ON SPECTRAL MEASURES OF RANDOM
JACOBI MATRICES

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Abstract

The paper studies the limiting behaviour of spectral measures of random Jacobi matrices of Gaussian, Wishart and MANOVA beta ensembles. We show that the spectral measures converge weakly to a limit distribution which is the semicircle distribution, Marchenko-Pastur distributions or Kesten-McKay distributions, respectively. The Gaussian fluctuation around the limit is then investigated.

1. Introduction

Three classical random matrix ensembles on the real line, Gaussian beta ensembles, Wishart beta ensembles and MANOVA beta ensembles, are now realized as eigenvalues of certain random Jacobi matrices. For instance, the following random Jacobi matrices whose components are independent and distributed as

$$
H_{n,\beta} = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{n-1} & a_n \end{pmatrix} \sim \frac{1}{\sqrt{n\beta}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{\beta} & \mathcal{N}(0,2) \end{pmatrix}
$$

are matrix models of (scaled) Gaussian beta ensembles for any $\beta > 0$. Here $\mathcal{N}(\mu, \sigma^2)$ denotes the normal (or Gaussian) distribution with mean μ and variance σ^2, and χ_k denotes the chi distribution with k degrees of freedom. Namely, the eigenvalues of $H_{n,\beta}$ are distributed as Gaussian beta ensembles,

$$
(\lambda_1, \ldots, \lambda_n) \propto |\Delta(\lambda)|^{\beta} \exp\left(-\frac{n\beta}{4}\sum_{i=1}^{n} \lambda_i^2\right),
$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ denotes the Vandermonde determinant.

Three special values of beta, $\beta = 1, 2$ and 4, correspond to Gaussian orthogonal, unitary and symplectic ensembles (GOE, GUE and GSE) in which the above formula describes the joint distribution of eigenvalues of random matrices with real, complex and quaternion entries, respectively. As a generalization, Gaussian beta ensembles were originally defined as ensembles of points on the real line whose joint density function is given as above. They can be also viewed as the equilibrium measure of a one dimensional Coulomb log-gas at

the inverse temperature β. Using the idea of tridiagonalizing a GOE matrix, Dumitriu and Edelman [6] introduced the model $H_{n,\beta}$ for Gaussian beta ensembles. In the same paper, they also gave a matrix model for Wishart beta ensembles. A model for MANOVA beta ensembles was discovered later by Killip and Nenciu [11].

One of main objects in random matrix theory is to study the limiting behaviour of the empirical distribution of eigenvalues

$$L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},$$

where δ denotes the Dirac measure. For Gaussian beta ensembles, as n tends to infinity, the empirical distributions converge weakly, almost surely, to the semicircle distribution, which is well known as Wigner's semicircle law. The convergence means that for any bounded continuous function f on $\mathbb{R}$,

$$\langle L_n, f \rangle = \frac{1}{n} \sum_{i=1}^{n} f(\lambda_i) \to \langle sc, f \rangle \text{ almost surely as } n \to \infty,$$

with sc denoting the semicircle distribution, a probability measure supported on $[-2, 2]$ with density $sc(x) = (2\pi)^{-1}\sqrt{4 - x^2}$. A fluctuation around the limit was also investigated. To be more precise, it was shown that for a 'nice' test function f,

$$n(\langle L_n, f \rangle - \mathbb{E}[\langle L_n, f \rangle]) = \sum_{i=1}^{n}(f(\lambda_i) - \langle sc, f \rangle) \xrightarrow{d} \mathcal{N}(0, a_f^2),$$

where a_f^2 can be written as a quadratic functional of f. There are several ways to prove those results. See Johansson [10] for an approach based on joint density function, Dumitriu and Edelman [7] and Dumitriu and Paquette [8] for a combinatorial approach based on the random Jacobi matrix models. Since GOE and GUE have their original matrix models, we can see more approaches in books [1, 15]. Note that the idea in the last section of this paper is also applicable to study such Gaussian fluctuation. Using the idea, we can show that the class of 'nice' test functions for which the above central limit theorem holds contains at least differentiable functions whose derivative is continuous of polynomial growth.

The spectral measures of random Jacobi matrices associated with those beta ensembles have been investigated recently. The weak convergence to a limit distribution, a central limit theorem for moments and large deviations have been established [5, 9, 13]. The spectral measure of a finite Jacobi matrix, a symmetric tridiagonal matrix of the form,

$$J = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & b_{n-1} & a_n \end{pmatrix}, (a_i \in \mathbb{R}, b_i > 0),$$

is defined to be a unique probability measure μ on $\mathbb{R}$ satisfying

$$\langle \mu, x^k \rangle = \langle J^k e_1, e_1 \rangle = J^k(1, 1), k = 0, 1, \ldots,$$

where $e_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^n$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the eigenvalues of J and $\{v_1, \ldots, v_n\}$ be the corresponding eigenvectors which are chosen to be an orthonormal basis of $\mathbb{R}^n$. Then

the spectral measure μ can be written as

$$\mu = \sum_{i=1}^{n} q_i^2 \delta_{\lambda_i}, \quad q_i = |v_i(1)|.$$

Note that the eigenvalues $\{\lambda_i\}$ are distinct and the weights $\{q_i^2\}$ are all positive. Moreover, a finite Jacobi matrix of size n is one-to-one correspondence with a probability measure supported on n real points.

Let μ_n be the spectral measure of $H_{n,\beta}$,

$$\mu_n = \sum_{i=1}^{n} q_i^2 \delta_{\lambda_i}.$$

In this case, and in all three beta ensembles in the paper, the weights $\{q_i^2\}$ are independent of eigenvalues and have Dirichlet distribution with parameters $(\beta/2, \ldots, \beta/2)$ (or symmetric Dirichlet distribution with parameter $\beta/2$). The distribution of $\{q_i^2\}$ is the same as that of the vector

$$\left(\frac{\chi_{\beta,1}^2}{\sum_{i=1}^{n} \chi_{\beta,i}^2}, \ldots, \frac{\chi_{\beta,n}^2}{\sum_{i=1}^{n} \chi_{\beta,i}^2} \right),$$

where $\{\chi_{\beta,i}^2\}_{i=1}^{n}$ is an i.i.d. sequence of random variables having chi-squared distributions with β degrees of freedom. In connection with empirical distributions, Nagel [14] showed that as n tends to infinity, the Kolmogorov distance between L_n and μ_n converges almost surely to zero. Thus the spectral measures and the empirical distributions converge to the same limit. Moreover, the limiting behaviour of spectral measures of Jacobi matrices can be read off from the convergence of their entries. For Gaussian beta ensembles, it is clear that

$$H_{n,\beta} \rightarrow \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} =: J_{free}, \text{ almost surely as } n \rightarrow \infty.$$

Here the convergence means the piecewise convergence of entries. The non-random Jacobi matrix J_{free} is called the free Jacobi matrix whose spectral measure is nothing but the semicircle distribution [16, Section 1.10]. Consequently the spectral measures μ_n, and hence, the empirical distributions L_n, converge weakly, almost surely, to the semicircle distribution. This result may be regarded as a strong law of large numbers for spectral measures.

A natural problem now is to study the fluctuation of spectral measures around the limit, or a central limit theorem for $\langle \mu_n, f \rangle$ with a 'nice' function f. In a work which is not so related to random matrix theory, Dette and Nagel [5] derived a central limit theorem for moments $\{\langle \mu_n, x^k \rangle\}$ of spectral measures. The result covers Gaussian, Wishart beta ensembles and MANOVA beta ensembles with fixed parameters. The aim of this paper is to reconsider the central limit theorem. We propose a universal approach which can be easily applied to all three beta ensembles. The idea is that for a polynomial test function, when n is large enough, $\langle \mu_n, p \rangle = p(H_{n,\beta})(1, 1)$ is a polynomial of finite variables. Then the central limit theorem follows from the limiting behaviour of entries of Jacobi matrices. Furthermore, by a relation between spectral measures and empirical measures, we obtain an explicit formula for the limit variance and can extend the central limit theorem to a large class of test functions. Our

main result for Gaussian beta ensembles can be stated as follows.

Theorem 1.1.   (i) *The spectral measures μ_n converge weakly, almost surely, to the semicircle distribution as $n \to \infty$, that is, for any bounded continuous function f,*

$$\langle \mu_n, f \rangle = \sum_{i=1}^{n} q_i^2 f(\lambda_i) \to \langle sc, f \rangle \text{ almost surely as } n \to \infty.$$

(ii) *For a function f with continuous derivative of polynomial growth,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, f \rangle - \mathbb{E}[\langle \mu_n, f \rangle]) \xrightarrow{d} \mathcal{N}(0, \sigma^2(f)) \text{ as } n \to \infty,$$

where $\sigma^2(f) = \langle sc, f^2 \rangle - \langle sc, f \rangle^2 = \mathrm{Var}_{sc}[f]$. Here '$\xrightarrow{d}$' denotes convergence in distribution or weak convergence of random variables.

