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# THE VERY-STRONG $C^\infty$ TOPOLOGY ON $C^\infty(M, N)$ AND $K$-EQUIVARIANT MAPS

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The main purpose of this paper is to prove Theorem C, given below. Let us however in this introduction first discuss the role of the very-strong $C^\infty$ topology and its relation to the strong $C^\infty$ topology.

Let $M$ and $N$ be $C^\infty$ manifolds. By $C^r(M, N)$, where $1 \leq r \leq \infty$, we denote the set of all $C^r$ maps from $M$ to $N$. In the case when $r$ is finite, and $M$ may be non-compact, there is a well established, standard choice of a topology for $C^r(M, N)$, namely the strong $C^r$ topology, also called the Whitney $C^r$ topology, see e.g. [5], Section 2.1, and [11], Section 2. We denote $C^r(M, N)$ with the strong $C^r$ topology by $C^r_S(M, N)$, $1 \leq r < \infty$. In the case when $r$ is finite the strong $C^r$ topology is clearly *the* right topology to use on $C^r(M, N)$. However, when $r = \infty$ the question concerning the right topology for $C^\infty(M, N)$ is more complex and interesting.

One possible choice of a topology for $C^\infty(M, N)$ is the strong $C^\infty$ topology, introduced by Mather in [11], Section 2, see also [5], Section 2.1. The strong $C^\infty$ topology on $C^\infty(M, N)$ has as a basis the union of all strong $C^r$ topologies on $C^\infty(M, N)$, $1 \leq r < \infty$. (The strong $C^r$ topology on $C^\infty(M, N)$ is the relative topology from $C^r_S(M, N)$, $1 \leq r < \infty$.) Mather calls the strong $C^\infty$ topology the Whitney $C^\infty$ topology, but as we shall see this choice of terminology is not well founded. It is only in the case when $r$ is finite that the strong $C^r$ topology should be named the Whitney topology. In fact du Plessis and Wall, see [15], p. 59, propose that the strong $C^\infty$ topology on $C^\infty(M, N)$ be named the Mather topology. Note that the strong $C^\infty$ topology is completely determined by the strong $C^r$ topologies on $C^\infty(M, N)$, $1 \leq r < \infty$, and in this sense the strong $C^\infty$ topology on $C^\infty(M, N)$ is not a genuine $C^\infty$ topology. We let $C^\infty_S(M, N)$ denote $C^\infty(M, N)$ with the strong $C^\infty$ topology.

There is however a genuine $C^\infty$ topology on $C^\infty(M, N)$, namely the *very-strong* $C^\infty$ *topology*. This topology was introduced by Cerf in [3], Definition I.4.3.1. We give the definition of the very-strong $C^\infty$ topology on $C^\infty(M, N)$ in Definition 1.1, in a slightly different way than Cerf does. Cerf does not give this topology any special name, but it is named the 'very strong topology', and denoted $\tau V^\infty$, by du Plessis and Wall, see [15], p. 59. We shall denote $C^\infty(M, N)$ with the very-strong $C^\infty$ topology by $C^\infty_{vS}(M, N)$.

There are some basic and clear facts which show that the very-strong $C^\infty$ topology is a better topology for $C^\infty(M,N)$ than the strong $C^\infty$ topology. Let us first present the following.

The very-strong $C^\infty$ topology on $C^\infty(M,N)$ provides the right means to express classical approximation results by Whitney [16] in a modern framework. Suppose $U$ is an open subset of some euclidean space $\mathbb{R}^m$, and let $C^\omega(U,\mathbb{R}^n)$ denote the set of all real analytic maps from $U$ to $\mathbb{R}^n$, $n \geq 1$. Using the $C^\infty$ case of Lemma 6 in [16] (see Lemma 4.1), we first prove that $C^\omega(U,\mathbb{R}^n)$ is dense in $C^\infty_{\mathrm{vS}}(U,\mathbb{R}^n)$, see Proposition 4.2. We then proceed to establish the following result, see Theorem 4.4.

**Theorem A.** *Let $M$ and $N$ be real analytic manifolds. Then the set $C^\omega(M,N)$, of all real analytic maps from $M$ to $N$, is dense in the space $C^\infty_{\mathrm{vS}}(M,N)$.*

We do not wish to claim originality concerning the result in Theorem A, but on the other hand we do not know of any place in the literature where it is proved. Theorem A is a stronger result than the well-known result that $C^\omega(M,N)$ is dense in $C^\infty_{\mathrm{S}}(M,N)$.

