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Author(s)	Harada, Manabu; Ishii, Tadamasa
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ON PERFECT RINGS AND THE EXCHANGE PROPERTY

Dedicated to Professor Kiiti Morita on his 60th birthday

Manabu HARADA and Tadamasa ISHII

Let R be a ring with unit element. We always consider unitary right R-modules. Let T be an R-module and η a cardinal number. If for any module K containing T as a direct summand and for any decomposition of K with η components: $K = \sum_{\alpha \in I} \oplus A_\alpha$, there exist submodules A_α' of A_α for all a such that $K = T \oplus \sum_{\alpha \in I} \oplus A_\alpha'$, then we say T has the η-*exchange property*[2]. If T has the η-exchange property for any η, we say T has the *exchange property*.

In this short note we shall show that R is a right perfect ring if and only if for every projective module P, P has the exchange property and $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ is a regular ring in the sense of Von Neumann. This is a refinement of Theorem 7 in [4] and we shall give its proof as an application of [6].

After submitting this paper to the journal, the authors have received a manuscript of Yamagata [13] and found that one of main theorems in this paper overlaped with one in [13]. The authors would like to express their thanks to Dr. Yamagata for his kindness.

1. Preliminaries

First we shall recall some definitions given in [3], [4] and [6]. Let T be an R-module. If $\mathrm{End}_R(T)$ is a local ring, T is called *completely indecomposable*. We take a set $\{M_\alpha\}_I$ of completely indecomposable modules and define the full additive subcategory $\mathfrak{A}$ of all right R-modules which is induced from $\{M_\alpha\}_I$, namely the objects in $\mathfrak{A}$ consist of all modules which are isomorphic to direct-sums of completely indecomposable modules in $\{M_\alpha\}_I$. We define an ideal $\mathfrak{J}'$ in $\mathfrak{A}$ as follows: let $A = \sum_{\alpha \in K} \oplus A_\alpha$, $B = \sum_{\beta \in L} \oplus B_\beta$ be in $\mathfrak{A}$, where A_α, B_β are isomorphic to some in $\{M_\alpha\}_I$, then $\mathfrak{J}' \cap [A, B] = \{f \mid \in \mathrm{Hom}_R(A, B), p_\beta f i_\alpha$ are not isomorphic for all $\alpha \in K, \beta \in L\}$, where $i_\alpha : A_\alpha \to A, p_\beta : B \to B_\beta$ are the inclusion and the projection, respectively. By $\bar{\mathfrak{A}}$ we denote the factor category of $\mathfrak{A}$ with respect to $\mathfrak{J}'$ [3]. For any object A and morphism f in $\mathfrak{A}$, by $\bar{A}$ and $\bar{f}$ we denote the residue classes of A and $/$ in $\bar{\mathfrak{A}}$, respectively.

We take a countable subset $\{M_{\alpha_i}\}_1^\infty$ of $\{M_\alpha\}_I$ (resp. we can take same modules M_{α_i} in $\{M_{\alpha_i}\}_1^\infty$ as many as we want) and a set of homomorphisms $f_i \in \mathfrak{F}' \Pi [M_{\alpha_i}, M_{\alpha_{i+1}}]$. If for any element m in M_{α_1} there exists n, which depends on m, such that $f_n f_{n-1} \ldots f_1(m)=0$, $\{f_i\}$ is called *locally semi-T-nilpotent* (resp. *T-nilpotent*). If every $\{f_i\}$ is locally semi-T-nilpotent (resp. T-nilpotent) for every subset $\{M_{\alpha_i}\}$, we say $\{M_\alpha\}_I$ is a *locally semi-T-nilpotent* (resp. *T-nilpotent*) system [4]. Finally, let $M \supset N$ be modules and $N = \sum_{\gamma \in J} \oplus N_\gamma$. If for any finite subset J' of J $\sum_{\gamma \in J'} \oplus N_\gamma$ is a direct summand of M, N is called a *locally direct summand* of M (with respect to the decomposition $\sum_{\gamma \in J} \oplus N_\gamma$) [6].