The results for Wishart and MANOVA beta ensembles are analogous where the semicircle distribution is replaced by Marchenko-Pastur distributions and Kesten-McKay distributions, respectively.

The Kolmogorov distance, the metric which implies the weak convergence, between two measures μ and ν on the real line with distribution functions F_μ and F_ν, respectively, is defined by

$$d_K(\mu, \nu) = \sup_{x \in \mathbb{R}} |F_\mu(x) - F_\nu(x)|.$$

Here $F_\mu(x) = \mu((-\infty, x])$. As mentioned above, for fixed $\beta > 0$, the Kolmogorov distance between the empirical distributions L_n and the spectral measures μ_n converges to zero almost surely as n tends to infinity. For the proof, we only need properties that both measures are support on the set of eigenvalues and that the weights $\{q_i^2\}$ have symmetric Dirichlet distribution with parameter $\beta/2$ [14, Theorem 4.2]. Consequently, the following strong law of large numbers for empirical distributions holds.

Corollary 1.2. *For all three beta ensembles in this paper, as $n \to \infty$, the empirical distributions L_n converge weakly, almost surely, to the same limit as the spectral measures.*

The paper is organized as follows. In the next section, we consider general random Jacobi matrices and derive the weak convergence of spectral measures as well as the central limit theorem for polynomial test functions. Applications to Gaussian, Wishart and MANOVA beta ensembles are then investigated in turn. The last section is devoted to extend the central limit theorem to a larger class of test functions.

2. Limiting behaviour of spectral measures of random Jacobi matrices

Let us begin by introducing some spectral properties of (non random) Jacobi matrices. A semi-infinite Jacobi matrix is a symmetric tridiagonal matrix of the form

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \text{ where } a_i \in \mathbb{R}, b_i > 0.$$

To a Jacobi matrix J, there exists a probability measure μ such that

$$\langle \mu, x^k \rangle = \int_{\mathbb{R}} x^k d\mu = \langle J^k e_1, e_1 \rangle, k = 0, 1, \dots,$$

where $e_1 = (1, 0, \dots,)^t \in \ell^2$. Then μ is unique, or μ is determined by its moments, if and only if, J is an essentially self-adjoint operator on ℓ^2. If the parameters $\{a_i\}$ and $\{b_i\}$ are bounded, or more generally, if $\sum b_i^{-1} = \infty$, then J is essentially self-adjoint [16, Corollary 3.8.9]. In case of uniqueness, we call μ the spectral measure of J, or of (J, e_1). See [4, Chapter 2] or [16, Section 3.8] for more details on Jacobi matrices.

When J is a finite Jacobi matrix of order n, then the spectral measure μ is supported on the eigenvalues $\{\lambda_i\}$ of J with weights $\{q_i^2\} = \{v_i(1)^2\}$,

$$\mu = \sum_{i=1}^{n} q_i^2 \delta_{\lambda_i}.$$

Here $\{v_1, \dots, v_n\}$ are the corresponding eigenvectors which are chosen to be an orthogonal basis of $\mathbb{R}^n$. The eigenvalues $\{\lambda_i\}_{i=1}^{n}$ are distinct and the weights $\{q_i^2\}_{i=1}^{n}$ are all positive [4, Proposition 2.40].

We are now in a position to study the convergence of spectral measures of Jacobi matrices. Recall that spectral measures are defined by their moments. Does the convergence of moments imply the weak convergence of probability measures? The following lemma gives us the answer. It is a classical result which can be found in some textbooks in probability theory.

Lemma 2.1. *Assume that $\{\mu_n\}_{n=1}^{\infty}$ and μ are probability measures on $\mathbb{R}$ such that for all $k = 0, 1, \dots,$*

$$\langle \mu_n, x^k \rangle \to \langle \mu, x^k \rangle \text{ as } n \to \infty.$$

Assume further that the measure μ is determined by its moments. Then μ_n converges weakly to μ as $n \to \infty$. Moreover, if f is a continuous function of polynomial growth, that is, there is a polynomial p such that $|f(x)| \leq p(x)$ for all $x \in \mathbb{R}$, then we also have

$$\langle \mu_n, f \rangle \to \langle \mu, f \rangle \text{ as } n \to \infty.$$

Proof. The first part of this lemma is a well-known moment problem [2, Theorem 30.2]. The second part follows by a truncated argument. For the sake of completeness, we give proof here. Assume that the sequence $\{\mu_n\}$ converges weakly to μ and that $\langle \mu_n, p \rangle$ converges to $\langle \mu, p \rangle$ for all polynomials p. Let f be a continuous function which is dominated by a polynomial p, $|f(x)| \leq p(x)$ for all $x \in \mathbb{R}$. For $M > 0$, write f_M for the truncated function

$$f_M(x) = \begin{cases} -M, & \text{if } f(x) \leq -M, \\ f(x), & \text{if } |f(x)| \leq M, \\ M, & \text{if } f(x) \geq M. \end{cases}$$

Then it is clear that $|f - f_M| \leq p - p_M$, where p_M is the truncated function of p. Thus by the triangle inequality,

$$|\langle \mu_n, f \rangle - \langle \mu, f \rangle| \leq |\langle \mu_n, f - f_M \rangle| + |\langle \mu_n, f_M \rangle - \langle \mu, f_M \rangle| + |\langle \mu, f - f_M \rangle|$$

$$\leq \langle \mu_n, p - p_M \rangle + |\langle \mu_n, f_M \rangle - \langle \mu, f_M \rangle| + \langle \mu, p - p_M \rangle$$

$$= \langle \mu_n, p \rangle - \langle \mu_n, p_M \rangle + |\langle \mu_n, f_M \rangle - \langle \mu, f_M \rangle| + \langle \mu, p - p_M \rangle.$$

As $n \to \infty$, the first term converges to $\langle \mu, p \rangle$ by the assumption, the second term converges to $\langle \mu, p_M \rangle$ and the third term converges to 0 because p_M and f_M are bounded continuous functions. Therefore

$$\limsup_{n \to \infty} |\langle \mu_n, f \rangle - \langle \mu, f \rangle| \le 2\langle \mu, p - p_M \rangle.$$

Finally, by letting $M \to \infty$, $\langle \mu, p - p_M \rangle \to 0$ by the monotone convergence theorem. The lemma is proved.   □

To random probability measures, we deal with two types of convergence, almost sure convergence and convergence in probability. A result for almost sure convergence is a direct consequence of the above deterministic result. However, it is not the case for convergence in probability. When the limit measure has compact support, the following result on convergence in probability of random measures can be derived by a method of polynomials approximation, see subsection 2.1.2 in [1], for instance.

Lemma 2.2. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of random probability measures and μ be a non-random probability measure which is determined by its moments. Assume that any moment of μ_n converges almost surely to that of μ, that is, for any $k = 0, 1, \ldots,$*

$$\langle \mu_n, x^k \rangle \to \langle \mu, x^k \rangle \text{ a.s. as } n \to \infty.$$

Then as $n \to \infty$, the sequence of measures $\{\mu_n\}$ converges weakly, almost surely, to μ, namely, for any bounded continuous function f,

$$\langle \mu_n, f \rangle \to \langle \mu, f \rangle \text{ a.s. as } n \to \infty.$$

The convergence still holds for a continuous function f of polynomial growth. An analogous result holds for convergence in probability.

Proof. The case of almost sure convergence is a direct consequence of Lemma 2.1. Indeed, for $k \ge 1$, let

$$A_k = \{\omega : \langle \mu_n(\omega), x^k \rangle \to \langle \mu, x^k \rangle \text{ as } n \to \infty\}.$$

Then $\mathbb{P}(A_k) = 1$ by the assumption. Therefore $\mathbb{P}(A := \bigcap_{k=1}^{\infty} A_k) = 1$. Applying Lemma 2.1 to the sequence of probability measures $\{\mu_n(\omega)\}$, for $\omega \in A$, yields the desired result.