The $C^\infty$ case of Lemma 6 in [16] proves the existence of an approximation of a given $C^\infty$ map $f\colon U \to \mathbb{R}^n$ by a real analytic map $h\colon U \to \mathbb{R}^n$, and the established approximation involves approximation of the partial derivatives of $f$ by the partial derivatives of $h$, of increasingly high order as one approaches the boundary of $U$, i.e., as one moves out towards infinity. This phenomenon is captured by the very-strong $C^\infty$ topology, but *not* by the strong $C^\infty$ topology. Here lies the reason why it is the very-strong $C^\infty$ topology on $C^\infty(M,N)$ that should be called the Whitney $C^\infty$ topology on $C^\infty(M,N)$, and the strong $C^\infty$ topology on $C^\infty(M,N)$ could well be named the Mather topology on $C^\infty(M,N)$, as was already pointed out above. It is however usually very difficult to change accepted mathematical terminology, even in cases where there is a good reason for a change, so we have mentioned these terminology questions here without any real expectation of a change. In this paper we only use the terminology "strong $C^\infty$ topology" and "very-strong $C^\infty$ topology", and refrain from using the terminology "Mather topology" and "Whitney topology" in order to avoid misunderstandings.

Another important property that the very-strong $C^\infty$ has, but which the strong $C^\infty$ topology lacks, is that by using the very-strong $C^\infty$ topology one can glue together two $C^\infty$ maps, as in Lemma B, see [3], I.4.3.4.4.

**Lemma B.** *Let $f\colon M \to N$ be a $C^\infty$ map between $C^\infty$ manifolds, and let $U$ be an open subset of $M$. Then there exists an open neighborhood $\mathcal{N}$ of $f|U$ in $C^\infty_{\mathrm{vS}}(U,N)$ such that for each $h \in \mathcal{N}$ the map $E(h)\colon M \to N$ defined by*

$$E(h)(x) = \begin{cases} h(x), & x \in U \\ f(x), & x \in M - U, \end{cases}$$

*is a* $C^\infty$ *map. Moreover* $E\colon \mathcal{N} \to C^\infty_{vS}(M,N)$, $h \mapsto E(h)$, *is continuous.*

For $1 \le r < \infty$, the $C^r$ case of Lemma B holds for the strong $C^r$ topology, see [5], Lemma 2.2.8, or [15], Lemma I.3.4.18. However, in the $C^\infty$ case Lemma B does not hold if we use the strong $C^\infty$ topology, see the remark after Corollary I.3.4.19 in [15]. Thus it is necessary that one uses the very-strong $C^\infty$ topology, and not the strong $C^\infty$ topology, in Lemma B.

Now let $K$ be a compact Lie group. By a $C^\infty$ $K$-manifold $M$ we mean a $C^\infty$ smooth manifold $M$ on which $K$ acts by a $C^\infty$ smooth action. If $M$ and $N$ are $C^\infty$ $K$-manifolds we let $C^{\infty,K}(M,N)$ denote the set of all $K$-equivariant $C^\infty$ maps from $M$ to $N$. We give $C^{\infty,K}(M,N)$ the very-strong $C^\infty$ topology, i.e., the relative topology from $C^\infty_{vS}(M,N)$, and we denote the obtained topological space by $C^{\infty,K}_{vS}(M,N)$. If $M$ and $N$ are real analytic $K$-manifolds, we denote the set of all $K$-equivariant real analytic maps from $M$ to $N$ by $C^{\omega,K}(M,N)$.

Suppose $\mathbb{R}^n(\theta)$ is a linear representation space for $K$. Using the invariant Haar integral on $K$ one obtains an averaging map $A$, which is a retraction map from the set $C^\infty(M,\mathbb{R}^n(\theta))$ onto the subset $C^{\infty,K}(M,\mathbb{R}^n(\theta))$. We prove in Theorem 6.4 that the averaging map $A$ is continuous in the very-strong $C^\infty$ topology, i.e., that

$$A\colon C^\infty_{vS}(M,\mathbb{R}^n(\theta)) \to C^{\infty,K}_{vS}(M,\mathbb{R}^n(\theta))$$

is continuous. This result plays a key role in the proof of our main result, namely Theorem C below.

As we have already mentioned the main purpose of this paper is to prove the following.

