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COMPACT MINIMAL CR-SUBMANIFOLDS WITH THE LEAST NULLITY IN A COMPLEX PROJECTIVE SPACE

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1. Introduction

A minimal submanifold M of a Riemannian manifold $\tilde{M}$ is nothing but a critical point of the volume functional induced from the Riemannian measure of $\tilde{M}$. The first variational formula then states, roughly speaking, that the gradient of the volume functional is the negative of the mean curvature vector field of M. Concerning with the second variational formula for the volume functional, the nullity and the index play an important role in the theory of minimal submanifolds. Geometrically, the nullity means how many deformations of M preserving the volume there are, while the index means how many deformations of M decreasing the volume there are. For precise definition, see Section 2.

In this paper we investigate the following problem: estimate the nullity from below and further determine minimal submanifolds with the least nullity. The first results for this problem are found in [12], where Simons has proved the following (cf. Exmple 3.6): the nullity of a compact minimal submanifold M in the Euclidean sphere is bounded from below by $(\dim M+1)\operatorname{codim} M$, and furthermore it attains the lower bound if and only if M is totally geodesic. The method that Simons used for Euclidean sphere will be explained in more general form in Section 3. Next Kimura [6] and Ohnita [10] have obtained respectively results for complex submanifolds and for totally real submanifolds in a complex projective space (cf. Example 3.8). On the other hand, Ohnita [10] has computed the nullities of totally geodesic submanifolds in compact symmetric spaces of rank one. Especially he obtained that the nullity of every compact totally geodesic submanifold in a compact rank one symmetric space is equal to its Killing nullity. Recently the present author [4] has obtained a result for real hypersurfaces in a complex projective space (cf. Example 3.7, Fact 7.13).

The purpose in this paper is to investigate the problem for some wider class of submanifolds in a complex projective space, that is, the class of minimal CR-submanifolds. The following theorem summarizes the main results (Theorem 5.9, 5.10, 7.6);

Theorem. *Let M be an m-dimensional compact minimal CR-submanifold immersed in CP^n.*

(1) *Suppose m is even. Then the nullity of M is estimated as*

$$\mathrm{nul}(M) \geq 2\left(\frac{m}{2}+1\right)\left(n-\frac{m}{2}\right).$$

Moreover equality holds when and only when $M = CP^{\frac{m}{2}}$ (totally geodesic).

(2) *Suppose m is odd and equal to n. Then the nullity of M is estimated as*

$$\mathrm{nul}(M) \geq \frac{n(n+3)}{2}.$$

Moreover equality holds when and only when $M = RP^n$ (totally geodesic).

(3) *Suppose m is odd and not equal to n. Then the nullity of M is estimated as*

$$\mathrm{nul}(M) \geq m+1+2\left(\frac{m+1}{2}+1\right)\left(n-\frac{m+1}{2}\right).$$

Moreover equality holds when and only when $M = M^C_{0,\frac{m-1}{2}} \subset CP^{\frac{m+1}{2}} \underset{\text{totally geodesic}}{\subset} CP^n$.

It is surprised that the nullity on the class of compact minimal CR-submanifolds of CP^n achives its minimum at a non-totally geodesic minimal submanifold even the class contains a totally geodesic one, contrary to the results of Simons, Kimura and Ohnita.

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2. The Jacobi operator and the nullity

First of all, we explain the notation which will be used in what follows. For a Riemannian manifold M, its tangent bundle, the Levi-Civita connection and the curvature tensor are denoted respectively by TM, ∇^M and R^M. If M is a submanifold immersed isometrically in a Riemannian manifold $\tilde{M}$, we denote by NM the normal bundle of M. Then the second fundamental form B, the shape operator A^ξ in the direction ξ and the normal connection ∇^{NM} are defined by

$$\nabla^{\tilde{M}}_X Y = \nabla^M_X Y + B(X, Y), \tag{2.1}$$

$$\nabla^{\tilde{M}}_X \xi = -A^\xi(X) + \nabla^{NM}_X \xi \tag{2.2}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(NM)$. Here, in each equation, the right hand side indicates the decomposition of the left hand side into the tangential and the normal part to M, and $\Gamma(E)$ denotes the space of all C^∞-sections of a bundle E. These two equations (2.1) and (2.2) are known as the formulae of Gauss and Weingarten respectively.

Let $E \to M$ be a Riemannian vector bundle with the fiber metric $\langle \bullet, \bullet \rangle$ over a Riemannian manifold M. Suppose there is given a metric connection ∇^E on E. Then an operator Δ^E acting on $\Gamma(E)$, called the *rough Laplacian*, is definde by

$$\Delta^E := \sum_{j=1}^{m} (\nabla^E_{e_j} \nabla^E_{e_j} - \nabla^E_{\nabla^M_{e_j} e_j}),$$

where $\{e_1, \cdots, e_m\}$, $m = \dim M$, is a field of locally defined orthonormal frame on M. Then it is known that the rough Laplacian Δ^E is self-adjoint with respect to the L^2-inner product on $\Gamma(E)$, that is, the equality

$$\int_M \langle \Delta^E \xi, \eta \rangle dV_M = \int_M \langle \xi, \Delta^E \eta \rangle dV_M$$

holds for all sections ξ, η of E, where dV_M denotes the Riemannian measure of M.

Now let M be a compact minimal submanifold immersed in a Riemannian manifold $\tilde{M}$. For a tangent vector X to $\tilde{M}$, we denote by X^{TM} (resp. X^{NM}) the orthogonal projection to TM (resp. NM). Define two endmorphisms $\mathscr{R}_M$ and $\mathscr{A}_M$ of the normal bundle NM by

$$\mathscr{R}_M \xi := \sum_{j=1}^{m} (R^{\tilde{M}}(e_j, \xi) e_j)^{NM}, \qquad \mathscr{A}_M \xi := \sum_{j=1}^{m} B(A^{\xi} e_j, e_j),$$

where $\{e_1, \cdots, e_m\}$ is, as above, a field of locally defined orthonormal frame on M. Using these operators together with the rough Laplacian, we define an operator $\mathfrak{J}_M : \Gamma(NM) \to \Gamma(NM)$ by

$$\mathfrak{J}_M := -\Delta^{NM} + \mathscr{R}_M - \mathscr{A}_M,$$

which is called the *Jacobi operator* of a minimal submanifold M. The Jacobi operator $\mathfrak{J}_M$ is related to the second variation formula for the volume functional Vol as follows ([12], [8]): for a (normal) variation M_t of M with $M = M_0$, we have

$$\left. \frac{d^2 \mathrm{Vol}(M_t)}{dt^2} \right|_{t=0} = \int_M \langle \mathfrak{J}_M \xi, \xi \rangle dV_M,$$

where $\xi \in \Gamma(NM)$ is the variational vector field of M_t given by $\xi = \frac{dM_t}{dt}|_{t=0}$. It is then known that the Jacobi operator $\mathfrak{J}_M$ is a self-adjoint strongly elliptic differential operator, and so it has discrete eigenvalues bounded from below and diverging to infinity. Furthermore each eigenspace of $\mathfrak{J}_M$ is of finite dimensional. We define the *nullity* of M as the dimension of 0-eigenspace of the Jacobi operator and denote it by nul(M); nul(M):$=\dim \operatorname{Ker} \mathfrak{J}_M$. On the other hand, the *index* of M, denoted by ind(M), is defined as the sum of all the dimensions of eigenspaces with negative eigenvalues of the Jacobi operator.

Remark 2.3. If it seems necessary to emphasize the ambient manifold $\tilde{M}$, we use the notation $N(M,\tilde{M})$, $\mathfrak{J}_{M,\tilde{M}}$ and nul($M,\tilde{M}$) in stead of NM, $\mathfrak{J}_M$ and nul(M) respectively, for instance.

Suppose now we have two submanifolds $M \subset L \subset \tilde{M}$. Then we have an orthogonal splitting

$$N(M,\tilde{M}) = N(M,L) \oplus N(L,\tilde{M}) \mid M, \tag{2.4}$$

where $E|M$ denotes the restriction of a bundle E to M. According to this decomposition, a normal vector $\xi \in N(M,\tilde{M})$ is decomposed as $\xi = \xi^{N(M,L)} + \xi^{N(L,\tilde{M})|M}$.