Next we consider the case of convergence in probability. The idea here is to use the following criterion for convergence in probability [2, Theorem 20.5]: a sequence $\{X_n\}$ converges to X in probability if and only if for every subsequence $\{X_{n(m)}\}$, there is a further subsequence $\{X_{n(m_k)}\}$ that converges almost surely to X. Let f be a continuous function which is dominated by some polynomial. Given a subsequence $\{n(m)\}$, the aim now is to find a subsequence $n(m_k)$ such that $\{\langle \mu_{n(m_k)}, f \rangle\}$ converges almost surely to $\langle \mu, f \rangle$. Let $\{n(0, m) = n(m)\}$. For $k \ge 1$, using the necessary condition in the criterion, we can find a subsequence $\{n(k, m)\}$ of $\{n(k-1, m)\}$ such that

$$\langle \mu_{n(k,m)}, x^k \rangle \to \langle \mu, x^k \rangle \text{ a.s. as } n \to \infty.$$

By selecting the diagonal, we get a subsequence $\{n(m_k) = n(k, k)\}$ for which all moments

of $\{\mu_{n(m_k)}\}$ converge almost surely to the corresponding moments of μ. Consequently, the sequence $\{\mu_{n(m_k)}\}$ converges weakly, almost surely, to μ by the first part of this lemma, which implies that $\{\langle \mu_{n(m_k)}, f \rangle\} \to \langle \mu, f \rangle$ almost surely. The proof is complete. $\quad\square$

Let us now explain the main idea of this paper. Consider the sequence of random Jacobi matrices

$$
J_n = \begin{pmatrix}
a_1^{(n)} & b_1^{(n)} & & \\
b_1^{(n)} & a_2^{(n)} & b_2^{(n)} & \\
& \ddots & \ddots & \ddots \\
& & b_{n-1}^{(n)} & a_n^{(n)}
\end{pmatrix},
$$

and let μ_n be the spectral measure of (J_n, e_1). Assume that each entry of J_n converges almost surely to a non random limit as $n \to \infty$, that is, for any fixed i, as $n \to \infty$,

$$(1) \qquad\qquad\qquad a_i^{(n)} \to \bar{a}_i; \quad b_i^{(n)} \to \bar{b}_i \text{ a.s.}$$

Here we require that $\bar{a}_i$ and $\bar{b}_i$ are non random and $\bar{b}_i > 0$. Assume further that the spectral measure of (J_∞, e_1), denoted by μ_∞, is unique, where J_∞ is the infinite Jacobi matrix consisting of $\{\bar{a}_i\}$ and $\{\bar{b}_i\}$,

$$
J_\infty = \begin{pmatrix}
\bar{a}_1 & \bar{b}_1 & & \\
\bar{b}_1 & \bar{a}_2 & \bar{b}_2 & \\
& \ddots & \ddots & \ddots
\end{pmatrix}.
$$

Then the measure μ_∞ is determined by its moments, and hence we get the following result.

Theorem 2.3. *The spectral measures μ_n converge weakly, almost surely, to the limit measure μ_∞ as $n \to \infty$. If in the assumption* (1)*, convergence in probability is assumed instead of almost sure convergence, then the spectral measures μ_n converge weakly, in probability, to μ_∞ as $n \to \infty$.*

REMARK 2.4. These results may be referred to as weak and strong laws of large numbers for spectral measures of random Jacobi matrices. They are natural results which may be found somewhere. For instance, the strong law was mentioned in [14].

Proof. Let p be a polynomial of degree m. When n is large enough, $\langle \mu_n, p \rangle = p(J_n)(1, 1)$ is a polynomial of $\{a_i^{(n)}, b_i^{(n)}\}_{i=1,\ldots,\lceil \frac{m}{2} \rceil}$. Therefore, as $n \to \infty$,

$$\langle \mu_n, p \rangle \to \langle \mu_\infty, p \rangle \text{ almost surely (resp. in probability)},$$

which implies the weak convergence of μ_n by Lemma 2.2. $\quad\square$

Next, we consider the second order of the convergence of spectral measures, or a type of central limit theorem. It turns out that the central limit theorem for polynomial test functions is a direct consequence of a joint central limit theorem for entries of Jacobi matrices. Indeed, assume that there are random variables $\{\eta_i\}$ and $\{\zeta_i\}$ defined on the same probability space such that for some fixed $r > 0$, for any i, as $n \to \infty$,

$$\begin{aligned}
\tilde{a}_i^{(n)} &= n^r(a_i^{(n)} - \bar{a}_i) \xrightarrow{d} \eta_i, \\
\tilde{b}_i^{(n)} &= n^r(b_i^{(n)} - \bar{b}_i) \xrightarrow{d} \zeta_i.
\end{aligned}$$

(2)

Moreover, we assume that the joint weak convergence holds. This means that any finite linear combination of $\tilde{a}_i^{(n)}$ and $\tilde{b}_i^{(n)}$ converges weakly to the corresponding linear combination of η_i and ζ_i as $n \to \infty$, namely, for any real numbers c_i and d_i,

$$\sum_{finite} (c_i \tilde{a}_i^{(n)} + d_i \tilde{b}_i^{(n)}) \xrightarrow{d} \sum_{finite} (c_i \eta_i + d_i \zeta_i).$$

From now on, both conditions (1) and (2) will be written in a compact form

$$J_n \approx \begin{pmatrix} \bar{a}_1 & \bar{b}_1 & \\ \bar{b}_1 & \bar{a}_2 & \bar{b}_2 \\ & \ddots & \ddots & \ddots \end{pmatrix} + \frac{1}{n^r} \begin{pmatrix} \eta_1 & \zeta_1 & \\ \zeta_1 & \eta_2 & \zeta_2 \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

or in term of entries

$$\begin{aligned}
a_i^{(n)} &\approx \bar{a}_i + \frac{1}{n^r}\eta_i, \\
b_i^{(n)} &\approx \bar{b}_i + \frac{1}{n^r}\zeta_i.
\end{aligned}$$

Let f be a polynomial of $2k$ variables $(a_1, \ldots, a_k, b_1, \ldots, b_k)$. For simplicity, we write $f(a_i, b_i)$ instead of $f(a_1, \ldots, a_k, b_1, \ldots, b_k)$.

Lemma 2.5.   (i) *As $n \to \infty$,*

$$n^r \left(f(a_i^{(n)}, b_i^{(n)}) - f(\bar{a}_i, \bar{b}_i) \right) - \sum_{i=1}^{k} \left(\frac{\partial f}{\partial a_i}(\bar{a}_i, \bar{b}_i)\tilde{a}_i^{(n)} + \frac{\partial f}{\partial b_i}(\bar{a}_i, \bar{b}_i)\tilde{b}_i^{(n)} \right) \to 0$$

in probability.

(ii) *As $n \to \infty$,*

$$n^r \left(f(a_i^{(n)}, b_i^{(n)}) - f(\bar{a}_i, \bar{b}_i) \right) \xrightarrow{d} \sum_{i=1}^{k} \left(\frac{\partial f}{\partial a_i}(\bar{a}_i, \bar{b}_i)\eta_i + \frac{\partial f}{\partial b_i}(\bar{a}_i, \bar{b}_i)\zeta_i \right).$$

Proof. Write

$$a_i^{(n)} = \bar{a}_i + \frac{1}{n^r}\tilde{a}_i^{(n)}; b_i^{(n)} = \bar{b}_i + \frac{1}{n^r}\tilde{b}_i^{(n)}.$$

Then use the Taylor expansion of $f(a_i^{(n)}, b_i^{(n)})$ at $(\bar{a}_i, \bar{b}_i)$ with noting that the Taylor expansion of a polynomial consists of finitely many terms,

$$f(a_i^{(n)}, b_i^{(n)}) = f(\bar{a}_i, \bar{b}_i) + \frac{1}{n^r} \sum_{i=1}^{k} \left(\frac{\partial f}{\partial a_i}(\bar{a}_i, \bar{b}_i)\tilde{a}_i^{(n)} + \frac{\partial f}{\partial b_i}(\bar{a}_i, \bar{b}_i)\tilde{b}_i^{(n)} \right) + \sum\nolimits^{*}.$$

Each term in the finite sum $\sum^{*}$ has the following form,

$$c(\alpha,\beta) \prod_{i=1}^{k} (a_i^{(n)} - \bar{a}_i)^{\alpha_i} (b_i^{(n)} - \bar{b}_i)^{\beta_i},$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are non negative integers and $\sum_{i=1}^{k} (\alpha_i + \beta_i) \geq 2$. Therefore, when that term is multiplied by n^r, it converges to 0 in distribution, and hence, in probability by Slutsky's theorem.

By using Slutsky's theorem again, we see that (ii) is a consequence of (i). The proof is complete. □

Let p be a polynomial of degree $m > 0$. Then there is a polynomial of $2\lceil \frac{m}{2} \rceil$ variables such that for $n > m/2$,

$$\langle \mu_n, p \rangle = p(J_n)(1,1) = f(a_1^{(n)}, \ldots, a_{\lceil \frac{m}{2} \rceil}^{(n)}, b_1^{(n)}, \ldots, b_{\lceil \frac{m}{2} \rceil}^{(n)}).$$

Therefore, by Lemma 2.5, we obtain the central limit theorem for polynomial test functions.

Theorem 2.6. *For any polynomial p, $n^r (\langle \mu_n, p \rangle - \langle \mu_\infty, p \rangle)$ converges weakly to a limit $\xi_\infty(p)$ as $n \to \infty$.*

Since we do not assume that all moments of $\{a_i^{(n)}\}$ and $\{b_i^{(n)}\}$ are finite, even the expectation of $\langle \mu_n, p \rangle$, $\mathbb{E}[\langle \mu_n, p \rangle]$ may not exist. Thus we need further assumptions to ensure the convergence of mean and variance in the central limit theorem above. Our assumptions are based on the following basic result in probability theory (the corollary following Theorem 25.12 in [2]).

