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Osaka University
SELF-INTERSECTIONS OF CURVES ON A SURFACE AND BERNOULLI NUMBERS

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Abstract
We study an operation which measures self-intersections of curves on an oriented surface. It turns out that a certain computation on this topological operation is related to the Bernoulli numbers $B_m$, and our study yields a family of explicit formulas for $B_m$. As a special case, this family contains the celebrated formula for $B_m$ due to Kronecker.

1. Introduction

The Bernoulli numbers $B_m$ ($m \geq 0$) are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m.$$ 

We have: $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, ..., and $B_m = 0$ for all odd $m \geq 3$. The appearance of the Bernoulli numbers is ubiquitous in mathematics, and a large number of identities involving the Bernoulli numbers has been known [3] [4] [9] [10].

In this article, we show that the Bernoulli numbers arise naturally from the topology of surfaces, i.e., 2-manifolds. In more detail, by studying self-intersections of curves on an oriented surface, we obtain the following family of explicit formulas for $B_m$:

**Theorem 1.** Let $m \geq 2$. For any integers $a$ and $n$ satisfying $0 \leq a \leq m \leq n$, we have

$$B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \sum_{i=1}^{k-1} i^a (k - i)^{m-a}. \quad (1)$$

Notice that the formula above has two parameters $a$ and $n$. When $a = 0$ and $n = m$, the formula (1) reduces to the celebrated formula for $B_m$ due to Kronecker ([7], see also [4] [5] [9] [10]): for $m \geq 2$,

$$B_m = \sum_{k=1}^{m+1} \frac{(-1)^{k+1}}{k} \binom{m+1}{k} \sum_{i=1}^{k-1} i^m. \quad (2)$$

In fact, using the classical formula for the sum of powers (known as Faulhaver’s formula) and a property of binomial coefficients (see Lemma 2), one can derive the formula (1) from

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the Kronecker formula (2). However, our derivation of the formula (1) is self-contained and more direct.

Our proof of Theorem 1 is motivated by a topological consideration on an oriented surface. In §2, we consider an operation $\mu$ to a curve on the surface. This operation was introduced in [6] inspired by a construction of Turaev [11]. and, among other things, it computes self-intersections of curves. The key is to compute $\mu(\log \gamma)$ for a simple loop $\gamma$ and we find that it involves the Bernoulli numbers (Theorem 2). Here, we work with a suitable completion to be able to consider $\log \gamma$. In §3, we formalize the topological argument in §2 and prove the main results. In §4, we give another self-contained proof of Theorem 1 by introducing a certain generating function.

The Bernoulli numbers have already appeared in the study of intersections of two curves on an oriented surface [8]. Our formula provides yet another evidence for a close connection between the topology of surfaces and the Bernoulli numbers. This connection has been developed in [1] to an unexpected connection between the operation $\mu$, or equivalently, the Turaev cobracket, and the Kashiwara-Vergne problem in the formulation by Alekseev-Torossian [2].

2. Self-intersection map and Bernoulli numbers

Let $S$ be a compact connected oriented surface with $\partial S \neq \emptyset$. Fix a basepoint $\ast \in \partial S$ and set $\pi_1(S) := \pi_1(S, \ast)$. We denote by $\hat{\pi}(S)$ the set of free homotopy classes of oriented loops on $S$. For any $p \in S$, we denote by $|: \pi_1(S, p) \to \hat{\pi}(S)$ the forgetful map of the basepoint.

We recall the operation $\mu: \mathbb{Q}\pi_1(S) \to \mathbb{Q}\pi_1(S) \otimes (\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1)$, which has been introduced in [6] inspired by a construction of Turaev [11]. Here, 1 is the class of a constant loop. Let $\gamma: [0, 1] \to S$ be an immersed based loop. We arrange so that the pair of tangent vectors $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of the tangent space $T_\gamma S$, and that the self-intersections of $\gamma$ (except for the base point $\ast$) lie in the interior $\text{Int}(S)$ and consist of transverse double points. Let $\Gamma$ be the set of such double points of $\gamma$. For $p \in \Gamma$ we denote $\gamma^{-1}(p) = \{t_1^p, t_2^p\}$, so that $0 < t_1^p < t_2^p < 1$. We define

$$\mu(\gamma) := -\sum_{p \in \Gamma} e(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0|t_1^p} \gamma_{t_2^p|1}) \otimes |\gamma_{t_1^p t_2^p}| \in \mathbb{Q}\pi_1(S) \otimes (\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1).$$