Proposition 2.5. *Suppose, in addition to the above, that M and L are compact, M is minimal in $\tilde{M}$ and further L is totally geodesic in $\tilde{M}$. Then we have the following relation between Jacobi operators $\mathfrak{J}_{M,\tilde{M}}$ and $\mathfrak{J}_{M,L}$:*

$$\begin{aligned}\mathfrak{J}_{M,\tilde{M}}(\xi) = {} & \mathfrak{J}_{M,L}(\xi^{N(M,L)}) \\ & - \Delta^{N(L,\tilde{M})|M}(\xi^{N(L,\tilde{M})}) + \sum_{j=1}^{m} (R^{\tilde{M}}(e_j, \xi^{N(L,\tilde{M})})e_j)\end{aligned}$$

for $\xi \in \Gamma(N(M,\tilde{M}))$, where $\{e_1,\cdots,e_m\}$, $m = \dim M$, *is a field of locally defined orthonormal frame on M.*

Proof. This relation can be proved by means of routine calculations using Gauss and Weingarten formulae (2.1) and (2.2). In fact we can show

$$\Delta^{N(M,\tilde{M})}\xi = \Delta^{N(M,L)}\xi^{N(M,L)} + \Delta^{N(L,\tilde{M})|M}\xi^{N(L,\tilde{M})|M},$$

$$\mathscr{R}_{M,\tilde{M}}\xi = \mathscr{R}_{M,L}\xi^{N(M,L)} + \sum_{j=1}^{m} (R^{\tilde{M}}(e_j, \xi^{N(L,\tilde{M})})e_j)^{N(M,\tilde{M})},$$

$$\mathscr{A}_{M,\tilde{M}}\xi = \mathscr{A}_{M,L}\xi^{N(M,L)}.$$

On the other hand, since L is totally geodesic by assumption, we have $\langle R^{\tilde{M}}(X,v)Y,\eta\rangle = \langle R^{\tilde{M}}(Y,\eta)X,v\rangle = \langle R^L(Y,\eta)X,v\rangle = 0$ for $X,Y \in T_xM$, $v \in N_x(L,\tilde{M})$ and $\eta \in T_xL$, $x \in M$. Thus $R^{\tilde{M}}(X,v)Y$ belongs to $N_x(L,\tilde{M})$ for each $X,Y \in T_xM$ and $v \in N_x(L,\tilde{M})$, $x \in M$. Especially we have

$$\sum_{j=1}^{m} (R^{\tilde{M}}(e_j, \xi^{N(L,\tilde{M})})e_j)^{N(L,\tilde{M})} = \sum_{j=1}^{m} (R^{\tilde{M}}(e_j, \xi^{N(L,\tilde{M})})e_j) \in N_x(L,\tilde{M})$$

in the second equality above. Therefore those three equaliyties above imply the relation between Jacobi operators stated above. $\square$

For later use, for the triple M, L, $\tilde{M}$ satisfying the assumption in Proposotion 2.5, we define an operator $\mathscr{L}_{M,L,\tilde{M}}$ acting on $\Gamma(N(L,\tilde{M})\,|\,M)$ by

$$\mathscr{L}_{M,L,\tilde{M}}(v) := -\Delta^{N(L,\tilde{M})|M}(v) + \sum_{j=1}^{m} (R^{\tilde{M}}(e_j, v)e_j) \tag{2.6}$$

for $v \in \Gamma(N(L,\tilde{M})\,|\,M)$. It is known from the proof of Proposition 2.5 that this operator $\mathscr{L}_{M,L,\tilde{M}}$ acts actually on $\Gamma(N(L,\tilde{M})\,|\,M)$. Hence we can restate Proposition 2.5 as follows: according to the splitting (2.4), the Jacobi operator $\mathfrak{J}_{M,\tilde{M}}$ is decomposed as

$$\mathfrak{J}_{M,\tilde{M}} = \mathfrak{J}_{M,L} \oplus \mathscr{L}_{M,L,\tilde{M}}. \tag{2.7}$$

3. Simons' method for estimating the nullity

In this section, we give a method for estimating the nullity of a minimal submanifold, which has been established essentially by Simons.

Let M be a compact minimal submanifold immersed in a Riemannian manifold $\tilde{M}$. We denote by $\mathfrak{i}(\tilde{M})$ the Lie algebra consisting of all the Killing vector fields of $\tilde{M}$. We also put $\mathfrak{i}(\tilde{M})^{NM} := \{Z^{NM} \in \Gamma(NM)\,|\,Z \in \mathfrak{i}(\tilde{M})\}$. Then Simons proved

Theorem 3.1 (J. Simons [12]). $\mathfrak{i}(\tilde{M})^{NM} \subset \operatorname{Ker} \mathfrak{J}_M$.

From this theorem we call $\dim \mathfrak{i}(\tilde{M})^{NM}$ the *Killing nullity* of M and denote it by $\mathrm{nul}_K(M)$; $\mathrm{nul}_K(M) := \dim \mathfrak{i}(\tilde{M})^{NM}$. In his paper [12], Simons has estimated the Killing nullity of a compact minimal submanifold in a Euclidean sphere. His method can be generalized easily as follows: fix a point, say x in M, and define a linear map

$$\Phi_x : \mathfrak{i}(\tilde{M})^{NM} \to N_x M \oplus \mathrm{Hom}(T_x M, N_x M)$$

by

$$\Phi_x(Z^{NM}) := (Z_x^{NM}, (\nabla^{NM} Z^{NM})_x).$$

Hence the inequality $\mathrm{nul}_K(M) \geq \dim \operatorname{Im} \Phi_x$ holds. This together with Theorem 3.1 implies the inequality

$$\mathrm{nul}(M) \geq \mathrm{nul}_K(M) \geq \max_{x \in M}(\dim \operatorname{Im} \Phi_x). \tag{3.2}$$

This estimation seems to be particularly useful if the dimension of $\operatorname{Im}\Phi_x$ does not depend on x. The following proposition gives a criterion when this is the case:

Proposition 3.3. *For each pair of points x and y in M, suppose there exists an isometry f of $\tilde{M}$ such that $f(x)=y$ and $f_*(T_xM)=T_yM$. Then $\dim\operatorname{Im}\Phi_x$ does not depend on x.*

Proof. Let x, y and f be as above. Define an linear mapping F from $\operatorname{Hom}(T_xM,N_xM)$ to $\operatorname{Hom}(T_yM,N_yM)$ by $F(\omega):=f_*\circ\omega\circ f_*^{-1}$, for $\omega\in\operatorname{Hom}(T_xM, N_xM)$. Note that F is well-defined by the assumption $f_*(T_xM)=T_yM$. Define also an endomorphism ϕ of $\mathfrak{i}(\tilde{M})$ by $\phi(Z):=f_*Z$, for $Z\in\mathfrak{i}(\tilde{M})$. Let $P:\mathfrak{i}(\tilde{M})\to\mathfrak{i}(\tilde{M})^{NM}$ be the projection. Now consider the following diagram:

$$\begin{array}{ccccc}
\mathfrak{i}(\tilde{M}) & \xrightarrow{P} & \mathfrak{i}(\tilde{M})^{NM} & \xrightarrow{\Phi_x} & N_xM\oplus\operatorname{Hom}(T_xM,N_xM) \\
\phi\downarrow & & & & \downarrow (f_*|N_xM)\oplus F \\
\mathfrak{i}(\tilde{M}) & \xrightarrow[P]{} & \mathfrak{i}(\tilde{M})^{NM} & \xrightarrow[\Phi_y]{} & N_yM\oplus\operatorname{Hom}(T_yM,N_yM).
\end{array} \tag{3.4}$$

It is easy to see that both the two maps on the columns in this diagram are linear isomorphisms. Therefore if the diagram is commutative, we obtain $\dim\operatorname{Im}\Phi_x=\dim\operatorname{Im}\Phi_y$.

We now show the commutativity of the diagram. By definition,

$$(f_*\oplus F)\circ\Phi_x\circ P(Z)=(f_*(Z_x^{NM}), f_*\circ(\nabla^{NM}Z^{NM})_x\circ f_*^{-1}),$$
$$\Phi_y\circ P\circ\phi(Z)=((f_*Z)_y^{NM},(\nabla^{NM}(f_*Z)^{NM})_y).$$

It is clear that $f_*(Z_x^{NM})=(f_*Z)_y^{NM}$. So we only have to prove

$$f_*\circ(\nabla^{NM}Z^{NM})_x\circ f_*^{-1}=(\nabla^{NM}(f_*Z)^{NM})_y. \tag{3.5}$$

For each tangent vector X to M, we have

$$\begin{aligned}
\nabla_X^{NM}Z^{NM} &= (\nabla_X^{\tilde{M}}Z^{NM})^{NM} \\
&= (\nabla_X^{\tilde{M}}Z-\nabla_X^{\tilde{M}}Z^{TM})^{NM} \\
&= (\nabla_X^{\tilde{M}}Z)^{NM}-(\nabla_X^{\tilde{M}}Z^{TM})^{NM} \\
&= (\nabla_X^{\tilde{M}}Z)^{NM}-B(X,Z^{TM}).
\end{aligned}$$

Therefore, for $Y\in T_yM$, we get

$$f_*\circ(\nabla^{NM}Z^{NM})_x\circ f_*^{-1}(Y)=f_*(\nabla^{NM}_{f_*^{-1}(Y)}Z^{NM})$$

$$=f_*(\nabla^{\tilde{M}}_{f_*^{-1}(Y)}Z)^{NM}-f_*B(f_*^{-1}(Y),Z^{TM})$$
$$=(\nabla^{\tilde{M}}_Y(f_*Z))^{NM}-B(Y,f_*Z^{TM}),$$

because f is an isometry of $\tilde{M}$. The similar computation shows

$$(\nabla^{NM}(f_*Z)^{NM})_y Y=(\nabla^{\tilde{M}}_Y(f_*Z))^{NM}-B(Y,f_*Z^{TM}).$$

Thus we have the equation (3.5). □

Here are some examples:

EXAMPLE 3.6 (J. Simons [12]). The case $\tilde{M}=S^n$. In this case, Φ_x is surjective. Therefore, (3.2) implies

$$\mathrm{nul}(M)\geq\dim(N_xM\oplus\mathrm{Hom}(T_xM,N_xM))=(m+1)(n-m),$$

where $\dim M=m$.