Lemma 2.7. *Assume that the sequence $\{X_n\}$ converges weakly to a random variable X. If for some $\delta > 0$,*

$$\sup_n \mathbb{E}[|X_n|^{2+\delta}] < \infty,$$

then $\mathbb{E}[X_n] \to \mathbb{E}[X]$ and $\mathrm{Var}[X_n] \to \mathrm{Var}[X]$ as $n \to \infty$. In general, if X_n converges to X in probability and for $q \geq 1$ and $\delta > 0$,

$$\sup_n \mathbb{E}[|X_n|^{q+\delta}] < \infty,$$

then X_n converges to X in L_q.

We make the following assumptions
   (i) all moments of $\{a_i^{(n)}\}$ and $\{b_i^{(n)}\}$ are finite and in addition, the convergences in (1) also hold in L_q for all $q < \infty$, which is equivalent to the following conditions

$$(3) \qquad \sup_n \mathbb{E}[|a_i^{(n)}|^k] < \infty, \quad \sup_n \mathbb{E}[|b_i^{(n)}|^k] < \infty, \text{ for all } k = 1, 2, \ldots;$$

   (ii) $\mathbb{E}[\eta_i] = 0, \mathbb{E}[\zeta_i] = 0$, and for some $\delta > 0$,

$$(4) \qquad \sup_n \mathbb{E}[|\tilde{a}_i^{(n)}|^{2+\delta}] < \infty, \quad \sup_n \mathbb{E}[|\tilde{b}_i^{(n)}|^{2+\delta}] < \infty.$$

Lemma 2.8. *As $n \to \infty$,*

$$(5) \qquad n^{2r} \mathbb{E}\left[\left(f(a_i^{(n)}, b_i^{(n)}) - f(\bar{a}_i, \bar{b}_i) \right)^2 \right] \to \mathrm{Var}\left[\sum_{i=1}^{k} \left(\frac{\partial f}{\partial a_i}(\bar{a}_i, \bar{b}_i)\eta_i + \frac{\partial f}{\partial b_i}(\bar{a}_i, \bar{b}_i)\zeta_i \right) \right],$$

(6) $\qquad n^r \left(\mathbb{E}[f(a_i^{(n)}, b_i^{(n)})] - f(\bar{a}_i, \bar{b}_i) \right) \to 0,$

(7) $\qquad n^{2r} \operatorname{Var}\left[f(a_i^{(n)}, b_i^{(n)}) \right] \to \operatorname{Var}\left[\sum_{i=1}^{k} \left(\frac{\partial f}{\partial a_i}(\bar{a}_i, \bar{b}_i)\eta_i + \frac{\partial f}{\partial b_i}(\bar{a}_i, \bar{b}_i)\zeta_i \right) \right].$

Proof. It is just a direct consequence of Lemma 2.7. $\qquad\qquad\qquad\square$

We state now a slightly different form of the central limit theorem for polynomial test functions.

Theorem 2.9. *Under assumptions* (1)–(4), *for any polynomial p, as* $n \to \infty$,

$$\langle \mu_n, p \rangle \to \langle \mu_\infty, p \rangle \text{ almost surely and in } L^q \text{ for all } q < \infty;$$

$$n^r(\langle \mu_n, p \rangle - \mathbb{E}[\langle \mu_n, p \rangle]) \xrightarrow{d} \xi_\infty(p).$$

Moreover, $\mathbb{E}[\xi_\infty(p)] = 0$ *and*

$$n^{2r} \operatorname{Var}[\langle \mu_n, p \rangle] \to \operatorname{Var}[\xi_\infty(p)] \text{ as } n \to \infty.$$

3. Gaussian beta ensembles or β-Hermite ensembles

Let $H_{n,\beta}$ be a random Jacobi matrix whose elements are independent (up to the symmetric constraint) and are distributed as

$$H_{n,\beta} = \frac{1}{\sqrt{n\beta}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & & \chi_{\beta} & \mathcal{N}(0,2) \end{pmatrix}.$$

Then the eigenvalues $\{\lambda_i\}$ of $H_{n,\beta}$ have Gaussian beta ensembles [6], that is,

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \propto |\Delta(\lambda)|^\beta \exp\left(-\frac{n\beta}{4} \sum_{i=1}^{n} \lambda_j^2 \right).$$

The weights $\{w_i\} = \{q_i^2\}$ are independent of $\{\lambda_i\}$ and have Dirichlet distribution with parameters $(\beta/2, \ldots, \beta/2)$, that is,

$$(w_1, \ldots, w_{n-1}) \propto \prod_{i=1}^{n} w_i^{\frac{\beta}{2}-1} \mathbf{1}_{\{w_1+\cdots+w_{n-1}<1, w_i>0\}}, \quad w_n = 1 - (w_1 + \cdots + w_{n-1}).$$

Recall that the distribution of $\{w_i\}$ is the same as that of the vector

$$\left(\frac{\chi_{\beta,1}^2}{\sum_{i=1}^{n} \chi_{\beta,i}^2}, \ldots, \frac{\chi_{\beta,n}^2}{\sum_{i=1}^{n} \chi_{\beta,i}^2} \right),$$

with $\{\chi_{\beta,i}^2\}_{i=1}^{n}$ being i.i.d. sequence of random variables having chi-squared distributions with β degrees of freedom.

Lemma 3.1. $\qquad$ (i) *As* $k \to \infty$,

$$\frac{\chi_k}{\sqrt{k}} \to 1 \text{ in probability and in } L_q \text{ for all } q < \infty.$$

A sequence $\{\chi_{k_n}/\sqrt{k_n}\}$ converges almost sure to 1, if $\sum_{n=1}^{\infty} e^{-\varepsilon k_n} < \infty$ for any $\varepsilon > 0$.

(ii) *As $k \to \infty$,*

$$\sqrt{k}\left(\frac{\chi_k}{\sqrt{k}} - 1\right) = (\chi_k - \sqrt{k}) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{2}\right).$$

Proof. The convergence in probability and a central limit theorem are standard results. Let us prove the almost sure convergence part. For almost sure convergence, usually there is some relation between random variables in the sequence. If there is no relation, the following criterion, a direct consequence of the Borel-Cantelli lemma, becomes useful. The sequence X_n converges almost surely to x as $n \to \infty$, if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - x| > \varepsilon) < \infty.$$

For the proof here and later for Beta distributions, we use the following bounds (see Lemma 4.1 in [14])

$$\mathbb{P}\left(\left|\frac{\chi_k^2}{k} - 1\right| > \varepsilon\right) \le 2e^{-\frac{k\varepsilon^2}{8}},$$

$$\mathbb{P}\left(|\text{Beta}(x, y) - \mathbb{E}[\text{Beta}(x, y)]| > \varepsilon\right) \le 4e^{-\frac{\varepsilon^2}{128}\frac{x^3+y^3}{xy}}.$$

Here $\text{Beta}(x, y)$ denotes the beta distribution with parameters x and y. Then the proof follows immediately from the criterion above. □

Since the Jacobi parameters $\{a_i, b_i\}$ of $H_{n,\beta}$ are independent, it follows that joint convergence in distribution holds. Thus we can write

$$H_{n,\beta} \approx \begin{pmatrix} 0 & 1 & \\ 1 & 0 & 1 \\ & \ddots & \ddots & \ddots \end{pmatrix} + \frac{1}{\sqrt{\beta n}} \begin{pmatrix} \mathcal{N}(0, 2) & \mathcal{N}(0, \frac{1}{2}) & \\ \mathcal{N}(0, \frac{1}{2}) & \mathcal{N}(0, 2) & \mathcal{N}(0, \frac{1}{2}) \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

The nonrandom Jacobi matrix in the above expression is called the free Jacobi matrix, denoted by J_{free}, whose spectral measure is the semicircle distribution

$$sc(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, (-2 \le x \le 2).$$

There are several ways to derive that fact. For example, in the theory of orthogonal polynomials on the real line, it is derived from the relation of Chebyshev polynomials of the second kind [16, Section 1.10]. One can also calculate moments and show that they match moments of the semicircle distribution. However, we introduce here a method to find the spectral measure by calculating its Stieltjes transform, which is applicable to the next two cases.

Let J be a Jacobi matrix with bounded coefficients,

$$J = \begin{pmatrix} a_1 & b_1 & \\ b_1 & a_2 & b_2 \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then the spectral measure μ is unique and has compact support. Let S_μ be the Stieltjes transform of μ,

$$S_\mu(z) = \int \frac{d\mu(x)}{x - z}.$$

In the theory of Jacobi matrices, the Stieljes transform S_μ is called an m-function,

$$S_\mu(z) = m(z) = \langle (J - z)^{-1}e_1, e_1 \rangle = (J - z)^{-1}(1, 1).$$

For bounded Jacobi matrix J, the m-function is an analytic function on $\mathbb{C} \setminus I$, for some bounded interval $I \subset \mathbb{R}$, $\operatorname{Im} m(z) > 0$, if $\operatorname{Im} z > 0$, and $m(\bar{z}) = \overline{m(z)}$.