2. Perfect rings

Let $\{M_\alpha\}$ be a set of completely indecomposable modules and $M = \sum_{\alpha \in I} \oplus M_\alpha$. We understand p_α means the projection of M to M_α in the decomposition if there are no confusions. Let N be a submodole of M, which is isomorphic to one in $\{M_\alpha\}_I$. We shall consider a strong condition:

each N above is a direct summand of $M \cdots$(*)

Lemma 1. *Let M and $\{M_\alpha\}_I$ be as above. We assume $\{M_\alpha\}_I$ is a locally semi-T-nilpotent system and M satisfies*(*). *Let A be a submodule of M. Then we have $A = A_1 \oplus A_2$, where A_1 is a direct summand of M (and hence $A_1 \in \mathfrak{A}$) and A_2 does not contain any submodules which are isomorphic to some in $\{M_\alpha\}_I$.*

Proof.   Let $\mathfrak{S}$ be the set of submodules A' in A as follows: A is in $\mathfrak{A}$, say $A' = \sum_{\alpha \in J} \oplus A_\alpha$; A_α are isomorphic to some in $\{M_\alpha\}_I$ and A' is a locally direct summand of M with respect to this decomposition. We can define a partial order in $\mathfrak{S}$ by members of direct components (cf. [6]). Then we obtain a maximal one in $\mathfrak{S}$ by Zorn's lemma, say A_1. Since $\{M_\alpha\}_I$ is locally semi-T-nilpotent, A_1 is a direct summand of M: $M = A_1 \oplus M_1$ by Theorem 9 in [3], Theorem in [7] and Lemma 3 and Corollary 2 to Lemma 2 in [6]. Hence, $A = A_1 \oplus (A \cap M_1)$ and $A \cap M_1$ does not contain any submodules in $\mathfrak{A}$ from the assumption and the maximality of A_1.

The following lemma is a modification of one part of Theorem 2.6 in [12].

Lemma 2.   *Let $\{M_\alpha\}_I$ and M be as above. We assume M satisfies*(*). *Then M has the exchange property if and only if $\{M_\alpha\}_I$ is a locally semi-T-nilpotent system.*

Proof.   If M has the exchange property, then $\{M_\alpha\}_I$ is a locally semi-Γ-nilpotent system by [4]', Corollary 2 to Proposition 1. Conversely, we assume that $\{M_\alpha\}_I$ is semi-Γ-nilpotent. Let $A = M \oplus N = \sum_{\alpha \in J} \oplus A_\alpha$ We may assume

from [2], Theorem 8.2 that all A_α are isomorphic to submodules in M, in order to show that M has the exchange property. Then from the assumption and Lemma 1, $A_\alpha = A_\alpha' \oplus A_\alpha''$, where $A_\alpha' \in \mathfrak{A}$ and A_α'' does not contain any submodules, isomorphic to some in $\{M_\alpha\}_I$. Put $A' = \sum_{\alpha \in J} \oplus A_\alpha'$ and $A'' = \sum_{\alpha \in J} \oplus A_\alpha''$, then $A = A' \oplus A''$. Let $\varphi : A \to A/A''$ be the natural epimorphism. We shall show that M is a locally direct summand of A/A'' through φ. Let I' be a finite subset of I and $M' = \sum_{\alpha \in I'} \oplus M_\alpha$. Since M' has the exchange property by [11], Proposition 1 and [2], Lemma 3.10, $A = M' \oplus A_0' \oplus A_0''$, where $A_0' \subset A'$ and $A_0'' \subset A''$. Then $A'' = A_0'' \oplus K''$ and K'' is isomorphic to a direct summand of M'. If $K'' \neq 0$, K'' contains a completely indecomposable module K_1 (isomorphic to one in $\{M_\alpha\}_I$) as a direct summand by Krull-Remak-Schmidt theorem. Since K_1 has the exchange property, we know from the argument above that some A_α'' contains a submodule isomorphic to K_1. Which is a contradiction. Hence, $A = M' \oplus A_0' \oplus A''$ and $\varphi(M) \approx M$ is a locally direct summand of A/A''. Since $A/A'' \approx A' \in \mathfrak{A}$ and $\{M_\alpha\}_I$ is locally semi-T-nilpotent, $\varphi(M)$ is a direct summand of A/A'' by [6], Lemma 3; $A/A'' = \varphi(M) \oplus \varphi(K)$ and $K \subset A'$. Furthermore, $\varphi(M)$ has the exchange property in $\mathfrak{A}$ by [4], Corollary 2 to Proposition 1 and hence $A/A'' = \varphi(M) \oplus \sum_{\alpha \in J} \oplus \varphi(A_\alpha''')$ where $A_\alpha''' \subset A_\alpha'$. Therefore,

$$A = M \oplus \sum_{\alpha \in J} \oplus (A_\alpha''' \oplus A_\alpha'').$$

Next, we shall consider some cases where M satisfies (*).

