subset C$,

where $I, I' \in \mathscr{J}(P)$ with $I \neq I'$ and $C, C' \in S(G)$ with $C \neq C'$, and $\mathscr{M}_{\mathscr{O}_P}$ and $\mathscr{M}_{\mathscr{Q}_G}$ the minimal sets of squarefree monomial generators of $\text{in}_{<_{\mathscr{O}_P}}(I_{\mathscr{O}_P})$ and $\text{in}_{<_{\mathscr{Q}_G}}(I_{\mathscr{Q}_G})$. Then it follows that

$$\mathscr{M}_{\mathscr{O}_P} = \{x_I x_{I'} : I, I' \in \mathscr{J}(P), I \not\subseteq I', I \not\supseteq I'\}. \tag{11.2}$$

Let $<$ be a reverse lexicographic order on $K[\mathscr{O}\mathscr{Q}]$ satisfying

- $z < y_C < x_I$;
- $x_{I'} < x_I$ if $I' \subset I$;
- $y_{C'} < y_C$ if $C' \subset C$,

where $I, I' \in \mathscr{J}(P)$ with $I \neq I'$ and $C, C' \in S(G)$ with $C \neq C'$, and set

$$\mathscr{M} = \mathscr{M}_{\mathscr{O}_P} \cup \mathscr{M}_{\mathscr{Q}_G} \cup \{x_I y_C : I \in \mathscr{J}(P), C \in S(G), \max(I) \cap C \neq \emptyset\} \cup \{x_\emptyset y_\emptyset\}.$$

Let $\mathscr{G}$ be a finite set of binomials belonging to $I_{\Omega(\mathscr{O}_P, \mathscr{Q}_G)}$ with $\mathscr{M} = \{\mathrm{in}_<(g) : g \in \mathscr{G}\}$.

Now, we prove that $\mathscr{G}$ is a Gröbner base of $I_{\Omega(\mathscr{O}_P, \mathscr{Q}_G)}$ with respect to $<$. Suppose that there exists a nonzero irreducible binomial $f = u - v$ be belonging to $I_{\Omega(\mathscr{O}_P, \mathscr{Q}_G)}$ such that $u \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$ and $v \notin \langle \{\mathrm{in}_<(g) : g \in \mathscr{G}\} \rangle$. Write

$$u = \left(\prod_{1 \leq i \leq a} x_{I_i}^{\mu_{I_i}} \right) \left(\prod_{1 \leq j \leq b} y_{C_j}^{\nu_{C_j}} \right), \quad v = z^\alpha \left(\prod_{1 \leq i \leq a'} x_{I'_i}^{\mu'_{I'_i}} \right) \left(\prod_{1 \leq j \leq b'} y_{C'_j}^{\nu'_{C'_j}} \right),$$

where

- $I_1, \ldots, I_a, I'_1, \ldots, I'_{a'} \in \mathscr{J}(P)$;
- $C_1, \ldots, C_b, C'_1, \ldots, C'_{b'} \in S(G)$;
- a, a', b, b' and α are nonnegative integers with $(a, a') \neq 0$;
- $\mu_I, \mu'_{I'}, \nu_C, \nu'_{C'}$ are positive integers.

By (11.2), we may assume that $I_1 \subsetneq \cdots \subsetneq I_a$ and $I'_1 \subsetneq \cdots \subsetneq I'_{a'}$

By the same way of the proof of Proposition 11.2, we know that $a = 0$ and $I_{a'} = \emptyset$, or $a' = 0$ and $I_a = \emptyset$. Suppose that $a = 0$ and $I_{a'} = \emptyset$. Then by focusing on the degree of t^{d+1} and s of $\pi(u)$ and $\pi(v)$, we have

$$-\sum_{1 \leq j \leq b} \nu_{C_j} = \mu'_\emptyset - \sum_{1 \leq j \leq b'} \nu'_{C'_j},$$

$$\sum_{1 \leq j \leq b} \nu_{C_j} = \alpha + \mu'_\emptyset + \sum_{1 \leq j \leq b'} \nu'_{C'_j}.$$

Hence $0 = \alpha + 2\mu'_\emptyset > 0$, a contradiction.

Suppose that $a' = 0$ and $I_a = \emptyset$. Then we have

$$\mu_\emptyset - \sum_{1 \leq j \leq b} \nu_{C_j} = -\sum_{1 \leq j \leq b'} \nu'_{C'_j},$$

$$\mu_\emptyset + \sum_{1 \leq j \leq b} \nu_{C_j} = \alpha + \sum_{1 \leq j \leq b'} \nu'_{C'_j}.$$

Hence one obtains $2\mu_\emptyset = \alpha$. By focusing on $y_\emptyset^{\mu_\emptyset} \cdot f$, it is easy to show that

$$f' = \left(\prod_{1 \leq j \leq b} y_{C_j}^{\nu_{C_j}} \right) - y_\emptyset^{\mu_\emptyset} \left(\prod_{1 \leq j \leq b'} y_{C'_j}^{\nu'_{C'_j}} \right) \in I_{\Omega(\mathscr{O}_P, \mathscr{Q}_G)}.$$

Since $x_\emptyset y_\emptyset \in \mathscr{M}$, for each i, $C_i \neq \emptyset$. Hence $\mathrm{in}_<(f') = \prod_{1\leq j\leq b} y_{C_j}^{v_{C_j}}$ and $\mathrm{in}_<(f')$ divides u, a contradiction. Therefore, $\mathscr{G}$ is a Gröbner base of $I_{\Omega(\mathscr{O}_P,\mathscr{Q}_G)}$ with respect to $<$.

Thus, by Lemma 8.4, it follows that $\Omega(\mathscr{O}_P,\mathscr{Q}_G)$ is reflexive and possesses the integer decomposition property.

((i) $\Rightarrow$ (iii)) We can prove this by the same way of Theorem 10.3. □

Remark 11.5. Proposition 11.4 is a generalization of Theorem 9.32 (2).

11.3 δ-polynomials

In this section, we discuss the δ-polynomials of $\Gamma(\mathscr{O}_P,\mathscr{Q}_G)$ and $\Omega(\mathscr{O}_P,\mathscr{Q}_G)$ for a finite poset $P=\{p_1,\dots,p_d\}$ and a perfect graph G on $[d]$.

The following theorem is the main result of this section.

Theorem 11.6 ([51, Theorem 1.4])**.** *Let* $P=\{p_1,\dots,p_d\}$ *be a finite poset and* G *a perfect graph on* $[d]$*. Then we have*

$$\delta(\Gamma(\mathscr{O}_P,\mathscr{Q}_G),t) = \delta(\Gamma(\mathscr{C}_P,\mathscr{Q}_G),t),$$

$$\delta(\Omega(\mathscr{O}_P,\mathscr{Q}_G),t) = \delta(\Omega(\mathscr{C}_P,\mathscr{Q}_G),t),$$

$$\delta(\Omega(\mathscr{O}_P,\mathscr{Q}_G),t) = (1+t)\cdot\delta(\Gamma(\mathscr{O}_P,\mathscr{Q}_G),t).$$

Proof. Let

$$K[\mathscr{C}\mathscr{Q}] = K[\{x_{\max(I)}\}_{\emptyset\neq I\in\mathscr{J}(P)} \cup \{y_C\}_{\emptyset\neq C\in S(G)} \cup \{z\}]$$

denote the polynomial ring over K and define the surjective ring homomorphism $\pi : K[\mathscr{C}\mathscr{Q}] \to K[\Gamma(\mathscr{C}_P,\mathscr{Q}_G)] \subset K[t_1^{\pm1},\dots,t_d^{\pm1}]$ by the following:

- $\pi(x_{\max(I)}) = \mathbf{t}^{\rho(\max(I))}s$, where $\emptyset \neq I \in \mathscr{J}(P)$;
- $\pi(y_C) = \mathbf{t}^{-\rho(C)}s$, where $\emptyset \neq C \in S(G)$;
- $\pi(z) = s$.

Then the toric ideal $I_{\Gamma(\mathscr{C}_P,\mathscr{Q}_G)}$ of $\Gamma(\mathscr{C}_P,\mathscr{Q}_G)$ is the kernel of π.

Let $<_{\mathscr{C}_P}$ and $<_{\mathscr{Q}_G}$ denote reverse lexicographic orders on $K[\mathscr{C}] = K[\{x_{\max(I)}\}_{\emptyset\neq I\in\mathscr{J}(P)} \cup \{z\}]$ and $K[\mathscr{Q}] = K[\{y_C\}_{\emptyset\neq C\in S(G)} \cup \{z\}]$ satisfying

- $z <_{\mathscr{C}_P} x_{\max(I)}$ and $z <_{\mathscr{Q}_G} y_C$;

- $x_{\max(I')} <_{\mathscr{C}_P} x_{\max(I)}$ if $I' \subset I$;
- $y_{C'} <_{\mathscr{Q}_G} y_C$ if $C' \subset C$,

where $I, I' \in \mathscr{J}(P) \setminus \{\emptyset\}$ with $I \neq I'$ and $C, C' \in S(G) \setminus \{\emptyset\}$ with $C \neq C'$. Then from Lemma 9.3, we know that $\text{in}_{<_{\mathscr{C}_P}}(I_{\mathscr{C}_P})$ is squarefree. Let $\mathscr{M}_{\mathscr{C}_P}$ and $\mathscr{M}_{\mathscr{Q}_G}$ be the minimal sets of squarefree monomial generators of $\text{in}_{<_{\mathscr{C}_P}}(I_{\mathscr{C}_P})$ and $\text{in}_{<_{\mathscr{Q}_G}}(I_{\mathscr{Q}_G})$. Then from 9.3, it follows that

$$\mathscr{M}_{\mathscr{C}_P} = \{x_{max(I)} x_{\max(I')} : I, I' \in \mathscr{J}(P), I \not\subseteq I', I \not\supseteq I'\}. \tag{11.3}$$

Let $<_{\mathscr{C}\mathscr{Q}}$ be a reverse lexicographic order on $K[\mathbf{x}, \mathbf{y}, z]$ satisfying

- $z <_{\mathscr{C}\mathscr{Q}} y_C <_{\mathscr{C}\mathscr{Q}} x_{\max(I)}$;
- $x_{\max(I')} <_{\mathscr{C}\mathscr{Q}} x_{\max(I)}$ if $I' \subset I$;
- $y_{C'} <_{\mathscr{C}\mathscr{Q}} y_C$ if $C' \subset C$,

where $I, I' \in \mathscr{J}(P) \setminus \{\emptyset\}$ with $I \neq I'$ and $C, C' \in S(G) \setminus \{\emptyset\}$ with $C \neq C'$, and set

$$\mathscr{M}_{\mathscr{C}\mathscr{Q}} = \mathscr{M}_{\mathscr{C}_P} \cup \mathscr{M}_{\mathscr{Q}_G} \cup \{x_{\max(I)} y_C : I \in \mathscr{J}(P), C \in S(G), \max(I) \cap C \neq \emptyset\}.$$

Let $\mathscr{G}$ be a finite set of binomials belonging to $I_{\Gamma(\mathscr{C}_P, \mathscr{Q}_G)}$ with $\mathscr{M}_{\mathscr{C}\mathscr{Q}} = \{\text{in}_{<_{\mathscr{C}\mathscr{Q}}}(g) : g \in \mathscr{G}\}$. By the same way of the proof of Proposition 11.2, we can prove that $\mathscr{G}$ is a Gröbner base of $I_{\Omega(\mathscr{C}_P, \mathscr{Q}_G)}$ with respect to $<_{\mathscr{C}\mathscr{Q}}$.