EXAMPLE 3.7 (T. Gotoh [4]). The case $\tilde{M}=\boldsymbol{C}P^n$ and $\dim M=2n-1$. Also in this case, Φ_x is surjective. Therefore, as same as above,

$$\mathrm{nul}(M)\geq 2n=\dim M+1.$$

EXAMPLE 3.8 (Y. Kimura [6]). The case $\tilde{M}=\boldsymbol{C}P^n$ and M is a complex submanifold. In this case, Φ_x is not surjective, but $\mathrm{Im}\,\Phi_x=N_xM\oplus\mathrm{Hom}_{\boldsymbol{C}}(T_xM,N_xM)$. Therefore, (3.2) implies

$$\mathrm{nul}(M)\geq\dim_{\boldsymbol{R}}(N_xM\oplus\mathrm{Hom}_{\boldsymbol{C}}(T_xM,N_xM))=2(m+1)(n-m),$$

where $\dim_{\boldsymbol{C}} M=m$.

REMARK 3.9. Each estimation of the nullity obtained in these examples above is optimal. In fact, the compact minimal submanifold whose nullity realizes the equality in the inequality above is completely determined as follows:

In Example 3.4, $M=S^m$ (totally geodesic) only,

In Example 3.5, $M=M^{\boldsymbol{C}}_{0,n-1}$ (geodesic hypersphere) only,

In Example 3.6, $M=\boldsymbol{C}P^m$ (totally geodesic) only.

For $M^{\boldsymbol{C}}_{0,n-1}$, see Section 7 below.

4. The case of symmetric spaces

In the present section, we apply the method for estimating the nullity, explained in Section 3, to the case where the ambient manifold $\tilde{M}$ is a symmetric space.

Let $\tilde{M}$ be a symmetric space, which is represented as $\tilde{M}=G/H$ as a homogeneous space. Here G is a Lie group and H its closed subgroup. As usual, their Lie algebras are denoted by the corresponding Germann letters $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $\mathfrak{g}=\mathfrak{h}+\tilde{\mathfrak{m}}$ be the canonical decomposition and hence $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}$, $[\mathfrak{h},\tilde{\mathfrak{m}}]\subset\tilde{\mathfrak{m}}$ and $[\tilde{\mathfrak{m}},\tilde{\mathfrak{m}}]\subset\mathfrak{h}$. In what follows we identify $\tilde{\mathfrak{m}}$ naturally with the tangent space $T_o\tilde{M}$, where $o=H$ is the origine of $\tilde{M}$. For $Z\in\mathfrak{g}$, we denote by Z^* the Killing vector field on $\tilde{M}$ generated by Z; $Z^*_x:=\frac{d(\exp tZ)x}{dt}|_{t=0}$, $x\in\tilde{M}$. Then the covariant derivatives are computed at the origin o as

$$Z\in\tilde{\mathfrak{m}}\Rightarrow(\nabla^{\tilde{M}}Z^*)_o=0, \tag{4.1}$$

$$Z\in\mathfrak{h}\Rightarrow(\nabla^{\tilde{M}}Z^*)_o=\mathrm{ad}_{\mathfrak{g}}(Z). \tag{4.2}$$

We now suppose that there is given a compact submanifold M immersed minimally in a symmetric space $\tilde{M}$. We may assume that M contains the origin o, and then identify T_oM with $\mathfrak{m}$ by regarding it as a subspace of $\tilde{\mathfrak{m}}$. The orthogonal complement of $\mathfrak{m}$ in $\tilde{\mathfrak{m}}$ is denoted by $\mathfrak{m}^\perp$. Then each $Z\in\mathfrak{g}$ can be decomposed as $Z=Z_{\mathfrak{h}}+Z_{\mathfrak{m}}+Z^\perp$ with respect to the decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}+\mathfrak{m}^\perp$. A projiection $\Pi:\mathfrak{g}\to\mathfrak{i}(\tilde{M})^{NM}$ is defined by $\Pi(Z):=Z^{*NM}$. Defining two maps $\Psi_1:\mathfrak{g}\to\mathfrak{m}^\perp$ and $\Psi_2:\mathfrak{g}\to\mathrm{Hom}(\mathfrak{m},\mathfrak{m}^\perp)$ by

$$\Psi_1(Z):=Z^\perp,\quad \Psi_2(Z)(X):=(\mathrm{ad}_{\mathfrak{g}}(Z_{\mathfrak{h}})X)^\perp-B(X,Z_m),$$

then we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Psi:=\Psi_1\oplus\Psi_2} & \mathfrak{m}^\perp\oplus\mathrm{Hom}(\mathfrak{m},\mathfrak{m}^\perp) \\ {\scriptstyle\Pi}\downarrow & & \downarrow\cong \\ \mathfrak{i}(\tilde{M})^{NM} & \xrightarrow[\Phi_o]{} & N_oM\oplus\mathrm{Hom}(T_oM,N_oM). \end{array} \tag{4.3}$$

We now states the theorem (cf. Proposition 1.1 in [10]):

Theorem 4.4. *Let M be a compact minimal submanifold immersed in a symmetric space $\tilde{M}=G/H$. Then, using the notation above, the nullity of M satisfies the inequality*

$$\mathrm{nul}(M)\geq\mathrm{nul}_K(M)\geq\mathrm{codim}\,M+\dim\mathrm{Im}\,\Psi_2\,|\,\mathfrak{h}.$$

Proof. Because of (4.1), (4.2) and by definition of Ψ_1 and Ψ_2, we obtain

$$\Psi_1(\mathfrak{h})=0,\quad \Psi_2(\mathfrak{m})=0,\quad \Psi_1(\tilde{\mathfrak{m}})=\mathfrak{m}^\perp,$$

$$\Psi_2(Z)(X)=-B(X,Z)\quad\text{for}\quad X\in\mathfrak{m}\quad\text{and}\quad Z\in\tilde{\mathfrak{m}},$$

$$\Psi_2(Z)(X)=(\mathrm{ad}_{\mathfrak{g}}(Z)(X))^{\perp} \quad \text{for} \quad X\in\mathfrak{m} \quad \text{for} \quad Z\in\mathfrak{h},$$

which imply

$$\mathrm{Im}\,\Psi=\mathfrak{m}^{\perp}\oplus\mathrm{Im}\,\Psi_2\,|\,(\mathfrak{h}+\mathfrak{m})\supset\mathfrak{m}^{\perp}\oplus\mathrm{Im}\,\Psi_2\,|\,\mathfrak{h}. \tag{4.5}$$

On the other hand, because the projection Π is surjective, we obtain $\dim\mathrm{Im}\,\Phi_o=\dim\mathrm{Im}\,\Psi$ from the diagram (4.3). Therefore, by virture of (3.2) and (4.5), we can conclude

$$\begin{aligned}\mathrm{nul}(M)\geq\mathrm{nul}_K(M)\geq\dim\mathrm{Im}\,\Phi_o&=\dim\mathrm{Im}\,\Psi\\ &\geq\mathrm{codim}\,M+\dim\mathrm{Im}\,\Psi_2\,|\,\mathfrak{h}.\end{aligned}$$

□

5. Minimal CR-submanifolds in CP^n

We begin this section with recalling the definition of CR-submaifolds. For details, we refer [5], [2], for instance. Let W be a complex vector space with a complex structure J and with an inner product. For a real subspace V of W, we set $V_H:=V\cap JV$ and $V_R:=$the orthogonal complement of V_H in V. On the contrary, we denote by $V^{\perp}$ the orthogonal complement of V in W. We call V a *CR-subspace* of W if $JV_R\subset V^{\perp}$ holds.

DEFINITION 5.1. Let $(\tilde{M},J)$ be an Hermitian manifold with a complex structure J. Then a real submanifold M of $\tilde{M}$ is called a CR-submanifold provided that the following three conditions are satisfied:

(1) $\dim_{\boldsymbol{R}}(T_xM)_H$ does not depend on each $x\in M$.

(2) The distribution $M\ni x\mapsto(T_xM)_H$ is smooth on M.

(3) Each tangent space T_xM is a CR-subspace of $T_x\tilde{M}$, namely $J(T_xM)_R\subset N_xM$.

We call a CR-submanifold M is *of type* (h,r) provided that $\dim_{\boldsymbol{C}}(T_xM)_H=h$ and $\dim_{\boldsymbol{R}}(T_xM)_R=r$. The most typical examples of CR-submanifolds are complex submanifolds (of type $(\dim_{\boldsymbol{C}}M,0)$) and totally-real submanifolds (of type $(0,\dim M)$). Any (real) codimension one submanifold, namely any real hypersurface is also CR-submanifold (of type $(\frac{\dim M-1}{2},1)$). From a linear algebraic point of view, CR-subspaces of fixed type are determined uniquely in a sence below; setting $(z,w)_n=\Sigma z_j\bar{w}_j$ for $z=(z_1,\cdots,z_n)$, $w=(w_1,\cdots,w_n)\in\boldsymbol{C}^n$, we give the canonical Euclidean metric $\langle\cdot,\cdot\rangle_n:=\mathrm{Re}(\cdot,\cdot)_n$ to $\boldsymbol{C}^n$:

Lemma 5.2. *Let V be a CR-subspace of type (h,r) in the complex Euclidean space $\boldsymbol{C}^n$. Then there exists an unitary matrix $u\in U(n)$ such that*

$$u(V_H)=\boldsymbol{C}e_1\oplus\cdots\oplus\boldsymbol{C}e_h,\qquad u(V_R)=\boldsymbol{R}e_{h+1}\oplus\cdots\oplus\boldsymbol{R}e_{h+r},$$

where $e_j = (\underbrace{0,\cdots,0,\overset{j}{1},0,\cdots,0}_{n \text{ times}})$, $j=1,\cdots,n$.