Lemma 3. *Let $\{M_\alpha\}_I$, M and N be as in (*) and $i : N \to M$ the inclusion. Then N is a direct summand of M if and only if $p_\alpha i$ is isomorphic for some α in I.*

Proof. It is clear from the definition of $\mathfrak{F}'$.

Lemma 4. *Let M_1 be a completely indecomposable module. We assume M_1 is a locally T-nilpotent system itself. Then $M = \sum_{\alpha \in I} \oplus M_\alpha$; $M_\alpha \approx M_1$ has the exchange property for any set I.*

Proof. We shall show that M satisfies (*). We may assume $N = M_1$. We put $f_\alpha = p_\alpha i$ and assume that f_α are not isomorphic for all $\alpha \in I$. Let $m \neq 0 \in N$ and $i(m) = \sum_{i=1}^n f_{\alpha_i}(m)$. Since i is monomorphic, we may assume $f_{\alpha_1}(m) = m_2 \neq 0$. Let $i(m_2) = \sum_{i=1}^{n'} f_{\alpha_i'}(m_2)$. Repeating this argument, we obtain a sequence $\{f_{\beta_i}\}_1^\infty$ such that $f_{\beta_n} f_{\beta_{n-1}} \cdots f_{\beta_1}(m) \neq 0$ for any n, which contradicts the T-nilpotency of $\{M_1\}$. Therefore, M satisfies (*) by Lemma 3.

Let A, B be R-modules and $f \in \mathrm{Hom}_R(A, B)$. If Im f is small in B, f is called a *small homomorphism*. We note that if $A = B$ are R-projective, then

the Jacobson radical $J(\mathrm{End}_R(A))$ of $\mathrm{End}_R(A)$ consists of all small homomorphisms by [10], Lemma 1.

Lemma 5. *Let* $\{P_\alpha\}_I$ *be a set of R-modules and* $P = \sum_{\alpha \in I} \oplus P_\alpha$. *If P has the* $\aleph_0$-*exchange property, then any sequence of small homomorphisms* $\{n_{\alpha_i} : P_{\alpha_i} \to P_{\alpha_{i+1}}\}$ *is locally semi-T-nilpotent for any countable subset* $\{P_{\alpha_i}\}_1^\infty$ *of* $\{P_\alpha\}_I$.