**Theorem C.** *Let $M$ and $N$ be real analytic $K$-manifolds, where $K$ is a compact Lie group, and assume that the number of $K$-isotropy types in $N$ is finite. Then $C^{\omega,K}(M,N)$ is dense in $C^{\infty,K}_{vS}(M,N)$.*

T. Matumoto and M. Shiota [12], Theorem 1.2, prove, under the same assumptions as in Theorem C, that $C^{\omega,K}(M,N)$ is dense in $C^{\infty,K}_{S}(M,N)$. By [10] the set $C^{\omega,K}(M,N)$ is dense in $C^{\infty,K}_{S}(M,N)$, even without the assumption of only a finite number of $K$-isotropy types in $N$. It would be interesting to know if one also in Theorem C can drop the assumption of only a finite number of $K$-isotropy types in $N$. A student of mine is at present working on this question.

In Section 9 we correct a mistake in [6]. The author wants to thank Sarah Packman for pointing out this mistake and for her very careful reading of [6].

This paper is written in such a way that all manifolds are assumed to be Hausdorff and second countable. However, the assumption of second countability is made purely for convenience, the results hold for paracompact manifolds, compare with Remark 6.3 in [7]. Here we content ourselves with recalling the well-known fact that a manifold $M$ is paracompact if and only if each connected component of $M$ is second countable.

## 1. The very-strong $C^\infty$ topology on $C^\infty(M, N)$

Let $U$ be an open subset of $\mathbb{R}^m$, and let $f\colon U \to \mathbb{R}^n$ be a $C^\infty$ map. Let $r$ be an integer, $1 \leq r < \infty$, and let $A \subset U$. Then we define

$$\|f\|_A^r = \sup\ \{|D^\alpha f_j(a)| \mid a \in A, \quad 1 \leq j \leq n,\ 0 \leq |\alpha| \leq r\},$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is an $m$-tuple of non-negative integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Here

$$D^\alpha f_j(a) = \frac{\partial^{|\alpha|} f_j(a)}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}.$$

Now suppose that $M$ and $N$ are $C^\infty$ manifolds. By $C^\infty(M, N)$ we denote the set of all $C^\infty$ maps from $M$ to $N$. Suppose that $f \in C^\infty(M, N)$. Let $(U, \varphi)$ be a chart in $M$, and $B$ a compact subset of $U$. Furthermore we let $(V, \psi)$ be a chart in $N$, such that $f(B) \subset V$, and $\varepsilon > 0$ is a positive real number, or $\varepsilon = \infty$. We then denote

$$\begin{aligned}(1)\qquad &\mathcal{N}^r(f; B, (U, \varphi), (V, \psi), \varepsilon)\\ &= \{h \in C^\infty(M, N) \mid h(B) \subset V \quad \text{and} \quad \|\psi \circ h \circ \varphi^{-1} - \psi \circ f \circ \varphi^{-1}\|_{\varphi(B)}^r < \varepsilon\}.\end{aligned}$$

Note that here $\psi \circ h \circ \varphi^{-1} - \psi \circ f \circ \varphi^{-1}\colon \varphi(f^{-1}(V) \cap h^{-1}(V) \cap U) \to \mathbb{R}^n$ is a $C^\infty$ map, and that $\varphi(B) \subset \varphi(f^{-1}(V) \cap h^{-1}(V) \cap U)$. We call such a set $\mathcal{N}^r(f; B, (U, \varphi), (V, \psi), \varepsilon)$ as in (1) *an elementary* $C^r$ *neighborhood of* $f$ *in* $C^\infty(M, N)$.

*Throughout this paper the notation* $\mathcal{N}^r(f; B, (U, \varphi), (V, \psi), \varepsilon)$ *will denote a set of the form given in* (1).

In the case when $N = \mathbb{R}^n$, and the chart $(V, \psi)$ in $\mathbb{R}^n$ is $(\mathbb{R}^n, \mathrm{id})$, we instead of the full notation $\mathcal{N}^r(f; B, (U, \varphi), (\mathbb{R}^n, \mathrm{id}), \varepsilon)$ use the simpler notation $\mathcal{N}^r(f; B, (U, \varphi), \varepsilon)$. If furthermore $U$ is an open subset of $\mathbb{R}^m$, and the chart $(U, \varphi)$ equals $(U, \mathrm{id})$, we denote $\mathcal{N}^r(f; B, (U, \mathrm{id}), \varepsilon)$ by $\mathcal{N}^r(f; B, \varepsilon)$. Thus

$$\mathcal{N}^r(f; B, \varepsilon) = \{h \in C^\infty(U, \mathbb{R}^n) \mid \|h - f\|_B^r < \varepsilon\},$$

where $B \subset U \subset \mathbb{R}^n$, and $B$ is compact.

