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## ON PERFECT RINGS AND THE EXCHANGE PROPERTY

Dedicated to Professor Kiiti Morita on his 60th birthday

### MANABU HARADA AND TADAMASA ISHII

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Let R be a ring with unit element. We always consider unitary right R-modules. Let T be an R-module and  $\eta$  a cardinal number. If for any module K containing T as a direct summand and for any decomposition of K with  $\eta$  components:  $K = \underset{\substack{\alpha \in I \\ \alpha \in I}}{2} \bigoplus A_{\alpha}$ , there exist submodules  $A_{\alpha}'$  of  $A_{\alpha}$  for all a such that  $K = T \bigoplus_{\substack{\alpha \in I \\ \alpha \in I}} \bigoplus A_{\alpha}'$ , then we say T has the  $\eta$ -exchange property[2]. If T has the  $\eta$ -exchange property for any  $\eta$ , we say T has the exchange property.

In this short note we shall show that R is a right perfect ring if and only if for every projective module P, P has the exchange property and  $\operatorname{End}_R(P)/J(\operatorname{End}_R(P))$  is a regular ring in the sense of Von Neumann. This is a refinement of Theorem 7 in [4] and we shall give its proof as an application of [6].

After submitting this paper to the journal, the authors have received a manuscript of Yamagata [13] and found that one of main theorems in this paper overlaped with one in [13]. The authors would like to express their thanks to Dr. Yamagata for his kindness.

#### 1. Preliminaries

First we shall recall some definitions given in [3], [4] and [6]. Let T be an R-module. If  $\operatorname{End}_{R}(T)$  is a local ring, T is called *completely indecomposable*. We take a set  $\{M_{\alpha}\}_{I}$  of completely indecomposable modules and define the full additive subcategory  $\mathfrak{A}$  of all right R-modules which is induced from  $\{M_{\alpha}\}_{I}$ , namely the objects in  $\mathfrak{A}$  consist of all modules which are isomorphic to direct-sums of completely indecomposable modules in  $\{M_{\alpha}\}_{I}$ . We define an ideal  $\mathfrak{I}'$  in  $\mathfrak{A}$  as follows: let  $A = \sum_{\alpha \in \mathcal{K}} \bigoplus A_{\alpha}, B = \sum_{\beta \in \mathcal{L}} \bigoplus B_{\beta}$  be in  $\mathfrak{A}$ , where  $A_{\alpha}, B_{\beta}$  are isomorphic to some in  $\{M_{\alpha}\}_{I}$ , then  $\mathfrak{I}' \cap [A, B] = \{f \models \operatorname{Hom}_{R}(A, B), p_{\beta} f_{i_{\alpha}}$  are not isomorphic for all  $\alpha \in K, \beta \in L\}$ , where  $i_{\alpha}: A_{\alpha} \to A, p_{\beta}: B \to B_{\beta}$  are the inclusion and the projection, respectively. By  $\overline{\mathfrak{A}}$  we denote the factor category of  $\mathfrak{A}$  with respect to  $\mathfrak{I}'$  [3]. For any object A and morphism f in  $\mathfrak{A}$ , by  $\overline{A}$  and  $\overline{f}$  we denote the residue classes of A and / in  $\mathfrak{A}$ , respectively.

We take a countable subset  $\{M_{\omega_i}\}_{1}^{\infty}$  of  $\{M_{\omega}\}_{I}$  (resp. we can take same modules  $M_{\omega_i}$  in  $\{M_{\omega_i}\}_{1}^{\infty}$  as many as we want) and a set of homomorphisms  $f_i \in \mathfrak{I}' \prod [M_{\omega_i}, M_{\omega_{i+1}}]$ . If for any element *m* in  $M_{\omega_1}$  there exists *n*, which depends on *m*, such that  $f_n f_{n-1} f_1(m) = 0$ ,  $\{f_i\}$  is called *locally semi-T-nilpotent* (resp. *T-nilpotent*). If every  $\{f_i\}$  is locally semi-*T*-nilpotent (resp. *T-nilpotent*) for every subset  $\{M_{\omega_i}\}$ , we say  $\{M_{\omega}\}_{I}$  is a *locally semi-T-nilpotent* (resp. *T-nilpotent*) system [4]. Finally, let  $M \supset N$  be modules and  $N = \sum_{\gamma \in J} \bigoplus N_{\gamma}$ . If for any finite subset J' of  $J \sum_{\gamma \in J'} \bigoplus N_{\gamma}$  is a direct summand of M, N is called a *locally direct* summand of M (with respect to the decomposition  $\sum_{\gamma \in J} \bigoplus N_{\gamma}$ ) [6].