Proof.   We make use of the same argument in [3], Lemma 9.   Since $P^* = \sum_{i=1}^\infty \oplus P_{\alpha_i}$ has the $\aleph_0$-exchange property by [2], Lemma 3.10, we may assume $I = \{\alpha_i\}^\infty$.   Let $\{n_i\}$ be the given small homomorphisms.   Put $P_i' = \{p_i + n_i p_i \mid p_i \in P_i\} \subset P_i \oplus P_{i+1}$.   Then $P = P_1 \oplus P_2' \oplus P_3 \oplus P_4' \oplus \cdots = P_1' \oplus P_2 \oplus P_3' \oplus P_4 \oplus \cdots$. Since $P_i' \approx P_i$, $\sum_{n=0}^\infty \oplus P_{2n+1}'$ has the $\aleph_0$-exchange property.   Hence, $P = \sum_{\alpha=0}^\infty \oplus P_{2n+1}' \oplus P_1^{(1)} \oplus P_2'^{(1)} \oplus P_3^{(1)} \oplus P_4'^{(1)} \oplus \cdots$, where $P_{2n+1}^{(1)}$ and $P_{2n+2}'^{(1)}$ are direct summands of P_{2n+1} and P_{2n+2}', respectively.   Since $P_{2n+2}' \approx P_{2n+2}$, $P_{2n+2}'^{(1)} = P_{2n+2}^{(1)}{}'$, where $P_{2n+2} = P_{2n+2}^{(1)} \oplus P_{2n+2}^{(2)}$.   Let p_{2n} be the projection of P to P_{2n} with respect to the decomposition $P = \sum_{i=1}^\infty \oplus P_i$ Then $P_{2n} = p_{2n}(P) = n_{2n-1}(P_{2n-1}) + P_{2n}^{(1)}$ from the latest decomposition.   On the other hand, $n_{2n-1}(P_{2n-1})$ is small in P_{2n} by the definition and hence $P_{2n}^{(1)} = P_{2n}$.   We consider the two decompositions $P = (P_1' \oplus P_1^{(1)}) \oplus \{P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \cdot\} = \sum_{i=1}^\infty \oplus P_i$.   We shall show $P_1^{(1)} = (0)$.   Let x be in $P_1^{(1)}$.   If $n_1 x$ is contained in $\{P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \}$, then $x = x + n_1 x + (-n_1 x) \in (P_1' \oplus \{P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \}) \cap P_1^{(1)} = (0)$ from the former decomposition and so $x = 0$, which implies that $n_{1\,P_1^{(1)}}$ is monomorphic.   Let y be any element in P_2 in the latter decomposition.   Consider the expression of y in the former, then $y = x_1 + x_1' + n_1 x_1' + x_2' + n_2 x_2' + y'$, where $x_1 \in P_1^{(1)}$, $x_1' \in P_1$, $x_2' \in P_2$ and $y \in (P_3' \oplus P_3^{(1)} \oplus P_4' \oplus)$.   We consider this expression in the latter, then $x_1 = -x_1'$, $n_2 x_2' = -y'$ and $y = n_1 x_1' + x_2'$.   We define a submodule N in P_2 as follows: $N = \{z \mid \in P_2,\ n_2 z \in (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \}$.   Then we obtain $P_2 = n_1(P_1^{(1)}) \oplus N$ from the above arguments.   On the other hand, $n_1(P_1^{(1)})$ is small in P_2 and hence, $n_1(P_1^{(1)}) = 0$.   We have already known that $n_{1\,P_1^{(1)}}$ is monomorphic.   Therefore, $P_1^{(1)} = (0)$ and $P = P_1' \oplus P_2' \oplus (P_3' \oplus P_3^{(1)}) \oplus P_4' \oplus \cdots$.   Consider an expression of element x_3 in P_3 in the above decomposition, then $x_3 = x_3^{(1)} + x_3' + n_3 x_3' + y$, $x_3^{(1)} \in P_3^{(1)}$, $x_3' \in P_3$ and $y \in \{P_4' \oplus \cdot\}$.   Hence if we repeat the same argument on the direct summand $P_3 \oplus P_4$ instead of the direct summand $P_1 \oplus P_2$, we know $P_3^{(1)} = (0)$.   Similarly, we obtain $P_{2n+1}^{(1)} = (0)$ for all n.   Thus, we have $P = \sum_{i=1}^\infty \oplus P_i'$.   It is easy from this fact to prove the lemma (cf. [3], Lemma 9).

Theorem 1. *Let R be a ring.*   *Then the following statements are equivalent.*
1) *R is a right perfect ring* (see [1]).

2) *For every projective module P,*

  i *P has the exchange property,*

  ii $\mathrm{End}_R(P)/J(\mathrm{End}_R(P))$ *a regular ring in the sense of Von Neumann.*

3) *Put* $P_0 = \sum_1^\infty \oplus R.$

  i P_0 *has the exchange property,*

  ii $\mathrm{End}_R(P_0)/J(\mathrm{End}_R(P_0))$ *is a regular ring.*

4) i P_0 *has the exchange property,*

  ii *R/J(R) is artinian.*

Proof.   1)→2)   Let R be perfect and $R = \sum_{i=1}^n \oplus e_i R,$ where $fa\}$ is a complete set of mutually orthogonal primitive idempotents (see [1]). Let P be a projective R-module.   Then $P \approx \sum (\sum_{J_i} \oplus e_i R)$ and $faR\}$ is a T-nilpotent system of completely indecomposable modules by [1]. Hence, $\sum_{J_i} \oplus e_i R$ has the exchange property by Lemma 4 and so does P from [2], Lemma 3.10.   ii is obtained by [8].

