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ON PERFECT RINGS AND THE EXCHANGE PROPERTY

Dedicated to Professor Kiiti Morita on his 60th birthday

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Let R be a ring with unit element. We always consider unitary right R-modules. Let T be an R-module and η a cardinal number. If for any module K containing T as a direct summand and for any decomposition of K with η components: $K=\sum_{\alpha\in I}\oplus A_\alpha$, there exist submodules A_α' of A_α for all α such that $K=T\oplus\sum_{\alpha\in I}\oplus A_\alpha'$, then we say T has the *η-exchange property* [2]. If T has the η-exchange property for any η, we say T has the *exchange property.*

In this short note we shall show that R is a right perfect ring if and only if for every projective module P, P has the exchange property and $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ is a regular ring in the sense of Von Neumann. This is a refinement of Theorem 7 in [4] and we shall give its proof as an application of [6].

After submitting this paper to the journal, the authors have received a manuscript of Yamagata [13] and found that one of main theorems in this paper overlaped with one in [13]. The authors would like to express their thanks to Dr. Yamagata for his kindness.

1. Preliminaries

First we shall recall some definitions given in [3], [4] and [6]. Let T be an R-module. If $\mathrm{End}_R(T)$ is a local ring, T is called *completely indecomposable.* We take a set $\{M_\alpha\}_I$ of completely indecomposable modules and define the full additive subcategory $\mathfrak{A}$ of all right R-modules which is induced from $\{M_\alpha\}_I$, namely the objects in $\mathfrak{A}$ consist of all modules which are isomorphic to direct-sums of completely indecomposable modules in $\{M_\alpha\}_I$. We define an ideal $\mathfrak{J}'$ in $\mathfrak{A}$ as follows: let $A=\sum_{\alpha\in K}\oplus A_\alpha$, $B=\sum_{\beta\in L}\oplus B_\beta$ be in $\mathfrak{A}$, where A_α, B_β are isomorphic to some in $\{M_\alpha\}_I$, then $\mathfrak{J}'\cap[A, B]=\{f\in\mathrm{Hom}_R(A, B), p_\beta f i_\alpha$ are not isomorphic for all $\alpha\in K, \beta\in L\}$, where $i_\alpha: A_\alpha\to A, p_\beta: B\to B_\beta$ are the inclusion and the projection, respectively. By $\bar{\mathfrak{A}}$ we denote the factor category of $\mathfrak{A}$ with respect to $\mathfrak{J}'$ [3]. For any object A and morphism f in $\mathfrak{A}$, by $\bar{A}$ and $\bar{f}$ we denote the residue classes of A and f in $\bar{\mathfrak{A}}$, respectively.

We take a countable subset $\{M_{\alpha_i}\}_1^\infty$ of $\{M_\alpha\}_I$ (resp. we can take same modules M_{α_i} in $\{M_{\alpha_i}\}_1^\infty$ as many as we want) and a set of homomorphisms $f_i \in \mathfrak{J}' \Pi [M_{\alpha_i}, M_{\alpha_{i+1}}]$. If for any element m in M_{α_1} there exists n, which depends on m, such that $f_n f_{n-1} \quad f_1(m)=0$, $\{f_i\}$ is called *locally semi-T-nilpotent* (resp. *T-nilpotent*). If every $\{f_i\}$ is locally semi-T-nilpotent (resp. T-nilpotent) for every subset $\{M_{\alpha_i}\}$, we say $\{M_\alpha\}_I$ is a *locally semi-T-nilpotent* (resp. *T-nilpotent*) *system* [4]. Finally, let $M \supset N$ be modules and $N=\sum_{\gamma\in J}\oplus N_\gamma$. If for any finite subset J' of J $\sum_{\gamma\in J'}\oplus N_\gamma$ is a direct summand of M, N is called a *locally direct summand* of M (with respect to the decomposition $\sum_{\gamma\in J}\oplus N_\gamma$) [6].