#### 2. Perfect rings

Let  $\{M_{\alpha}\}$  be a set of completely indecomposable modules and  $M = \sum_{\alpha \in I} \bigoplus M_{\alpha}$ . We understand  $p_{\alpha}$  means the projection of M to  $M_{\alpha}$  in the decomposition if there are no confusions. Let N be a submodule of M, which is isomorphic to one in  $\{M_{\alpha}\}_{I}$ . We shall consider a strong condition:

each N above is a direct summand of  $M \cdots (*)$ 

**Lemma 1.** Let M and  $\{M_{\alpha}\}_{I}$  be as above. We assume  $\{M_{\alpha}\}_{I}$  is a locally semi-T-nilpotent system and M satisfies (\*). Let A be a submodule of M. Then we have  $A = A_1 \oplus A_2$ , where  $A_1$  is a direct summand of M (and hence  $A_1 \in \mathfrak{A}$ ) and  $A_2$  does not contain any submodules which are isomorphic to some in  $\{M_{\alpha}\}_{I}$ .

Proof. Let  $\mathfrak{S}$  be the set of submodules A' in A as follows: A is in  $\mathfrak{A}$ , say  $A' = \sum_{\alpha \in J} \bigoplus A_{\alpha}; A_{\alpha}$  are isomorphic to some in  $\{M_{\alpha}\}_{I}$  and A' is a locally direct summand of M with respect to this decomposition. We can define a partial order in  $\mathfrak{S}$  by members of direct components (cf. [6]). Then we obtain a maximal one in  $\mathfrak{S}$  by Zorn's lemma, say  $A_{1}$ . Since  $\{M_{\alpha}\}_{I}$  is locally semi-T-nilpotent,  $A_{1}$  is a direct summand of  $M: M = A_{1} \oplus M_{1}$  by Theorem 9 in [3], Theorem in [7] and Lemma 3 and Corollary 2 to Lemma 2 in [6]. Hence,  $A = A_{1} \oplus (A \cap M_{1})$  and  $A \cap M_{1}$  does not contain any submodules in  $\mathfrak{A}$  from the assumption and the maximality of  $A_{1}$ .

The following lemma is a modification of one part of Theorem 2.6 in [12].

**Lemma** 2. Let  $\{M_{\alpha}\}_{I}$  and M be as above. We assume M satisfies (\*). Then M has the exchange property if and only if  $\{M_{\alpha}\}_{I}$  is a locally semi-T-nilpotent system.

Proof. If M has the exchange property, then  $\{M_{\alpha}\}_{I}$  is a locally semi-Γnilpotent system by [4'], Corollary 2 to Proposition 1. Conversely, we assume that  $\{M_{\alpha}\}_{I}$  is semi-Γ-nilpotent. Let  $A=M\oplus N=\sum_{\alpha\in r}\oplus A_{\alpha}$  We may assume

from [2], Theorem 8.2 that all  $A_{\alpha}$  are isomorphic to submodules in M, in order to show that M has the exchange property. Then from the assumption and Lemma 1,  $A_{\alpha} = A_{\alpha}' \oplus A_{\alpha}''$ , where  $A_{\alpha}' \in \mathfrak{A}$  and  $A_{\alpha}''$  does not contain any submodules, isomorphic to some in  $\{M_{\alpha}\}_{I}$ . Put  $A' = \sum_{\alpha \in J} \bigoplus A_{\alpha}'$  and  $A'' = \sum_{\alpha \in J} \bigoplus A_{\alpha}''$ , then  $A = A' \oplus A''$ . Let  $\varphi: A \rightarrow A/A''$  be the natural epimorphism. We shall show that M is a locally direct summand of A/A'' through  $\varphi$ . Let I' be a finite subset of / and  $M' = \sum_{\alpha \in I'} \bigoplus M_{\alpha}$ . Since M' has the exchange property by [11], Proposition 1 and [2], Lemma 3.10,  $A = M' \oplus A_0' \oplus A_0''$ , where  $A_0' \subset A'$  and  $A_0'' \subset A''$ . Then  $A'' = A_0'' \oplus K''$  and K'' is isomorphic to a direct summand of M'. If  $K'' \neq 0$ , K'' contains a completely indecomposable module  $K_1$  (isomorphic to one in  $\{M_{\alpha}\}_{I}$ ) as a direct summand by Krull-Remak-Schmidt theorem. Since  $K_1$  has the exchange property, we know from the argument above that some  $A_{\omega}^{\prime\prime}$  contains a submodule isomorphic to  $K_1$ . Which is a contradiction. Hence,  $A = M' \oplus A_0' \oplus A''$  and  $\varphi(M) \approx M$  is a locally direct summand of AA''. Since  $A/A'' \approx A' \in \mathfrak{A}$  and  $\{M_{a}\}_{I}$  is locally semi-T-nilpotent,  $\varphi(M)$  is a direct summand of A/A'' by [6], Lemma 3;  $A/A'' = \varphi(M) \oplus \varphi(K)$  and  $K \subset A'$ . Furthermore,  $\varphi(M)$  has the exchange property in  $\mathfrak{A}$  by [4], Corollary 2 to Proposition 1 and hence  $A/A'' = \varphi(M) \bigoplus_{\alpha \in J} \bigoplus \varphi(A_{\alpha}'')$  where  $A_{\alpha}''' \subset A_{\alpha}'$ . Therefore,  $A = M \oplus \sum_{\alpha \in I} \oplus (A_{\alpha}^{\prime \prime \prime} \oplus A_{\alpha}^{\prime \prime}).$ 