2)→3)   It is clear.

3)→4)   Since P_0 has the exchange property, $J(R)$ is right T-nilpotent from Lemma 5 and [10], Lemma 1.   It is well known that $\mathrm{End}_R(P_0)$ is isomorphic to the ring of column finite matrices over R with degree $\aleph_0$.   Since $J(R)$ is right T-nilpotent, $J(\mathrm{End}_R(P_0))$ is isomorphic to the subring of column finite matrices over $J(R)$ by [9] or [5], Corollary 1 to Proposition 1.   Hence, $\mathrm{End}_R(P_0)/J(\mathrm{End}_R(P_0))$ is isomorphic to the ring of column finite matrices over $R/J(R)$.   Therefore, $R/J(R)$ is artinian by [5], Corollary to Lemma 2.

4)→1)   It is clear from Lemma 5 and [1].

Proposition 1.   *Let R be a semi-perfect ring (see* [1]) *and P a projective R-module.   Then P is semi-perfect (see* [8]) *if and only if P has the exchange property.*

Proof.   Since R is semi-perfect, P is isomorphic to a module $\sum_{i=1}^n 2 \oplus e_i R$ by [11].   If P is semi-perfect, $\{e_i R\}_{J_i}$ is semi-T-nilpotent by [8] or [4], Theorem 7.   Hence, P has the exchange property from Lemma 4.   The converse is clear from [4], Theorem 7.

Finally, we shall add here some remarks concerned with (*).

Lemma 6.   *Let* $\{M_\alpha\}_I$, *M and N be as in* (*).   *// N is uniform, $p_\alpha i$ is monomorphic for some* α.

Proof.   Let $ra\varphi O$ be in N and $n = \sum_{i=l}^n m_{\alpha_i}.$ Put $M_0 = \sum_{I-\{\alpha_i\}} \oplus M_\alpha$ and let $p_0 : M \to M_0$ be its projection.   Then $0 = (\cap \mathrm{Ker}(p_{\alpha_j} i)) \cap \mathrm{Ker}(p_0 i).$ Since N is uniform and $\mathrm{Ker}(p_0 i) \neq 0,$ $\mathrm{Ker}(p_{\alpha_j} i) = 0$ for some α_j.

Corollary 1 (cf. [12])   *Let $\{M_\alpha\}_I$ and M be as above. We assume that all M_α are uniform and each M_α is not isomorphic to a proper submodule in M_β for all α, β (e.g. all M_α are injective). Then M has the exchange property if and only if $\{M_\alpha\}_I$ is a locally semi-T-nilpotent system.*

Proof.   It is clear from Lemma 6 and [4].

For an R-module L we denote its composition length by $||L||$.

Corollary 2.   *We assume all M_α are uniform and of $|M_\alpha|| \leqslant n < \infty$ for all a. Then $M = \sum_{\alpha \in I} \oplus M_\alpha$ has the exchange property.*

Proof.   Put $M(i) = \sum_{T_i} \oplus M_\gamma$, where $||M_\gamma|| = i$.   Then $M(i)$ satisfies (*) by Lemma 6.   On the other hand, $\{M_\alpha\}_I$ is T-nilpotent by [3], Corollary to Lemma 12.   Hence, M has the exchange property by Lemma 2 and [2], Lemma 3.10.

3. $\aleph_0$-exchange property

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $\mathfrak{A}$ the induced category from $\{M_\alpha\}_I$. We have shown in [4] that every object in $\mathfrak{A}$ has the exchange property in $\mathfrak{A}$ if and only if $\{M_\alpha\}_I$ is a locally T-nilpotent system.

In this section we shall study a similar theorem to the above. We rearrange $\{M_\alpha\}_I$ as follows: $\{M_{\alpha\beta}\}_{\alpha \in \mathbf{K}, \beta \in J_\alpha}$ such that $M_{\alpha\beta} \approx M_{\alpha\beta'}$ and $M_{\alpha\beta} \not\approx M_{\alpha'\beta'}$ if $\alpha \neq \alpha'$. Put $\mathbf{K}^{(1)} = \{\alpha | \in \mathbf{K}, |J_\alpha| < \aleph_0\}$, $\mathbf{K}^{(2)} = \{\alpha | \in \mathbf{K}, |J_\alpha| \geqslant \aleph_0\}$ and $M^{(i)} = \sum_{\alpha \in \mathbf{K}^{(i)}} \sum_{\beta \in J_\alpha} \oplus M_{\alpha\beta}$, where $|\mathbf{K}|$ means the cardinal of $\mathbf{K}$.   Then $M = M^{(1)} \oplus M^{(2)}$. In Section 2, we have mainly studied a case $M = M^{(2)}$. We shall consider here a case $M = M^{(1)}$.