From now on we consider the case where the ambient manifold is the complex projective space CP^n with the Fubini-Study metric of constant holomorphic sectional curvature 4, and M is a compact minimal CR-submanifold of CP^n. If we adopt the expression $CP^n = U(n+1)/U(1)\times U(n) = G/H$ as an symmetric space, then the canonical decomposition $\mathfrak{g} = \mathfrak{h} + \tilde{\mathfrak{m}}$ is given by $\tilde{\mathfrak{m}} = \{(\begin{smallmatrix} 0 & -{}^t\bar{\zeta} \\ \zeta & 0 \end{smallmatrix}) \in \mathfrak{u}(n+1) \mid \zeta \in C^n\}$. We identify $\tilde{\mathfrak{m}}$ with C^n naturally. Then the linear isotropy representation $U(1)\times U(n) \to GL(n,C)$ is given by $(\begin{smallmatrix} \lambda & 0 \\ 0 & A \end{smallmatrix}) \mapsto \bar{\lambda}A$. By virture of this fact combined with the use Lemma 5.2, if M is of type (h,r), we may always assume without loss of generality that M containes the origin o of CP^n and furthermore the tangent space to M at the origin is of the form

$$T_oM = \mathfrak{m} = Ce_1 \oplus \cdots \oplus Ce_h \oplus Re_{h+1} \oplus \cdots \oplus Re_{h+r}. \tag{5.3}$$

Consequently Proposition 3.3 implies

Proposition 5.4. *Suppose M is a compact minimal CR-submanifold in CP^n. Then the dimension of* $\operatorname{Im}\Phi_x$ *does not depend on $x \in M$.*

We now give an estimation of the nullity:

Theorem 5.5. *Let M be an m-dimensional compact minimal CR-submanifold of type (h,r) in CP^n. Then its nullity satisfies the following inequality*:

$$\operatorname{nul}(M) \geq \operatorname{nul}_K(M) \geq -4h^2 + (4m-2n-1)h + \frac{4n+4mn-3m^2-m}{2}.$$

Proof. In this case, we have $\mathfrak{h} = R\sqrt{-1} \oplus \mathfrak{u}(n)$. It is easy to see $\operatorname{Im}\Psi_2 \mid \mathfrak{h} = \operatorname{Im}\Psi_2 \mid \mathfrak{u}(n)$, and so we compute the dimension of $\operatorname{Im}\Psi_2 \mid \mathfrak{u}(n)$. Note that $\Psi_2(A)\zeta = (A\zeta)^\perp$ for $A \in \mathfrak{u}(n)$ and $\zeta \in \mathfrak{m}$ under the identification $\tilde{\mathfrak{m}} = C^n$. Therefore, keeping the assumption (5.3) in mind, we find that

$$\operatorname{Ker}\Psi_2 \mid \mathfrak{u}(n) = \left\{ \begin{pmatrix} S & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & T \end{pmatrix} \,\middle|\, S \in \mathfrak{u}(h),\ T \in \mathfrak{u}(n-h-r),\ W \in \mathfrak{o}(r) \right\}. \tag{5.6}$$

Thus, we obtain

$$\begin{aligned} \dim \operatorname{Im}\Psi_2 \mid \mathfrak{u}(n) &= \dim \mathfrak{u}(n) - \dim \operatorname{Ker}\Psi_2 \mid \mathfrak{u}(n) \\ &= \dim \mathfrak{u}(n) - (\dim \mathfrak{u}(h) + \dim \mathfrak{o}(r) + \dim \mathfrak{u}(n-h-r)) \end{aligned}$$

$$= -4h^2 + (4m - 2n - 1)h - \frac{3m^2 - 4mn - m}{2}.$$

The inequality in the assertion now comes from Theorem 4.4. □

In order to estimate the nullity of M only by its dimension, we consider the quadratic function

$$L_{n,m}(h) := -4h^2 + (4m - 2n - 1)h + \frac{4n + 4mn - 3m^2 - m}{2}.$$

Note that h runs over integers with $\max\{0, m-n\} \leq h \leq [\frac{m}{2}]$.

Lemma 5.7. *The quadratic function $L_{n,m}$ attains its minimum as follows:*

(1) *If m is even, $L_{n,m}$ attains its minimum only when $h = \left[\frac{m}{2}\right] = \frac{m}{2}$ and the value is $L_{n,m}\left(\frac{m}{2}\right) = 2\left(n - \frac{m}{2}\right)\left(\frac{m}{2} + 1\right)$.*

(2) *If m is odd and $m = n$, $L_{n,n}$ attains its minimum only when $h = 0$ and the value is $L_{n,n}(0) = \frac{n(n+3)}{2}$.*

(3) *If m is odd and $m \neq n$, $L_{n,m}$ attains its minimum only when $h = \left[\frac{m}{2}\right] = \frac{m-1}{2}$ and the value is $L_{n,m}\left(\frac{m-1}{2}\right) = m + 1 + 2\left(n - \frac{m+1}{2}\right)\left(\frac{m+1}{2} + 1\right)$.*

Summarizing Theorem 5.5 and Lemma 5.7, we obtain

Theorem 5.8. *Let M be an m-dimensional compact minimal CR-submanifold immersed in CP^n. Then its nullity can be estimated as follows:*

$$\mathrm{nul}(M) \geq \begin{cases} 2\left(n - \dfrac{m}{2}\right)\left(\dfrac{m}{2} + 1\right), & m \text{ is even}, \\[2ex] \dfrac{n(n+3)}{2}, & m \text{ is odd and } m = n, \\[2ex] m + 1 + 2\left(n - \dfrac{m+1}{2}\right)\left(\dfrac{m+1}{2} + 1\right), & m \text{ is odd and } m \neq n. \end{cases}$$

It is known ([6], [9], [10]) that a complex projective space $CP^{\frac{m}{2}}$ and a real projective space $\boldsymbol{R}P^n$, both imbedded totally geodesically in $\boldsymbol{C}P^n$, realize the equalities of the first two cases in Theorem 5.8 respectively. Speaking of conclusion, we find that the estimation obtained in the last case in Theorem 5.8 is also optimal; this case will be investigated in Section 7 below.

We conclude this section by determining M whose nullity satisfies the equality in each first two cases in Theorem 5.8:

Theorem 5.9. *Let M be an m-dimensional compact minimal CR-submanifold immersed in $\boldsymbol{C}P^n$. Suppose the dimension of M is even. Then an inequality*

$$\mathrm{nul}(M) \geq 2\left(n-\frac{m}{2}\right)\left(\frac{m}{2}+1\right)$$

holds. Moreover equality holds when and only when $M=\boldsymbol{C}P^{\frac{m}{2}}$ (totally geodesic).

Theorem 5.10. *Let M be a compact minimal CR-submanifold immersed in $\boldsymbol{C}P^n$. Suppose the dimension of M is odd and equal to n. Then an inequality*

$$\mathrm{nul}(M) \geq \frac{n(n+3)}{2}$$

holds. Moreover equality holds when and only when $M=\boldsymbol{R}P^n$ (totally geodesic).

Because similar argument works in proving both these Theorems, we only give here a proof of Theorem 5.9:

Proof of Theorem 5.9. The inequality have been obtained in Theorem 5.8, so we consider M with $\mathrm{nul}(M)=2(n-\frac{m}{2})(\frac{m}{2}+1)$. Then according to Lemma 5.7(1), M must be of type $(\frac{m}{2},0)$, namely M is a complex submanifold. Our assumption (5.3) then becomes $T_oM=\mathfrak{m}=\boldsymbol{C}e_1\oplus\cdots\oplus\boldsymbol{C}e_h$, where we put $h=\frac{m}{2}$. On the other hand, (5.6) becomes

$$\begin{aligned}\mathrm{Ker}\,\Psi_2\,|\,\mathfrak{u}(n) &= \left\{\begin{pmatrix} S & 0 \\ 0 & T\end{pmatrix}\middle|\, S\in\mathfrak{u}(h),\ T\in\mathfrak{u}(n-h)\right\} \\ &\cong \mathfrak{u}(h)\oplus\mathfrak{u}(n-h),\end{aligned}$$

and hence the Lie subgroup of $U(n+1)$ generated by $\mathrm{Ker}\,\Psi_2\,|\,\mathfrak{u}(n)$ is

$$U_{h,n-h}=\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B\end{pmatrix}\middle|\, A\in U(h),\ B\in U(n-h)\right\}.$$

The point of proof is

Claim 5.11. *If* $\mathrm{nul}(M)=2\left(n-\frac{m}{2}\right)\left(\frac{m}{2}+1\right)$, *the action of* $U_{h,n-h}$ *on* CP^n *leaves* M *invariant.*

In fact, $\Psi(Z)=0$ for $Z\in\operatorname{Ker}\Psi_2\,|\,\mathfrak{u}(n)$, which implies $\Phi_o(Z^{*NM})=0$. However Φ_o must be injective by assumption on the nullity. As a result $Z^{*NM}=0$ for all $Z\in\operatorname{Ker}\Psi_2\,|\,\mathfrak{u}(n)$, namely each Z^* is tangent to M everywhere. This means $U_{h,n-h}(M)\subset M$.