2. Perfect rings

Let $\{M_\alpha\}$ be a set of completely indecomposable modules and $M=\sum_{\alpha\in I}\oplus M_\alpha$. We understand p_α means the projection of M to M_α in the decomposition if there are no confusions. Let N be a submodole of M, which is isomorphic to one in $\{M_\alpha\}_I$. We shall consider a strong condition:

each N above is a direct summand of M $\cdots$(*)

Lemma 1. *Let M and* $\{M_\alpha\}_I$ *be as above. We assume* $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent system and M satisfies* (*). *Let A be a submodule of M. Then we have* $A=A_1\oplus A_2$, *where* A_1 *is a direct summand of M (and hence* $A_1\in\mathfrak{A}$) *and* A_2 *does not contain any submodules which are isomorphic to some in* $\{M_\alpha\}_I$.

Proof. Let $\mathfrak{S}$ be the set of submodules A' in A as follows: A is in $\mathfrak{A}$, say $A'=\sum_{\alpha\in J}\oplus A_\alpha$; A_α are isomorphic to some in $\{M_\alpha\}_I$ and A' is a locally direct summand of M with respect to this decomposition. We can define a partial order in $\mathfrak{S}$ by members of direct components (cf. [6]). Then we obtain a maximal one in $\mathfrak{S}$ by Zorn's lemma, say A_1. Since $\{M_\alpha\}_I$ is locally semi-T-nilpotent, A_1 is a direct summand of M: $M=A_1\oplus M_1$ by Theorem 9 in [3], Theorem in [7] and Lemma 3 and Corollary 2 to Lemma 2 in [6]. Hence, $A=A_1\oplus(A\cap M_1)$ and $A\cap M_1$ does not contain any submodules in $\mathfrak{A}$ from the assumption and the maximality of A_1.

The following lemma is a modification of one part of Theorem 2.6 in [12].

Lemma 2. *Let* $\{M_\alpha\}_I$ *and M be as above. We assume M satisfies* (*). *Then M has the exchange property if and only if* $\{M_\alpha\}_I$ *is a locally semi-T-nilpotent system.*

Proof. If M has the exchange property, then $\{M_\alpha\}_I$ is a locally semi-Γ-nilpotent system by [4′], Corollary 2 to Proposition 1. Conversely, we assume that $\{M_\alpha\}_I$ is semi-Γ-nilpotent. Let $A=M\oplus N=\sum_{\alpha\in J}\oplus A_\alpha$ We may assume from [2], Theorem 8.2 that all A_α are isomorphic to submodules in M, in order to show that M has the exchange property. Then from the assumption and Lemma 1, $A_\alpha=A_\alpha'\oplus A_\alpha''$, where $A_\alpha'\in\mathfrak{A}$ and A_α'' does not contain any submodules, isomorphic to some in $\{M_\alpha\}_I$. Put $A'=\sum_{\alpha\in J}\oplus A_\alpha'$ and $A''=\sum_{\alpha\in J}\oplus A_\alpha''$, then $A=A'\oplus A''$. Let $\varphi: A\to A/A''$ be the natural epimorphism. We shall show that M is a locally direct summand of A/A'' through φ. Let I' be a finite subset of I and $M'=\sum_{\alpha\in I'}\oplus M_\alpha$. Since M' has the exchange property by [11], Proposition 1 and [2], Lemma 3.10, $A=M'\oplus A_0'\oplus A_0''$, where $A_0'\subset A'$ and $A_0''\subset A''$. Then $A''=A_0''\oplus K''$ and K'' is isomorphic to a direct summand of M'. If $K''\neq 0$, K'' contains a completely indecomposable module K_1 (isomorphic to one in $\{M_\alpha\}_I$) as a direct summand by Krull-Remak-Schmidt theorem. Since K_1 has the exchange property, we know from the argument above that some A_α'' contains a submodule isomorphic to K_1. Which is a contradiction. Hence, $A=M'\oplus A_0'\oplus A''$ and $\varphi(M)\approx M$ is a locally direct summand of A/A''. Since $A/A''\approx A'\in\mathfrak{A}$ and $\{M_\alpha\}_I$ is locally semi-T-nilpotent, $\varphi(M)$ is a direct summand of A/A'' by [6], Lemma 3; $A/A''=\varphi(M)\oplus\varphi(K)$ and $K\subset A'$. Furthermore, $\varphi(M)$ has the exchange property in $\mathfrak{A}$ by [4], Corollary 2 to Proposition 1 and hence $A/A''=\varphi(M)\oplus\sum_{\alpha\in J}\oplus\varphi(A_\alpha''')$ where $A_\alpha'''\subset A_\alpha'$. Therefore, $A=M\oplus\sum_{\alpha\in J}\oplus(A_\alpha'''\oplus A_\alpha'')$.