Let J_1 be a Jacobi matrix obtained by removing the top row and left-most column of J and let $m_1(z)$ be the m-function of J_1. Then the following relation holds (see [16, Theorem 3.2.4])

$$(8) \qquad\qquad -\frac{1}{m(z)} = z - a_1 + b_1^2 m_1(z).$$

When $a_i \equiv a$ and $b_i \equiv b$, then $m_1(z) = m(z)$, and hence $m(z)$ satisfies the following equation

$$-\frac{1}{m(z)} = z - a + b^2 m(z),$$

which is equivalent to a quadratic form

$$b^2 m(z)^2 + (z - a)m(z) + 1 = 0.$$

In particular, by solving the above equation with $a = 0$ and $b = 1$, we obtain the explicit formula for the m-function of the free Jacobi matrix,

$$m_{free}(z) = -\frac{1}{2}\left(z - \sqrt{z^2 - 4}\right),$$

where the branch of the square-root is chosen to ensure that $\operatorname{Im} m(z) > 0$, if $\operatorname{Im} z > 0$. Then the spectral measure μ of J_∞ can be easily calculated by an inverse formula ([1, Theorem 2.4.3])

$$\mu(I) = \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \int_I \operatorname{Im} m(x + i\varepsilon)dx,$$

provided that I is an open interval with neither endpoint on an atom of μ. Moreover, when the limit $\lim_{\varepsilon \searrow 0} \operatorname{Im} m(x + i\varepsilon)$ exists and is finite for all $x \in \mathbb{R}$, the measure μ has density given by

$$\mu(x) = \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \operatorname{Im} m(x + i\varepsilon).$$

The spectral measure of the free Jacobi matrix is now easily derived from the above formula.

Back to the case $a_i \equiv a$ and $b_i \equiv b$, and denote by $m_{a,b}(z)$ its m-function. Then

$$(9) \qquad m_{a,b}(z) = (J_{a,b} - z)^{-1}(1, 1) = (bJ_{free} + a - z)^{-1}(1, 1) = \frac{1}{b}m_{free}(\frac{z - a}{b}).$$

Here

$$J_{a,b} = \begin{pmatrix} a & b & \\ b & a & b & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad J_{free} = J_{0,1}.$$

Here is the main result in this section.

Theorem 3.2. *The spectral measure μ_n of $H_{n,\beta}$ converges weakly, almost surely, to the semicircle distribution as $n \to \infty$. Moreover, for any polynomial p, as $n \to \infty$,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, p \rangle - \langle sc, p \rangle) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2),$$

where σ_p^2 is a constant.

REMARK 3.3. Here we implicitly assume that the parameter β is fixed. However, the result still holds if β varies and $n\beta \to \infty$. In this case, by our method, the spectral measures μ_n converge weakly, in probability, to the semicircle distribution because we do not require the rate of convergence of $n\beta$ to infinity.

4. Wishart beta ensembles or β-Laguerre ensembles

Let G be an $m \times n$ matrix whose entries are i.i.d. standard Gaussian $\mathcal{N}(0, 1)$ random variables. Then for $m \geq n$, $(m^{-1}G^tG)$ is a Wishart real matrix whose eigenvalues are distributed as

$$(\lambda_1, \ldots, \lambda_n) \propto |\Delta(\lambda)| \prod_{i=1}^{n} \lambda_i^{\frac{1}{2}(m-n+1)-1} \exp\left(-\frac{m}{2}\sum_{i=1}^{n}\lambda_i\right), \quad \lambda_i > 0.$$

Note that Wishart complex matrices are defined similarly by considering i.i.d. complex Gaussian matrices G. The eigenvalues also a nice density formula similar as above. Then Wishart beta ensembles are defined to be ensembles of n points on $(0, \infty)$ with the following joint probability density function

$$(\lambda_1, \ldots, \lambda_n) \propto |\Delta(\lambda)|^\beta \prod_{i=1}^{n} \lambda_i^{\frac{\beta}{2}(m-n+1)-1} \exp\left(-\frac{m\beta}{2}\sum_{i=1}^{n}\lambda_i\right).$$

Here we only require that $n - 1 < m \in \mathbb{R}$.

Dumitriu and Edelman [6] gave the following matrix model for Wishart beta ensembles. For $n \in \mathbb{N}$ and $m > n - 1$, let B_β be a bidiagonal matrix whose elements are independent and are distributed as

$$B_\beta = \frac{1}{\sqrt{m\beta}}\begin{pmatrix} \chi_{\beta m} & & & \\ \chi_{\beta(n-1)} & \chi_{\beta m-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{\beta m-\beta(n-1)} \end{pmatrix}.$$

Let $L_{m,n,\beta} = B_\beta B_\beta^t$. Then the eigenvalues of $L_{m,n,\beta}$ are distributed as Wishart beta ensembles. Moreover, the weights $\{w_i\}$ in the spectral measures of $L_{m,n,\beta}$ have Dirichlet distribution with parameters $(\beta/2, \ldots, \beta/2)$ and are independent of $\{\lambda_i\}$. For empirical distributions, the weak convergence to Marchenko-Pastur distributions and a central limit theorem for polynomial

test functions were established, see [7] and the references therein.

Denote by $\{c_i\}_{i=1}^n$ and $\{d_j\}_{j=1}^{n-1}$ the diagonal and the sub-diagonal of the matrix $\sqrt{m\beta}B_\beta$. Then

$$
L_{m,n,\beta} = \frac{1}{m\beta}
\begin{pmatrix}
c_1^2 & c_1 d_1 & & & \\
c_1 d_1 & c_2^2 + d_1^2 & c_2 d_2 & & \\
& \ddots & \ddots & \ddots & \\
& & c_{n-1}d_{n-1} & c_n^2 + d_{n-1}^2 &
\end{pmatrix}.
$$

Lemma 4.1. *For fixed i, as $n \to \infty$ with $n/m \to \gamma \in (0,1)$,*

$$
\frac{c_i}{\sqrt{m\beta}} = \frac{\chi_{\beta(m-i+1)}}{\sqrt{m\beta}} \approx 1 + \frac{\sqrt{\gamma}}{\sqrt{n\beta}}\eta_i, \quad \eta_i \sim \mathcal{N}(0, \tfrac{1}{2}),
$$

$$
\frac{d_i}{\sqrt{m\beta}} = \frac{\chi_{\beta(n-i)}}{\sqrt{m\beta}} \approx \sqrt{\gamma} + \frac{\sqrt{\gamma}}{\sqrt{n\beta}}\zeta_i, \quad \zeta_i \sim \mathcal{N}(0, \tfrac{1}{2}),
$$

$$
\frac{c_i^2}{m\beta} \approx 1 + \frac{\sqrt{\gamma}}{\sqrt{n\beta}}2\eta_i, \quad \frac{d_i^2}{m\beta} \approx \gamma + \frac{\sqrt{\gamma}}{\sqrt{n\beta}}2\zeta_i,
$$

$$
\frac{c_i d_i}{m\beta} \approx \sqrt{\gamma} + \frac{\sqrt{\gamma}}{\sqrt{n\beta}}(\sqrt{\gamma}\eta_i + \zeta_i).
$$

Consequently, as $n \to \infty$ and $n/m \to \gamma \in (0,1)$, we can write

$$
L_{m,n,\beta} \approx
\begin{pmatrix}
1 & \sqrt{\gamma} & & \\
\sqrt{\gamma} & 1+\gamma & \sqrt{\gamma} & \\
& \ddots & \ddots & \ddots
\end{pmatrix}
+ \frac{\sqrt{\gamma}}{\sqrt{n\beta}}
\begin{pmatrix}
2\eta_1 & \sqrt{\gamma}\eta_1 + \zeta_1 & & \\
\sqrt{\gamma}\eta_1 + \zeta_1 & 2(\eta_2 + \zeta_1) & \sqrt{\gamma}\eta_2 + \zeta_2 & \\
& \ddots & \ddots & \ddots
\end{pmatrix}.
$$

Here $\{\eta_i\}$ and $\{\zeta_i\}$ are two i.i.d. sequences of $\mathcal{N}(0, \tfrac{1}{2})$ random variables.