Next, we shall consider some cases where M satisfies (\*).

**Lemma 3.** Let  $\{M_{\alpha}\}_{I}$ , M and N be as in (\*) and  $i: N \rightarrow M$  the inclusion. Then N is a direct summand of M if and only if  $p_{\alpha}$  i is isomorphic for some a in I.

Proof. It is clear from the definition of  $\mathfrak{F}'$ .

**Lemma 4.** Let  $M_1$  be a completely indecomposable module. We assume  $M_1$  is a locally T-nilpotent system itself. Then  $M = \sum_{\alpha \in I} \bigoplus M_{\alpha}; M_{\alpha} \approx M_1$  has the exchange property for any set I.

Proof. We shall show that M satisfies (\*). We may assume  $N=M_1$ . We put  $f_{\alpha}=p_{\alpha}i$  and assume that  $f_{\alpha}$  are not isomorphic for all  $\alpha \in I$ . Let  $m \neq 0 \in N$  and  $i(m)=\sum_{i=1}^{n} f_{\alpha_i}(m)$ . Since *i* is monomorphic, we may assume  $f_{\alpha_1}(m)=m_2 \neq 0$ . Let  $i(m_2)=\sum_{i=1}^{n} f_{\alpha_i}(m_2)$ . Repeating this argument, we obtain a sequence  $\{f_{\beta_i}\}_{1}^{\infty}$  such that  $f_{\beta_n}f_{\beta_{n-1}}\cdots f_{\beta_1}(m) \neq 0$  for any *n*, which contradicts the *T*-nilpotency of  $\{M_1\}$ . Therefore, *M* satisfies (\*) by Lemma 3.

Let A, B be R-modules and  $f \in \operatorname{Hom}_{R}(A, B)$ . If Im / is small in B, f is called a *small homomorphism*. We note that if A=B are R-projective, then

the Jacobson radical  $J(\operatorname{End}_R(A))$  of  $\operatorname{End}_R(A)$  consists of all small homomorphisms by [10], Lemma 1.

**Lemma** 5. Let  $\{P_{\omega}\}_{I}$  be a set of *R*-modules and  $P = \underset{\alpha \in I}{=} \bigoplus P_{\omega}$  If *P* has the  $\aleph_{0}$ -exchange property, then any sequence of small homomorphisms  $\{n_{\omega}: P_{\omega_{i}} \rightarrow P_{\omega_{i+1}}\}$  is locally semi-*T*-nilpotent for any countable subset  $\{P_{\omega_{i}}\}_{1}^{\sim}$  of  $\{P_{\omega}\}_{I}$ .