Lemma 7.   *Let $M = M^{(1)} \oplus M^{(2)}$ be as above.   We assume $\{M_\alpha\}_I$ is a locally T-nilpotent system.   If either $M^{(2)} = 0$ or $M^{(1)} = 0$ and $|\mathbf{K}^{(2)}| = 1$, then every monomorphism f in $Hom_R(M, M)$ has a left inverse, namely $Im f$ is a direct summand of M, (cf. [13], Proposition 6).*

Proof.   We first note that there exist indices a, $\beta \in I$ such that $p_\beta f |_{M_\alpha}$ is isomorphic from the proof of Lemma 4.   Let $\mathfrak{S}$ be the set of direct summands $\sum_{L'} \oplus M_\delta$ of M such that $\sum_{L'} \oplus f(M_\delta)$ are locally direct summands of M.   Then $\mathfrak{S}$ contains a maximal element $K = \sum_L \oplus M_\delta$ with respect to the inclusion.   Since $\{f(M_\delta)\}_L$ is locally T-nilpotent, $f(K)$ is a direct summand of M and $M = f(K) \oplus \sum_{I'} \oplus M_\varepsilon'$ by Lemma 10 in [3] and Lemma 3 in [6], where M_ε' are isomorphic to some in $\{M_\alpha\}_I$.   We assume $K \neq M$.   Then $f(M) = f(K) \oplus f(M) \cap (\sum_{I'} \oplus M_\varepsilon')$. Let p be the projection of M to $\sum_{I'} \oplus M_\varepsilon'$, then $pf |_{\sum_{I-L} \oplus M_\alpha}$ is monomorphic.   On

the other hand, if $M^{(2)}=0$, $\sum_{I-L}\oplus M_\alpha \approx \sum_{I'}\oplus M_\varepsilon'$ and we may assume $pf|_{\sum\oplus M_\alpha}$ is
a monomorphism in $\text{Hom}_R(\sum_{I'}\oplus M_\varepsilon', \sum_{I'}\oplus M_\varepsilon')$. Hence, in either case $M^{(1)}=0$
or $M^{(2)}=0$, there exists α' in $I-L$ such that $pf(M_{\alpha'})$ is a direct summand of
$\sum_{I'}\oplus M_\varepsilon'$ from the first argument and Lemma 3. Therefore, $f(K\oplus M_{\alpha'})$ is a
direct summand of M, which contradicts the maximality of K. Thus, we have
proved the lemma.

REMARK. Lemma 7 is not true for the different cases from the assump-
tion.

The following lemma is substantially due to [2], Lemma 3.11.

Lemma 8. *We assume that an R-module T has the finite exchange property
and* $T\oplus T'=\sum_{i=1}^{\infty}\oplus A_i=A$. *Then each* A_i *contains a direct summand* A_i' *such that*
$T\cap(\sum_{i=1}^{\infty}\oplus A_i')=0$ *and* $T\oplus\sum_{i=1}^{n}\oplus A_i'$ *is a direct summand of A for any n.*