Here,

- the sign $e(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))$ is +1 if the pair $(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))$ is a positive basis of $T_p S$, and is −1 otherwise,
- the based loop $\gamma_{0|t_1^p} \gamma_{t_2^p|1}$ is the conjunction of the paths $\gamma_{|0|t_1^p}$ and $\gamma|_{t_2^p|1}$,
- the element $\gamma_{t_1^p t_2^p} \in \pi_1(S, p)$ is the restriction of $\gamma$ to $[t_1^p, t_2^p]$ and we understand that $|\gamma_{t_1^p t_2^p}| = 0$ if the loop $\gamma_{t_1^p t_2^p}$ is homotopic to a constant loop.

**Remark 1.** The operation $\mu$ is essentially the same as Turaev’s operation $\mu^T: \pi_1(S) \to \mathbb{Q}\pi_1(S)$ in [11]. In fact, we have $\mu^T(\gamma)\gamma = -(id \otimes e)\mu(\gamma)$ for any $\gamma \in \pi_1(S)$, where $e(\alpha) = 1$ for any $\alpha \in \hat{\pi}(S) \setminus \{1\}$. Conversely, one can express $\mu$ in terms of $\mu^T$. The alternating part of $(| \otimes 1)\mu(\gamma)$ is exactly the Turaev cobracket [12] of the free loop $|\gamma|$.

We observe that if $\gamma$ is simple and the pair $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of $T_\gamma S$, then for any integer $k \in \mathbb{Z}$,
The filtration ideal of $I$ (resp. $FX$) gives a filtration of (total) degree $\geq Z$. Let $\gamma$ be the set of formal power series in $X$, $Y$, and assume that the pair $(\hat{\gamma}(0), \hat{\gamma}(1))$ is a positive basis of the tangent space $T(1)$. Then the Laurent polynomial ring $Q[p]\hat{\pi}(1)$ is defined by $Q[p]\hat{\pi}(1)\mathfrak{g}$ gives a filtration $I(p)$ of $Q[p]\hat{\pi}(1)$, and the Goldman-Turaev Lie bialgebra $\hat{\pi}\mathfrak{g}$, respectively, with respect to the augmentation ideal of $Q[p]\hat{\pi}(1)$. Then we can consider $\log \gamma = \sum_{i=1}^{\infty} ((-1)^{i+1}/i)(\gamma - 1)^i \in Q[p]\hat{\pi}(1)$.

As the following result shows, if $\gamma$ is simple then one can compute $\mu(\log \gamma)$ explicitly and the formula involves the Bernoulli numbers.

**Theorem 2.** Let $\gamma \in \pi$ be represented by a simple loop, and assume that the pair $(\hat{\gamma}(0), \hat{\gamma}(1))$ is a positive basis of the tangent space $T, S$. Then we have

$$\mu(\log \gamma) = -\sum_{m=0}^{\infty} \frac{B_m}{m!} \sum_{p=0}^{\infty} (-1)^p \binom{m}{p} (\log \gamma)^p \hat{\otimes}(\log \gamma)^{m-p}. \tag{4}$$

**3. Proof of Theorem 1 and Theorem 2**

First of all, we describe a preliminary construction.

Let $Q[[Z]]$ (resp. $Q[[X, Y]]$) be the commutative ring of formal power series in an indeterminate $Z$ (resp. in indeterminates $X$ and $Y$). For a non-negative integer $p$, let $F^Z_p$ (resp. $F^{X,Y}_p$) be the set of formal power series in $Q[[Z]]$ (resp. $Q[[X, Y]]$) which has only terms of (total) degree $\geq p$. We have natural isomorphisms $Q[[Z]] \cong \lim_{\leftarrow p} Q[[Z]]/F^Z_p$ and $Q[[X, Y]] \cong \lim_{\leftarrow p} Q[[X, Y]]/F^{X,Y}_p$.