Before we give the definition of the very-strong $C^\infty$ topology on $C^\infty(M, N)$ we recall, for comparison, the definition of the strong $C^\infty$ topology on $C^\infty(M, N)$. First recall that for $1 \leq r < \infty$ the *strong* $C^r$ *topology* on $C^\infty(M, N)$ is the topology, which as a basis has the family of all sets of the form

$$(2)\qquad \mathcal{S} = \bigcap_{i \in \Lambda} \mathcal{N}^r(f; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i),$$

where $f \in C^\infty(M, N)$, and the family $\{B_i\}_{i\in\Lambda}$ is locally finite in $M$. In the definition of the strong $C^r$ topology some authors require that the family $\{U_i\}_{i\in\Lambda}$ is locally finite, see e.g. [5], Section 2.1. However, this definition and the one given above are equivalent, see Section 1 in [8].

Then recall that the *strong* $C^\infty$ *topology* on $C^\infty(M, N)$, introduced by Mather in [11], Section 2, is the topology which as a basis has the union of all strong $C^r$ topologies on $C^\infty(M, N)$, $1 \leq r < \infty$, see also [5], Section 2.1. We denote $C^\infty(M, N)$ with the strong $C^\infty$ topology by $C^\infty_S(M, N)$.

Let us now turn to the definition of the *very-strong* $C^\infty$ *topology* on $C^\infty(M, N)$. Suppose $f \in C^\infty(M, N)$. By a *basic very-strong* $C^\infty$ *neighborhood* of $f$ we mean a set of the form

$$\mathcal{U} = \bigcap_{i\in\Lambda} \mathcal{N}^{r_i}(f; B_i, (U_i, \varphi), (V, \psi_i), \varepsilon_i), \tag{3}$$

where $1 \leq r_i < \infty$, $i \in \Lambda$, and the family $\{B_i\}$ is locally finite in $M$. By Lemma 1.3 below the family of all sets of the form (3), i.e., the family of all basic very-strong $C^\infty$ neighborhoods of all $f \in C^\infty(M, N)$, is a basis for a topology on $C^\infty(M, N)$. Thus we can give the following definition.

DEFINITION 1.1. The *very-strong* $C^\infty$ *topology* on $C^\infty(M, N)$ is the topology which as a basis has the family of all sets of the form (3), for all $f \in C^\infty(M, N)$. We denote the set $C^\infty(M, N)$ with the very-strong $C^\infty$ topology by $C^\infty_{vS}(M, N)$.

The very-strong $C^\infty$ topology is defined in [3], Definition I.4.3.1, in a slightly different way. Note that in (3) we may very well have that

$$\sup\{r_i \mid i \in \Lambda\} = \infty.$$

It is this property that is the crucial one, and this property makes the very-strong $C^\infty$ topology to differ from the strong $C^\infty$ topology, in the case when $M$ is non-compact. The very-strong $C^\infty$ topology on $C^\infty(M, N)$ is always at least as strong (i.e., at least as fine) as the strong $C^\infty$ topology. In other words, the map

$$\mathrm{id}\colon C^\infty_{vS}(M, N) \to C^\infty_S(M, N)$$

is continuous.

**Lemma 1.2.** *Let* $\mathcal{N} = \mathcal{N}^r(f; B, (U, \varphi), (V, \psi), \varepsilon)$ *be an elementary* $C^r$ *neighborhood of* $f \in C^\infty(M, N)$, *and let* $f_0 \in \mathcal{N}$. *Then there exists* $\varepsilon_0 > 0$ *such that if we set* $\mathcal{N}_0 = \mathcal{N}^r(f_0; B, (U, \varphi), (V, \psi), \varepsilon_0)$, *then* $\mathcal{N}_0 \subset \mathcal{N}$.

Proof. We have that $d = \|\psi \circ f_0 \circ \varphi^{-1} - \psi \circ f \circ \varphi\|^r_{\varphi(B)} < \varepsilon$, and by choosing $\varepsilon_0 = \varepsilon - d$ the claim follows. ☐

**Lemma 1.3.** *Let $\mathcal{U}$ and $\mathcal{U}'$ be basic very-strong* $C^\infty$ *neighborhoods of $f$ and $f'$ respectively, where $f, f' \in C^\infty(M, N)$. If $f_0 \in \mathcal{U} \cap \mathcal{U}'$, then there exists a basic very-strong* $C^\infty$ *neighborhood $\mathcal{U}_0$ of $f_0$, such that $\mathcal{U}_0 \subset \mathcal{U} \cap \mathcal{U}'$.*