Now, use the same notation as in the proof of Proposition 11.2. Set

$$R_{\mathscr{O}\mathscr{Q}} = \frac{K[\mathscr{O}\mathscr{Q}]}{\langle \mathscr{M}_{\mathscr{O}\mathscr{Q}} \rangle}, R_{\mathscr{C}\mathscr{Q}} = \frac{K[\mathscr{C}\mathscr{Q}]}{\langle \mathscr{M}_{\mathscr{C}\mathscr{Q}} \rangle}.$$

Then the Hilbert function of $K[\Gamma(\mathscr{O}_P, \mathscr{Q}_G)]$ equals that of $R_{\mathscr{O}\mathscr{Q}}$, and the Hilbert function of $K[\Gamma(\mathscr{C}_P, \mathscr{Q}_G)]$ equals that of $R_{\mathscr{C}\mathscr{Q}}$. Moreover, it is easy to see that the ring homomorphism $\varphi : R_{\mathscr{O}\mathscr{Q}} \to R_{\mathscr{C}\mathscr{Q}}$ by setting $\varphi(x_I) = x_{\max(I)}, \varphi(y_C) = y_C$ and $\varphi(z) = z$ is an isomorphism. Hence since $\Gamma(\mathscr{O}_P, \mathscr{Q}_G)$ and $\Gamma(\mathscr{C}_P, \mathscr{Q}_G)$ possess the integer decomposition property, we have

$$\delta(\Gamma(\mathscr{O}_P, \mathscr{Q}_G), t) = \delta(\Gamma(\mathscr{C}_P, \mathscr{Q}_G), t).$$

Similarly, we obtain

$$\delta(\Omega(\mathscr{O}_P, \mathscr{Q}_G), t) = \delta(\Omega(\mathscr{C}_P, \mathscr{Q}_G), t).$$

Moreover, since the chain polytope $\mathscr{C}_P$ is a stable set polytope of a perfect graph, by Theorem 10.6, it follows that

$$\delta(\Omega(\mathscr{O}_P, \mathscr{Q}_G), t) = (1+t) \cdot \delta(\Gamma(\mathscr{O}_P, \mathscr{Q}_G), t),$$

as desired. □

Remark 11.7. In Theorems 9.8 and 9.36, for any finite posets P and Q on $[d]$, it is proved that $\delta(\Gamma(\mathscr{O}_P, \mathscr{C}_Q), t) = \delta(\Gamma(\mathscr{C}_P, \mathscr{C}_Q), t)$ and $\delta(\Omega(\mathscr{O}_P, \mathscr{C}_Q), t) = \delta(\Omega(\mathscr{C}_P, \mathscr{C}_Q), t)$. Therefore, Theorem 11.6 is a generalization of those results.

Chapter 12

Reflexive polytopes arising from finite simple graphs

It is known that every lattice polytope is unimodularly equivalent to a face of some reflexive polytope ([29]). From this fact, in [29], the *reflexive dimension* of a lattice polytope is introduced. For a lattice polytope $\mathscr{P}$, its *reflexive dimension* is the smallest integer r such that $\mathscr{P}$ is unimodularly equivalent to a face of a reflexive polytope of dimension r. Computing the reflexive dimension of a lattice polytope is hard problem in general. However, it is reasonable to determine the reflexive dimension of a $(0,1)$-polytope. In fact, it follows from Chapters 9, 10 and 11 that if $\mathscr{P}$ is an order polytope, a chain polytope or a stable set polytope, then its reflexive dimension equals $\dim(\mathscr{P})+1$. Hence, we ask the following question:

Question 12.1. *For any $(0,1)$-polytope of dimension d, is its reflexive dimension equal to $d+1$?*

In Chapters 9, 10 and 11, by using technique on Grönbner bases, we give several classes of reflexive polytopes. In fact, the order polytopes and the cain polytopes of finite posets and the stable set polytopes of perfect graphs are compressed and, in particular, possess the integer decomposition property. However, a $(0,1)$-polytope may not possess the integer decomposition property. Hence we cannot solve this question for any $(0,1)$-polytope by using the same methods. In this chapter, by using matrix theory, we show this question is true for the edge polytopes of finite simple graphs, which necessarily do not possess the integer decomposition property.

This chapter is organized as follows. In Section 12.1, we recall the definition of the edge polytopes of finite simple graphs. In Section 12.2, we will give a new class of reflexive polytopes arising from some class of lattice polytopes (Theorem 12.4). From this result, we can show that every edge polytope is unimodularly equivalent to a facet of some reflexive polytope (Corollary 12.6). Finally, in Section 12.3, we will give a criterion to discuss when the reflexive polytopes arising from the edge polytope of connected finite simple graph described in Corollary 12.6 (Theorem 12.8) possess the integer decomposition property.

12.1 Edge polytopes

In this section, we introduce the edge polytopes of finite simple graphs. Throughout this section, we assume that every finite simple graph has no isolated vertices. Let G be a simple graph on the vertex set $V(G) = [d]$ and denote $E(G)$ the edge set of G. The *edge polytope* $\mathscr{P}_G \subset \mathbb{R}^d$ of G is the convex hull of all vectors $\mathbf{e}_i + \mathbf{e}_j$ such that $\{i, j\} \in E(G)$. This means that the edge polytope of $\mathscr{P}_G$ of G is the convex hull of all row vectors of the *incidence matrix* A_G of G, where A_G is the matrix in $\{0,1\}^{E(G)\times[d]}$ with

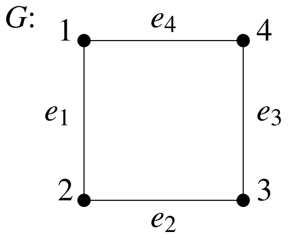
$$a_{e,v} = \begin{cases} 1 & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the dimension of P_G equals $\mathrm{rank}(A) - 1$. In fact,

Lemma 12.2 ([94, p. 57]). *Let G be a finite simple graph on $[d]$ and $c_0(G)$ the number of connected bipartite components of G. Then the dimension of the edge polytope $\mathscr{P}_G$ of G equals $d - c_0(G) - 1$.*

Every lattice polytope $\mathscr{P}$ is unimodularly equivalent to a full-dimensional lattice polytope $\mathscr{Q}$. We say that $\mathscr{Q}$ is a *full-dimensional unimodularly equivalent copy* of $\mathscr{P}$. Sometimes it is convenient to work with full-dimensional lattice polytopes, i.e., lattice polytopes embedded in a space of their same dimension. However, the edge polytopes of finite simple graphs are not full-dimensional from Lemma 12.2. Given an edge polytope $\mathscr{P}_G$, one can easily get a full-dimensional unimodularly equivalent copy of $\mathscr{P}_G$ by considering the lattice polytope defined as the convex hull of the row vectors of A_G with some columns deleted. Indeed, let $G_1, \dots, G_k$ be the connected bipartite components of G. If $k = 0$, we can get a full-dimensional unimodularly equivalent copy of $\mathscr{P}_G$ by considering the lattice polytope defined as the convex hull of the rows of A_G with one column deleted. Assume that $k \geq 1$ and $V_1 \sqcup V_2$ the bipartition of G_1. Then we can get a full-dimensional unimodularly equivalent copy of $\mathscr{P}_G$ by considering the lattice polytopes defined as the convex hull of the row vectors of A_G with the the columns $i_0, \dots, i_k$ deleted, where $i_0 \in V_1$, $i_1 \in V_2$ and for $2 \leq j \leq k$, i_j is a vertex of G_j. An example of this can be observed in Example 12.3.

Example 12.3. Let G be the following finite simple graph with the incidence matrix of A_G.



$$A_G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Then by Lemma 12.2, $\mathscr{P}_G \subset \mathbb{R}^6$ is a lattice polytope of dimension 3. Let $\mathscr{Q}_G \subset \mathbb{R}^3$ be the lattice polytope which is the convex hull of the row vectors of A_G with the first, second and fifth column deleted. Namely, $\mathscr{Q}_G$ is the convex hull of the row vectors of

$$A'_G = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathscr{Q}_G$ is a full-dimensional unimodularly equivalent copy of $\mathscr{P}_G$.

12.2 Reflexive polytopes arising from edge polytopes

In this section, we construct reflexive polytopes which arise from the edge polytopes of finite simple graphs. We show the following theorem:

Theorem 12.4 ([63, Theorem 2.1]). *Let $\mathscr{P}, \mathscr{Q} \subset \mathbb{R}^d$ be $(0,1,2)$-polytopes of dimension d such that all of their vertices belong to*

$$\{\mathbf{0}\} \cup \{\mathbf{e}_i : 1 \leq i \leq d\} \cup \{\mathbf{e}_i + \mathbf{e}_j : 1 \leq i \leq j \leq d\} \subset \mathbb{R}^d.$$

If the origin of $\mathbb{R}^{d+1}$ belongs to the interior of $\Omega(\mathscr{P}, \mathscr{Q})$, then $\Omega(\mathscr{P}, \mathscr{Q})$ is reflexive. In particular, $\Omega(\mathscr{P})$ and $\Omega(\mathscr{Q})$ are reflexive.

For two $d \times d$ integer matrices A, B, we write $A \sim B$ if B can be obtained from A by some row and column operations over $\mathbb{Z}$. In order to prove Theorem 12.4, we will need the following proposition.

Proposition 12.5. *Let $A = (a_{ij})_{1 \leq i,j \leq d} \in \{0,1,2\}^{d \times d}$ be a $d \times d$ integer matrix such that each row vector $\mathbf{a}_i = (a_{i1}, \ldots, a_{id})$ of A satisfies the following conditions:*

- $a_{id} = 1$;
- $|a_{i1} + \cdots + a_{id-1}| \leq 2$.

If $\det(A) \neq 0$*, then, for some integer* $0 \leq s \leq d$*,*

$$A \sim \left(\begin{array}{cccccc} 1 & & & & & \\ & \ddots & & & \mathbf{0} & \\ & & 1 & & & \\ & & & 2 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & 2 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} s \\ \\ \\ \end{array}.$$

In this case, one has $2A^{-1} \in \mathbb{Z}^{d \times d}$.

Since the proof of Proposition 12.5 is rather technical, we save it until the end of this section. We now prove Theorem 12.4 and discuss a quick corollary.