Next, we shall consider some cases where M satisfies (*).

Lemma 3. *Let $\{M_\alpha\}_I$, M and N be as in (*) and $i: N\to M$ the inclusion. Then N is a direct summand of M if and only if $p_\alpha i$ is isomorphic for some α in I.*

Proof. It is clear from the definition of $\mathfrak{J}'$.

Lemma 4. *Let M_1 be a completely indecomposable module. We assume M_1 is a locally T-nilpotent system itself. Then $M=\sum_{\alpha\in I}\oplus M_\alpha; M_\alpha\approx M_1$ has the exchange property for any set I.*

Proof. We shall show that M satisfies (*). We may assume $N=M_1$. We put $f_\alpha=p_\alpha i$ and assume that f_α are not isomorphic for all $\alpha\in I$. Let $m\neq 0\in N$ and $i(m)=\sum_{i=1}^{n}f_{\alpha_i}(m)$. Since i is monomorphic, we may assume $f_{\alpha_1}(m)=m_2\neq 0$. Let $i(m_2)=\sum_{i=1}^{n'}f_{\alpha_i'}(m_2)$. Repeating this argument, we obtain a sequence $\{f_{\beta_i}\}_1^\infty$ such that $f_{\beta_n}f_{\beta_{n-1}}\cdots f_{\beta_1}(m)\neq 0$ for any n, which contradicts the T-nilpotency of $\{M_1\}$. Therefore, M satisfies (*) by Lemma 3.

Let A, B be R-modules and $f\in\mathrm{Hom}_R(A, B)$. If $\mathrm{Im}\, f$ is small in B, f is called a *small homomorphism*. We note that if $A=B$ are R-projective, then the Jacobson radical $J(\mathrm{End}_R(A))$ of $\mathrm{End}_R(A)$ consists of all small homomorphisms by [10], Lemma 1.

Lemma 5. *Let* $\{P_\alpha\}_I$ *be a set of R-modules and* $P=\sum_{\alpha\in I}\oplus P_\alpha$. *If* P *has the* $\aleph_0$*-exchange property, then any sequence of small homomorphisms* $\{n_{\alpha_i}: P_{\alpha_i}\to P_{\alpha_{i+1}}\}$ *is locally semi-T-nilpotent for any countable subset* $\{P_{\alpha_i}\}_1^\infty$ *of* $\{P_\alpha\}_I$.