Let us now complete the proof of Theorem 5.9. The linear isotropy action of $U_{h,n-h}$ on $\mathfrak{m}$ is transitive on its unit sphere because the action is essentially the same to the natural action of $U(h)$ on C^h. On the other hand, Claim 5.11 shows that the group $U_{h,n-h}$ acts isometrically not only on CP^n but also on M. Therefore $B(X,X)=B(Y,Y)$ for all $X,Y\in\mathfrak{m}$ with $\|X\|=\|Y\|=1$, together with which minimality of M implies $B=0$. □

6. The nullity of a non-full minimal submanifold in CP^n

Let M be an m-dimensional compact minimal submanifold immersed in a complex projective space CP^l. Then imbedding CP^l totally geodesically into CP^n, we shall investigate a relation between $\mathrm{nul}(M,CP^l)$ and $\mathrm{nul}(M,CP^n)$.

If $\mathscr{M}\to\mathscr{B}$ is a Riemannian submersion and X is a vector field on $\mathscr{B}$, we denote by $\hat{X}$ its horizontal lift to $\mathscr{M}$.

Consider now the following system consisting of minimal submanifolds and Hopf fibrations:

$$\begin{array}{ccccc} \hat{M} & \rightarrow & S^{2l+1} & \rightarrow & S^{2n+1} \\ {\scriptstyle S^1}\downarrow & & \downarrow{\scriptstyle S^1} & & \downarrow{\scriptstyle S^1} \\ M & \rightarrow & CP^l & \rightarrow & CP^n. \end{array} \tag{6.1}$$

The action of the circle group $S^1=\{e^{\sqrt{-1}\theta}\in C\,|\,\theta\in R\}$ is given by $\tau_\theta(z)=ze^{\sqrt{-1}\theta}$ for $z\in S^{2n+1}$. Set

$$C^\infty(\hat{M})_{S^1}:=\{f\in C^\infty(\hat{M})\otimes C\,|\,f(ze^{\sqrt{-1}\theta})=f(z),\ \forall z\in\hat{M},\ \forall\theta\in R\},$$

$$C^\infty(\hat{M})^{S^1}:=\{f\in C^\infty(\hat{M})\otimes C\,|\,f(ze^{\sqrt{-1}\theta})=f(z)e^{\sqrt{-1}\theta},\ \forall z\in\hat{M},\ \forall\theta\in R\},$$

and further

$$E(\lambda;\Delta_{\hat{M}})_{S^1}:=E(\lambda;\Delta_{\hat{M}})\otimes C\cap C^\infty(\hat{M})_{S^1},$$

$$E(\lambda;\Delta_{\hat{M}})^{S^1}:=E(\lambda;\Delta_{\hat{M}})\otimes C\cap C^\infty(\hat{M})^{S^1},$$

Here and in what follows, $E(\lambda;\Delta_{\hat{M}})$ indicates the λ-eigenspace of the operator $\Delta_{\hat{M}}$.

An aim in the present section is to prove

Theorem 6.2. *Let M be an m-dimensional compact minimal submanifold in CP^l. Then, imbedding CP^l totally geodesically into CP^n, we get the following equality:*

$$\mathrm{nul}(M,CP^n)=\mathrm{nul}(M,CP^l)+2(n-l)\dim_C E(m+1;\Delta_{\hat{M}})^{S^1}.$$

Proof. First of all we note the curvature tensor R^{CP^n} is given by $R^{CP^n}(X,Y)Z = \langle Y,Z\rangle X-\langle X,Z\rangle Y+\langle JY,Z\rangle JX-\langle JX,Z\rangle JY+2\langle X,JY\rangle JZ$. Hence the operator $\mathscr{L}=\mathscr{L}_{M,CP^l,CP^n}$ defined in (2.6) is given as

$$\mathscr{L}=-\Delta^{N(CP^l,CP^n)|M}-mI_{N(CP^l,CP^n)|M}, \tag{6.3}$$

where $I_{N(CP^l,CP^n)|M}$ is the identity endomorphism of the bundle $N(CP^l,CP^n)\,|\,M$. The decomposition (2.7) then implies $\mathrm{Ker}\mathfrak{J}_{M,CP^n}=\mathrm{Ker}\mathfrak{J}_{M,CP^l}\oplus\mathrm{Ker}\mathscr{L}$. Especially we have

$$\mathrm{nul}(M,CP^n)=\mathrm{nul}(M,CP^l)+\dim\mathrm{Ker}\mathscr{L}. \tag{6.4}$$

We shall henceforth investigate $\mathrm{Ker}\mathscr{L}$ in what follows.

Suppose $z\in\hat{M}$ is a point of the fiber over $x\in M$. Then there are natural isomorphisms $N_z(\hat{M},S^{2n+1})\cong N_x(M,CP^n)$, $N_z(\hat{M},S^{2l+1})\cong N_x(M,CP^l)$ and $N_z(S^{2l+1},S^{2n+1})\cong N_x(CP^l,CP^n)$, since each normal space of the total space in the fibrations (6.1) is horizontal with respect to the Riemannian submarsion. Thus if we set

$$\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1}:=\{\xi\in\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M}\otimes C)\,|\,\tau_{\theta*}\xi=\xi,\ \forall\theta\in\boldsymbol{R}\},$$

this space $\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1}$ is isomorphic to $\Gamma(N(CP^l,CP^n)\,|\,M)$. In order to investigate $\mathrm{Ker}\mathscr{L}$, we consider an operator $\hat{\mathscr{L}}$ defined by the commutative diagram

$$\begin{array}{ccc}
\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1} & \xrightarrow{\hat{\mathscr{L}}} & \Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1} \\
\cong\downarrow & & \downarrow\cong \\
\Gamma(N(CP^l,CP^n)\,|\,M\otimes C) & \xrightarrow[\mathscr{L}\otimes C]{} & \Gamma(N(CP^l,CP^n)\,|\,M\otimes C).
\end{array}$$

The first step is to describe the operator $\hat{\mathscr{L}}$ as follows:

Lemma 6.5.

$$\hat{\mathscr{L}}=-\Delta^{N(S^{2l+1},S^{2n+1})|\hat{M}}-(m+1)I_{N(S^{2l+1},S^{2n+1})|\hat{M}}.$$

Proof. Let $\{e_1,\cdots,e_m\}$ be a field of locally defined orthonormal frame on

M. Define a vertical unit vector field V on $\hat{M}$ by $V_z=\sqrt{-1}z$, $z\in\hat{M}$. Then we have $\{\hat{e}_1,\cdots,\hat{e}_m,V\}$ as a field of locally defined orthonormal frame on $\hat{M}$. By definition of the rough Laplacian,

$$\text{(6.6)}\quad \Delta^{N(S^{2l+1},S^{2n+1})|\hat{M}}=\sum_{j=1}^{m}(\nabla_{\hat{e}_j}^{N(S^{2l+1},S^{2n+1})|\hat{M}}\nabla_{\hat{e}_j}^{N(S^{2l+1},S^{2n+1})|\hat{M}}-\nabla_{\nabla^{\hat{M}}_{\hat{e}_j}\hat{e}_j}^{N(S^{2l+1},S^{2n+1})|\hat{M}})$$
$$+\nabla_V^{N(S^{2l+1},S^{2n+1})|\hat{M}}\nabla_V^{N(S^{2l+1},S^{2n+1})|\hat{M}}-\nabla_{\nabla^{\hat{M}}_V V}^{N(S^{2l+1},S^{2n+1})|\hat{M}}.$$

Note that $\eta\in\Gamma(N(S^{2l+1},S^{2n+1})|\hat{M})$ belongs to $\Gamma(N(S^{2l+1},S^{2n+1})|\hat{M})_{S^1}$ if and only if $\eta(\tau_\theta z)=\eta(z)e^{\sqrt{-1}\theta}$ for all $z\in\hat{M}$ and $\theta\in\boldsymbol{R}$. This fact implies

$$\text{(6.7)}\quad \nabla_V^{N(S^{2l+1},S^{2n+1})|\hat{M}}\nabla_V^{N(S^{2l+1},S^{2n+1})|\hat{M}}\eta=-\eta$$

for $\eta\in\Gamma(N(S^{2l+1},S^{2n+1})|\hat{M})$. We also note V is generated by the geodesic flow on $\tilde{M}$:

$$\text{(6.8)}\quad \nabla_V^{\hat{M}}V=0.$$

Now it is shown that $\nabla_{\hat{e}_j}^{N(S^{2l+1},S^{2n+1})|\hat{M}}\xi=(\nabla_{e_j}^{N(CP^l,CP^n)}\xi)^\wedge$ for $\xi\in\Gamma(N(CP^l,CP^n)|M)$, and hence

$$\text{(6.9)}\quad \nabla_{\hat{e}_j}^{N(S^{2l+1},S^{2n+1})|\hat{M}}\nabla_{\hat{e}_j}^{N(S^{2l+1},S^{2n+1})|\hat{M}}\xi=(\nabla_{e_j}^{N(CP^l,CP^n)}\nabla_{e_j}^{N(CP^l,CP^n)}\xi)^\wedge.$$