Proof of Theorem 12.4. Let $\mathscr{F}$ be a facet of $\Omega(\mathscr{P},\mathscr{Q})$. Then there exist $d+1$ vertices $\mathbf{v}_1,\dots,\mathbf{v}_{d+1}$ of $\Omega(\mathscr{P},\mathscr{Q})$ such that $\mathrm{aff}(\{\mathbf{v}_1,\dots,\mathbf{v}_{d+1}\}) \cap \Omega(\mathscr{P},\mathscr{Q}) = \mathscr{F}$, where for a subset $A \subset \mathbb{R}^{d+1}$, $\mathrm{aff}(A)$ is the affine space generated by A. Since the origin of $\mathbb{R}^{d+1}$ belongs to the interior of $\Omega(\mathscr{P},\mathscr{Q})$, there exist rational numbers $a_1,\dots,a_{d+1}$ such that

$$\mathrm{aff}(\{\mathbf{v}_1,\dots,\mathbf{v}_{d+1}) = \{(x_1,\dots,x_{d+1})\} \in \mathbb{R}^{d+1} : a_1x_1 + \cdots + a_{d+1}x_{d+1} = 1\}.$$

By Lemma 2.7, we need to show that $a_1,\dots,a_{d+1} \in \mathbb{Z}$. Let V be the $(d+1)\times(d+1)$ integer matrix whose ith row vector is $\mathbf{v}_i$. Then we have $\det(V) \neq 0$ and

$$V \begin{pmatrix} a_1 \\ \vdots \\ a_{d+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Hence each a_i is the sum of all entries in the ith row vector of V^{-1}. If $\mathbf{v}_1,\dots,\mathbf{v}_{d+1}$ are vertices of $\mathscr{P} \times \{1\}$, then one has $a_1 = \cdots = a_d = 0$ and $a_{d+1} = 1$. Similarly, if $\mathbf{v}_1,\dots,\mathbf{v}_{d+1}$ are vertices of $-\mathscr{Q} \times \{-1\}$, then one has $a_1 = \cdots = a_d = 0$ and $a_{d+1} = -1$.

Now, we assume that for some positive integer $1 \leq k \leq d$, $\mathbf{v}_1,\dots,\mathbf{v}_k$ are vertices of $\mathscr{P} \times \{1\}$ and $\mathbf{v}_{k+1},\dots,\mathbf{v}_{d+1}$ are vertices of $-\mathscr{Q} \times \{-1\}$. Let W be a $(d+1)\times(d+1)$ integer matrix such that for $1 \leq i \leq k$, the ith row vector is $\mathbf{v}_i$ and for $k+1 \leq j \leq d+1$, the jth row vector is $-\mathbf{v}_j$. Then one has $\det(W) \neq 0$ and

$$W \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

Hence for any $1 \leq i \leq d+1$, the sum of all entries in the ith row vector of W^{-1} is an integer. Moreover, for $k+1 \leq i \leq d+1$, the ith column vector of V^{-1} is coincide with the ith column vector of W^{-1} times (-1). Thus, since $2W^{-1}$ is an integer matrix from Proposition 12.5, we know that for any $1 \leq i \leq d+1$, the sum of all entries in the ith row vector of V^{-1} is an integer. Therefore, $\Omega(\mathscr{P}, \mathscr{Q})$ is reflexive, as desired. □

By Theorem 12.4, we can give a new class of reflexive polytopes arising from the edge polytopes of finite simple graphs and we can determine the reflexive dimensions of the edge polytopes of finite simple graphs.

Corollary 12.6. *Let G be a finite simple graph on $[N]$. Then $\Omega(\mathscr{P}_G)$ is unimodularly equivalent to some reflexive polytope. Moreover, $\mathscr{P}_G$ is unimodularly equivalent to a facet of $\Omega(\mathscr{P}_G)$.*

Proof. Let d be the dimension of $\mathscr{P}_G$ and $\mathscr{P}'_G \subset \mathbb{R}^d$ a full-dimensional unimodularly equivalent copy of $\mathscr{P}_G$ as defined in Section 12.1 (see Example 12.3). Then all of its vertices of $\mathscr{P}'_G$ belong to

$$\{\mathbf{0}\} \cup \{\mathbf{e}_i : 1 \leq i \leq d\} \cup \{\mathbf{e}_i + \mathbf{e}_j : 1 \leq i \leq j \leq d\} \subset \mathbb{R}^d.$$

Hence, it follows from Theorem 12.4 that $\Omega(\mathscr{P}'_G)$ is reflexive. Moreover it is easy to see that $\Omega(\mathscr{P}'_G)$ is a full-dimensional unimodularly equivalent copy of $\Omega(\mathscr{P}_G)$. Therefore, we know that $\Omega(\mathscr{P}_G)$ is unimodularly equivalent to some reflexive polytope and $\mathscr{P}_G$ is unimodularly equivalent to a facet of $\Omega(\mathscr{P}_G)$. □

We now end the section with a proof of Proposition 12.5.

Proof of Proposition 12.5. We prove this proposition by induction on d, i.e., the size of A. When $d = 1$, the claim is trivial. Suppose that $d > 1$. We should show that for a given $d \times d$ integer matrix A satisfying the assumption of the proposition, one can obtain a matrix from A by some row and column operations over $\mathbb{Z}$ as the following:

$$A \sim \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right),$$

where A' is a $(d-1) \times (d-1)$ integer matrix satisfying the assumption of the proposition.

We will divide the proof into some cases. Note that by the assumption, no two row vectors in A are the same, and each row vector $\mathbf{a}_i$ of A is one of the following:

(Type 1) $\mathbf{a}_i = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1)$, i.e., for some $1 \leq j_1 < j_2 \leq d-1, a_{ij_1} = a_{ij_2} = a_{id} = 1$, and for other $j, a_{ij} = 0$;

(Type 2) $\mathbf{a}_i = (0,\dots,0,1,0,\dots,0,1)$, i.e., for some $1 \le j_1 \le d-1, a_{ij_1} = a_{id} = 1$, and for other $j, a_{ij} = 0$;

(Type 3) $\mathbf{a}_i = (0,\dots,0,2,0,\dots,0,1)$, i.e., for some $1 \le j_1 \le d-1, a_{ij_1} = 2, a_{id} = 1$, and for other $j, a_{ij} = 0$;

(Type 4) $\mathbf{a}_i = (0,\dots,0,1)$, i.e., for any $1 \le j \le d-1, a_{ij} = 0$, and $a_{id} = 1$.

We can then divide the proof into the following cases:

(1) A does not have any row vectors of Type 1 and Type 2;

(2) A has at least one row vector of Type 2, but A has no row vectors of Type 1;

(3) A has at least one row vector of Type 1.

The case (1): Since each row vector of A is either Type 3 or Type 4 and since $\det(A) \neq 0$, one can obtain the following matrices subsequently from A by some row and column operations over $\mathbb{Z}$:

$$A \sim \begin{pmatrix} 2 & & & 1 \\ & \ddots & \mathbf{0} & \vdots \\ & & \ddots & \vdots \\ \mathbf{0} & & 2 & 1 \\ & & & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & & \\ & 2 & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & \ddots \\ & & & & 2 \end{pmatrix}.$$

Thus, in this case, we can get the desired matrix from A by some row and column operations over $\mathbb{Z}$.

The case (2): Since A has at least one row vector of Type 2 and A does not have any row vector of Type 1, one can obtain the following matrix $\tilde{A}$ from A by interchanging some rows and columns:

$$\tilde{A} = \left(\begin{array}{cccccc} 1 & 0 & \cdots & 0 & & 1 \\ 0 & 1 & \cdots & 0 & & 1 \\ \vdots & & \ddots & \vdots & \mathbf{0} & \vdots \\ 0 & 0 & \cdots & 1 & & 1 \\ \hline & & & C & & \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} r \\ \\ \end{array},$$

where $r \ge 1$ and C is a $(d-r) \times d$ integer matrix such that each row vector is either Type 3 or Type 4. Now, by interchanging some row vectors of C if necessarily, we can assume the first column vector of C is $(0,\ldots,0)^\top$ or $(2,0,\ldots,0)^\top$.If the first column vector of C is $(0,\ldots,0)^\top$, we can obtain the below matrix from $\tilde{A}$ by a column operation:

$$
\tilde{A} = \left(\begin{array}{cccc}
 & & & 1 \\
 & & & 1 \\
 & I_r & 0 & \vdots \\
 & & & 1 \\ \hline
0 & & & 1 \\
0 & & & 1 \\
\vdots & C' & & \vdots \\
0 & & & 1
\end{array}\right) \sim \left(\begin{array}{cccc}
 & & & 0 \\
 & & & 1 \\
 & I_r & 0 & \vdots \\
 & & & 1 \\ \hline
0 & & & 1 \\
0 & & & 1 \\
\vdots & C' & & \vdots \\
0 & & & 1
\end{array}\right)
$$

$$
=: \left(\begin{array}{c|ccc}
1 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & A' & \\
0 & & &
\end{array}\right),
$$

where I_r is the unit matrix of size r. Moreover, it is clear that A' satisfies the assumption of the proposition. Hence, by the inductive hypothesis, one obtains the desired matrix from A.

Assume that the first column vector of C is $(2,0,\ldots,0)^\top$. Then we can obtain the below matrices from $\tilde{A}$ subsequently by some row and column operations over $\mathbb{Z}$:

$$
\tilde{A} = \left(\begin{array}{ccccc}
 & & & & 1 \\
 & & & & 1 \\
 & I_r & & 0 & \vdots \\
 & & & & 1 \\ \hline
2 & 0 & \cdots & 0 & 1 \\ \hline
0 & & & & 1 \\
\vdots & & C'' & & \vdots \\
0 & & & & 1
\end{array}\right) \sim \left(\begin{array}{ccccc}
 & & & & 0 \\
 & & & & 1 \\
 & I_r & & 0 & \vdots \\
 & & & & 1 \\ \hline
2 & 0 & \cdots & 0 & -1 \\ \hline
0 & & & & 1 \\
\vdots & & C'' & & \vdots \\
0 & & & & 1
\end{array}\right)
$$

$$
\sim \left(\begin{array}{ccccc}
 & & & & 0 \\
 & & & & 1 \\
 & I_r & & 0 & \vdots \\
 & & & & 1 \\ \hline
0 & 0 & \cdots & 0 & -1 \\ \hline
0 & & & & 1 \\
\vdots & & C'' & & \vdots \\
0 & & & & 1
\end{array}\right) \sim \left(\begin{array}{ccccc}
 & & & & 0 \\
 & & & & 1 \\
 & I_r & & 0 & \vdots \\
 & & & & 1 \\ \hline
0 & 0 & \cdots & 0 & 1 \\ \hline
0 & & & & 1 \\
\vdots & & C'' & & \vdots \\
0 & & & & 1
\end{array}\right)
$$

$$=: \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A'' & \\ 0 & & & \end{array}\right).$$

Moreover, it is clear that A'' satisfies the assumption of the proposition. Hence, by induction hypothesis, one obtain a desired matrix from A.