Proof. We make use of the same argument in [3], Lemma 9. Since $P^*=\sum_{i=1}^\infty\oplus P_{\alpha_i}$ has the $\aleph_0$-exchange property by [2], Lemma 3.10, we may assume $I=\{\alpha_i\}^\infty$. Let $\{n_i\}$ be the given small homomorphisms. Put $P_i'=\{p_i+n_ip_i \mid p_i\in P_i\}\subset P_i\oplus P_{i+1}$. Then $P=P_1\oplus P_2'\oplus P_3\oplus P_4'\oplus\cdots=P_1'\oplus P_2\oplus P_3'\oplus P_4\oplus\cdots$. Since $P_i'\approx P_i$, $\sum_{n=0}^\infty\oplus P_{2n+1}'$ has the $\aleph_0$-exchange property. Hence, $P=\sum_{n=0}^\infty\oplus P_{2n+1}'\oplus P_1^{(1)}\oplus P_2'^{(1)}\oplus P_3^{(1)}\oplus P_4'^{(1)}\oplus\cdots$, where $P_{2n+1}^{(1)}$ and $P_{2n+2}'^{(1)}$ are direct summands of P_{2n+1} and P_{2n+2}', respectively. Since $P_{2n+2}'\approx P_{2n+2}$, $P_{2n+2}'^{(1)}=P_{2n+2}^{(1)'}$, where $P_{2n+2}=P_{2n+2}^{(1)}\oplus P_{2n+2}^{(2)}$. Let p_{2n} be the projection of P to P_{2n} with respect to the decomposition $P=\sum_{i=1}^\infty\oplus P_i$. Then $P_{2n}=p_{2n}(P)=n_{2n-1}(P_{2n-1})+P_{2n}^{(1)}$ from the latest decomposition. On the other hand, $n_{2n-1}(P_{2n-1})$ is small in P_{2n} by the definition and hence $P_{2n}^{(1)}=P_{2n}$. We consider the two decompositions $P=(P_1'\oplus P_1^{(1)})\oplus\{P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\cdots\}=\sum_{i=1}^\infty\oplus P_i$. We shall show $P_1^{(1)}=(0)$. Let x be in $P_1^{(1)}$. If n_1x is contained in $\{P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\cdots\}$, then $x=x+n_1x+(-n_1x)\in(P_1'\oplus\{P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\cdots\})\cap P_1^{(1)}=(0)$ from the former decomposition and so $x=0$, which implies that $n_1|P_1^{(1)}$ is monomorphic. Let y be any element in P_2 in the latter decomposition. Consider the expression of y in the former, then $y=x_1+x_1'+n_1x_1'+x_2'+n_2x_2'+y'$, where $x_1\in P_1^{(1)}$, $x_1'\in P_1$, $x_2'\in P_2$ and $y\in(P_3'\oplus P_3^{(1)}\oplus P_4'\oplus\cdots)$. We consider this expression in the latter, then $x_1=-x_1'$, $n_2x_2'=-y'$ and $y=n_1x_1'+x_2'$. We define a submodule N in P_2 as follows: $N=\{z\mid\in P_2, n_2z\in(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\cdots\}$. Then we obtain $P_2=n_1(P_1^{(1)})\oplus N$ from the above arguments. On the other hand, $n_1(P_1^{(1)})$ is small in P_2 and hence, $n_1(P_1^{(1)})=0$. We have already known that $n_1|P_1^{(1)}$ is monomorphic. Therefore, $P_1^{(1)}=(0)$ and $P=P_1'\oplus P_2'\oplus(P_3'\oplus P_3^{(1)})\oplus P_4'\oplus\cdots$. Consider an expression of element x_3 in P_3 in the above decomposition, then $x_3=x_3^{(1)}+x_3'+n_3x_3'+y$, $x_3^{(1)}\in P_3^{(1)}$, $x_3'\in P_3$ and $y\in\{P_4'\oplus\cdots\}$. Hence if we repeat the same argument on the direct summand $P_3\oplus P_4$ instead of the direct summand $P_1\oplus P_2$, we know $P_3^{(1)}=(0)$. Similarly, we obtain $P_{2n+1}^{(1)}=(0)$ for all n. Thus, we have $P=\sum_{i=1}^\infty\oplus P_i'$. It is easy from this fact to prove the lemma (cf. [3], Lemma 9).

Theorem 1. *Let R be a ring. Then the following statements are equivalent.*

1) *R is a right perfect ring* (see [1]).

2) *For every projective module P,*

i *P has the exchange property,*

ii $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ *is a regular ring in the sense of Von Neumann.*

3) *Put* $P_0=\sum_1^\infty \oplus R$.

i P_0 *has the exchange property,*

ii $\mathrm{End}_R(P_0)/J(\mathrm{End}_R(P_0))$ *is a regular ring.*

4) i P_0 *has the exchange property,*

ii *R/J(R) is artinian.*

Proof. 1)→2) Let R be perfect and $R=\sum_{i=1}^n \oplus e_iR$, where $\{e_i\}$ is a complete set of mutually orthogonal primitive idempotents (see [1]). Let P be a projective R-module. Then $P\approx\sum_i(\sum_{J_i}\oplus e_iR)$ and $\{e_iR\}$ is a T-nilpotent system of completely indecomposable modules by [1]. Hence, $\sum_{J_i}\oplus e_iR$ has the exchange property by Lemma 4 and so does P from [2], Lemma 3.10. ii is obtained by [8].