On the other hand, $\langle\nabla^{\hat{M}}_{\hat{e}_j}\hat{e}_j,V\rangle=-\langle\hat{e}_j,\nabla^{\hat{M}}_{\hat{e}_j}V\rangle=-\langle\hat{e}_j,\nabla^{S^{2l+1}}_{\hat{e}_j}V\rangle=-\langle\hat{e}_j,\sqrt{-1}\hat{e}_j\rangle=0$, which means each $\nabla^{\hat{M}}_{\hat{e}_j}\hat{e}_j$ is horizontal. This shows

$$\text{(6.10)}\quad (\nabla_{\nabla^M_{e_j}e_j}^{N(CP^l,CP^n)|M}\xi)^\wedge=\nabla_{\widehat{\nabla^M_{e_j}e_j}}^{N(S^{2l+1},S^{2n+1})|\hat{M}}\xi=\nabla_{\nabla^{\hat{M}}_{\hat{e}_j}\hat{e}_j}^{N(S^{2l+1},S^{2n+1})|\hat{M}}\xi.$$

Substituting (6.7), (6.8), (6.9), (6.10) into (6.6), and by definition of the rough Laplacian, we get

$$\text{(6.11)}\quad \Delta^{N(S^{2l+1},S^{2n+1})|\hat{M}}\xi=(\Delta^{N(CP^l,CP^n)|M}\xi)^\wedge-\xi.$$

By virture of (6.3) and (6.11), we obtain

$$\widehat{\mathscr{L}\xi}=-\Delta^{N(S^{2l+1},S^{2n+1})|\hat{M}}\xi-(m+1)\xi,$$

for $\xi\in\Gamma(N(CP^l,CP^n)|M)$. This completes the proof of Lemma 6.5. □

The second step is to describe $\Gamma(N(S^{2l+1},S^{2n+1})|\hat{M})_{S^1}$. For the canonical complex basis $\{e_1,\cdots,e_n\}$ of C^n, we set $v_\alpha=e_{l+1+\alpha}$, $v_{\bar{\alpha}}=\sqrt{-1}e_{l+1+\alpha}$, $\alpha=1,\cdots,n-l$. These v_α's and $v_{\bar{\alpha}}$'s are regarded naturally as elements in $\Gamma(N(S^{2l+1},S^{2n+1})|\hat{M})$. Then the followings are easily verified:

(6.12) v_α and $v_{\bar{\alpha}}$ are horizontal with respect to the Riemannian submarsion

(6.1),

(6.13) v_α and $v_{\bar{\alpha}}$ are parallel with respect to the normal connection $\nabla^{N(S^{2l+1},S^{2n+1})|\hat{M}}$,

(6.14) $\tau_{\theta *}v_\alpha(z)=v_\alpha(z)e^{\sqrt{-1}\theta}$, $\tau_{\theta *}v_{\bar{\alpha}}(z)=v_{\bar{\alpha}}(z)e^{\sqrt{-1}\theta}$ for all $z\in\hat{M}$, $\theta\in\boldsymbol{R}$ and $\alpha=1,\cdots,n-l$,

(6.15) $v_\alpha(\tau_\theta(z))=v_\alpha(z)$, $v_{\bar{\alpha}}(\tau_\theta(z))=v_{\bar{\alpha}}(z)$ for all $z\in\hat{M}$, $\theta\in\boldsymbol{R}$ and $\alpha=1,\cdots,n-l$.

The last (6.15) is due exactly to the definition. Note that (6.14) and (6.15) show that v_α's and $v_{\bar{\alpha}}$'s do not belong to $\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1}$. However we also find from those (6.12), (6.14) and (6.15) that $\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1}$ can be described as follows:

$$\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1}=\left\{\xi=\sum_{\alpha=1}^{n-l}f^\alpha v_\alpha+\sum_{\bar{\alpha}=1}^{n-l}f^{\bar{\alpha}}v_{\bar{\alpha}}\,\middle|\,f^\alpha,\,f^{\bar{\alpha}}\in C^\infty(\hat{M})^{S^1}\right\}. \tag{6.16}$$

Let us now complete the proof of Theorem 6.2. Let $\xi=\Sigma f^\alpha v_\alpha+\Sigma f^{\bar{\alpha}}v_{\bar{\alpha}}$ be an element of $\Gamma(N(S^{2l+1},S^{2n+1})\,|\,\hat{M})_{S^1}$ expressed as in (6.16). Then by means of (6.13), we have the following equality:

$$-\Delta^{N(S^{2l+1},S^{2n+1})|\hat{M}}\xi=\sum_{\alpha=1}^{n-l}(\Delta_{\hat{M}}f^\alpha)v_\alpha+\sum_{\bar{\alpha}=1}^{n-l}(\Delta_{\hat{M}}f^{\bar{\alpha}})v_{\bar{\alpha}}. \tag{6.17}$$

Combining Lemma 6.5 and (6.17), we get

$$\hat{\mathscr{L}}\xi=\sum_{\alpha=1}^{n-l}\{\Delta_{\hat{M}}f^\alpha-(m+1)f^\alpha\}v_\alpha+\sum_{\bar{\alpha}=1}^{n-l}\{\Delta_{\hat{M}}f^{\bar{\alpha}}-(m+1)f^{\bar{\alpha}}\}v_{\bar{\alpha}}.$$

Consequently the kernel of $\hat{\mathscr{L}}$ is given by

$$\begin{aligned}\mathrm{Ker}\hat{\mathscr{L}}&=\left\{\xi=\sum_{\alpha=1}^{n-l}f^\alpha v_\alpha+\sum_{\bar{\alpha}=1}^{n-l}f^{\bar{\alpha}}v_{\bar{\alpha}}\,\middle|\,f^\alpha,f^{\bar{\alpha}}\in E(m+1\,;\Delta_{\hat{M}})^{S^1}\right\}\\&\cong_{\boldsymbol{C}}(E(m+1\,;\Delta_{\hat{M}})^{S^1})^{2(n-l)},\end{aligned}$$

and in particular we find

$$\dim_{\boldsymbol{C}}\mathrm{Ker}\hat{\mathscr{L}}=2(n-l)\dim_{\boldsymbol{C}}E(m+1\,;\Delta_{\hat{M}})^{S^1}.$$

Because $\dim\mathrm{Ker}\mathscr{L}=\dim_{\boldsymbol{C}}\mathrm{Ker}\hat{\mathscr{L}}$ by definition, (6.4) deduces

$$\mathrm{nul}(M,\boldsymbol{C}P^n)=\mathrm{nul}(M,\boldsymbol{C}P^l)+2(n-l)\dim_{\boldsymbol{C}}E(m+1\,;\Delta_{\hat{M}})^{S^1},$$

which is the conclusion. □

Corollary 6.18. *In addition to the above, we suppose the immersion* $M\to\boldsymbol{C}P^l$

is full. Then the following inequality holds:

$$\mathrm{nul}(M, CP^n) \geq \mathrm{nul}(M, CP^l) + 2(l+1)(n-l).$$

Note that $2(l+1)(n-l)$ on the right is just equal to $\mathrm{nul}(CP^l, CP^n)$.

Proof. On account of Theorem 6.2, we have only to show the inequality

$$\dim_C E(m+1; \Delta_{\hat{M}})^{S^1} \geq l+1.$$

We denote the canonical complex coordinate on C^{l+1} by $z = (z_1, \cdots, z_{l+1})$. In our setting, $\hat{M}$ is also minimal in S^{2l+1} ([7]). Then by a well-known theorem of Takahashi ([13]), we know that z_j's and $\bar{z}_j$'s, when restricted on $\hat{M}$, are eigenfunctions with common eigenvalue $m+1 (= \dim \hat{M})$ of $\Delta_{\hat{M}}$. The z_j's are elements in $C^\infty(\hat{M})^{S^1}$, while the $\bar{z}_j$'s neither in $C^\infty(\hat{M})^{S^1}$ nor in $C^\infty(\hat{M})_{S^1}$. Hence the z_j's belong to $E(m+1; \Delta_{\hat{M}})^{S^1}$. Moreover because $M \to CP^l$ is full by assumption, $z_1, \cdots, z_{l+1}$ are linearly independent over C as functions on $\hat{M}$. Thus we conclude $\dim_C E(m+1; \Delta_{\hat{M}})^{S^1} \geq l+1$. □

REMARK 6.19. According to Kimura ([6]), if M is a Kähler C-space fully imbedded in CP^l, then the equality

$$\mathrm{nul}(M, CP^n) = \mathrm{nul}(M, CP^l) + 2(l+1)(n-l)$$

holds. The author does not know an example of M whose nullity does not satisfy the equality in Corollary 6.18.

7. The final case

In this section we shall investigate the final case of Theorem 5.8. We begin with a construction of an example of a compact minimal CR-submanifold M in CP^n whose nullity equals to $m+1+2(n-\frac{m+1}{2})(\frac{m+1}{2}+1)$, where $m = \dim M$ is odd and $m \neq n$.