Proof. We make use of the same argument in [3], Lemma 9. Since  $P^* = \sum_{\alpha_i}^{\infty} \oplus P_{\alpha_i}$  has the  $\aleph_0$ -exchange property by [2], Lemma 3.10, we may assume  $I = \{\alpha_i\}^{\infty}$ . Let  $\{n_i\}$  be the given small homomorphisms. Put  $P_i = \{p_i + n_i p_i\}$  $p_i \in P_i$   $\subset P_i \oplus P_{i+1}$ . Then  $P = P_1 \oplus P_2' \oplus P_3 \oplus P_4' \oplus \cdots = P_1' \oplus P_2 \oplus P_3' \oplus P_4 \oplus \cdots$ . Since  $P_i \approx P_i$ ,  $\sum_{n=0}^{\infty} \oplus P_{2n+1}$  has the  $\aleph_0$ -exchange property. Hence,  $P = \sum_{n=0}^{\infty} \oplus P_{2n+1} \oplus P_{2n+1}$  $P_1^{(1)} \oplus P_2^{\prime(1)} \oplus P_3^{(1)} \oplus P_4^{\prime(1)} \oplus \cdots$ , where  $P_{2n+1}^{(1)}$  and  $P_{2n+2}^{\prime(1)}$  are direct summands of  $P_{2n+1}$  and  $P_{2n+2}'$ , respectively. Since  $P_{2n+2}' \approx P_{2n+2}$ ,  $P_{2n+2}'^{(1)} = P_{2n+2}^{(1)'}$ , where  $P_{2n+2}$  $=P_{2n+2}^{(1)} \oplus P_{2n+2}^{(2)}$ . Let  $p_{2n}$  be the projection of P to  $P_{2n}$  with respect to the decomposition  $P = \sum_{i=1}^{\infty} \bigoplus P_i$  Then  $P_{2n} = p_{2n}(P) = n_{2n-1}(P_{2n-1}) + P_{2n}^{(1)}$  from the latest decomposition. On the other hand,  $n_{2n-1}(P_{2n-1})$  is small in  $P_{2n}$  by the definition and hence  $P_{2n}^{(1)} = P_{2n}$ . We consider the two decompositions  $P = (P_1^{\prime} \oplus P_1^{(1)}) \oplus$  $\{P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \cdot\} = \sum_{i=1}^{\infty} \oplus P_i$ . We shall show  $P_1^{(1)} = (0)$ . Let x be in  $P_1^{(1)}$ . If  $n_1x$  is contained in  $\{P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \}$ , then  $x = x + n_1x + n_2x + n_3x + n_3x$  $(-n_1x) \in (P_1' \oplus \{P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \}) \cap P_1^{(1)} = (0)$  from the former decomposition and so x=0, which implies that  $n_1 p_1^{(1)}$  is monomorphic. Let y be any element in  $P_2$  in the latter decomposition. Consider the expression of y in the former, then  $y = x_1 + x_1' + n_1 x_1' + x_2' + n_2 x_2' + y'$ , where  $x_1 \in P_1^{(1)}, x_1' \in P_1$ ,  $x_2' \in P_2$  and  $y \in (P_3' \oplus P_3^{(1)} \oplus P_4' \oplus )$ . We consider this expression in the latter, then  $x_1 = -x_1', n_2 x_2' = -y'$  and  $y = n_1 x_1' + x_2'$ . We define a submodule N in  $P_2$  as follows:  $N = \{z \mid \in P_2, n_2 z \in (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \}$ . Then we obtain  $P_2 = n_1(P_1^{(1)})$  $\oplus N$  from the above arguments. On the other hand,  $n_1(P_1^{(1)})$  is small in  $P_2$  and hence,  $n_1(P_1^{(1)})=0$ . We have already known that  $n_1 P_1^{(1)}$  is monomorphic. Therefore,  $P_1^{(1)} = (0)$  and  $P = P_1' \oplus P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \cdots$ . Consider an expression of element  $x_3$  in  $P_3$  in the above decomposition, then  $x_3 = x_3^{(1)} + x_3' + x_3$  $n_3x_3'+y, x_3^{(1)} \in P_3^{(1)}, x_3' \in P_3$  and  $y \in \{P_4' \oplus \cdot\}$ . Hence if we repeat the same argument on the direct summand  $P_3 \oplus P_4$  instead of the direct summand  $P_1 \oplus P_2$ , we know  $P_3^{(1)} = (0)$ . Similarly, we obtain  $P_{2n+1}^{(1)} = (0)$  for all *n*. Thus, we have  $P = \sum_{i=1}^{\infty} \bigoplus P_i'$ . It is easyfrom this fact to prove the lemma (cf. [3], Lemma 9).

**Theorem 1.** Let *R* be a ring. Then the following statements are equivalent. 1) *R* is a right perfect ring (see [1]).

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- 2) For every projective module P,
  - i *P* has the exchange property,
  - ii  $\operatorname{End}_{R}(P)/J(\operatorname{End}_{R}(P))$ ; a regular ring in the sense of Von Neumann.
- 3) Put  $P_0 = \sum_{1}^{\infty} \oplus R$ .
  - i  $P_0$  has the exchange property,
  - ii  $\operatorname{End}_{R}(P_{0})/J(\operatorname{End}_{R}(P_{0}))$  is a regular ring.
- 4) i  $P_0$  has the exchange property,
  - ii R/J(R) is artinian.

Proof. 1) $\rightarrow$ 2) Let *R* be perfect and  $R = \sum_{i=1}^{n} \bigoplus e_i R$ , where fa is a complete set of mutually orthogonal primitive idempotents (see [1]). Let P be a projective *R*-module. Then  $P \approx \sum_{j} (\sum_{J_i} \bigoplus e_i R)$  and faR is a *T*-nilpotent system of completely indecomposable modules by [1]. Hence,  $\sum_{J_i} \bigoplus e_i R$  has the exchange property by Lemma 4 and so does *P* from [2], Lemma 3.10. ii is obtained by [8].