The case (3): Since A has at least one row vector of Type 1, we can obtain the following matrix from A by interchanging some rows and columns of A:

$$\tilde{A} := \left(\begin{array}{cccccc} 1 & 1 & 0 & \cdots & 0 & 1 \\ \hline & & & B & & \\ \hline & & & C & & \end{array}\right),$$

where B is a $m \times d$ integer matrix for some $m \geq 0$ such that each row vector is either Type 1 or Type 2, and C is a $(d-m-1) \times d$ integer matrix such that each row vector is either Type 3 or Type 4. We set $l := d-m-1$. Let $B_{\{1,2\}}$ (resp. $C_{\{1,2\}}$) denote the submatrix consisting of the first and second column vectors of B (resp. C). Here, to prove the claim, we divide into the following subcases:

(3-1) $B_{\{1,2\}}$ is a zero matrix;

(3-2) $B_{\{1,2\}}$ is not a zero matrix.

Note, in both subcases, by a permutation of first row and second row, we can assume

$$C_{\{1,2\}} = \begin{pmatrix} c_0 & c_1 \\ 0 & c_2 \\ \vdots & \vdots \\ 0 & c_l \end{pmatrix},$$

where either $c_0 = 2$ and $c_1 = 0$, or $c_0 = 0$.

The subcase (3-1): In this case, if $c_0 = 0$, then we can obtain the below matrix from $\tilde{A}$ by some column operations:

$$
\tilde{A} = \left(\begin{array}{cc|ccc|c}
1 & 1 & 0 & \cdots & 0 & 1 \\ \hline
0 & 0 & & & & 1 \\
\vdots & \vdots & & B' & & \vdots \\
0 & 0 & & & & 1 \\ \hline
0 & c_1 & & & & 1 \\
\vdots & \vdots & & C' & & \vdots \\
0 & c_l & & & & 1
\end{array}\right)
\sim
\left(\begin{array}{cc|ccc|c}
1 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & 0 & & & & 1 \\
\vdots & \vdots & & B' & & \vdots \\
0 & 0 & & & & 1 \\ \hline
0 & c_1 & & & & 1 \\
\vdots & \vdots & & C' & & \vdots \\
0 & c_l & & & & 1
\end{array}\right)
$$

$$
=: \left(\begin{array}{c|ccc}
1 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & A' & \\
0 & & &
\end{array}\right).
$$

Moreover, it is clear that A' satisfies the assumption of the proposition. Hence, by the inductive hypothesis, we obtain the desired matrix from A. Next, if $c_0 = 2$ and $c_1 = 0$, then we can obtain the below matrices from $\tilde{A}$ subsequently by some row and column operations:

$$
\tilde{A} = \left(\begin{array}{cc|ccc|c}
1 & 1 & 0 & \cdots & 0 & 1 \\ \hline
0 & 0 & & & & 1 \\
\vdots & \vdots & & B'' & & \vdots \\
0 & 0 & & & & 1 \\ \hline
2 & 0 & 0 & \cdots & 0 & 1 \\ \cline{3-5}
0 & c_2 & & & & 1 \\
\vdots & \vdots & & C'' & & \vdots \\
0 & c_l & & & & 1
\end{array}\right)
\sim
\left(\begin{array}{cc|ccc|c}
1 & 1 & 0 & \cdots & 0 & 1 \\ \hline
0 & 0 & & & & 1 \\
\vdots & \vdots & & B'' & & \vdots \\
0 & 0 & & & & 1 \\ \hline
0 & -2 & 0 & \cdots & 0 & -1 \\ \cline{3-5}
0 & c_2 & & & & 1 \\
\vdots & \vdots & & C'' & & \vdots \\
0 & c_l & & & & 1
\end{array}\right)
$$

$$
\sim \left(\begin{array}{cc|ccc|c}
1 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & 0 & & & & 1 \\
\vdots & \vdots & & B'' & & \vdots \\
0 & 0 & & & & 1 \\ \hline
0 & -2 & 0 & \cdots & 0 & -1 \\ \cline{3-5}
0 & c_2 & & & & 1 \\
\vdots & \vdots & & C'' & & \vdots \\
0 & c_l & & & & 1
\end{array}\right)
\sim
\left(\begin{array}{cc|ccc|c}
1 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & 0 & & & & 1 \\
\vdots & \vdots & & B'' & & \vdots \\
0 & 0 & & & & 1 \\ \hline
0 & 2 & 0 & \cdots & 0 & 1 \\ \cline{3-5}
0 & c_2 & & & & 1 \\
\vdots & \vdots & & C'' & & \vdots \\
0 & c_l & & & & 1
\end{array}\right)
$$

$$
=: \left(\begin{array}{c|ccc}
1 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & A'' & \\
0 & & &
\end{array}\right).
$$

Moreover, it is clear that A'' satisfies the assumption of the proposition. Hence, by the inductive hypothesis, we obtain the desired matrix from A.

The subcase (3-2): Note, since $\det(A) \neq 0$, each row vector of $B_{\{1,2\}}$ is $(0,1)$, $(1,0)$, or $(0,0)$. Thus, by some row permutations of $\tilde{A}$, and if necessary a permutation of the first and second columns, we can assume

$$B_{\{1,2\}} = \left(\begin{array}{cc} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \vdots\\0\end{array}}\right\} p \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\0\end{array}}\right\} q-p, \\ \left.\vphantom{\begin{array}{c}0\\ \vdots\\0\end{array}}\right\} m-q \end{array}$$

where $p \geq 1$ and $q-p \geq 0$.

Let $\tilde{\mathbf{a}}_2$ be the 2nd row vector of $\tilde{A}$. Then $\tilde{\mathbf{a}}_2$ is either Type 1 or Type 2. If $\tilde{\mathbf{a}}_2$ is Type 2, we can obtain the below matrices from $\tilde{A}$ subsequently by some row operations over $\mathbb{Z}$ as the following:

$$\tilde{A} = \left(\begin{array}{cc|ccc|c} 1 & 1 & 0 & \cdots & 0 & 1 \\ \hline 0 & 1 & 0 & \cdots & 0 & 1 \\ \cline{3-5} 0 & 1 & & & & 1 \\ \vdots & \vdots & & & & \vdots \\ 0 & 1 & & & & 1 \\ 1 & 0 & & & & 1 \\ \vdots & \vdots & & B' & & \vdots \\ 1 & 0 & & & & 1 \\ 0 & 0 & & & & 1 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & & & 1 \\ \hline c_0 & c_1 & & & & 1 \\ 0 & c_2 & & & & 1 \\ \vdots & \vdots & & C' & & \vdots \\ 0 & c_l & & & & 1 \end{array}\right) \sim \left(\begin{array}{cc|ccc|c} 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 & 1 \\ \cline{3-5} 0 & 1 & & & & 1 \\ \vdots & \vdots & & & & \vdots \\ 0 & 1 & & & & 1 \\ 1 & 0 & & & & 1 \\ \vdots & \vdots & & B' & & \vdots \\ 1 & 0 & & & & 1 \\ 0 & 0 & & & & 1 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & & & 1 \\ \hline c_0 & c_1 & & & & 1 \\ 0 & c_2 & & & & 1 \\ \vdots & \vdots & & C' & & \vdots \\ 0 & c_l & & & & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|ccc|c}
1 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & 1 & 0 & \cdots & 0 & 1 \\ \cline{3-5}
0 & 1 & & & & 1 \\
\vdots & \vdots & & & & \vdots \\
0 & 1 & & & & 1 \\
0 & 0 & & & & 1 \\
\vdots & \vdots & & B' & & \vdots \\
0 & 0 & & & & 1 \\
0 & 0 & & & & 1 \\
\vdots & \vdots & & & & \vdots \\
0 & 0 & & & & 1 \\ \hline
0 & c_1 & & & & 1 \\
0 & c_2 & & & & 1 \\
\vdots & \vdots & & C' & & \vdots \\
0 & c_l & & & & 1
\end{array}\right) =: \left(\begin{array}{c|ccc}
1 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & A' & \\
0 & & &
\end{array}\right).$$

Moreover, it is clear that A' satisfies the assumption of the proposition. Hence, by the inductive hypothesis, we obtain the desired matrix from A.

Next assume that $\tilde{\mathbf{a}}_2$ is Type 1. Note that $\tilde{\mathbf{a}}_2$ is a row vector like $(0,1,0,\dots,0,1,0,\dots,0,1)$. By interchanging columns, we can assume $\tilde{\mathbf{a}}_2 = (0,1,1,0,\dots,0,1)$. Then, we can obtain the below matrices from $\tilde{A}$ subsequently by some row and column operations over $\mathbb{Z}$ as follows:

$$\tilde{A} = \left(\begin{array}{ccc|ccc|c}
1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ \hline
0 & 1 & 1 & 0 & \cdots & 0 & 1 \\ \cline{4-6}
0 & 1 & b_2 & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 1 & b_p & & & & 1 \\
1 & 0 & b_{p+1} & & & & 1 \\
\vdots & \vdots & \vdots & & B'' & & \vdots \\
1 & 0 & b_q & & & & 1 \\
0 & 0 & b_{q+1} & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 0 & b_m & & & & 1 \\ \hline
c_0 & c_1 & c'_1 & & & & 1 \\
0 & c_2 & c'_2 & & & & 1 \\
\vdots & \vdots & \vdots & & C'' & & \vdots \\
0 & c_l & c'_l & & & & 1
\end{array}\right) \sim \left(\begin{array}{ccc|ccc|c}
1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \hline
0 & 1 & 1 & 0 & \cdots & 0 & 1 \\ \cline{4-6}
0 & 1 & b_2 & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 1 & b_p & & & & 1 \\
1 & 0 & b_{p+1} & & & & 1 \\
\vdots & \vdots & \vdots & & B'' & & \vdots \\
1 & 0 & b_q & & & & 1 \\
0 & 0 & b_{q+1} & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 0 & b_m & & & & 1 \\ \hline
c_0 & c_1 & c'_1 & & & & 1 \\
0 & c_2 & c'_2 & & & & 1 \\
\vdots & \vdots & \vdots & & C'' & & \vdots \\
0 & c_l & c'_l & & & & 1
\end{array}\right)$$

$$
\sim \left(\begin{array}{ccc|ccc|c}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 1 & 1 & 0 & \cdots & 0 & 1 \\
\cline{4-6}
0 & 1 & b_2 & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 1 & b_p & & & & 1 \\
1 & 0 & b_{p+1}+1 & & & & 1 \\
\vdots & \vdots & \vdots & & B'' & & \vdots \\
1 & 0 & b_q+1 & & & & 1 \\
0 & 0 & b_{q+1} & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 0 & b_m & & & & 1 \\
\cline{1-6}
c_0 & c_1 & c_1'+c_0 & & & & 1 \\
0 & c_2 & c_2' & & & & 1 \\
\vdots & \vdots & \vdots & & C'' & & \vdots \\
0 & c_l & c_l' & & & & 1
\end{array}\right)
\sim \left(\begin{array}{ccc|ccc|c}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 1 & 1 & 0 & \cdots & 0 & 1 \\
\cline{4-6}
0 & 1 & b_2 & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 1 & b_p & & & & 1 \\
0 & 0 & b_{p+1}+1 & & & & 1 \\
\vdots & \vdots & \vdots & & B'' & & \vdots \\
0 & 0 & b_q+1 & & & & 1 \\
0 & 0 & b_{q+1} & & & & 1 \\
\vdots & \vdots & \vdots & & & & \vdots \\
0 & 0 & b_m & & & & 1 \\
\cline{1-6}
0 & c_1 & c_1'+c_0 & & & & 1 \\
0 & c_2 & c_2' & & & & 1 \\
\vdots & \vdots & \vdots & & C'' & & \vdots \\
0 & c_l & c_l' & & & & 1
\end{array}\right)
$$

$$
=: \left(\begin{array}{c|ccc}
1 & 0 & \cdots & 0 \\
\hline
0 & & & \\
\vdots & & A'' & \\
0 & & &
\end{array}\right).
$$

One can easily show A'' also satisfies the assumption of the proposition. Therefore, this completes the proof by induction. □

12.3 When do they possess the integer decomposition property?

In this section, we discuss when the reflexive polytopes arising from the edge polytopes of connected finite simple graphs described in Corollary 12.6 possesses the integer decomposition property. First, we introduce a criterion to determine when the edge polytopes of connected finite simple graphs possess the integer decomposition property.