Proof. Put $K_n=\sum_{i\geq n}\oplus A_i$. We assume $A=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus K_{n+1}'$,
where A and K_{n+1}' are direct summands of A_i and K_{n+1}, respectively, say $A_i=$
$A_i'\oplus A_i''$, $K_{n+1}=K_{n+1}'\oplus K_{n+1}''$. Since $T\approx\sum_{i=1}^{n}\oplus A_i''\oplus K_{n+1}''$, K_{n+1}'' has the finite
exchange property by [2], Lemma 3.10. We may assume $K_{n+1}''=(T\oplus A_1'\oplus A_2'\oplus$
$\cdots\oplus A_n')\cap K_{n+1}$ and hence $T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'$ contains K_{n+1}'' as a direct
summand; $T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'=K_{n+1}''\oplus P$. Put $K_{n+1}=A_{n+1}\oplus K_{n+2}$. Then
$K_{n+1}=K_{n+1}''\oplus A_{n+1}'\oplus K_{n+2}'$, since K_{n+1}'' has the finite exchange property. Thus,
$A=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus K_{n+1}'=K_{r+1}''\oplus P\oplus K_{n+1}'=K_{n+1}\oplus P=K_{n+1}''\oplus P\oplus$
$A_{n+1}'\oplus K_{n+2}'=T\oplus A_1'\oplus A_2'\oplus\cdots\oplus A_n'\oplus A_{n+1}'\oplus K_{n+2}'$.

In Lemma 8 we have obtained $A_i=A_i'\oplus A_i''$ and $A=(\sum_{i=1}^{\infty}\oplus A_i')\oplus(\sum_{i=1}^{\infty}\oplus A_i'')$.
Let p_n be the projection of A to $\sum_{i=1}^{n}\oplus A_i''$ in the decomposition above.

Lemma 9. $p_n(T)=\sum_{i=1}^{n}\oplus A_i$ *for any n.*

Proof. Let p be the projection of A to $\sum_{i=1}^{\infty}\oplus A_i''$ Since $A=\sum_{i=1}^{n}\oplus A_i'\oplus$
$T\oplus K_{n+1}'$ and $K_{n+1}=\sum_{i>n}\ominus A_i'\oplus\sum_{i>n}\oplus A_i''$, $p(A)=p(T\oplus K_{n+1}')\subseteq p(T)+p(K_{n+1})=$
$p(T)+\sum_{i>n}\oplus A_i''\subseteq p(A)$. Hence, $p_n(T)=\sum_{i=1}^{n}\oplus A_i''$.

Lemma 10. *If* $\{M_\alpha\}_I$ *is fo^α/fy T-nilpotent and* $M^{(2)}=0$, *then M has the*
$\aleph_0$-*exchange property.*

Proof. Let $M=M^{(1)}=\sum_{\alpha\in K}\sum_{\beta\in J_\alpha}\oplus M_{\alpha\beta}$ as above and $M\oplus N=\sum_{i=1}^{\infty}\oplus A_i=A$.
Then M has the finite exchange property by [12], Proposition 1.7. Hence, we

obtain direct summands A of A_i such that $M \cap (\underset{i=1}{\Sigma} \oplus A_i')=(0)$ from Lemma 8. Since

$$\sum_{i=1}^{n} \oplus A_i'' \oplus K_{n+1}'' \approx M \cdots (**),$$

$A_i'' \approx \underset{\alpha \in K^{(i)}}{\Sigma} \underset{\beta \in J_\alpha^{(i)}}{\Sigma} \oplus M_{\alpha\beta}$ and $J_\alpha^{(i)} \subseteq J_\alpha$, $K^{(i)} \subseteq K$ by [3], Theorem 9. Since $|J_\alpha| < \aleph_0$, $\sum_1^\infty |J_\alpha^{(i)}|$ is finite and $|J_\alpha| \geqslant \sum |J_\alpha^{(i)}|$ from (**). Now, $\sum_1^\infty \oplus A_i = \sum_1^\infty \oplus A_i' \oplus \sum_1^\infty \oplus A_i''$ and let p be the projection to $\sum_1^\infty \oplus A_i''$ Since $M \cap (\sum_1 \oplus A_i')=(0)$, M is isomorphic to $p(M) \subseteq \sum_1^\infty \oplus A_i''$. From the above argument we may assume that $p \backslash_M$ is a monomorphism in $\mathrm{Hom}_R(M, M)$. Then $p(M)$ is a direct summand of $\sum_1^\infty \oplus A_i''$ by Lemma 7. Hence, $\sum_1^\infty \oplus A_i'' = p(M) \oplus \sum_1^\infty \oplus A_i'''$; $A_i''' \subseteq A_i''$, by [4'], Corollary 2 to Proposition 1. Thus, $\sum_1^\infty \oplus A_i = M \oplus \sum_1^\infty \oplus (A_i' \oplus A_i''')$.