2) $\rightarrow$ 3) It is clear.

3) $\rightarrow$ 4) Since  $P_0$  has the exchange property, J(R) is right T-nilpotent from Lemma 5 and [10], Lemma 1. It is well known that  $\operatorname{End}_R(P_0)$  is isomorphic to the ring of column finite matrices over R with degree  $\aleph_0$ . Since J(R) is right T-nilpotent,  $J(\operatorname{End}_R(P_0))$  is isomorphic to the subring of column finite matrices over J(R) by [9] or [5], Corollary 1 to Proposition 1. Hence,  $\operatorname{End}_R(P_0)/J(\operatorname{End}_R(P_0))$  is isomorphic to the ring of column finite matrices over R/J(R). Therefore, R/J(R) is artinian by [5], Corollary to Lemma 2. 4) $\rightarrow$ 1) It is clear from Lemma 5 and [1].

**Proposition 1.** Let R be a semi-perfectring (see [1]) and P a projective R-module. Then P is semi-perfect(see [8]) if and only if P has the exchange property.

Proof. Since *R* is semi-perfect, P is isomorphic to a module  $\sum_{i=1}^{n} \sum_{J_i} \bigoplus e_i R$  by [11]. If *P* is semi-perfect,  $\{e_i R\}_{J_i}$  is semi-*T*-nilpotent by [8] or [4], Theorem 7. Hence, P has the exchange property from Lemma 4. The converse is clear from [4], Theorem 7.

Finally, we shall add here some remarks concerned with (\*).

**Lemma 6.** Let  $\{M_{\alpha}\}_{I}$ , M and N be as in (\*). // N is uniform,  $p_{\alpha}i$  is monomorphic for some  $\alpha$ .

Proof. Let rapObe in N and  $n = \sum_{i=1}^{n} m_{\alpha_i}$ . Put  $M_0 = \sum_{I - \{\alpha_i\}} \bigoplus M_{\alpha}$  and let  $p_0$ :  $M \to M_0$  be its projection. Then  $0 = (\cap \operatorname{Ker}(p_{\alpha_j}i)) \cap \operatorname{Ker}(p_0i)$ . Since N is uniform and  $\operatorname{Ker}(p_0i) \neq 0$ ,  $\operatorname{Ker}(p_{\alpha_j}i) = 0$  for some  $\alpha_j$ . **Corollary 1** (cf. [12]) Let  $\{M_{\alpha}\}_{I}$  and M be as above. We assume that all  $M_{\alpha}$  are uniform and each  $M_{\alpha}$  is not isomorphic to a proper submodule in  $M_{\beta}$ for all  $\alpha$ ,  $\beta$  (e.g. all  $M_{\alpha}$  are injective). Then M has the exchange property if and only if  $\{M_{\alpha}\}_{I}$  is a locally semi-T-nilpotentsystem.

Proof. It is clear from Lemma 6 and [4].

For an *R*-module *L* we denote its composition length by ||L||.

**Corollary** 2. We assume all  $M_{\alpha}$  are uniform and of  $|M_{\alpha}| \le n < \infty$  for all a. Then  $M = \sum_{\alpha \in I} \bigoplus M_{\alpha}$  has the exchange property.

Proof. Put  $M(i) = \sum_{T_i} \bigoplus M_{\gamma}$ , where  $||M_{\gamma}|| = i$ . Then M(i) satisfies (\*) by Lemma 6. On the other hand,  $\{M_{\alpha}\}_{I}$  is *T*-nilpotent by [3], Corollary to Lemma 12. Hence, *M* has the exchange property by Lemma 2 and [2], Lemma 3.10.

### 3. $\aleph_0$ -exchange property

Let  $\{M_{\alpha}\}_{I}$  be a set of completely indecomposable modules and  $\mathfrak{A}$  the induced category from  $\{M_{\alpha}\}_{I}$ . We have shown in [4] that every object in  $\mathfrak{A}$  has the exchange property in  $\mathfrak{A}$  if and only if  $\{M_{\alpha}\}_{I}$  is a locally *T*-nilpotentsystem.

In this section we shall study a similar theorem to the above. We rearrange  $\{M_{\alpha}\}_{I}$  as follows:  $\{M_{\alpha\beta}\}_{\alpha\in\mathbb{K},\beta\in J_{\alpha}}$  such that  $M_{\alpha\beta}\approx M_{\alpha\beta'}$  and  $M_{\alpha\beta}\approx M_{\alpha'\beta'}$  if  $\alpha \neq \alpha'$ . Put  $K^{(1)} = \{\alpha \in K, |J_{\alpha}| < \aleph_0\}, K^{(2)} = \{\alpha \in K, |J_{\alpha}| \geq \aleph_0\}$  and  $M^{(i)} = \sum_{\alpha\in\mathbb{K}^{(i)}}\sum_{\beta\in J_{\alpha}} \oplus M_{\alpha\beta}$ , where |K| means the cardinal of K. Then  $M = M^{(1)} \oplus M^{(2)}$ . In Section 2, we have mainly studied a case  $M = M^{(2)}$ . We shall consider here a case  $M = M^{(1)}$ .