Theorem 12.7 ([71, Corollarly 2.3])**.** *Let G be a connected finite simple graph on $[n]$. Then $\mathscr{P}_G$ possesses the integer decomposition property if and only if for any two odd cycles C and C' of G having no common vertex, there exists an edge of G joining a vertex of C with a vertex of C'.*

The following theorem gives a criterion to determine when the reflexive polytopes arising from the edge polytopes of connected finite simple graphs described in Corollary 12.6 possess the integer decomposition property.

Theorem 12.8 ([63, Theorem 3.2]). *Let G be a connected finite simple graph on $[N]$. Then $\Omega(\mathscr{P}_G)$ possesses the integer decomposition property if and only if G does not contain two disjoint odd cycles.*

In order to prove this theorem, we need the following lemma. Recall that a lattice polytope is called *unimodular* if all its triangulations are unimodular, that is each simplex has the normalized volume equal to 1. In particular, unimodular lattice polytopes are compressed.

Lemma 12.9 ([70, Example 3.6 b]). *Let G be a connected finite simple graph on $[N]$. Then $\mathscr{P}_G$ is unimodular if and only if G done not contain two disjoint odd cycles.*

Now, we prove Theorem 12.8.

Proof of Theorem 12.8. First, let us assume that G has two disjoint odd cycles C_1 and C_2. Then it follows from Lemma 12.2 that the dimension of $\mathscr{P}_G$ equals $N-1$. Moreover, we can assume that $V(C_1)=[2k+1]$ and $V(C_2)=\{2k+2,\dots,2k+2\ell+2\}$ with some positive integers k and ℓ. Let $\mathscr{P}'_G \subset \mathbb{R}^{N-1}$ be the full-dimensional unimodularly equivalent copy of $\mathscr{P}_G$ which is the convex hull of the row vectors of the incidence matrix A_G of G with $(2k+2\ell+2)$nd column deleted. Then $\Omega(\mathscr{P}'_G)$ is a full-dimensional unimodularly equivalent copy of $\Omega(\mathscr{P}_G)$ and one has

$$\Omega(\mathscr{P}'_G)\cap\mathbb{Z}^N=((\mathscr{P}'_G\times\{1\})\cap\mathbb{Z}^N)\cup((-\mathscr{P}'_G\times\{-1\})\cap\mathbb{Z}^N)\cup\{\mathbf{0}_N\}.$$

We show that $\Omega(\mathscr{Q}_G)$ does not possess the integer decomposition property. Set

$$\mathbf{x}=\mathbf{e}_1+\cdots+\mathbf{e}_{2k+1}-(\mathbf{e}_{2k+2}+\cdots+\mathbf{e}_{2k+2\ell+1})+(k-\ell)\mathbf{e}_N\in\mathbb{Z}^N.$$

Since $\mathbf{e}_1+\mathbf{e}_2,\dots,\mathbf{e}_{2k}+\mathbf{e}_{2k+1}$ and $\mathbf{e}_1+\mathbf{e}_{2k+1}$ are vertices of $\mathscr{P}'_G$ and since $-\mathbf{e}_{2k+2}-\mathbf{e}_{2k+3},\dots,-\mathbf{e}_{2k+2\ell}-\mathbf{e}_{2k+2\ell+1},-\mathbf{e}_{2k+2\ell+1}$ and $-\mathbf{e}_{2k+2}$ are vertices of $-\mathscr{P}'_G$, it follows that $\mathbf{x}\in(k+\ell+1)\Omega(\mathscr{P}'_G)\cap\mathbb{Z}^N$. Suppose that $\Omega(\mathscr{P}'_G)$ possesses the integer decomposition property. Then there exist just $k+m+1$ lattice points $\mathbf{x}_1,\dots,\mathbf{x}_{k+m+1}\in\Omega(\mathscr{P}'_G)\cap\mathbb{Z}^N$ such that $\mathbf{x}=\mathbf{x}_1+\cdots+\mathbf{x}_{k+\ell+1}$. For any vertex $\mathbf{v}$ of $\mathscr{P}'_G\times\{1\}$, one has $\langle\mathbf{v},\mathbf{e}_1+\cdots+\mathbf{e}_{2k+1}\rangle\in\{0,1,2\}$ and $\langle\mathbf{v},\mathbf{e}_{2k+2}+\cdots+\mathbf{e}_{2k+2\ell+1}\rangle\in\{0,1,2\}$. Hence since $\langle\mathbf{x},\mathbf{e}_1+\cdots+\mathbf{e}_{2k+1}\rangle=2k+1$ and $\langle\mathbf{x},\mathbf{e}_{2k+2}+\cdots+\mathbf{e}_{2k+2\ell+1}\rangle=2\ell$, we can assume that $\mathbf{x}_1,\dots,\mathbf{x}_{k+1}\in\mathscr{P}'_G\times\{1\}$ and $\mathbf{x}_{k+2},\dots,\mathbf{x}_{k+\ell+1}\in-\mathscr{Q}_G\times\{-1\}$. Then one has $\langle\mathbf{x}_1+\cdots+\mathbf{x}_{k+\ell+1},\mathbf{e}_n\rangle=k-\ell+1$. Thus, $\mathbf{x}\neq\mathbf{x}_1+\cdots+\mathbf{x}_{k+\ell+1}$, a contradiction. Therefore, $\Omega(\mathscr{P}'_G)$ does not possess the integer decomposition property.

Conversely, assume that G does not have two disjoint odd cycles. Let $\mathscr{P}'_G\subset\mathbb{R}^d$ be a full-dimensional unimodularly equivalent copy of $\mathscr{P}_G$ defined in Section 12.1 (see Example 12.3). Hence $\Omega(\mathscr{P}'_G)$ is a full-dimensional unimodularly equivalent copy of $\Omega(\mathscr{P}_G)$ and one has

$$\Omega(\mathscr{P}'_G)\cap\mathbb{Z}^{d+1}=((\mathscr{P}'_G\times\{1\})\cap\mathbb{Z}^{d+1})\cup((-\mathscr{P}'_G\times\{-1\})\cap\mathbb{Z}^{d+1})\cup\{\mathbf{0}_{d+1}\}.$$

Let Δ be a pulling triangulation of $\Omega(\mathscr{P}'_G)$ such that all its maximal simplices contain the origin of $\mathbb{R}^{d+1}$. We will show that Δ is unimodular. Let σ be a maximal simplex of Δ. Then there exist $d+1$ lattice points $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$ belonging to $\mathscr{P}'_G \times \{1\}$ such that

$$\sigma = \mathrm{conv}(\{\mathbf{0}_{d+1}, \mathbf{v}_1, \ldots, \mathbf{v}_t, -\mathbf{v}_{t+1}, \ldots, -\mathbf{v}_{d+1}\})$$

with some integer $0 \leq t \leq d+1$. It follows that the normalized volume of σ equals $|\det(V)|$, where V is the $(d+1) \times (d+1)$ integer matrix whose ith row vector is $\mathbf{v}_i$. Set

$$\sigma' = \mathrm{conv}(\{\mathbf{0}_{d+1}, \mathbf{v}_1, \ldots, \mathbf{v}_{d+1}\}).$$

Then the normalized volume of σ is equal to that of σ'. We show that the normalized volume of σ' is 1. Let τ be the lattice simplex which is the convex hull of the row vectors of V with the last column deleted. Then τ is a simplex of dimension d all of whose vertices belonging to $\mathscr{P}'_G$. By Lemma 12.9 and [39, Theorem 5.6.3], the normalized volume of any maximal simplex all of whose vertices belonging to $\mathscr{P}'_G$ is 1, that of τ is also 1. Since $\mathrm{Pyr}(\tau)$ is unimodularly equivalent to σ', the normalized volume of σ' equals 1. Hence Δ is unimodular. □

Chapter 13

New classes of self dual reflexive polytopes

A reflexive polytope $\mathscr{P}$ is called *self dual* if $\mathscr{P}$ is unimodularly equivalent to its dual polytope $\mathscr{P}^\vee$. This is an extremely rare property in reflexive polytopes, especially for reflexive simplices. In this chapter, we give new classes of self dual reflexive polytopes.

This chapter is organized as follows. In Section 13.1, we will give a higher-dimensional construction of self dual reflexive polytopes. In Sections 13.2 and 13.3, we will give two classes of self dual reflexive simplices.

13.1 A higher-dimensional construction of self dual reflexive polytopes

In this section, we give a higher-dimensional construction of self dual reflexive polytopes.

For $d \geq 2$ and a lattice polytope $\mathscr{P} \subset \mathbb{R}^{d-1}$ of dimension $d-1$, we set

$$\mathscr{A}(\mathscr{P}) = \mathscr{P} \times [-1,1] \subset \mathbb{R}^d,$$
$$\mathscr{B}(\mathscr{P}) = \mathrm{conv}(\{\mathscr{P} \times \{0\}, \mathbf{e}_d, -\mathbf{e}_d\}) \subset \mathbb{R}^d,$$
$$\mathscr{D}(\mathscr{P}) = \mathrm{conv}(\{\mathscr{P} \times [-1,0], \mathbf{e}_d\}) \subset \mathbb{R}^d.$$

We recall that if $\mathscr{P}$ is reflexive, then $\mathscr{A}(\mathscr{P})$ and $\mathscr{B}(\mathscr{P})$ are also reflexive. Moreover, we have $\mathscr{A}(\mathscr{P})^\vee = \mathscr{B}(\mathscr{P}^\vee)$ and $\mathscr{B}(\mathscr{P})^\vee = \mathscr{A}(\mathscr{P}^\vee)$. $\mathscr{D}(\mathscr{P})$ is an analogy between $\mathscr{A}(\mathscr{P})$ and $\mathscr{B}(\mathscr{P})$.