For non-negative integers p, q with $p+q = \frac{m-1}{2}$, we set $r_p := \sqrt{\frac{2p+1}{m+1}}$ and $r_q := \sqrt{\frac{2q+1}{m+1}}$. Then imbedding the Riemannian product $S^{2p+1}(r_p) \times S^{2q+1}(r_q)$ minimally in S^{m+2}, we define a real hypersurface $M_{p,q}^C$ in $CP^{\frac{m+1}{2}}$ by the commutative diagram

$$\begin{array}{ccc} S^{2p+1}(r_p) \times S^{2q+1}(r_q) & \rightarrow & S^{m+2}(1) \\ \downarrow & & \downarrow \\ M_{p,q}^C & \rightarrow & CP^{\frac{m+1}{2}}. \end{array} \tag{7.1}$$

By making a composition with a totally geodesic imbedding $CP^{\frac{m+1}{2}} \to CP^n$, we regard $M^C_{p,q}$ as a minimal submanifold in CP^n. Then $M^C_{p,q}$ is a CR-submanifold in CP^n of type $(\frac{m-1}{2},1)$.

Claim 7.2. *The nullity of $M^C_{p,q}$ in CP^n is given as follows:*

$$\mathrm{nul}(M^C_{p,q}, CP^n) = 2(p+1)(q+1) + 2\left(n - \frac{m+1}{2}\right)\left(\frac{m+1}{2}+1\right).$$

Especially we have

$$\mathrm{nul}(M^C_{0,\frac{m-1}{2}}, CP^n) = m+1+2\left(n - \frac{m+1}{2}\right)\left(\frac{m+1}{2}+1\right).$$

Proof. It has been shown in our previous paper [4] that $\mathrm{nul}(M^C_{p,q}, CP^{\frac{m+1}{2}}) = 2(p+1)(q+1)$. Therefore Theorem 6.2 shows

$$\mathrm{nul}(M^C_{p,q}, CP^n) = 2(p+1)(q+1) + 2\left(n - \frac{m+1}{2}\right)\dim_C E(m+1; \Delta_{\widehat{M^C_{p,q}}})^{S^1}. \tag{7.3}$$

Note in the present case that $\widehat{M^C_{p,q}} = S^{2p+1}(r_p)\times S^{2q+1}(r_q)$. The eigenspaces of $\Delta_{\widehat{M^C_{p,q}}}$ are described as follows (cf. [1]): set $\lambda^{2p+1}_\alpha := \frac{\alpha(\alpha+2p)(m+1)}{2p+1}$, $\lambda^{2q+1}_\beta := \frac{\beta(\beta+2q)(m+1)}{2q+1}$, $\alpha,\beta = 0,1,\cdots$. Then $\mathrm{Spec}(\Delta_{\widehat{M^C_{p,q}}}) = \{\lambda^{2p+1}_\alpha + \lambda^{2q+1}_\beta \mid \alpha,\beta = 0,1,\cdots\}$, the set consisting of all the eigenvalues of $\Delta_{\widehat{M^C_{p,q}}}$. Let $(z_1,\cdots,z_{p+1})$ (resp. $(w_1,\cdots,w_{q+1})$) be the canonical complex coordinate on C^{p+1} (resp. C^{q+1}), and denote by $\mathscr{H}^{2p+1}_{s,t}(z,\bar{z})$ (resp. $\mathscr{H}^{2q+1}_{u,v}(w,\bar{w})$) the vector space consisting of all homogeneous harmonic polynomials with coefficients in C and of bidegree (s,t) (resp. (u,v)) with respect to z_j and $\bar{z}_j$ (resp. w_j and $\bar{w}_j$). Then it is known that

$$E(\lambda^{2p+1}_\alpha + \lambda^{2q+1}_\beta; \Delta_{\widehat{M^C_{p,q}}}) = \sum_{\substack{s+t=\alpha \\ u+v=\beta}} \mathscr{H}^{2p+1}_{s,t}(z,\bar{z}) \otimes \mathscr{H}^{2q+1}_{u,v}(w,\bar{w}). \tag{7.4}$$

Since $\lambda^{2p+1}_\alpha + \lambda^{2q+1}_\beta = m+1$ if and only if $(\alpha,\beta) = (1,0)$ or $(0,1)$, (7.4) implies especially $E(m+1; \Delta_{\widehat{M^C_{p,q}}}) = \mathscr{H}^{2p+1}_{1,0}(z,\bar{z}) + \mathscr{H}^{2p+1}_{0,1}(z,\bar{z}) + \mathscr{H}^{2q+1}_{1,0}(w,\bar{w}) + \mathscr{H}^{2q+1}_{0,1}(w,\bar{w})$. Thus by definition we have

$$E(m+1; \Delta_{\widehat{M^C_{p,q}}})^{S^1} = \mathscr{H}^{2p+1}_{1,0}(z,\bar{z}) + \mathscr{H}^{2q+1}_{1,0}(w,\bar{w}),$$

and hence $\dim_C E(m+1; \Delta_{\widehat{M^C_{p,q}}})^{S^1} = \frac{m+1}{2}+1$. Consequently (7.3) shows the assertion. □

Remark 7.5. As for the index, we can show

$$\mathrm{ind}(M^C_{p,q}, CP^n) = 2(n-m)+1 = \mathrm{codim}(M^C_{p,q}, CP^n).$$

Now we state the main result in this section:

Theorem 7.6. *Let M be an m-dimensional compact minimal CR-submanifold immersed in CP^n. Suppose m is odd and not equal to n. Then an inequality*

$$\mathrm{nul}(M) \geq m+1+2\left(n-\frac{m+1}{2}\right)\left(\frac{m+1}{2}+1\right)$$

holds. Moreover equality holds when and only when $M = M^C_{0,\frac{m-1}{2}} \subset CP^{\frac{m+1}{2}} \underset{\text{totally geodesic}}{\subset} CP^n$.

Because the inequality have been established in Theorem 5.8, we assume

$$\mathrm{nul}(M) = m+1+2\left(n-\frac{m+1}{2}\right)\left(\frac{m+1}{2}+1\right) \tag{7.7}$$

in what follows.

First of all we know from Lemma 5.7 and the assumption (7.7) that the type of M must be $(\frac{m-1}{2}, 1)$. Hence our assumption (5.3) becomes $T_oM = \mathfrak{m} = \boldsymbol{C}e_1 \oplus \cdots \boldsymbol{C}e_h \oplus \boldsymbol{R}e_{h+1}$, where we put $h = \frac{m-1}{2}$. On the other hand, (5.6) becomes

$$\begin{aligned}\mathrm{Ker}\,\Psi_2 \mid \mathfrak{u}(n) &= \left\{ \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{pmatrix} \,\middle|\, S \in \mathfrak{u}(h),\ T \in \mathfrak{u}(n-h-1) \right\} \\ &\cong \mathfrak{u}(h) \oplus \mathfrak{u}(n-h-1),\end{aligned}$$

which generates the following Lie subgroup of $U(n+1)$:

$$\begin{aligned}U_{h,n-h-1} &= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \,\middle|\, A \in U(h),\ B \in U(n-h-1) \right\} \\ &\cong U(h) \times U(n-h-1).\end{aligned}$$

The follwing is a key claim, which is proved similarly to Claim 5.11:

Claim 7.8. *Under the assumption* (7.7), *the action of $U_{h,n-h-1}$ on CP^n leaves M invariant.*

Moreover we shall prove

Lemma 7.9. *$B(X,Y)\in \boldsymbol{R}\sqrt{-1}e_{h+1}$ for all $X,Y\in\mathfrak{m}$.*

Proof. Consider an element u in $U_{h,n-h-1}$ such that

$$u=\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{h+1} & 0 \\ 0 & 0 & -I_{n-h-1} \end{pmatrix}.$$

Then the isotropy action $u_*:\tilde{\mathfrak{m}}\to\tilde{\mathfrak{m}}$ is given as

$$u_*=\begin{cases} \text{identity on } \mathfrak{m}\oplus\boldsymbol{R}\sqrt{-1}e_{h+1}=\bigoplus\limits_{j=1}^{h+1}\boldsymbol{C}e_j, \\ -\text{identity on } \bigoplus\limits_{\alpha=h+2}^{n}\boldsymbol{C}e_\alpha. \end{cases} \tag{7.10}$$

According to the direct sum $\mathfrak{m}^\perp=\boldsymbol{R}\sqrt{-1}e_{h+1}\oplus\left(\bigoplus\limits_{\alpha=h+2}^{n}\boldsymbol{C}e_\alpha\right)$, we decompose B as $B(X,Y)=B'(X,Y)+B''(X,Y)$. Then (7.10) implies

$$u_*B(X,Y)=B'(X,Y)-B''(X,Y). \tag{7.11}$$

On the other hand, by virture of Claim 7.8, we know that u is an isometry of both M and $\boldsymbol{C}P^n$. This combined with (7.10) shows

$$\begin{aligned} u_*B(X,Y)&=B(u_*X,u_*Y)\\ &=B(X,Y)\\ &=B'(X,Y)+B''(X,Y). \end{aligned} \tag{7.12}$$

Those (7.11) and (7.12) imply $B(X,Y)=B'(X,Y)\in\boldsymbol{R}\sqrt{-1}e_{h+1}$. □

We now complete the proof of the Theorem 7.6. The idea is to reduce to the following result which treats the case of codimension one:

Fact 7.13 (T. Gotoh [4]). *Let M be a compact minimal real hypersurface in $\boldsymbol{C}P^l$. Then an inequality $\mathrm{nul}(M,\boldsymbol{C}P^l)\geq 2l$ holds. Moreover the equality holds when and only when $M=M^{\boldsymbol{C}}_{0,l-1}$.*