**Lemma** 7. Let  $M = M^{(1)} \oplus M^{(2)}$  be as above. We assume  $\{M_{a}\}_{I}$  is a locally *T*-nilpotent system. If either  $M^{(2)} = 0$  or  $M^{(1)} = 0$  and  $|\mathbf{K}^{(2)}| = 1$ , then every monomorphism fin  $Hom_{R}(M, M)$  has a left inverse, namely Imf is a direct summand of M, (cf. [13], Proposition 6).

Proof. We first note that there exist indices  $a, \beta \in I$  such that  $p_{\beta}f|_{M_{\alpha}}$  is isomorphic from the proof of Lemma 4. Let  $\mathfrak{S}$  be the set of direct summands  $\Sigma \bigoplus M_{\delta}$  of M such that  $\Sigma \bigoplus f(M_{\delta})$  are locally direct summands of M. Then  $\mathfrak{S}$ contains a maximal element  $K = \sum_{L} \bigoplus M_{\delta}$  with respect to the inclusion. Since  $\{f(M_{\delta})\}_{L}$  is locally T-nilpotent, f(K) is a direct summand of M and  $M = f(K) \oplus$  $\sum_{I'} \bigoplus M_{\epsilon'}$  by Lemma 10 in [3] and Lemma 3 in [6], where  $M_{\epsilon'}$  are isomorphic to some in  $\{M_{\alpha}\}_{I}$ . We assume  $K \neq M$ . Then  $f(M) = f(K) \oplus f(M) \cap (\sum_{I'} \bigoplus M_{\epsilon'})$ . Let p be the projection of M to  $\Sigma \bigoplus_{I'} M_{\epsilon'}$ , then  $pf|_{\Sigma \bigoplus M_{\alpha}}$  is monomorphic. On the other hand, if  $M^{(2)}=0$ ,  $\sum_{I,L} \oplus M_{\alpha} \approx \sum_{I'} \oplus M_{\varepsilon}'$  and we may assume  $pf|_{\substack{\Sigma \oplus M_{\alpha} \\ I-L}}$  is a monomorphism in  $\operatorname{Hom}_{R}(\sum_{I'} \oplus M_{\varepsilon}', \sum_{I'} \oplus M_{\varepsilon}')$ . Hence, in either case  $M^{(1)}=0$ or  $M^{(2)}=0$ , there exists  $\alpha'$  in I-L such that  $pf(M_{\alpha'})$  is a direct summand of  $\sum_{I'} \oplus M_{\varepsilon}'$  from the first argument and Lemma 3. Therefore,  $f(K \oplus M_{\alpha'})$  is a direct summand of M, which contradicts the maximality of K. Thus, we have proved the lemma.

REMARK. Lemma 7 is not true for the different cases from the assumption.

The following lemma is substantially due to [2], Lemma 3.11.

**Lemma 8.** We assume that an *R*-module *T* has the finite exchange property and  $T \oplus T' = \sum_{i=1}^{\infty} \oplus A_i = A$ . Then each  $A_i$  contains a direct summand  $A_i'$  such that  $T \cap (\sum_{i=1}^{\infty} \oplus A_i') = 0$  and  $T \circledast \sum_{i=1}^{n} \oplus A_i'$  is a direct summand of *A* for any *n*.

Proof. Put  $K_n = \sum_{i \ge n} \oplus A_i$ . We assume  $A = T \oplus A_1' \oplus A_2' \oplus \cdots \oplus A_n' \oplus K_{n+1'}$ , where A and  $K_{n+1}'$  are direct summands of  $A_i$  and  $K_{n+1}$ , respectively, say  $A_i = A_i' \oplus A_i'', K_{n+1} = K_{n+1'} \oplus K_{n+1''}$ . Since  $T \approx \sum_{i=1}^n \oplus A_i'' \oplus K_{n+1''}, K_{n+1''}$  has the finite exchange property by [2], Lemma 3.10. We may assume  $K_{n+1''} = (T \oplus A_1' \oplus A_2' \oplus \cdots \oplus A_n') \cap K_{n+1}$  and hence  $T \oplus A_1' \oplus A_2' \oplus \cdots \oplus A_n'$  contains  $K_{n+1''}$  as a direct summand;  $T \oplus A_1' \oplus A_2' \oplus \cdots \oplus A_n' = K_{n+1''} \oplus P$ . Put  $K_{n+1} = A_{n+1} \oplus K_{n+2}$ . Then  $K_{n+1} = K_{n+1'} \oplus A_{n+1'} \oplus K_{n+2'}$ , since  $K_{n+1''}$  has the finite exchange property. Thus,  $A = T \oplus A_1' \oplus A_2' \oplus \cdots \oplus A_n' \oplus K_{n+1'} = K_{n+1'} \oplus P \oplus K_{n+1'} = K_{n+1} \oplus P \oplus K_{n+1'} \oplus P \oplus A_{n+1'} \oplus K_{n+2'} = T \oplus A_1' \oplus A_2' \oplus \cdots \oplus A_n' \oplus A_{n+1'} \oplus K_{n+2'}$ .