2)→3) It is clear.

3)→4) Since P_0 has the exchange property, $J(R)$ is right T-nilpotent from Lemma 5 and [10], Lemma 1. It is well known that $\mathrm{End}_R(P_0)$ is isomorphic to the ring of column finite matrices over R with degree $\aleph_0$. Since $J(R)$ is right T-nilpotent, $J(\mathrm{End}_R(P_0))$ is isomorphic to the subring of column finite matrices over $J(R)$ by [9] or [5], Corollary 1 to Proposition 1. Hence, $\mathrm{End}_R(P_0)/J(\mathrm{End}_R(P_0))$ is isomorphic to the ring of column finite matrices over $R/J(R)$. Therefore, $R/J(R)$ is artinian by [5], Corollary to Lemma 2.

4)→1) It is clear from Lemma 5 and [1].

Proposition 1. *Let R be a semi-perfect ring (see* [1]) *and P a projective R-module. Then P is semi-perfect (see* [8]) *if and only if P has the exchange property.*

Proof. Since R is semi-perfect, P is isomorphic to a module $\sum_{i=1}^n\sum_{J_i}\oplus e_iR$ by [11]. If P is semi-perfect, $\{e_iR\}_{J_i}$ is semi-T-nilpotent by [8] or [4], Theorem 7. Hence, P has the exchange property from Lemma 4. The converse is clear from [4], Theorem 7.

Finally, we shall add here some remarks concerned with (*).

Lemma 6. *Let* $\{M_\alpha\}_I$, *M and N be as in* (*). *If N is uniform,* $p_\alpha i$ *is monomorphic for some* α.

Proof. Let $x\neq 0$ be in N and $n=\sum_{i=1}^n m_{\alpha_i}$. Put $M_0=\sum_{I-\{\alpha_i\}}\oplus M_\alpha$ and let p_0: $M\to M_0$ be its projection. Then $0=(\cap\,\mathrm{Ker}(p_{\alpha_j}i))\cap\mathrm{Ker}(p_0i)$. Since N is uniform and $\mathrm{Ker}(p_0i)\neq 0$, $\mathrm{Ker}(p_{\alpha_j}i)=0$ for some α_j.

Corollary 1 (cf. [12]) *Let $\{M_\alpha\}_I$ and M be as above. We assume that all M_α are uniform and each M_α is not isomorphic to a proper submodule in M_β for all α, β (e.g. all M_α are injective). Then M has the exchange property if and only if $\{M_\alpha\}_I$ is a locally semi-T-nilpotent system.*

Proof. It is clear from Lemma 6 and [4].

For an R-module L we denote its composition length by $||L||$.

Corollary 2. *We assume all M_α are uniform and of $|M_\alpha|| \leqslant n < \infty$ for all a. Then $M=\sum_{\alpha\in I}\oplus M_\alpha$ has the exchange property.*

Proof. Put $M(i)=\sum_{J_i}\oplus M_\gamma$, where $||M_\gamma||=i$. Then $M(i)$ satisfies (*) by Lemma 6. On the other hand, $\{M_\alpha\}_I$ is T-nilpotent by [3], Corollary to Lemma 12. Hence, M has the exchange property by Lemma 2 and [2], Lemma 3.10.

3. $\aleph_0$-exchange property

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $\mathfrak{A}$ the induced category from $\{M_\alpha\}_I$. We have shown in [4] that every object in $\mathfrak{A}$ has the exchange property in $\mathfrak{A}$ if and only if $\{M_\alpha\}_I$ is a locally T-nilpotent system.