Proof. Here $\mathcal{U} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ and $\mathcal{U}' = \bigcap_{j\in\Gamma} \mathcal{N}'_j$, where

$$\mathcal{N}_i = \mathcal{N}^{r_i}(f; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i), i \in \Lambda,$$

and

$$\mathcal{N}'_j = \mathcal{N}^{s_j}(f'; B'_j, (U'_j, \varphi'_j), (V'_j, \psi'_j), \varepsilon'_j), \quad j \in \Gamma,$$

are elementary neighborhoods of $f$ and $f'$, respectively, and $1 \le r_i < \infty$, for all $i \in \Lambda$, and $1 \le s_j < \infty$ for all $j \in \Gamma$, and the families $\{B_i\}_{i\in\Lambda}$ and $\{B'_j\}_{j\in\Gamma}$ are locally finite in $M$. By Lemma 1.2 there exists for each $i \in \Lambda$ an $\varepsilon_{0,i} > 0$ such that $\mathcal{N}_{0,i} = \mathcal{N}^{r_i}(f_0; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_{0,i}) \subset \mathcal{N}_i$, and also for each $j \in \Gamma$ an $\varepsilon'_{0,j} > 0$ such that $\mathcal{N}'_{0,j} = \mathcal{N}^{s_j}(f_0; B'_j, (U'_j, \varphi'_j), (V'_j, \psi'_j), \varepsilon'_{0,j}) \subset \mathcal{N}'_j$. Since the family $\{B_i, B'_j\}_{i\in\Lambda, j\in\Gamma}$ is locally finite in $M$, it follows that $\mathcal{N}_0 = \bigcap_{i\in\Lambda} \mathcal{N}_{0,i} \cap \bigcap_{j\in\Gamma} \mathcal{N}'_{0,j}$ is a basic very-strong $C^\infty$ neighborhood of $f_0$, and we have that $\mathcal{U}_0 \subset \mathcal{U} \cap \mathcal{U}'$. □

The following easy lemma will be used later on in the paper.

**Lemma 1.4.** *Let $M$ and $N$ be* $C^\infty$ *manifolds, and let $W$ be an open subset of $N$. Then the set $C^\infty(M, W)$ is open in $C^\infty_{vS}(M, N)$.*

Proof. Let $f \in C^\infty(M, W)$. We choose a locally finite family $\{B_i\}_{i\in\Lambda}$ of compact subsets of $M$ such that:
(a) $\bigcup_{i\in\Lambda} B_i = M$,
(b) $B_i \subset U_i$, where $(U_i, \varphi_i)$ is a chart in $M$, $i \in \Lambda$,
(c) $f(B_i) \subset V_i \subset W$, where $(V_i, \psi_i)$ is a chart in $N$, $i \in \Lambda$.
Let $\varepsilon_i > 0$ be arbitrary, e.g., $\varepsilon_i = \infty$, $i \in \Lambda$. Then

$$\mathcal{U}^* = \bigcap_{i\in\Lambda} \mathcal{N}^1(f; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i)$$

is a basic $C^1$ neighborhood of $f$. Furthermore $\mathcal{U}^* \subset C^\infty(M, W)$, since if $f' \in \mathcal{U}^*$, then $f'(M) = \bigcup_{i\in\Lambda} f'(B_i) \subset \bigcup_{i\in\Lambda} V_i \subset W$. Thus $C^\infty(M, W)$ is an open subset of $C^\infty(M, N)$ in the strong $C^1$ topology, and therefore also in the strong $C^\infty$ topology and hence also in the very-strong $C^\infty$ topology. □

## 2. Continuity of induced maps in the very-strong $C^\infty$ topology

*Throughout this section $M$, $N$ and $P$ denote* $C^\infty$ *manifolds.* The main purpose of this section is to prove Propositions 2.5 and 2.6, both of which are about the continuity of induced maps in the very-strong $C^\infty$ topology. We also establish Proposition 2.3, which is concerned with the continuity of the composition map

$$\Gamma\colon C^\infty(M,N)\times C^\infty(N,P)\to C^\infty(M,P),\quad (f,h)\mapsto h\circ f.$$

We begin by proving the following lemma.