For $\gamma \in (0,1)$, let

$$
MP_\gamma =
\begin{pmatrix}
1 & \sqrt{\gamma} & & \\
\sqrt{\gamma} & 1+\gamma & \sqrt{\gamma} & \\
& \ddots & \ddots & \ddots
\end{pmatrix},
$$

and denote its m-function by $m_\gamma(z)$. When the first row and the left-most column are removed, we obtain the Jacobi matrix $J_{1+\gamma, \sqrt{\gamma}}$. Then it follows from the relations (8) and (9) that

$$
m_\gamma(z) = -\frac{1}{z - 1 + \sqrt{\gamma}m_{free}(\frac{z-(1+\gamma)}{\sqrt{\gamma}})}
$$

$$
= -\frac{1}{z - 1 - \frac{\sqrt{\gamma}}{2}\left(\frac{z-(1+\gamma)}{\sqrt{\gamma}} - \sqrt{(\frac{z-(1+\gamma)}{\sqrt{\gamma}})^2 - 4}\right)}
$$

$$
= \frac{1 - \gamma - z + \sqrt{(z-\lambda_+)(z-\lambda_-)}}{2\gamma z},
$$

where $\lambda_\pm = (1 \pm \sqrt{\gamma})^2$. By taking the limit $\lim_{\varepsilon \searrow 0} \pi^{-1} \operatorname{Im} m_\gamma(x + i\varepsilon)$, we get the density of the spectral measure of MP_γ, which coincides with the density of the Marchenko-Pastur distribution with parameter $\gamma \in (0,1)$,

$$mp_\gamma(x) = \frac{1}{2\pi\gamma x}\sqrt{(\lambda_+ - x)(x - \lambda_-)}, (\lambda_- < x < \lambda_+).$$

Theorem 4.2. *The spectral measure μ_n of $L_{m,n,\beta}$ conveges weakly, almost surely, to the Marchenko-Pastur distribution with parameter γ as $n \to \infty$, $n/m \to \gamma \in (0, 1)$. Moreover for any polynomial p,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, p\rangle - \langle mp_\gamma, p\rangle) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2),$$

where σ_p^2 is a constant.

5. MANOVA beta ensembles or β-Jacobi ensembles

Let $W_1 = G_1^t G_1$ and $W_2 = G_2^t G_2$ be two independent real Wishart matrices with G_1 (resp. G_2) being an $m_1 \times n$ (resp. $m_2 \times n$) matrix whose entries are i.i.d. standard Gaussian $\mathcal{N}(0, 1)$ random variables. Assume that $m_1, m_2 \geq n$. Then the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of the matrix $W_1(W_1 + W_2)^{-1}$ (or of the Hermitian matrix $(W_1 + W_2)^{-1/2}W_1(W_1 + W_2)^{-1/2}$) are distributed according to the following joint probability density function with $\beta = 1$ ([12, Theorem 3.3.4])

$$(10) \qquad (\lambda_1, \dots, \lambda_n) \propto |\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^{\frac{\beta}{2}(m_1-n+1)-1}(1 - \lambda_i)^{\frac{\beta}{2}(m_2-n+1)-1},$$

$$= |\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^a(1 - \lambda_i)^b, \quad \lambda_i \in [0, 1].$$

For general $\beta > 0$, the above joint density is referred to as MANOVA beta ensembles or β-Jacobi ensembles with parameters (a, b), where $a, b > -1$.

A matrix model for β-Jacobi ensembles is given as follows. Let $a, b > -1$ be fixed. Let $p_1, \dots, p_{2n-1}$ be independent random variables distributed as

$$p_k \sim \begin{cases} \text{Beta}\left(\frac{2n-k}{4}\beta, \frac{2n-k-2}{4}\beta + a + b + 2\right), & k \text{ even,} \\ \text{Beta}\left(\frac{2n-k-1}{4}\beta + a + 1, \frac{2n-k-1}{4}\beta + b + 1\right), & k \text{ odd.} \end{cases}$$

Here recall that $\text{Beta}(x, y)$ denotes the beta distribution with parameters x, y. Define

$$a_k = p_{2k-2}(1 - p_{2k-3}) + p_{2k-1}(1 - p_{2k-2}),$$

$$b_k = \sqrt{p_{2k-1}(1 - p_{2k-2})p_{2k}(1 - p_{2k-1})},$$

where $p_{-1} = p_0 = 0$, and form a random Jacobi matrix $J_{n,\beta}(a, b)$ as

$$J_{n,\beta}(a, b) = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{n-1} & a_n \end{pmatrix}.$$

Then the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of $J_{n,\beta}(a, b)$ are distributed as the β-Jacobi ensembles (10) (cf. [11]). The weights $\{w_i\} = \{q_i^2\}$ are independent of $\{\lambda_i\}$ and have Dirichlet distribution with parameters $(\beta/2, \dots, \beta/2)$. We are interested in studying β-Jacobi ensembles in the regime that $a(n)/(\frac{n\beta}{2}) \to \kappa_a \in [0, \infty)$ and $b(n)/(\frac{n\beta}{2}) \to \kappa_b \in [0, \infty)$. The limiting behaviour

of empirical distributions in that regime was studied in [8].

We need the following properties of beta distributions.

Lemma 5.1. *Let $\{x_k\}$ and $\{y_k\}$ be two sequences of positive real numbers such that as $k \to \infty$*

$$\frac{x_k}{k} \to x > 0, \frac{y_k}{k} \to y > 0.$$

Then the following asymptotic behaviours of the beta distribution $\text{Beta}(x_k, y_k)$ *hold.*

(i) *As $k \to \infty$,*

$$\text{Beta}(x_k, y_k) \to \frac{x}{x + y} \text{ in probability and in } L_q \text{ for all } q < \infty.$$

The almost sure convergence also holds.

(ii) *As $k \to \infty$,*

$$\sqrt{k}\left(\text{Beta}(x_k, y_k) - \frac{x_k}{x_k + y_k}\right) \xrightarrow{d} \mathcal{N}(0, xy(x + y)^{-3}).$$

Moreover, if $(x_k y - y_k x)/\sqrt{k} \to 0$ as $k \to \infty$, then

$$\sqrt{k}\left(\text{Beta}(x_k, y_k) - \frac{x}{x + y}\right) \xrightarrow{d} \mathcal{N}(0, xy(x + y)^{-3}).$$

Proof. Let X_k and Y_k be two independent random variables distributed as $\chi^2_{2x_k}$ and $\chi^2_{2y_k}$, respectively. Then

$$\text{Beta}(x_k, y_k) \stackrel{d}{=} \frac{X_k}{X_k + Y_k}.$$

For chi-squared distribution, we have

$$\frac{\chi^2_k}{k} \to 1 \text{ in probability as } k \to \infty.$$

Therefore

$$\frac{X_k}{X_k + Y_k} = \frac{\frac{X_k}{2x_k}\frac{2x_k}{2k}}{\frac{X_k}{2x_k}\frac{2x_k}{2k} + \frac{Y_k}{2y_k}\frac{2y_k}{2k}} \to \frac{x}{x + y} \text{ in probability as } k \to \infty.$$

The convergence in L_q is clear because beta distributions are bounded by 1. For the almost sure convergence, see the proof of Lemma 3.1.

Next we consider the central limit theorem for beta distributions. It also follows from the following central limit theorem for chi-squared distribution

$$\sqrt{\frac{k}{2}}\left(\frac{\chi^2_k}{k} - 1\right) \xrightarrow{d} \mathcal{N}(0, 1) \text{ as } k \to \infty.$$

Indeed, if we write

$$\sqrt{k}\left(\frac{X_k}{X_k + Y_k} - \frac{x_k}{x_k + y_k}\right)$$

$$= \frac{\left(\frac{y_k}{k} \frac{\sqrt{x_k}}{\sqrt{k}}\right) \sqrt{x_k} \left(\frac{X_k}{2x_k} - 1\right) - \left(\frac{x_k}{k} \frac{\sqrt{y_k}}{\sqrt{k}}\right) \sqrt{y_k} \left(\frac{Y_k}{2y_k} - 1\right)}{\left(\frac{x_k}{k} + \frac{y_k}{k}\right)\left(\frac{X_k}{2k} + \frac{Y_k}{2k}\right)},$$

then as $k \to \infty$, the numerator converges in distribution to $\mathcal{N}(0, xy(x+y))$ because X_k and Y_k are independent while the denominator converges in probability to $(x + y)^2$. Thus we obtain (ii). □

Lemma 5.2. *As $n \to \infty$ with $a(n) = \frac{n\beta}{2}\kappa_a + o((n\beta)^{1/2})$ and $b(n) = \frac{n\beta}{2}\kappa_b + o((n\beta)^{1/2})$,*

$$p_{2k} \approx \mathrm{Beta}\left(\frac{n\beta}{2}, \frac{n\beta}{2}(1 + \kappa_a + \kappa_b)\right) \approx \frac{1}{2 + \kappa_a + \kappa_b} + \frac{1}{\sqrt{n\beta}}\mathcal{N}(0, \sigma_{even}^2),$$

$$p_{2k-1} \approx \mathrm{Beta}\left(\frac{n\beta}{2}(1 + \kappa_a), \frac{n\beta}{2}(1 + \kappa_b)\right) \approx \frac{1 + \kappa_a}{2 + \kappa_a + \kappa_b} + \frac{1}{\sqrt{n\beta}}\mathcal{N}(0, \sigma_{odd}^2).$$