In this section we shall study a similar theorem to the above. We rearrange $\{M_\alpha\}_I$ as follows: $\{M_{\alpha\beta}\}_{\alpha\in\mathrm{K},\beta\in J_\alpha}$ such that $M_{\alpha\beta}\approx M_{\alpha\beta'}$ and $M_{\alpha\beta}\not\approx M_{\alpha'\beta'}$ if $\alpha\neq\alpha'$. Put $\mathrm{K}^{(1)}=\{\alpha|\in\mathrm{K},\ |J_\alpha|<\aleph_0\}$, $\mathrm{K}^{(2)}=\{\alpha|\in\mathrm{K},\ |J_\alpha|\geqslant\aleph_0\}$ and $M^{(i)}=\sum_{\alpha\in\mathrm{K}^{(i)}}\sum_{\beta\in J_\alpha}\oplus M_{\alpha\beta}$, where $|\mathrm{K}|$ means the cardinal of K. Then $M=M^{(1)}\oplus M^{(2)}$. In Section 2, we have mainly studied a case $M=M^{(2)}$. We shall consider here a case $M=M^{(1)}$.

Lemma 7. *Let $M=M^{(1)}\oplus M^{(2)}$ be as above. We assume $\{M_\alpha\}_I$ is a locally T-nilpotent system. If either $M^{(2)}=0$ or $M^{(1)}=0$ and $|\mathrm{K}^{(2)}|=1$, then every monomorphism f in $Hom_R(M,M)$ has a left inverse, namely $Im f$ is a direct summand of M, (cf.* [13], Proposition 6).

Proof. We first note that there exist indices a, $\beta\in I$ such that $p_\beta f|_{M_\alpha}$ is isomorphic from the proof of Lemma 4. Let $\mathfrak{S}$ be the set of direct summands $\sum_{L'}\oplus M_\delta$ of M such that $\sum_{L'}\oplus f(M_\delta)$ are locally direct summands of M. Then $\mathfrak{S}$ contains a maximal element $K=\sum_{L}\oplus M_\delta$ with respect to the inclusion. Since $\{f(M_\delta)\}_L$ is locally T-nilpotent, $f(K)$ is a direct summand of M and $M=f(K)\oplus\sum_{I'}\oplus M_\varepsilon'$ by Lemma 10 in [3] and Lemma 3 in [6], where M_ε' are isomorphic to some in $\{M_\alpha\}_I$. We assume $K\neq M$. Then $f(M)=f(K)\oplus f(M)\cap(\sum_{I'}\oplus M_\varepsilon')$. Let p be the projection of M to $\sum_{I'}\oplus M_\varepsilon'$, then $pf|_{\sum_{I-L}\oplus M_\alpha}$ is monomorphic. On the other hand, if $M^{(2)}=0$, $\sum_{I-L}\oplus M_\alpha\approx\sum_{I'}\oplus M_\varepsilon'$ and we may assume $pf|_{\sum_{I-L}\oplus M_\alpha}$ is a monomorphism in $\mathrm{Hom}_R(\sum_{I'}\oplus M_\varepsilon', \sum_{I'}\oplus M_\varepsilon')$. Hence, in either case $M^{(1)}=0$ or $M^{(2)}=0$, there exists α' in $I-L$ such that $pf(M_{\alpha'})$ is a direct summand of $\sum_{I'}\oplus M_\varepsilon'$ from the first argument and Lemma 3. Therefore, $f(K\oplus M_{\alpha'})$ is a direct summand of M, which contradicts the maximality of K. Thus, we have proved the lemma.

REMARK. Lemma 7 is not true for the different cases from the assumption.

The following lemma is substantially due to [2], Lemma 3.11.

Lemma 8. *We assume that an R-module T has the finite exchange property and* $T\oplus T'=\sum_{i=1}^{\infty}\oplus A_i=A$. *Then each* A_i *contains a direct summand* A_i' *such that* $T\cap(\sum_{i=1}^{\infty}\oplus A_i')=0$ *and* $T\oplus\sum_{i=1}^{n}\oplus A_i'$ *is a direct summand of A for any n.*