At first, we show that if $\mathscr{P}$ is reflexive, then $\mathscr{D}(\mathscr{P})$ is a reflexive polytope of dimension d.

Proposition 13.1. *For $d \geq 2$, let $\mathscr{P} \subset \mathbb{R}^{d-1}$ be a reflexive polytope of dimension $d-1$. Then $\mathscr{D}(\mathscr{P})$ is a reflexive polytope of dimension d. Moreover, $\mathscr{D}(\mathscr{P}^\vee)$ and $\mathscr{D}(\mathscr{P})^\vee$ are unimodularly equivalent.*

Proof. Let $\mathscr{F}_1, \ldots, \mathscr{F}_s$ be facets of $\mathscr{P}$ and for $1 \leq i \leq s$, and let $\mathscr{H}_i$ be the hyperplane satisfying $\mathscr{F}_i = \mathscr{P} \cap \mathscr{H}_i$. Then $\mathscr{D}(\mathscr{P})$ has $2s+1$ facets. By Lemma 2.7, we can assume that for $1 \leq i \leq s$,

$$\mathscr{H}_i = \left\{ \mathbf{x} \in \mathbb{R}^{d-1} \mid \langle \mathbf{a}_i, \mathbf{x} \rangle = 1 \right\},$$

where $\mathbf{a}_i \in \mathbb{Z}^{d-1}$. Set

$$\mathscr{F}'_i = \begin{cases} \mathrm{conv}(\{\mathscr{F}_i \times \{0\}, \mathbf{e}_d\}) & i = 1, \ldots, s, \\ \mathscr{F}_{i-s} \times [-1, 0] & i = s+1, \ldots, 2s, \\ \mathscr{P} \times \{-1\} & i = 2s+1. \end{cases}$$

Then $\mathscr{F}'_1, \ldots, \mathscr{F}'_{2s+1}$ are facets of $\mathscr{D}(\mathscr{P})$. For $1 \leq i \leq 2s+1$ let $\mathscr{H}'_i$ be the hyperplane satisfying $\mathscr{F}'_i = \mathscr{D}(\mathscr{P}) \cap \mathscr{H}'_i$. Then

$$\mathscr{H}'_i = \begin{cases} \left\{ \mathbf{x} \in \mathbb{R}^d \mid \langle (\mathbf{a}_i, 1), \mathbf{x} \rangle = 1 \right\} & i = 1, \ldots, s, \\ \left\{ \mathbf{x} \in \mathbb{R}^d \mid \langle (\mathbf{a}_{i-s}, 0), \mathbf{x} \rangle = 1 \right\} & i = s+1, \ldots, 2s, \\ \left\{ \mathbf{x} \in \mathbb{R}^d \mid \langle -\mathbf{e}_d, \mathbf{x} \rangle = 1 \right\} & i = 2s+1. \end{cases}$$

Hence by Lemma 2.7, $\mathscr{D}(\mathscr{P})$ is a reflexive polytope of dimension d.

Moreover, since $\mathbf{a}_1, \ldots, \mathbf{a}_s$ are the vertices of $\mathscr{P}^\vee$, it clearly follows that $\mathscr{D}(\mathscr{P}^\vee)$ and $\mathscr{D}(\mathscr{P})^\vee$ are unimodularly equivalent. □

Next, we present a direct formula for the computation of the δ-vector of $\mathscr{D}(\mathscr{P})$ in terms of the δ-vector of $\mathscr{P}$.

Proposition 13.2. *For $d \geq 2$, let $\mathscr{P} \subset \mathbb{R}^{d-1}$ be a lattice polytope of dimension $d-1$ with the δ-vector $\delta(\mathscr{P}) = (\delta_0(\mathscr{P}), \delta_1(\mathscr{P}), \ldots, \delta_{d-1}(\mathscr{P}))$, and let*

$$\delta(\mathscr{D}(\mathscr{P})) = (\delta_0(\mathscr{D}(\mathscr{P})), \delta_1(\mathscr{D}(\mathscr{P})), \ldots, \delta_d(\mathscr{D}(\mathscr{P})))$$

be the δ-vector of $\mathscr{D}(\mathscr{P})$. Then for $i = 0, 1, \ldots, d$, we have

$$\delta_i(\Gamma(\mathscr{P})) = (i+1)\delta_i(\mathscr{P}) + (d-i+1)\delta_{i-1}(\mathscr{P}),$$

where $\delta_{-1}(\mathscr{P}) = \delta_d(\mathscr{P}) = 0$.

In order to prove Proposition 13.2, we use the following lemmas.

Lemma 13.3. *For $d \geq 2$, let $\mathscr{P} \subset \mathbb{R}^{d-1}$ be a lattice polytope of dimension $d-1$ with the δ-vector $\delta(\mathscr{P}) = (\delta_0(\mathscr{P}), \delta_1(\mathscr{P}), \ldots, \delta_{d-1}(\mathscr{P}))$. Set*

$$\mathscr{Q} = \mathscr{P} \times [0,1] \subset \mathbb{R}^d,$$

and we let $\delta(\mathscr{Q}) = (\delta_0(\mathscr{Q}), \delta_1(\mathscr{Q}), \ldots, \delta_d(\mathscr{Q}))$ be the δ-vector of $\mathscr{Q}$. Then for $i = 0, 1, \ldots, d$, we have

$$\delta_i(\mathscr{Q}) = (i+1)\delta_i(\mathscr{P}) + (d-i)\delta_{i-1}(\mathscr{P}),$$

where $\delta_{-1}(\mathscr{P}) = \delta_d(\mathscr{P}) = 0$.

Proof. We know $L_{\mathscr{Q}}(n) = (n+1) \cdot L_{\mathscr{P}}(n)$. Hence one has

$$\delta_j(\mathscr{Q}) = \sum_{k=0}^{j} \binom{d+1}{k} (-1)^k (j-k+1) \cdot L_{\mathscr{P}}(j-k).$$

Since

$$\delta_j(\mathscr{P}) = \sum_{k=0}^{j} \binom{d}{k} (-1)^k \cdot L_{\mathscr{P}}(j-k),$$

we obtain

$$(j+1)\delta_j(\mathscr{P}) + (d-j)\delta_{j-1}(\mathscr{P}) = \delta_j(\mathscr{Q}),$$

as desired. □

Now, we prove Propsition 13.2.

Proof of Proposition 13.2. We set $\mathscr{Q}_1 = \mathrm{Pyr}(\mathscr{P}) \subset \mathbb{R}^d$ and $\mathscr{Q}_2 = \mathscr{P} \times [-1,0]$. Then $\mathscr{Q}_1 \cup \mathscr{Q}_2 = \mathscr{D}(\mathscr{P})$ and $\mathscr{Q}_1 \cap \mathscr{Q}_2 = \mathscr{P} \times \{0\}$. Hence we have

$$\mathrm{Ehr}_{\mathscr{D}(\mathscr{P})}(t) = \mathrm{Ehr}_{\mathscr{Q}_1}(t) + \mathrm{Ehr}_{\mathscr{Q}_2}(t) - \mathrm{Ehr}_{\mathscr{P}}(t).$$

By Lemma 2.9, $(1-t) \cdot \mathrm{Ehr}_{\mathscr{Q}_1}(t) = \mathrm{Ehr}_{\mathscr{P}}(t)$. Hence we have

$$\mathrm{Ehr}_{\Gamma(\mathscr{P})}(t) = t \cdot \mathrm{Ehr}_{\mathscr{Q}_1}(t) + \mathrm{Ehr}_{\mathscr{Q}_2}(t).$$

Let $\delta(\mathscr{Q}_1) = (\delta_0(\mathscr{Q}_1), \delta_1(\mathscr{Q}_1), \ldots, \delta_d(\mathscr{Q}_1))$ and $\delta(\mathscr{Q}_2) = (\delta_0(\mathscr{Q}_2), \delta_1(\mathscr{Q}_2), \ldots, \delta_d(\mathscr{Q}_2))$ be the δ-vectors of $\mathscr{Q}_1$ and $\mathscr{Q}_2$. By Lemmas 13.3 and 2.9, we have

$$\delta_i(\mathscr{Q}_1) = \begin{cases} \delta_i(\mathscr{P}) & i = 0, \ldots, d-1, \\ 0 & i = d, \end{cases}$$

and for $i = 0, \ldots, d$, we have

$$\delta_i(\mathscr{Q}_2) = (i+1)\delta_i(\mathscr{P}) + (d-i)\delta_{i-1}(\mathscr{P}),$$

where $\delta_{-1}(\mathscr{P}) = \delta_d(\mathscr{P}) = 1$. Hence for $i = 0, \ldots, d$,

$$\begin{aligned}\delta_i(\mathscr{D}(\mathscr{P})) &= \delta_{i-1}(\mathscr{Q}_1) + \delta_i(\mathscr{Q}_2) \\ &= \delta_{i-1}(\mathscr{P}) + (i+1)\delta_i(\mathscr{P}) + (d-i)\delta_{i-1}(\mathscr{P}) \\ &= (i+1)\delta_i(\mathscr{P}) + (d-i+1)\delta_{i-1}(\mathscr{P}),\end{aligned}$$

as desired. □

We give a higher-dimensional construction of self dual reflexive polytopes. By following theorem, we obtain the construction.

Theorem 13.4 ([90, Theorem 2.7])**.** *For $d \geq 2$, let $\mathscr{P}$ and $\mathscr{Q} \subset \mathbb{R}^{d-1}$ be lattice polytopes of dimension $d-1$ such that each of them has at least one interior integer point. Then we have the following properties:*

(a) $\mathscr{P}$ and $\mathscr{Q}$ are unimodularly equivalent if and only if $\mathscr{D}(\mathscr{P})$ and $\mathscr{D}(\mathscr{Q})$ are unimodularly equivalent;

(b) $\delta(\mathscr{P}) = \delta(\mathscr{Q})$ if and only if $\delta(\mathscr{D}(\mathscr{P})) = \delta(\mathscr{D}(\mathscr{Q}))$,

where $\delta(\mathscr{P})$, $\delta(\mathscr{Q})$, $\delta(\mathscr{D}(\mathscr{P}))$ and $\delta(\mathscr{D}(\mathscr{Q}))$ are the δ-vectors of $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{D}(\mathscr{P})$ and $\mathscr{D}(\mathscr{Q})$.

Remark 13.5. For $d \geq 2$, let $\mathscr{P} \subset \mathbb{R}^{d-1}$ be a reflexive polytope of dimension $d-1$. Then by Proposition 13.1 and Theorem 13.4, we have the following properties:

(a) $\mathscr{P}$ is self dual if and only if $\mathscr{D}(\mathscr{P})$ is self dual;

(b) $\delta(\mathscr{P}) = \delta(\mathscr{P}^\vee)$ if and only if $\delta(\mathscr{D}(\mathscr{P})) = \delta(\mathscr{D}(\mathscr{P})^\vee)$,

where $\delta(\mathscr{P})$, $\delta(\mathscr{P}^\vee)$, $\delta(\mathscr{D}(\mathscr{P}))$ and $\delta(\mathscr{D}(\mathscr{P})^\vee)$ are the δ-vectors of $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{D}(\mathscr{P})$ and $\mathscr{D}(\mathscr{P})^\vee$.