Lemma 11. // $\{M_\alpha\}_i$ is locally semi-T-nilpotent and $M=M^{(1)}$ (i.e. $M^{(2)}=0$) is R-projective, then M has the tf $_0$-exchange property.

Proof. Let $M = M^{(1)} = \underset{\alpha \in K^{(1)}}{\Sigma} \underset{\beta \in J_\alpha^{(1)}}{\Sigma} \oplus M_{\alpha\beta}$. We shall use the same notations as in the proof of Lemma 10. We have obtained the monomorphism p of M to $\sum_1^\infty \oplus A_i''$ and A_i'' are in $\mathfrak{A}$ from (**) and [4], Theorem 4. Put $A_i'' = \underset{\alpha \in K^{(i)}}{\Sigma} \underset{\beta \in J^{(i)}}{\Sigma} \oplus M_{\alpha\beta}'$. Since $|J_\alpha^{(1)}| < \aleph_0$, $\sum_1^\infty |J_\alpha^{(i)}| \leqslant J_\alpha|$ from (**). We consider all $M_{\alpha\beta}'$ and p in $\overline{\mathfrak{A}}$. Since M is projective, so is A ' from (**). Furthermore, $p_n p_M$ is epimorphic to $\sum_1^\infty \oplus A_i''$ by Lemma 9 and so $p_n p \backslash_M$ splits. Therefore, $p_n \overline{p} \backslash_{\overline{M}}$ is epimorphic in $\overline{\mathfrak{A}}$. Now, $\sum_1^\infty \oplus A_i'' = \underset{\cup K^{(i)}}{\Sigma} \underset{\Sigma J_\alpha^{(i)}}{\Sigma} \oplus M_{\alpha\beta}'$. Since $\sum_i |J_\alpha^{(i)}| < \aleph_0$, $\underset{\Sigma \overline{J}_\alpha^{(i)}}{\Sigma} \oplus \overline{M}_{\alpha\beta}'$ is a direct summand of some $\sum_1^t \oplus A_i''$. Let q be the projection of $\sum_1^\infty \oplus A_i''$ to $\underset{\Sigma J_\alpha^{(i)}}{\Sigma} \oplus M_{\alpha\beta}'$. Then $\overline{qp}|_{\overline{M}}$ is epimorphic from the above. On the other hand, $\overline{qp}(\underset{\alpha' \neq \alpha}{\Sigma} \oplus \overline{M}_{\alpha'\beta})=(0)$, since $\overline{M}_{\alpha'\beta}$ are minimal and $\overline{M}_{\alpha'\beta} \approx \overline{M}_{\alpha\beta}$. Hence, $\overline{p}(\underset{\beta \in J_\alpha}{\Sigma} \oplus \overline{M}_{\alpha\beta}) = \underset{\Sigma \overline{J}_\alpha^{(i)}}{\Sigma} \oplus \overline{M}_{\alpha\beta}'$, which implies $\overline{p}|_{\overline{M}}$ is epimorphic. Since $\overline{\mathfrak{A}}$ is a regular abelian category from [3], Theorem 7, there exists $t: \sum_1^\infty \oplus A_i'' \to \overline{M}$ such that $pt = \overline{1}_{\Sigma \oplus \overline{A}_i''}$ Therefore, p is epimorphic as R-modules by [6] and [7]. Thus, $A = \sum_1^\infty \oplus A_i' \oplus M$.

Let $\mathfrak{A}(f)$ be the subadditive category of $\mathfrak{A}$, whose objects consist of all A

such that $A = A^{(1)} \oplus A^{(2)}$ and $|K^{(2)}| < \aleph_0$.

Summarizing the above we have

Proposition 2.   *Let* $\{M_\alpha\}_I$ *be a set of completely indecomposable modules and* $\mathfrak{A}(f)$ *as above.   Then* $\{M_\alpha\}_I$ *is a locally* T-*nilpotent system if and only if every module in* $\mathfrak{A}(f)$ *has the* $\aleph_0$-*exchange property.*

Proposition 3.   *Let* P *be a projective* R-*module in* $\mathfrak{A}(f)$.   *Then* P *is semi-perfect if and only if* P *has the* $\aleph_0$-*exchange property.*

OSAKA CITY UNIVERSITY
KINKI UNIVERSITY

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