As a consequence,

$$a_1^{(n)} = p_1 \approx \frac{1 + \kappa_a}{2 + \kappa_a + \kappa_b} + \frac{1}{\sqrt{n\beta}}\mathcal{N} =: A_1 + \frac{1}{\sqrt{n\beta}}\mathcal{N},$$

$$b_1^{(n)} = \sqrt{p_1 p_2 (1 - p_1)} \approx \frac{\sqrt{(1 + \kappa_a)(1 + \kappa_b)}}{(2 + \kappa_a + \kappa_b)^{3/2}} + \frac{1}{\sqrt{n\beta}}\mathcal{N} =: B_1 + \frac{1}{\sqrt{n\beta}}\mathcal{N},$$

$$a_k^{(n)} = p_{2k-2}(1 - p_{2k-3}) + p_{2k-1}(1 - p_{2k-2})$$
$$\approx \frac{1 + \kappa_b + (1 + \kappa_a)(1 + \kappa_a + \kappa_b)}{(2 + \kappa_a + \kappa_b)^2} + \frac{1}{\sqrt{n\beta}}\mathcal{N} =: A + \frac{1}{\sqrt{n\beta}}\mathcal{N}, \quad k \geq 2,$$

$$b_k^{(n)} = \sqrt{p_{2k-1}(1 - p_{2k-2})p_{2k}(1 - p_{2k-1})}$$
$$\approx \frac{\sqrt{(1 + \kappa_a)(1 + \kappa_b)(1 + \kappa_a + \kappa_b)}}{(2 + \kappa_a + \kappa_b)^2} + \frac{1}{\sqrt{n\beta}}\mathcal{N} =: B + \frac{1}{\sqrt{n\beta}}\mathcal{N}, \quad k \geq 2.$$

Here $\mathcal{N}$ denotes a normal distribution with mean 0 and positive variance. The joint asymptotic also holds.

Proof. For fixed k, as $n \to \infty$, the asymptotic for p_k follows from Lemma 5.1. The asymptotic for $a_k^{(n)}$ follows from Lemma 2.5 because it is a polynomial of $\{p_{2k-3}, p_{2k-2}, p_{2k-1}\}$.

The asymptotic for $b_k^{(n)}$ is a consequence of the following fact. If $X_n \to c \neq 0$ in probability and $\sqrt{n}(X_n - c) \overset{d}{\to} \mathcal{N}(0, \sigma^2)$ as $n \to \infty$, then

$$\sqrt{n}(\sqrt{X_n} - \sqrt{c}) = \frac{\sqrt{n}(X_n - c)}{\sqrt{X_n} + \sqrt{c}} \overset{d}{\to} \frac{\mathcal{N}(0, \sigma^2)}{2\sqrt{c}} = \mathcal{N}\left(0, \frac{\sigma^2}{4c}\right).$$

The joint asymptotic is clear. □

Therefore, in the regime that $a(n) = \frac{n\beta}{2}\kappa_a + o((n\beta)^{1/2})$ and $b(n) = \frac{n\beta}{2}\kappa_b + o((n\beta)^{1/2})$, the limit Jacobi matrix is of the form

$$J_\infty = \begin{pmatrix} A_1 & B_1 & & & \\ B_1 & A & B & & \\ & B & A & B & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where A_1, B_1, A and B are defined in the above lemma. We use relations (8) and (9) again to derive an explicit formula for the m-function of J_∞,

$$m(z) = \frac{\kappa_a}{2z} + \frac{\kappa_b}{2(z-1)} - \frac{(2 + \kappa_a + \kappa_b)\sqrt{(z - u_-)(z - u_+)}}{2z(z-1)},$$

where

(11) $$u_\pm = A \pm 2B = \left(\frac{\sqrt{(1 + \kappa_a)(1 + \kappa_a + \kappa_b)} \pm \sqrt{1 + \kappa_b}}{2 + \kappa_a + \kappa_b} \right)^2.$$

Note that $0 \le u_- < u_+ \le 1$. The density of the spectral measure of J_∞ can be easily calculated by the inverse formula,

$$km_{u_-,u_+}(x) = \frac{2 + \kappa_a + \kappa_b}{2\pi} \frac{\sqrt{(x - u_-)(u_+ - x)}}{x(1 - x)}, (u_- < x < u_+).$$

It coincides with the density of the Kesten-McKay distribution with parameters (u_-, u_+) (see [9, Section 7.6]) because

$$\frac{1}{2}\left(1 - \sqrt{u_- u_+} - \sqrt{(1 - u_-)(1 - u_+)}\right) = \frac{1}{2 + \kappa_a + \kappa_b}.$$

When $\kappa_a = \kappa_b = 0$, then $u_- = 0$ and $u_+ = 1$, we get the arcsine distribution.

Therefore, we obtain the limiting behaviour of the spectral measures of β-Jacobi ensembles.

Theorem 5.3. *As* $n \to \infty$ *with* $a(n) = \frac{n\beta}{2}\kappa_a + o((n\beta)^{1/2})$ *and* $b(n) = \frac{n\beta}{2}\kappa_b + o((n\beta)^{1/2})$, *the spectral measure* μ_n *of* $J_{n,\beta}(a(n), b(n))$ *converges weakly, almost surely, to the Kesten-McKay distribution with parameters* (u_-, u_+). *For any polynomial p,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, p\rangle - \langle km_{u_-,u_+}, p\rangle) \to \mathcal{N}(0, \sigma_p^2),$$

where σ_p^2 *is a constant.*

REMARK 5.4. In different regimes, the weak convergence of both empirical distributions and spectral measures was considered in [14], in which the limit distribution is the Marchenko-Pastur distribution or the semicircle distribution.

6. Extend the central limit theorem to a large class of test functions

For all three beta ensembles in this paper, the spectral measure μ_n can be written as

$$\mu_n = \sum_{i=1}^{n} w_i \delta_{\lambda_i},$$

where the weights $\{w_i\}$ are independent of the eigenvalues $\{\lambda_i\}$ and have Dirichlet distribution with parameters $(\beta/2, \ldots, \beta/2)$. Recall that as a consequence of Theorem 2.9, for any polynomial p, as $n \to \infty$,

$$\langle \mu_n, p\rangle \to \langle \mu_\infty, p\rangle \text{ almost surely and in } L^q \text{ for } q \ge 1,$$

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle\mu_n, p\rangle - \mathbb{E}[\langle\mu_n, p\rangle]) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2),$$

where

$$\sigma_p^2 = \lim_{n\to\infty}\frac{n\beta}{2}\,\text{Var}[\langle\mu_n, p\rangle],$$

and μ_∞ denotes the limit distribution.

One can easily show that

$$\mathbb{E}[w_i] = \frac{1}{n}, \mathbb{E}[w_i^2] = \frac{\beta+2}{n(n\beta+2)}, \mathbb{E}[w_i w_j] = \frac{\beta}{n(n\beta+2)}, (1 \le i \ne j \le n).$$

Therefore for any test function f, the following relations hold

(12) $\qquad \mathbb{E}[\langle\mu_n, f\rangle] = \mathbb{E}[\langle L_n, f\rangle],$

(13) $\qquad \text{Var}[\langle\mu_n, f\rangle] = \frac{\beta n}{\beta n + 2}\,\text{Var}[\langle L_n, f\rangle] + \frac{2}{n\beta+2}\left(\mathbb{E}[\langle\mu_n, f^2\rangle] - \mathbb{E}[\langle\mu_n, f\rangle]^2\right).$

The mean of a random measure μ, denoted by $\bar{\mu}$, is defined to be a probability measure satisfying

$$\langle\bar{\mu}, f\rangle = \mathbb{E}[\langle\mu, f\rangle],$$

for all bounded continuous function f. Moreover, the above relation still holds for any continuous function f with $\mathbb{E}[\langle\mu, |f|\rangle] < \infty$. It follows from (12) that $\bar{L}_n = \bar{\mu}_n$, and hence, $\bar{L}_n = \bar{\mu}_n$ converges weakly to μ_∞ as $n \to \infty$. Thus for any continuous function of polynomial growth,

$$\langle\bar{\mu}_n, f\rangle \to \langle\mu_\infty, f\rangle \text{ as } n \to \infty.$$

Denote by $C(\mathbb{R})$ the set of continuous functions on $\mathbb{R}$ and let

$$\mathcal{D} = \{f \in C(\mathbb{R}) : n\,\text{Var}[\langle L_n, f\rangle] \to 0, \langle\bar{\mu}_n, f\rangle \to \langle\mu_\infty, f\rangle, \langle\bar{\mu}_n, f^2\rangle \to \langle\mu_\infty, f^2\rangle\}.$$

Then $\mathcal{D}$ is a linear space containing all polynomials. It follows from the variance relation (13) that for $f \in \mathcal{D}$,

$$\lim_{n\to\infty}\frac{n\beta}{2}\,\text{Var}[\langle\mu_n, f\rangle] = \langle\mu_\infty, f^2\rangle - \langle\mu_\infty, f\rangle^2 =: \sigma^2(f).$$

Next, we use the following result to extend the central limit theorem to any test function in $\mathcal{D}$.