Set $\varepsilon \mathfrak{e} = \sum_{i=1}^{\infty} (1/i!) Z^i$. Then the Laurent polynomial ring $Q[z, \varepsilon^{-1}]$ is a subring of $Q[[Z]]$. The augmentation ideal $I$ is defined by

$$I = \ker(Q[z, \varepsilon^{-1}] \rightarrow Q, \sum_j a_j \varepsilon^j \mapsto \sum_j a_j).$$

Then $I$ gives a filtration $(I^p)_p$ of $Q[z, \varepsilon^{-1}]$. By the inclusion map $Q[z, \varepsilon^{-1}] \hookrightarrow Q[[Z]]$, the filtration $(I^p)_p$ restricts to $(I^p)_p$. Moreover, we have a natural isomorphism $Q[[Z]] \cong \lim_{\leftarrow p} Q[z, \varepsilon^{-1}]/I^p$. 

Fig. 1. computation of $\mu(\gamma^k)$ for a simple $\gamma$ ($k = 4$).
Motivated by the formula (3), we define a \( \mathbb{Q} \)-linear map \( \hat{\mu} : \mathbb{Q}[z, z^{-1}] \rightarrow \mathbb{Q}[X, Y] \) by

\[
\hat{\mu}(z^k) = \begin{cases} 
- \sum_{i=1}^{k} e^{iX} e^{(k-i)Y} & (k > 0) \\
0 & (k = 0) \\
\sum_{i=0}^{k-1} e^{-iX} e^{(k+i)Y} & (k < 0).
\end{cases}
\]

(5)

From the definition of \( \hat{\mu} \) it is easy to see that

\[
(e^{-X} e^{Y} - 1) \hat{\mu}(z^k) = e^{kX} - e^{kY}, \quad k \in \mathbb{Z}.
\]

Therefore, we have

\[
(e^{-X} e^{Y} - 1) \hat{\mu}(f(z)) = f(e^X) - f(e^Y)
\]

(6)

for any Laurent polynomial \( f(z) \in \mathbb{Q}[z, z^{-1}] \). Consider

\[
\Phi(X, Y) := \sum_{i=0}^{\infty} \frac{B_i}{i!} (-X + Y)^i.
\]

Then we have \((e^{-X} e^{Y} - 1)\Phi(X, Y) = -X + Y\). Multiplying \( \Phi(X, Y) \) to both sides of (6), we have

\[
(-X + Y) \hat{\mu}(f(z)) = (f(e^X) - f(e^Y)) \Phi(X, Y)
\]

(7)

for any \( f(z) \in \mathbb{Q}[z, z^{-1}] \).

**Lemma 1.** There is a unique continuous extension \( \hat{\mu} : \mathbb{Q}[[Z]] \rightarrow \mathbb{Q}[[X, Y]] \) of the map \( \hat{\mu} \) in (5).

Proof. It is sufficient to prove that \( \hat{\mu}(1^p) \subset F_{X^p}^{XY} \) for any \( p \geq 1 \). Suppose \( f(z) \in 1^p \). Then \( f(e^X) \) and \( f(e^Y) \) lie in \( F_{X^p}^{XY} \). This means that the right hand side of (7) is an element of \( F_{X^p}^{XY} \). Therefore, \( \hat{\mu}(f(z)) \in F_{X^p}^{XY} \). \( \Box \)

Now for each \( k \geq 1 \) we can put \( f(z) = (\log z)^k = Z^k \) in (7), and we obtain

\[
(-X + Y) \hat{\mu}(Z^k) = (X^k - Y^k) \Phi(X, Y).
\]

This shows that \( \hat{\mu}(Z^k) \in F_{X^k}^{X^k} \). Setting \( k = 1 \), we have

\[
\hat{\mu}(Z) = -\Phi(X, Y) = - \sum_{i=0}^{\infty} \frac{B_i}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} X^j Y^{i-j}.
\]

(8)

This formula is essentially the same as the assertion of Theorem 2:

Proof of Theorem 2. We identify the ring \( \mathbb{Q}[[X, Y]] \) with the complete tensor product \( \mathbb{Q}[[Z]] \otimes \mathbb{Q}[[Z]] \) by the map \( X \mapsto Z \otimes 1 \) and \( Y \mapsto 1 \otimes Z \). Then the computation (8) implies

\[
\hat{\mu}(\log z) = - \sum_{m=1}^{\infty} \frac{B_m}{m^2} \sum_{p=0}^{m} (-1)^p \binom{m}{p} (\log z)^p \otimes (\log z)^{m-p}.
\]

(9)

From (3) and (5) it follows that the substitution \( z \mapsto \gamma \) commutes with \( \mu \) and \( \hat{\mu} \). Thus we obtain (4). \( \Box \)
Further, by expanding the left hand side of (8) in terms of $\hat{\mu}(z^k)$’s modulo higher degree terms, we have the following:

**Proposition 1.** Let $m, n, a$ be integers satisfying $0 \leq a \leq m \leq n$. Then it holds that

$$B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \left[ \sum_{i=1}^{k-1} \hat{\mu}(k-i)^{m-a} + \delta_{a,m} k^m \right].$$

Here $\delta_{a,m}$ is the Kronecker delta.

**Proof.** In what follows, $\equiv$ means an equality in $\mathbb{Q}[[X, Y]]$ modulo $F_{n+1}^{X,Y}$. For $k = 1, \ldots, n + 1$, we have

$$\hat{\mu}(z^k) = \hat{\mu}(e^{kZ}) = \sum_{i=1}^{\infty} \frac{k^i}{i!} \hat{\mu}(Z^i) \equiv \sum_{i=1}^{n+1} \frac{k^i}{i!} \hat{\mu}(Z^i).$$

Consider the square matrix $D = (D_{ki})_{i,j}$ of order $n + 1$, where $D_{ki} = \frac{k^i}{i!}$. Then $D$ is invertible since $\det(D)$ is a non-zero multiple of Vandermonde’s determinant $\det(k^{i-1}_{i,j})$. The inverse matrix of $D$ has the first row $(a_1, \ldots, a_{n+1})$, where

$$a_k = \frac{(-1)^{k+1}}{k} \binom{n+1}{k}.$$

(To see this, for instance, one can use Lemma 2 below to get $(a_1, \ldots, a_{n+1})D = (1, \ldots, 0).$) From (10) we have

$$\hat{\mu}(Z) \equiv \sum_{k=1}^{n+1} a_k \hat{\mu}(z^k) = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \hat{\mu}(z^k).$$

Furthermore, for $k = 1, \ldots, n + 1$, from (5) we have

$$\hat{\mu}(z^k) = -\sum_{i=1}^{k-1} \sum_{a,b=0}^{\infty} \frac{i^a (k-i)^b}{a! b!} X^a Y^b - \sum_{a=0}^{\infty} \frac{k^a}{a!} X^a.$$

By (11) and (12), the coefficient of $X^a Y^{m-a}$ in $\hat{\mu}(Z)$ is

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \left[ \sum_{i=1}^{k-1} \frac{i^a (k-i)^{m-a}}{a!(m-a)!} + \delta_{m,a} k^m \right].$$

On the other hand, by (8), this coincides with

$$(-1)^{m+1} \frac{B_m}{m!} \frac{m^a}{a!(m-a)!} B_m.$$

This completes the proof. \(\Box\)

Now, we can derive Theorem 1 from Proposition 1 by applying the following lemma. Although it might be well known, we give its proof for the sake of completeness.

**Lemma 2.** Let $m, n$ be integers satisfying $0 \leq m \leq n$. Then it holds that

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} k^m = \begin{cases} 0 & \text{if } m \geq 1, \\ -1 & \text{if } m = 0. \end{cases}$$
Proof. Set $f(x) := (e^x - 1)^{n+1}$. Since $m \leq n$, the coefficient of $x^m$ in the series expansion of $f(x)$ is zero.