Proof. Put $K_n=\sum_{i\geqslant n}\oplus A_i$. We assume $A=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus K_{n+1}'$, where A and K_{n+1}' are direct summands of A_i and K_{n+1}, respectively, say $A_i=A_i'\oplus A_i''$, $K_{n+1}=K_{n+1}'\oplus K_{n+1}''$. Since $T\approx\sum_{i=1}^{n}\oplus A_i''\oplus K_{n+1}''$, K_{n+1}'' has the finite exchange property by [2], Lemma 3.10. We may assume $K_{n+1}''=(T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n')\cap K_{n+1}$ and hence $T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'$ contains K_{n+1}'' as a direct summand; $T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'=K_{n+1}''\oplus P$. *Put* $K_{n+1}=A_{n+1}\oplus K_{n+2}$. Then $K_{n+1}=K_{n+1}''\oplus A_{n+1}'\oplus K_{n+2}'$, since K_{n+1}'' has the finite exchange property. Thus, $A=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus K_{n+1}'=K_{r+1}''\oplus P\oplus K_{n+1}'=K_{n+1}\oplus P=K_{n+1}''\oplus P\oplus A_{n+1}'\oplus K_{n+2}'=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus A_{n+1}'\oplus K_{n+2}'$.

In Lemma 8 we have obtained $A_i=A_i'\oplus A_i''$ and $A=(\sum_{i=1}^{\infty}\oplus A_i')\oplus(\sum_{i=1}^{\infty}\oplus A_i'')$. Let p_n be the projection of A to $\sum_{i=1}^{n}\oplus A_i''$ in the decomposition above.

Lemma 9. $p_n(T)=\sum_{i=1}^{n}\oplus A_i''$ *for any n.*

Proof. Let p be the projection of A to $\sum_{i=1}^{\infty}\oplus A_i''$ Since $A=\sum_{i=1}^{n}\oplus A_i'\oplus T\oplus K_{n+1}'$ and $K_{n+1}=\sum_{i>n}\oplus A_i'\oplus\sum_{i>n}\oplus A_i''$, $p(A)=p(T\oplus K_{n+1}')\subseteq p(T)+p(K_{n+1})=p(T)+\sum_{i>n}\oplus A_i''\subseteq p(A)$. Hence, $p_n(T)=\sum_{i=1}^{n}\oplus A_i''$.

Lemma 10. *If* $\{M_\alpha\}_I$ *is locally T-nilpotent and* $M^{(2)}=0$, *then M has the* $\aleph_0$*-exchange property.*

Proof. Let $M=M^{(1)}=\sum_{\alpha\in K}\sum_{\beta\in J_\alpha}\oplus M_{\alpha\beta}$ as above and $M\oplus N=\sum_{i=1}^{\infty}\oplus A_i=A$. Then M has the finite exchange property by [12], Proposition 1.7. Hence, we

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obtain direct summands A of A_i such that $M\cap(\sum_{i=1}\oplus A_i')=(0)$ from Lemma 8. Since

$$\sum_{i=1}^{n}\oplus A_i''\oplus K_{n+1}''\approx M\cdots(**)\,,$$

$A_i''\approx\sum_{\alpha\in K^{(i)}}\sum_{\beta\in J_\alpha^{(i)}}\oplus M_{\alpha\beta}$ and $J_\alpha^{(i)}\subseteqq J_\alpha$, $K^{(i)}\subseteqq K$ by [3], Theorem 9. Since $|J_\alpha|<\aleph_0$, $\sum_1^\infty|J_\alpha^{(i)}|$ is finite and $|J_\alpha|\geqslant\sum|J_\alpha^{(i)}|$ from (**). Now, $\sum_1^\infty\oplus A_i=\sum_1^\infty\oplus A_i'\oplus\sum_1^\infty\oplus A_i''$ and let p be the projection to $\sum_1^\infty\oplus A_i''$ Since $M\cap(\sum_1\oplus A_i')=(0)$, M is isomorphic to $p(M)\subseteqq\sum_1^\infty\oplus A_i''$. From the above argument we may assume that $p\backslash_M$ is a monomorphism in $\mathrm{Hom}_R(M,\mathrm{M})$. Then $p(M)$ is a direct summand of $\sum_1^\infty\oplus A_i''$ by Lemma 7. Hence, $\sum_1^\infty\oplus A_i''=p(M)\oplus\sum_1^\infty\oplus A_i'''$; $A_i'''\subseteqq A_i''$, by [4'], Corollary 2 to Proposition 1. Thus, $\sum_1^\infty\Theta A_i=M\oplus\sum_1^\infty\oplus(A_i'\oplus A_i''')$.