Lemma 6.1 ([2, Theorem 25.5]). *Let $\{Y_n\}_n$ and $\{X_{n,k}\}_{k,n}$ be real-valued random variables. Assume that*

(i) $X_{n,k} \xrightarrow{d} X_k$ *as* $n \to \infty$;

(ii) $X_k \xrightarrow{d} X$ *as* $k \to \infty$;

(iii) *for any* $\varepsilon > 0$, $\lim_{k\to\infty}\limsup_{n\to\infty}\mathbb{P}(|X_{n,k} - Y_n| \ge \varepsilon) = 0$.

Then $Y_n \xrightarrow{d} X$ *as* $n \to \infty$.

Theorem 6.2. *For $f \in \mathcal{D}$,*

$$\frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, f \rangle - \mathbb{E}[\langle \mu_n, f \rangle]) \xrightarrow{d} \mathcal{N}(0, \sigma(f)^2),$$

where $\sigma^2(f) = \langle \mu_\infty, f^2 \rangle - \langle \mu_\infty, f \rangle^2 = \mathrm{Var}_{\mu_\infty}[f]$.

Proof. Let $f \in \mathcal{D}$. Since μ_∞ has compact support, we can find a sequence of polynomials $\{p_k\}$ converging to f uniformly in the support of μ_∞. Thus

$$\sigma^2(p_k) \to \sigma^2(f) \text{ as } k \to \infty.$$

Let

$$Y_n = \frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, f \rangle - \mathbb{E}[\langle \mu_n, f \rangle]),$$

$$X_{n,k} = \frac{\sqrt{n\beta}}{\sqrt{2}}(\langle \mu_n, p_k \rangle - \mathbb{E}[\langle \mu_n, p_k \rangle]).$$

We only need to check three conditions in Lemma 6.1. Conditions (i) and (ii) are clear. For the condition (iii), note that $(f - p_k) \in \mathcal{D}$, and thus

$$\lim_{n \to \infty} \mathrm{Var}[X_{n,k} - Y_n] = \langle \mu_\infty, (f - p_k)^2 \rangle - \langle \mu_\infty, (f - p_k) \rangle^2,$$

which tends to zero as $k \to \infty$. Therefore, for any $\varepsilon > 0$,

$$\lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{P}(|X_{n,k} - Y_n| \geq \varepsilon) \leq \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\varepsilon^2} \mathrm{Var}[X_{n,k} - Y_n] = 0.$$

The theorem is proved.   □

Lemma 6.3. *For the Gaussian and Wishart beta ensembles, the class $\mathcal{D}$ contains at least all functions whose derivative is continuous of polynomial growth. For the MANOVA beta ensembles, the class $\mathcal{D}$ contains at least all differentiable functions with continuous derivative on $[0, 1]$, provided that the parameters $a(n)$ and $b(n)$ are positive and $a(n) + b(n) \to \infty$ as $n \to \infty$.*

The idea of proof is taken from [8]. The key tool is the following result.

Lemma 6.4 ([3, Proposition 2.1])**.** *Let $d\nu = e^{-V} dx$ be a probability measure supported on an open convex set $\Omega \subset \mathbb{R}^n$. Assume that V is twice continuously differentiable and strictly convex on Ω. Then for any locally Lipschitz function F on Ω,*

$$\mathrm{Var}_\nu[F] = \int F^2 d\nu - \left(\int F d\nu\right)^2 \leq \int \langle \mathrm{Hess}(V)^{-1} \nabla F, \nabla F \rangle d\nu$$

$$\leq \int \frac{1}{\lambda_{\min}(\mathrm{Hess}(V))} |\nabla F|^2 d\nu,$$

where $\mathrm{Hess}(V)$ denotes the Hessian of V and $\lambda_{\min}(A)$ denote the smallest eigenvalue of A.

Proof of Lemma 6.3.   Let us consider the Gaussian beta ensembles case first. Here we consider the ordered eigenvalues $\lambda_1 < \cdots < \lambda_n$. Let $\Omega = \{(\lambda_1, \ldots, \lambda_n) : \lambda_1 < \lambda_2 < \cdots < \lambda_n\} \subset \mathbb{R}^n$. Then the joint probability density function can be written in the form e^{-V} on Ω

with

$$V = const + \frac{n\beta}{4} \sum_{i=1}^{n} \lambda_i^2 - \frac{\beta}{2} \sum_{i \neq j} \log |\lambda_j - \lambda_i|.$$

It follows that

$$\frac{\partial V}{\partial \lambda_i} = \frac{n\beta}{2} \lambda_i - \beta \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j},$$

and hence

$$\frac{\partial^2 V}{\partial \lambda_i^2} = \frac{n\beta}{2} + \beta \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2},$$

$$\frac{\partial^2 V}{\partial \lambda_i \partial \lambda_j} = -\beta \frac{1}{(\lambda_i - \lambda_j)^2}.$$

By the Gershgorin circle theorem, the smallest eigenvalue of Hess(V) is at least $n\beta/2$,

$$\lambda_{\min}(\text{Hess}(V)) \geq \frac{n\beta}{2}.$$

Therefore for any locally Lipschitz function F,

$$(14) \qquad \text{Var}_\nu[F] \leq \frac{2}{n\beta} \int |\nabla F|^2 d\nu.$$

Now let f be a continuous function on $\mathbb{R}$ with continuous derivative and let

$$F(\lambda_1, \ldots, \lambda_n) = \frac{1}{n} \sum_{i=1}^{n} f(\lambda_i)(= \langle L_n, f \rangle).$$

Then it follows from (14) that

$$(15) \qquad \text{Var}[\langle L_n, f \rangle] \leq \frac{2}{n^2\beta} \int \langle L_n, (f')^2 \rangle d\nu = \frac{2}{n^2\beta} \langle \bar{L}_n, (f')^2 \rangle = \frac{2}{n^2\beta} \langle \bar{\mu}_n, (f')^2 \rangle.$$

Therefore when f has continuous derivative of polynomial growth, as $n \to \infty$,

$$\langle \mu_n, (f')^2 \rangle \to \langle sc, (f')^2 \rangle.$$

Consequently, $n \text{Var}[\langle L_n, f \rangle] \to 0$, and thus the class $\mathcal{D}$ contains all functions f which have continuous derivative of polynomial growth.

For MANOVA beta ensembles, by similar argument we arrive at the following inequality

$$\text{Var}[\langle L_n, f \rangle] \leq \frac{1}{n(a(n) + b(n))} \langle \bar{\mu}_n, (f')^2 \rangle,$$

provided that the parameters $a(n)$ and $b(n)$ are positive. Thus if $a(n) + b(n)$ tends to infinity, we also have

$$n \text{Var}[\langle L_n, f \rangle] \to 0 \text{ as } n \to \infty,$$

for all functions f which have continuous derivative on $[0, 1]$.

The Wishart beta ensembles are little different because we do not have a uniform estimate for $\lambda_{\min}(\text{Hess}(V))$. In this case we are working on $\Omega = \{(\lambda_1, \ldots, \lambda_n) : 0 < \lambda_1 < \lambda_2 < \cdots <$

$\lambda_n\} \in \mathbb{R}^n$ with

$$V = const + \frac{m\beta}{2} \sum_{i=1}^{n} \lambda_i - a \sum_{i=1}^{n} \log \lambda_i - \frac{\beta}{2} \sum_{i \neq j} \log |\lambda_j - \lambda_i|.$$

Here $a = \frac{\beta}{2}(m - n + 1) - 1$. Therefore

$$\frac{\partial^2 V}{\partial \lambda_i^2} = \frac{a}{\lambda_i^2} + \beta \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2},$$

$$\frac{\partial^2 V}{\partial \lambda_i \partial \lambda_j} = -\beta \frac{1}{(\lambda_i - \lambda_j)^2}.$$

By the Gershgorin circle theorem again, we get the following bound

$$\lambda_{\min}(\text{Hess}(V)) \geq \frac{a}{\lambda_n^2}.$$

Consequently,

$$\text{Var}_\nu[F] \leq \frac{1}{a} \int \lambda_n^2 |\nabla F|^2 d\nu,$$

and hence,

$$\text{Var}[\langle L_n, f \rangle] \leq \frac{1}{a} \mathbb{E}\left[\frac{\lambda_{\max}^2}{n} \langle L_n, (f')^2 \rangle \right].$$

Here $\lambda_{\max}$ denotes the largest eigenvalue. Assume that f' is dominated by some polynomial. Then $\langle L_n, (f')^2 \rangle$ is bounded in L^2 because of (13). In addition,

$$\mathbb{E}\left[\frac{\lambda_{\max}^4}{n^2} \right] \leq \frac{1}{n} \mathbb{E}[\langle L_n, x^4 \rangle] = \frac{1}{n} \langle \bar{u}_n, x^4 \rangle \to 0 \text{ as } n \to \infty.$$

Finally, recall that $a = (\beta/2)(m - n + 1) - 1$, which behaves like $\frac{\beta}{2}(\frac{1}{\gamma} - 1)n$ in the regime that $n/m \to \gamma \in (0, 1)$. Thus

$$n \, \text{Var}[\langle L_n, f \rangle] \to 0 \text{ as } n \to \infty.$$

The proof is complete. $\square$

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