On the other hand, we compute

$$f(x) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kx},$$

$$= (-1)^{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} e^{kx} + 1,$$

$$= (-1)^{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \left( \sum_{a=0}^{\infty} \frac{k^a}{a!} x^a + 1 \right).$$

Since the coefficient of $x^m$ in the last expression is equal to

$$\begin{cases} 
\frac{(-1)^{n+1}}{m!} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k^m & \text{if } m \geq 1, \\
(-1)^{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} + 1 & \text{if } m = 0,
\end{cases}$$

the assertion follows. \[\square\]

4. Another proof of Theorem 1

Introducing a generating function of two variables, we give another self-contained proof of Theorem 1. Since we have Lemma 2, it is sufficient to prove Proposition 1.

Let $f(x, y)$ and $g(x, y)$ be functions in variables $x$ and $y$ defined by

$$f(x, y) := \int_x^y (e^t - 1)^{n+1} dt, \quad \text{and} \quad g(x, y) := \frac{f(x, y)}{e^{y-x} - 1}.$$

We will examine the coefficient of $x^m y^{m-a}$ in the series expansion of $g(x, y)$.

First we compute $f(x, y)$ as follows:

$$f(x, y) = \int_x^y (e^t - 1)^{n+1} dt$$

$$= \int_x^y \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kt} dt$$

$$= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} (e^{ky} - e^{kx}) + (-1)^{n+1} (y - x).$$

Since

$$\frac{e^{ky} - e^{kx}}{e^{y-x} - 1} = \frac{e^{kx} (e^{ky} - 1)}{e^{y-x} - 1} = \sum_{i=1}^{k-1} e^{ix} e^{(k-i) y} + e^{kx},$$

we have

$$f(x, y) = \sum_{a=0}^{\infty} \sum_{k=0}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} a^k e^{a(ky - x)} dt.$$
we can compute \( g(x, y) \) as follows:

\[
g(x, y) = \frac{f(x, y)}{e^{y-x} - 1}
\]

\[
= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \frac{e^{kx} - e^{ky}}{e^{y-x} - 1} + (-1)^{n+1} \frac{y-x}{e^{y-x} - 1}
\]

\[
= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \left( \sum_{i=1}^{k-1} e^{i(x-k+1)} + e^{k+1} \right)
\]

\[
+ (-1)^{n+1} \sum_{b=0}^{\infty} B_b \frac{b!}{b!} (y-x)^b.
\]

Then using the identities:

\[
e^{ix} e^{(k-i)y} = \sum_{b,c=0}^{\infty} \frac{i^b (k-i)^c}{b! c!} x^b y^c \quad \text{and} \quad e^{kx} = \sum_{b=0}^{\infty} \frac{k^b}{b!} x^b,
\]

we see that the coefficient of \( x^a y^{m-a} \) in \( g(x, y) \) is given by

\[
(-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left( \sum_{i=1}^{k-1} \frac{i^a (k-i)^{m-a}}{a! (m-a)!} + \delta_{a,m} \frac{k^m}{m!} \right)
\]

\[
+ (-1)^{n+1} \sum_{b=0}^{\infty} B_b \frac{b!}{b!} (y-x)^b.
\]

This is equal to \((-1)^{n+1+a}/m!\binom{m}{a}\) times

\[
(-1)^a \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left( \sum_{i=1}^{k-1} i^a (k-i)^{m-a} + \delta_{a,m} k^m \right) + B_m.
\]

Secondly, we expand \( g(x, y) \) in a different way. Put \( g_1(x, y) = f(x, y)/(y-x) \). Then we have

\[
g(x, y) = \frac{f(x, y)}{y-x} \frac{y-x}{e^{y-x} - 1} = g_1(x, y) \sum_{b=0}^{\infty} \frac{B_b}{b!} (y-x)^b.
\]

Writing \((e^t - 1)^{n+1} = \sum_{i\geq n+1} a_i t^i\), we have

\[
f(x, y) = \int_x^y (e^t - 1)^{n+1} dt = \sum_{i\geq n+1} \frac{a_i}{i+1} (y^{i+1} - x^{i+1}).
\]

Thus the series expansion of \( g_1(x, y) \) has all terms of degree \( \geq n+1 \), so does that of \( g(x, y) \). In particular, the coefficient of \( x^a y^{m-a} \) in this expansion is zero. Therefore, the expression (13) is zero, and we obtain Proposition 1. \(\square\)
References


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