In Lemma 8 we have obtained  $A_i = A_i' \oplus A_i''$  and  $A = (\sum_{i=1}^{\infty} \oplus A_i') \oplus (\sum_{i=1}^{\infty} \oplus A_i'')$ . Let  $p_n$  be the projection of A to  $\sum_{i=1}^{n} \oplus A_i''$  in the decomposition above.

**Lemma 9.**  $p_n(T) = \sum_{i=1}^n \bigoplus A_i$  for any n.

Proof. Let p be the projection of A to  $\sum_{i=1}^{\infty} \oplus A_i''$  Since  $A = \sum_{i=1}^{n} \oplus A_i' \oplus T \oplus K_{n+1}'$  and  $K_{n+1} = \sum_{i>n} \bigoplus A_i' \oplus \sum_{i>n} \oplus A_i'', p(A) = p(T \oplus K_{n+1}') \subseteq p(T) + p(K_{n+1}) = p(T) + \sum_{i>n} \oplus A_i'' \subseteq p(A)$ . Hence,  $p_n(T) = \sum_{i=1}^{n} \oplus A_i''$ .

**Lemma 10.** If  $\{M_{\alpha}\}_{I}$  is fo<sup> $\alpha$ </sup>/fy *T*-nilpotentand  $M^{(2)}=0$ , then *M* has the  $\aleph_{0}$ -exchange property.

Proof. Let  $M = M^{(1)} = \sum_{\alpha \in K} \sum_{\beta \in J_{\alpha}} \bigoplus M_{\alpha\beta}$  as above and  $M \oplus N = \sum_{i=1}^{\infty} \bigoplus A_i = A$ . Then *M* has the finite exchange property by [12], Proposition 1.7. Hence, we obtain direct summands A of  $A_i$  such that  $M \cap (\sum_{i=1} \bigoplus A_i) = (0)$  from Lemma 8. Since

$$\sum_{i=1}^{n} \oplus A_{i}^{\prime\prime} \oplus K_{n+1}^{\prime\prime} \approx M \cdots (**),$$

 $\begin{array}{l} A_{i}^{\prime\prime} \approx \sum_{\alpha \in \mathbb{R}^{(i)}} \sum_{\beta \in J_{\alpha}^{(i)}} \oplus M_{\alpha\beta} \quad \text{and} \quad J_{\alpha}^{(i)} \subseteq J_{\alpha}, \ \mathbb{K}^{(i)} \subseteq \mathbb{K} \ \text{by} \ [3], \ \text{Theorem 9. Since} \\ |J_{\alpha}| < \aleph_{0}, \ \sum_{1}^{\infty} |J_{\alpha}^{(i)}| \quad \text{is finite and} \quad |J_{\alpha}| \ge \sum |J_{\alpha}^{(i)}| \quad \text{from (**). Now,} \\ \sum_{1}^{\infty} \oplus A_{i} = \sum_{1}^{\infty} \oplus A_{i}^{\prime} \oplus \sum_{1}^{\infty} \oplus A_{i}^{\prime\prime} \quad \text{and let } p \ \text{be the projection to} \ \sum_{1}^{\infty} \oplus A_{i}^{\prime\prime} \quad \text{Since} \\ M \cap (\sum_{1}^{\infty} \oplus A_{i}^{\prime}) = (0), \ \mathbf{M} \ \text{is isomorphic to} \ p(M) \subseteq \sum_{1}^{\infty} \oplus A_{i}^{\prime\prime}. \ \text{From the above} \\ \text{argument we may assume that} \ p \setminus_{M} \text{is a monomorphism in } \operatorname{Hom}_{R}(M, M). \ \text{Then} \\ p(M) \oplus \sum_{1}^{\infty} \oplus A_{i}^{\prime\prime\prime}; \ A_{i}^{\prime\prime\prime\prime} \subseteq A_{i}^{\prime\prime}, \ \text{by } \ [4'], \ \text{Corollary 2 to Proposition 1. Thus,} \\ \sum_{1}^{\infty} \oplus A_{i} = M \oplus \sum_{1}^{\infty} \oplus (A_{i}^{\prime} \oplus A_{i}^{\prime\prime\prime\prime}). \end{array}$ 

**Lemma 11.** //  $\{M_{\alpha}\}_{I}$  is locally semi-T-nilpotentand  $M = M^{(1)}$  (i.e.  $M^{(2)} = 0$ ) is R-projective, then M has the tf  $_{0}$ -exchange property.