Now, we prove Theorem 13.4.

Proof of Theorem 13.4. (a) Clearly, if $\mathscr{P}$ and $\mathscr{Q}$ are unimodularly equivalent, then $\mathscr{D}(\mathscr{P})$ and $\mathscr{D}(\mathscr{Q})$ are unimodularly equivalent. Conversely, suppose that $\mathscr{D}(\mathscr{P})$ and $\mathscr{D}(\mathscr{Q})$ are unimodularly equivalent. We can assume that the origin of $\mathbb{R}^{d-1}$ belongs to the interior of $\mathscr{P}$ and the interior of $\mathscr{Q}$, and there exists a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ such that $\mathscr{D}(\mathscr{Q}) = f_U(\mathscr{D}(\mathscr{P}))$, where f_U is the linear transformation in $\mathbb{R}^d$ defined by U. Let $\mathbf{v}_1, \ldots, \mathbf{v}_s$ be the vertices of $\mathscr{P}$ and $\mathbf{w}_1, \ldots, \mathbf{w}_s$ be the vertices of $\mathscr{Q}$, and let

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1d} \\ u_{21} & u_{22} & \cdots & u_{2d} \\ \vdots & \vdots & & \vdots \\ u_{d1} & u_{d2} & \cdots & u_{dd} \end{pmatrix}.$$

Since$\mathbf{e}_d$ and $-\mathbf{e}_d$ belong to $\mathscr{D}(\mathscr{P})$ and since for each $(x_1,\dots,x_d) \in \mathscr{D}(\mathscr{Q})$, we have $-1 \le x_d \le 1$, we know $-1 \le u_{dd} \le 1$. If $u_{dd} = 0$, then $f_U((\mathbf{v}_i,0))$ and $f_U((\mathbf{v}_i,-1))$ have a common dth coordinate for $1 \le i \le s$. Since $\mathscr{D}(\mathscr{Q})$ has just one vertex whose dth coordinate equals 1 and since the dth coordinate of $f_U(\mathbf{e}_d)$ equals 0, there does not exist a vertex $\mathbf{v}$ of $\mathscr{D}(\mathscr{P})$ such that $f_U(\mathbf{v}) = \mathbf{e}_d$, a contradiction. If $v_{dd} = -1$, then $f_U(\mathbf{e}_d) = -\mathbf{e}_d$. However, $\mathbf{e}_d$ is a vertex of $\mathscr{D}(\mathscr{P})$ but $-\mathbf{e}_d$ is not a vertex of $\mathscr{D}(\mathscr{Q})$, a contradiction. Hence $u_{dd} = 1$. Since $f_U(\mathbf{e}_d) = \mathbf{e}_d$, we have

$$U = \begin{pmatrix} & U' & & * \\ & & & \vdots \\ & & & * \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $U' \in \mathbb{Z}^{(d-1)\times(d-1)}$ is a unimodular matrix. Then for each $\mathbf{v}_i$ there exists a vertex $\mathbf{w}_{j_i}$ of $\mathscr{Q}$ such that $f_U((\mathbf{v}_i,0)) = (\mathbf{w}_{j_i},-1)$ and $f_U((\mathbf{v}_i,1)) = (\mathbf{w}_{j_i},0)$. Hence for each $\mathbf{v}_i$ we have $f_{U'}(\mathbf{v}_i) = \mathbf{w}_{j_i}$, where $f_{U'}$ is the linear transformation in $\mathbb{R}^{d-1}$ defined by U'. Therefore, $\mathscr{P}$ and $\mathscr{Q}$ are unimodularly equivalent. (b) If $\delta(\mathscr{P}) = \delta(\mathscr{Q})$, by Proposition 13.2, we have $\delta(\mathscr{D}(\mathscr{P})) = \delta(\mathscr{D}(\mathscr{Q}))$. Suppose that $\delta(\mathscr{D}(\mathscr{P})) = \delta(\mathscr{D}(\mathscr{Q}))$. We set

$$\delta(\mathscr{P}) = (\delta_0(\mathscr{P}), \delta_1(\mathscr{P}), \dots, \delta_{d-1}(\mathscr{P})),$$

$$\delta(\mathscr{Q}) = (\delta_0(\mathscr{Q}), \delta_1(\mathscr{Q}), \dots, \delta_{d-1}(\mathscr{Q})).$$

By Proposition 13.2, for $i = 1,\dots,d-1$, we have

$$(i+1)(\delta_i(\mathscr{P}) - \delta_i(\mathscr{Q})) + (d-i+1)(\delta_{i-1}(\mathscr{P}) - \delta_{i-1}(\mathscr{Q})) = 0$$

Since $\delta_0(\mathscr{P}) = \delta_0(\mathscr{Q})$, for $i = 0,\dots,d-1$, we have $\delta_i(\mathscr{P}) = \delta_i(\mathscr{Q})$. Hence $\delta(\mathscr{P}) = \delta(\mathscr{Q})$. □

We let $\mathscr{P} \subset \mathbb{R}^2$ be a reflexive polytope of dimension 2. Then the δ-vector of $\mathscr{P}$ equals the δ-vector of $\mathscr{P}^\vee$ if and only if $\mathscr{P}$ is self dual. However, there exists a reflexive polytope of dimension 3 whose δ-vector equals the δ-vector of the dual polytope such that it is not self dual. We give an example of such a reflexive polytope.

Example 13.6 ([37, Example 35.11]). *Let $\mathscr{P} \subset \mathbb{R}^3$ be the reflexive polytope with the vertices $(-1,0,1)$, $(-1,0,-1)$, $(1,1,1)$, $(1,1,-1)$, $(0,-1,1)$ and $(0,-1,-1)$. Then $\mathscr{P}$ has 5 facets. Hence $\mathscr{P}^\vee$ has 5 vertices $(0,0,1)$, $(0,0,-1)$, $(2,-1,0)$, $(-1,2,0)$ and $(-1,-1,0)$. Therefore $\mathscr{P}$ and $\mathscr{P}^\vee$ are not unimodularly equivalent. However, $\delta(\mathscr{P}) = \delta(\mathscr{P}^\vee) = (1,8,8,1)$.*

By using Theorem 13.4 and Example 13.6, we obtain the following corollary.

Corollary 13.7. *For each $d \ge 3$, there exists a reflexive polytope of dimension d such that $\delta(\mathscr{P}) = \delta(\mathscr{P}^\vee)$ but $\mathscr{P}$ is not self dual.*

13.2 A new class of self dual reflexive simplices arising from Sylvester Sequence

For $d \geq 2$, we let $\mathscr{P}$ be a reflexive polytope of dimension d. Clearly, $\mathscr{D}(\mathscr{P})$ is not simplicial and not a simplex. In this section we consider self dual reflexive simplices.

First, we give an elementary number-theoretic notion.

Definition 13.8. The well-known recursive sequence ([79, A000058]) of pairwise coprime natural numbers $b_0 := 2$, $b_n := 1 + b_0 \cdots b_{n-1} (n \geq 1)$ is called *Sylvester Sequence*. It starts as $b_0 = 2$, $b_1 = 3$, $b_2 = 7$, $b_3 = 43$, $b_4 = 1807$.

It is known that a class of self dual reflexive simplices arising from Sylvester Sequence.

Example 13.9 ([66, Definition 4.6])**.** *For $d \geq 2$, we let $\mathscr{P}$ be the d-dimensional lattice simplex whose vertices $\mathbf{v}_i \in \mathbb{R}^d, i = 0, 1, \ldots, d$, are of the form:*

$$\mathbf{v}_i = \begin{cases} -(\mathbf{e}_1 + \cdots + \mathbf{e}_d) & i = 0, \\ b_{i-1}\mathbf{e}_i - \mathbf{v}_0 & i = 1, \ldots, d. \end{cases}$$

Then $\mathscr{P}$ is reflexive and we know that $\mathscr{P}$ and $\mathscr{P}^\vee$ are unimodularly equivalent, in particular, $\text{Vol}(\mathscr{P}) = b_d - 1$

For $d \geq 2$, we let $\mathscr{P}$ be a reflexive simplex of dimension d. It is known

$$(d+1)^{d+1} \leq \text{Vol}(\mathscr{P})\text{Vol}(\mathscr{P}^\vee) \leq (b_d - 1)^2,$$

and if $\text{Vol}(\mathscr{P}) = b_d - 1$, then $\mathscr{P}$ is unimodularly equivalent the lattice simplex described in Example 13.9, hence, $\mathscr{P}$ and $\mathscr{P}^\vee$ are unimodularly equivalent ([66, Theorem C]). This implies that if $\mathscr{P}$ is a self dual reflexive simplex, then we have $\text{Vol}(\mathscr{P}) \leq b_d - 1$.

In this section, we give a new class of self dual reflexive simplices arising from Selvester Sequence. In fact,

Theorem 13.10 ([90, Theorem 3.2.])**.** *For $d \geq 3$, let $\mathscr{P}$ be the d-dimensional lattice simplex whose vertices $\mathbf{v}_i \in \mathbb{R}^d, i = 0, 1, \ldots, d$, are of the form:*

$$\mathbf{v}_i = \begin{cases} -3\mathbf{e}_1 - 2\sum_{i=2}^{d} \mathbf{e}_i & i = 0, \\ \mathbf{e}_1 & i = 1, \\ \mathbf{e}_1 + 2\mathbf{e}_i & i = 2, 3, \\ \mathbf{e}_1 + 2b_{i-4}\mathbf{e}_i & i = 4, \ldots, d. \end{cases}$$

Then $\mathscr{P}$ is reflexive and we know that $\mathscr{P}$ and $\mathscr{P}^\vee$ are unimodularly equivalent, in particular, $\mathrm{Vol}(\mathscr{P}) < b_d - 1$.

In order to prove Theorem 13.10, we use the following lemma.

Lemma 13.11 ([79, A000058]). *For each $n \geq 0$*

$$\frac{1}{b_0}+\frac{1}{b_1}+\cdots+\frac{1}{b_n}=1-\frac{1}{b_0\cdots b_n}.$$

Now, we prove Thorem 13.10.