Lemma 11. *// $\{M_\alpha\}_I$ is locally semi-T-nilpotentand $M=M^{(1)}$ (i.e. $M^{(2)}=$ 0) is R-projective, then M has the* tf $_0$*-exchangeproperty.*

Proof. Let $M=M^{(1)}=\sum_{\alpha\in K^{(1)}}\sum_{\beta\in J_\alpha^{(1)}}\oplus M_{\alpha\beta}$. We shall use the same notations as in the proof of Lemma 10. We have obtained the monomorphism p of M to $\sum_1^\infty\oplus A_i''$and A_i'' are in $\mathfrak{A}$ from (**) and [4], Theorem 4. Put $A_i''=\sum_{\alpha\in K^{(i)}}\sum_{\beta\in J^{(i)}}\oplus M_{\alpha\beta}'$. Since $|J_\alpha^{(1)}|<\aleph_0$, $\sum_1^\infty|J_\alpha^{(i)}\leqslant J_\alpha|$ from (**). We consider all $M_{\alpha\beta}'$ and p in $\bar{\mathfrak{A}}$. Since M is projective, so is A $'$ from (**). Furthermore, $p_n p$ $_M$ is epimorphic to $\sum_1^n\oplus A_i''$ by Lemma 9 and so $p_n p\backslash_M$splits. Therefore, $\overline{p_n p}\backslash_{\bar{M}}$ is epimorphic in $\bar{\mathfrak{A}}$. Now, $\sum_1^\infty\oplus A_i''=\sum_{\cup K^{(i)}}\sum_{\sum_i J_\alpha^{(i)}}\oplus M_{\alpha\beta}'$. Since $\sum_i|J_\alpha^{(i)}|<\aleph_0$, $\sum_{\sum\bar{J}_\alpha^{(i)}}\oplus\bar{M}_{\alpha\beta}'$ is a direct summand of some $\sum_1^t\oplus\bar{A}_i''$. Let q be the projection of $\sum_1^\infty\oplus A_i''$ to $\sum_{\sum J_\alpha^{(i)}}\oplus M_{\alpha\beta}'$. Then $\overline{qp}|_{\bar{M}}$ is epimorphic from the above. On the other hand, $\overline{qp}(\sum_{\alpha'\neq\alpha}\oplus\bar{M}_{\alpha'\beta})=(0)$, since $\bar{M}_{\alpha'\beta}$are minimal and $\bar{M}_{\alpha'\beta}\not\approx\bar{M}_{\alpha\beta}$. Hence, $\bar{p}(\sum_{\beta\in J_\alpha}\oplus\bar{M}_{\alpha\beta})=\sum_{\sum_i J_\alpha^{(i)}}\oplus\bar{M}_{\alpha\beta}'$, which implies $\bar{p}|_{\bar{M}}$ is epimorphic. Since $\bar{\mathfrak{A}}$ is a regular abelian category from [3], Theorem 7, there exists $t\colon\sum_1^\infty\oplus A_i''\to M$such that $pt=1_{\Sigma\oplus\bar{A}_i''}$ Therefore, p is epimorphic as R-modules by [6] and [7]. Thus, $A=\sum_1^\infty\oplus A_i'\oplus M$.

Let $\mathfrak{A}(f)$ be the subadditive category of $\mathfrak{A}$, whose objects consist of all A

such that $A=A^{(1)}\oplus A^{(2)}$ and $|K^{(2)}|<\aleph_0$.

Summarizing the above we have

Proposition 2. *Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $\mathfrak{A}(f)$ as above. Then $\{M_\alpha\}_I$ is a locally T-nilpotent system if and only if every module in $\mathfrak{A}(f)$ has the $\aleph_0$-exchange property.*

Proposition 3. *Let P be a projective R-module in $\mathfrak{A}(f)$. Then P is semi-perfect if and only if P has the $\aleph_0$-exchange property.*

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