Proof. Let  $M = M^{(1)} = \sum_{\alpha \in \mathbb{R}^{(1)}} \sum_{\beta \in \overline{J}_{\alpha}} \bigoplus M_{\alpha\beta}$ . We shall use the same notations as in the proof of Lemma 10. We have obtained the monomorphism p of M to  $\sum_{1}^{\infty} \oplus A_i''$  and  $A_i''$  are in  $\mathfrak{A}$  from (\*\*) and [4], Theorem 4. Put  $A_i'' = \sum_{\alpha \in \mathbb{R}^{(1)}} \sum_{\beta \in \overline{J}^{(1)}} \bigoplus M_{\alpha\beta}'$ . Since  $|J_{\alpha}^{(1)}| < \aleph_0$ ,  $\sum_{1}^{\infty} |J_{\alpha}^{(1)}| \leq J_{\alpha}|$  from (\*\*). We consider all  $M_{\alpha\beta}'$ and p in  $\overline{\mathfrak{A}}$ . Since M is projective, so is A' from (\*\*). Furthermore,  $p_n p_M$ is epimorphic to  $\sum_{1}^{\infty} \oplus A_i''$  by Lemma 9 and so  $p_n p \setminus_M$  splits. Therefore,  $p_n \overline{p} \setminus_{\overline{M}}$ is epimorphic in  $\overline{\mathfrak{A}}$ . Now,  $\sum_{1}^{\infty} \bigoplus A_i'' = \sum_{\bigcup \overline{L}^{(1)}} \sum_{\substack{j \in J_{\alpha}^{(1)} \\ i \neq J_{\alpha}^{(j)}}} \bigoplus M_{\alpha\beta'}$ . Since  $\sum_{1}^{||J_{\alpha}^{(i)}|} < \aleph_0$ ,  $\sum_{1}^{\infty} \oplus M_{\alpha\beta'} = a$  direct summand of some  $\sum_{1}^{t} \oplus A_i''$ . Let q be the projection of  $\sum_{1}^{\infty} \oplus A_i''$  to  $\sum_{\sum J_{\alpha}^{(1)}} \bigoplus M_{\alpha\beta'}$ . Then  $\overline{qp}|_{\overline{M}}$  is epimorphic from the above. On the other hand,  $\overline{qp}(\sum_{\alpha \neq \mu} \oplus \overline{M}_{\alpha'\beta}) = (0)$ , since  $\overline{M}_{\alpha'\beta}$  are minimal and  $\overline{M}_{\alpha'\beta} \approx \overline{M}_{\alpha\beta}$ . Hence,  $\overline{p}(\sum_{\beta \in J_{\alpha}} \oplus \overline{M}_{\alpha\beta}) = \sum_{\substack{j J_{\alpha}^{(1)}} \oplus \overline{M}_{\alpha\beta'}}$ , which implies  $\overline{p}|_{\overline{M}}$  is epimorphic. Since  $\overline{\mathfrak{A}}$  is a regular abelian category from [3], Theorem 7, there exists  $t: \sum_{1}^{\infty} \oplus A_i'' \to M$ such that  $pt = \overline{\Gamma}_{\Sigma \oplus \overline{A}_i}'' \oplus M$ .

Let  $\mathfrak{A}(f)$  be the subadditive category of  $\mathfrak{A}$ , whose objects consist of all A

such that  $A = A^{(1)} \oplus A^{(2)}$  and  $|\mathbf{K}^{(2)}| < \aleph_0$ . Summarizing the above we have

**Proposition** 2. Let  $\{M_{\alpha}\}_{I}$  be a set of completely indecomposable modules and  $\mathfrak{A}(f)$  as above. Then  $\{M_{\alpha}\}_{I}$  is a locally T-nilpotent system if and only if every module in  $\mathfrak{A}(f)$  has the  $\aleph_{0}$ -exchange property.

**Proposition 3.** Let P be a projective R-module in  $\mathfrak{A}(f)$ . Then P is semiperfect if and only if P has the  $\aleph_0$ -exchange property.

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