Proof of Theorem 13.10. First, we show that $\mathscr{P}$ is reflexive. Let $\mathscr{F}_0,\dots,\mathscr{F}_d$ be facets of $\mathscr{P}$, which are of the form:

$$\mathscr{F}_i = \mathrm{conv}(\{\mathbf{v}_0,\dots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\dots,\mathbf{v}_d\}) \quad 0 \leq i \leq d,$$

and for $0 \leq i \leq d$, let $\mathscr{H}_i$ be a hyperplane satisfying $\mathscr{F}_i = \mathscr{P} \cap \mathscr{H}_i$. Then

$$\mathscr{H}_i = \begin{cases} \left\{(x_1,\dots,x_d) \in \mathbb{R}^d : x_1 = 1\right\} & i=0, \\ \left\{(x_1,\dots,x_d) \in \mathbb{R}^d : x_1 - 2x_i = 1\right\} & i=2,\dots,d. \end{cases}$$

Also $\mathscr{H}_1 = \left\{(x_1,\dots,x_d) \in \mathbb{R}^d : \sum_{i=1}^d a_i x_i = 1\right\}$, where

$$a_i = \begin{cases} -(4b_0\cdots b_{d-4}-1) & i=1, \\ \dfrac{4b_0\cdots b_{d-4}}{2} & i=2,3, \\ \dfrac{4b_0\cdots b_{d-4}}{2b_{i-4}} & i=4,\dots d. \end{cases}$$

In fact, $\mathbf{v}_0 \in \mathscr{H}_1$ since

$$\begin{aligned} &3(4b_0\cdots b_{d-4}-1)-4b_0\cdots b_{d-4}-4b_0\cdots b_{d-4}-\frac{4b_0\cdots b_{d-4}}{b_0}-\cdots-\frac{4b_0\cdots b_{d-4}}{b_{d-4}} \\ =&-3+4b_0\cdots b_{d-4}(1-(\frac{1}{b_0}+\frac{1}{b_1}+\cdots+\frac{1}{b_{d-4}})) \\ =&-3+4b_0\cdots b_{d-4}\frac{1}{b_0\cdots b_{d-4}} \quad \text{(Lemma 13.11)} \\ =&1. \end{aligned}$$

Hence since $a_i \in \mathbb{Z}$ $(1 \leq i \leq d)$, by Lemma 2.7, $\mathscr{P}$ is reflexive.

Next, we show that $\mathscr{P}$ and $\mathscr{P}^\vee$ are unimodularly equivalent. By Lemma 2.7, we obtain that $\mathbf{w}_0, \dots, \mathbf{w}_d$ are vertices of $\mathscr{P}^\vee$, where

$$w_i = \begin{cases} \mathbf{e}_1 & i = 0, \\ (a_1, \dots, a_d) & i = 1, \\ \mathbf{e}_1 - 2\mathbf{e}_i & i = 2, \dots, d \end{cases}$$

We set a $d \times d$ matrix

$$U = \begin{pmatrix} 1 & 2 & & & & \\ 2 & 2 & 1 & 1 & \cdots & 1 \\ & 1 & -1 & & & \\ & 1 & & -b_0 & & \\ & \vdots & & & \ddots & \\ & 1 & & & & -b_{d-4} \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

where all other terms are zero. Then by Lemma 13.11, we have

$$\begin{aligned} \det(U) &= \det \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} (-1)^{d-3} b_0 \cdots b_{d-4} + (-1)^{d-2} \sum_{i=0}^{d-4} \frac{b_0 \cdots b_{d-4}}{b_i} \\ &= (-1)^{d-3} b_0 \cdots b_{d-4} + (-1)^{d-2} \sum_{i=0}^{d-4} \frac{b_0 \cdots b_{d-4}}{b_i} \\ &= (-1)^{d-3} b_0 \cdots b_{d-4} (1 - \sum_{i=0}^{d-4} \frac{1}{b_i}) \\ &= (-1)^{d-3}. \end{aligned}$$

Hence U is a unimodular matrix. Let f_U be the linear transformation in $\mathbb{R}^d$ defined by U. Then

$$f_U(\mathbf{w}_i) = \begin{cases} \mathbf{v}_2 & i = 0, \\ \mathbf{v}_1 & i = 1, \\ \mathbf{v}_0 & i = 2, \\ \mathbf{v}_i & i = 3, \dots, d. \end{cases}$$

Hence $\mathscr{P} = f_U(\mathscr{P}^\vee)$. Therefore we have $\mathscr{P}$ and $\mathscr{P}^\vee$ are unimodularly equivalent.

Finally, we show that $\mathrm{Vol}(\mathscr{P}) < b_d - 1$. If $d = 3$, then $\mathrm{Vol}(\mathscr{P}) = 16 < 42 = b_3 - 1$. We assume that $d \geq 4$. Since for each $n \geq 1$, $b_n > b_0 = 2$, for each $n \geq 0$, we have $b_n > 2^n$. Hence since $d \geq 4$ and since $\mathrm{Vol}(\mathscr{P}) = |2^{d-1}(a_1 - 1)| = 2^{d+1} b_0 \cdots b_{d-4}$, we have

$$b_d - 1 = b_0 \cdots b_{d-1} > 2^{3d-6} b_0 \cdots b_{d-4} > 2^{d+1} b_0 \cdots b_{d-4} = \mathrm{Vol}(\mathscr{P}),$$

as desired. $\square$

13.3 Self dual reflexive simplices with Eulerian polynomials

In this section, we give another class of self dual reflexive simplices. In particular, their δ-polynomials coincide with Eulerian polynomials.

We recall the *descent set* of a permutation $\pi = i_1 i_2 \cdots i_d$ of $[d] = \{1, 2, \ldots, d\}$ is $D(\pi) = \{j : i_j > i_{j+1}\} \subset [d-1]$. The *Eulerian polynomial* of degree $d-1$ is the polynomial

$$A_d(t) = \sum_{\pi \in S_d} t^{\mathrm{des}(\pi)},$$

where S_d is the symmetric group on $[d]$ and $\mathrm{des}(\pi) = |D(\pi)|$. An alternative definition of the Eulerian polynomial is via

$$\sum_{k=0}^{\infty} (1+k)^d t^k = \frac{A_d(t)}{(1-t)^{d+1}}.$$

We now define a class of reflexive simplices which are self dual. For $d \geq 2$, let $\mathscr{Q}_d$ denote the $d-1$ dimensional lattice simplex which is the convex hull of the row vectors of

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1-d & 1 & 1 & \cdots & 1 \\ 0 & 2-d & 1 & \cdots & 1 \\ 0 & 0 & 3-d & \ddots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Let $\mathbf{v}_i$ denote the $(i-1)$st row vector of the matrix.

We have the following theorem.

Theorem 13.12 ([47, Theorem 1])**.** *For $d \geq 2$, we have $\mathscr{Q}_d$ and $\mathscr{Q}_d^\vee$ are unimodularly equivalent.*

It behooves us to give the equations of supporting hyperplanes of facets of the simplex to compute its dual polytope.

Proposition 13.13. *For $0 \leq i \leq d-1$, let $\mathscr{F}_i$ be the facet of $\mathscr{Q}_d$ whose vertices are $\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_{d-1}$ and $\mathscr{H}_i$ the supporting hyperplane of $\mathscr{F}_i$. Then one has*

- $\mathscr{H}_0 = \{\mathbf{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} : -\sum_{i=1}^{d-1} x_i \leq 1\}$;
- $\mathscr{H}_{d-k} = \{\mathbf{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} : kx_k - \sum_{i=1}^{k-1} x_i \leq 1\}$ *for* $1 \leq k \leq d-1$.

Proof. It is sufficient to show that the vertices of $\mathcal{Q}_d$ each satisfy precisely $d-1$ of the halfspace inequalities with equality and satisfies the other inequalty strictly. Let $f_k(\mathbf{x}) = kx_k - \sum_{i=1}^{k-1} x_i$, and $f_d(\mathbf{x}) = -\sum_{i=1}^{d-1} x_i$. For a vertex $\mathbf{v}_j$, we have that $f_k(\mathbf{v}_j) = 1$ for all $k \neq d-j$. This follows, because if $k < d-j$, we have $f_k(\mathbf{v}_j) = (k)(1) - \sum_{i=1}^{k-1} 1 = 1$, if $k > d-j$ with $k \neq d$, we have $f_k(\mathbf{v}_j) = (d-j) - \sum_{i=1}^{d-j-1} 1 = 1$, and if $k = d > d-j$, we have $f_d(\mathbf{v}_j) = -(j-d) - \sum_{i=1}^{d-j-1} 1 = 1$. In the case of $k = d-j$, $f_{d-j}(\mathbf{v}_j) = -(d-j)^2 - (d-1-j) < 1$ if $j \neq 0$ and $j \neq d-1$. For $j = 0$ we have $f_d(\mathbf{v}_0) = 1-d < 1$ and for $j = d-1$, we have $f_1(\mathbf{v}_{d-1}) = -1 < 1$. Thus, this completes the proof. □

By Lemma 2.7, and Proposition 13.13, it is clear that $\mathcal{Q}_d^\vee = -\mathcal{Q}_d$. Therefore, we have shown Theorem 13.12.

Remark 13.14. We should note that $\mathrm{Vol}(\mathcal{Q}_d) = d!$. For $d \geq 4$, it is immediate that these polytopes are different than previously known self dual reflexive simplices given in Example 13.9 and Theorem 13.10.

Moreover, the self dual reflexive simplex of $\mathcal{Q}_d$ has an interesting δ-polynomial and a special triangulation.

Theorem 13.15. *Let $d \geq 2$.*
(i) *We have $\delta(\mathcal{Q}_d, t) = A_d(t)$, where $A_d(t)$ is the Eulerian polynomial.*
(ii) *$\mathcal{Q}_d$ has a regular, flag, unimodular triangulation.*

Proof. It is well-known that $\mathcal{P}$ has a regular, flag, unimodular triangulation if and only if $\mathrm{Pyr}(\mathcal{P})$ has a regular, flag, unimodular triangulation (cf. [22, Section 4.2]).

Let $\mathcal{R}_n$ denote the d-dimensional lattice simplex which is the convex hull of the row vectors of

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ n & 0 & 0 & \cdots & 0 \\ n & n-1 & 0 & \cdots & 0 \\ n & n-1 & n-2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ n & n-1 & n-2 & \cdots & 1 \end{pmatrix}.$$

This polytope $\mathcal{R}_d$ is called a *lecture hall polytope*. Notice that $\mathrm{Pyr}(\mathcal{Q}_d)$ is unimodularly equivalent to $\mathcal{R}_d$. Let $\widetilde{\mathcal{R}_d}$ be the polytope defined from $\mathcal{R}_d$ by removing the $(d+1)$st row and dth column, let U_d denote the $(d-1)\times(d-1)$ upper triangular matrix defined by $(U_d)_{ij} = 1$ if $i \leq j$ and $(U_d)_{ij} = 0$ otherwise. Then we know that $\mathcal{Q}_d$ is unimodularly equivalent to $-f_{U_d}(\mathcal{Q}_d - (\mathbf{e}_1 + \cdots + \mathbf{e}_{d-1})) = \widetilde{\mathcal{R}_d}$. Hence it follows that $\mathrm{Pyr}(\mathcal{Q}_d)$ is unimodularly equivalent to $\mathcal{R}_d$.

It is known that for $d \geq 2$, $\delta(\mathcal{R}_d, t) = A_d(t)$ ([76]) and $\mathcal{R}_d$ has a regular, flag, unimodular triangulation ([11]). Therefore, by Lemma 2.3, the assertion follows. □

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