Proof of Theorem 7.6. The preceding Lemma 7.9 states that the first normal space (see Appendix below for definition) of M coincide with $(N_oM)_{\boldsymbol{R}}$ only at the origin

o. However, by means of Lemma 5.2, we find that $\mathcal{N}_x^1(M)=(N_xM)_R$ hold for all $x \in M$. As a result of Theorem A.3, which will be given in Appendix below, there exists a totally geodesic submanifold $CP^{\frac{m+1}{2}}$ containing M as a real hypersurface. Then since the immersion $M \to CP^{\frac{m+1}{2}}$ is full, Corollary 6.14 implies $\mathrm{nul}(M, CP^n) \geq \mathrm{nul}(M, CP^{\frac{m+1}{2}}) + 2(n-\frac{m+1}{2})(\frac{m+1}{2}+1)$. Hence, by assumption (7.7), we obtain $m+1 \geq \mathrm{nul}(M, CP^{\frac{m+1}{2}})$. By making use of Fact 7.13, we conclude $\mathrm{nul}(M, CP^{\frac{m+1}{2}}) = m+1$ and $M = M^C_{0,\frac{m-1}{2}} \subset CP^{\frac{m+1}{2}} \subset CP^n$. □

Appendix-a result on codimension reduction problem

To state the result, we introduce some terminology: let M be a submanifold immersed in a Riemannian manifold $\tilde{M}$. The second fundamental form of it is denoted by B_M. For each x in M, define a subspace $\mathcal{N}_x^1(M)$ in the normal space N_xM by

$$\text{(A.1)} \qquad \mathcal{N}_x^1(M) := \textit{linear span } \{B_M(X,Y) \in N_xM \mid X, Y \in T_xM\},$$

which we call the *first normal space*. It is easy to verify that $\mathcal{N}^1(M) := \cup_{x \in M} \mathcal{N}_x^1(M)$ becomes a subbundle of NM provided the dimension of $\mathcal{N}_x^1(M)$ does not depend on each x. In this case, it is also easy to see that the subbundle $\mathcal{N}^1(M)$ is invariant under parallel translations with respect to the normal connection on NM if and only if the following condition is satisfied: $X \in T_xM$, $\xi \in \Gamma(\mathcal{N}^1(M)) \Rightarrow \nabla_X^{NM}\xi \in \mathcal{N}_x^1(M)$. The first normal space is said to be *parallel* provided the condition is satisfied.

Erbacher has been proved the following theorem concerning with codimension reduction problem:

Fact A.2 (J. Erbacher [3]). *Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\tilde{M}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. If the first normal space $\mathcal{N}_x^1(M)$ is parallel and q is the constant dimension of $\mathcal{N}_x^1(M)$, then there exists a totally geodesic submanifold L of $\tilde{M}(\tilde{c})$ of dimension $n+q$ such that $M \subset L$.*

Now we state the theorem:

Theorem A.3. *Let M be an m-dimensional CR-submanifold of type (h,r) in CP^n. Suppose that the first normal space $\mathcal{N}_x^1(M)$ is equal to $(N_xM)_R$ for each x in M. Then there exists a totally geodesic submanifold $CP^{\frac{m+r}{2}}$ in CP^n which contains M.*

Proof. Considering the Hopf fibration $S^1 \to S^{2n+1} \to CP^n$, we denote by $\hat{x} \in S^{2n+1}$ a point on the fiber over $x \in CP^n$. For a tangent vector $X \in T_x CP^n$, $\hat{X}$ indicates its horizontal lift at a point on the fiber over x. Then we have easily

$$B_{\hat{M}}(\hat{X}, \hat{Y}) = (B_M(X, Y))\hat{}. \tag{A.4}$$

We shall show first

$$\mathcal{N}^1_{\hat{x}}(\hat{M}) = \widehat{\mathcal{N}^1_x(M)}. \tag{A.5}$$

Because $N_{\hat{x}}\hat{M} = \widehat{N_x M}$, we find $\{B_{\hat{M}}(\hat{X}, \hat{Y}) \mid X, Y \in T_x M\} = \widehat{\mathcal{N}^1_x M}$. Hence, on account of (A.4), we show that $B_{\hat{M}}(V, V)$ and $B_{\hat{M}}(\hat{X}, V)$ are contained in $\widehat{\mathcal{N}^1_x M}$. Here V is the vertical vector field on S^{2n+1} defined in section 6. Since $\nabla^{S^{2n+1}}_V V = 0$, we get $B_{\hat{M}}(V, V) = 0$. On the other hand, $\nabla^{S^{2n+1}}_{\hat{X}} V = \widehat{JX}$, which is tangent to $\hat{M}$ for $X \in (T_x M)_H$ and normal to $\hat{M}$ for $X \in (T_x M)_R$. Thus $B_{\hat{M}}(\hat{X}, V) = 0$ for $X \in (T_x M)_H$ and $= \widehat{JX}$ for $X \in (T_x M)_R$. Because if X belongs to $(T_x M)_R$, JX belongs to $(N_x M)_R = \mathcal{N}^1_x(M)$ by assumtion, we find $B_{\hat{M}}(\hat{X}, V)$ to be contained in $\widehat{\mathcal{N}^1_x M}$.

Secondly we show

$$X \in T_x M,\ \xi \in \Gamma(\mathcal{N}^1(M)) \Rightarrow \nabla^{NM}_X \xi \in \mathcal{N}^1_x(M). \tag{A.6}$$

In fact let η be an element in $(N_x M)_H$. Then we have, by assumption $\mathcal{N}^1_x(M) = (N_x M)_R$, that $J\xi \in \Gamma(TM)$, $J\eta \in (N_x M)_H$. Thus $\langle \nabla^{NM}_X \xi, \eta \rangle = \langle \nabla^{CP^n}_X \xi, \eta \rangle = \langle J\nabla^{CP^n}_X \xi, J\eta \rangle = \langle \nabla^{CP^n}_X (J\xi), J\eta \rangle = \langle B_M(X, J\xi), J\eta \rangle = 0$, and hence $\nabla^{NM}_X \xi \in \mathcal{N}^1_x(M)$.

Moreover we show

$$v \in \Gamma(\mathcal{N}^1(\hat{M})),\ W \in T_{\hat{x}}\hat{M} \Rightarrow \nabla^{N\hat{M}}_W v \in \mathcal{N}^1_{\hat{x}}(\hat{M}). \tag{A.7}$$

It suffices to consider v which is written as the horizontal lift of an element, say ξ in $\Gamma(\mathcal{N}^1(M)) : v = \hat{\xi}$. For $W = \hat{X}$ for some $X \in T_x M$, we have $\nabla^{N\hat{M}}_W v = \widehat{\nabla^{NM}_X \xi}$. By means of (A.5) and (A.6), this belongs to $\mathcal{N}^1_{\hat{x}}(\hat{M})$. For $W = V$, since $\hat{\xi}$ is invariant by S^1-action, we get $\nabla^{S^{2n+1}}_V \hat{\xi} = \widehat{J\xi}$, which is contained in $T_{\hat{x}}\hat{M}$ by assumption. Hence $\nabla^{N\hat{M}}_V \hat{\xi} = 0$.

Now on account of Fact A.2 and (A.7), we find that there exists a totally geodesic submanifold $S^{m+1+r} \to S^{2n+1}$ which contains $\hat{M}$. In order to complete the proof, we must show that this S^{m+1+r} is invariant by S^1-action. According to Erbacher's proof given in [3], the S^{m+1+r} above is constructed as follows: set $\mathcal{W} := T_{\hat{x}}\hat{M} \oplus \mathcal{N}^1_{\hat{x}}(\hat{M}) \oplus \boldsymbol{R}\hat{x}$. Then this defines a real subspace in $\boldsymbol{C}^{n+1}$ not depending on $\hat{x}$. Then the S^{m+1+r} above is obtained as $S^{m+1+r} = S^{2n+1} \cap \mathcal{W}$. For this reason we have only to show that $\mathcal{W}$ is a complex subspace in $\boldsymbol{C}^{n+1}$. By making use of the decompositons into horizontal and vertical subspace of $T_{\hat{x}}\hat{M}$, $T_{\hat{x}}\hat{M} = \widehat{T_x M} \oplus \boldsymbol{R}\sqrt{-1}\hat{x} = \widehat{(T_x M)_H} \oplus \widehat{(T_x M)_R} \oplus \boldsymbol{R}\sqrt{-1}\hat{x}$, $\mathcal{W}$ can be expressed as

$$\mathcal{W} = \{\widehat{(T_x M)_H}\} \oplus \{\widehat{(T_x M)_R} \oplus \widehat{(N_x M)_R}\} \oplus \{\boldsymbol{R}\hat{x} \oplus \boldsymbol{R}\sqrt{-1}\hat{x}\}.$$

Since each three factor in the above is complex subspace, so does $\mathscr{W}$. □

REMARK A.8. (1) In [3], the first normal space was defined as the orthonormal complement of $\{\xi \in N_xM \mid A^\xi = 0\}$ in N_xM. Our definition (A.1) coinsides with this definition.
(2) Okumura have studied the codimension reduction problem in a complex projective space. In fact, our Theorem A.3 can be proved by using his results in [11]; we gave a direct proof for the sake of completeness. Various results are found in [11].

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