**Lemma 2.1.** *Let $(f,h)\in C^\infty(M,N)\times C^\infty(N,P)$ and let $\mathcal{N}=\mathcal{N}^r(h\circ f;\ B,\ (U,\varphi),\ (W,\omega),\ \varepsilon)$ be an elementary $C^r$ neighborhood of $\Gamma(f,h)=h\circ f$ in $C^\infty(M,P)$, where $1\le r<\infty$. Then there exist finitely many elementary $C^r$ neighborhoods $\mathcal{M}_j=\mathcal{N}^r(f;\ B_j,\ (U,\varphi),\ (V_j,\psi_j),\ \varepsilon_j)$ of $f$, and $\mathcal{M}'_j=\mathcal{N}^r(h;\ \overline{V}_j,\ (V'_j,\psi'_j),\ (W,\omega),\ \varepsilon'_j)$ of $h$, $1\le j\le t$, such that $\Gamma(\bigcap_{j=1}^t(\mathcal{M}_j\times\mathcal{M}'_j))\subset\mathcal{N}$. Moreover, if $Q$ is any neighborhood of $f(B)$ in $N$, then we can choose the sets $V_j$ such that $\overline{V}_j\subset Q$, $1\le j\le t$.*

Proof. Since $f(B)$ is compact, and $Q\cap h^{-1}(W)$ is a neighborhood of $f(B)$, there are finitely many charts $(V'_j,\psi'_j)$ in $N$ and open subsets $V_j$ of $V'_j$, with $\overline{V}_j$ compact, and compact subsets $D_j$ of $V_j$, $1\le j\le t$, such that
(a) $f(B)\subset\bigcup_{j=1}^t D_j$,
(b) $D_j\subset V_j\subset\overline{V}_j\subset V'_j\subset Q\cap h^{-1}(W)$.
We set $B_j=B\cap f^{-1}(D_j)$, $\quad 1\le j\le t$. Then
(c) $\bigcup_{j=1}^t B_j=B$,
(d) $f(B_j)\subset D_j\subset V_j$, $\quad 1\le j\le t$.
We denote $\psi_j=\psi'_j\mid V_j$. Since $\psi_j(V_j)\subset\psi'_j(\overline{V}_j)$ and since $\psi'_j(\overline{V}_j)$ is compact, we have that $\|\omega\circ h\circ\psi_j^{-1}\|^r_{\psi_j(V_j)}<\infty$. Hence there exists $\varepsilon_j>0$ such that the following holds: If $\tilde f\colon M\to N$ is a $C^\infty$ map, with $\tilde f(B_j)\subset V_j$, then

$$\|\psi_j\circ\tilde f\circ\varphi^{-1}-\psi_j\circ f\circ\varphi^{-1}\|^r_{\varphi(B_j)}<\varepsilon_j \tag{1}$$

implies that $\|(\omega\circ h\circ\psi_j^{-1})\circ(\psi_j\circ\tilde f\circ\varphi^{-1}-\psi_j\circ f\circ\varphi^{-1})\|^r_{\varphi(B_j)}<\varepsilon/2$, i.e., that

$$\|\omega\circ h\circ\tilde f\circ\varphi^{-1}-\omega\circ h\circ f\circ\varphi^{-1}\|^r_{\varphi(B_j)}<\varepsilon/2. \tag{2}$$

Here above one should note the following. If (1) holds, then in particular

$$\hat B_j=(\psi_j\circ\tilde f\circ\varphi^{-1}-\psi_j\circ f\circ\varphi^{-1})(\varphi(B_j))\subset\mathring{E}^n(\varepsilon_j),$$

where $\mathring{E}^n(\varepsilon_j)=\{x\in\mathbb{R}^n\mid\|x\|<\varepsilon_j\}$. We may assume that $0\in\psi_j(V_j)$, $\quad 1\le j\le t$, and hence we have, for $\varepsilon_j$ small enough, that

$$\hat B_j\subset\mathring{E}^n(\varepsilon_j)\subset\psi_j(V_j),\quad 1\le j\le t.$$

We define

$$\mathcal{M}_j=\mathcal{N}^r(f;B_j,(U,\varphi),(V_j,\psi_j),\varepsilon_j),\quad 1\le j\le t.$$

Thus we have shown that if $\tilde{f} \in \mathcal{M}_j$, then (2) holds.

Since $\varphi(B_j)$ is compact we have that $\|\psi_j \circ f \circ \varphi^{-1}\|^r_{\varphi(B_j)} < \infty$. Furthermore we have that for all $\tilde{f} \in \mathcal{M}_j$ the inequality (1) holds, and hence there is a finite number $L_j$ such that

$$\|\psi_j \circ \tilde{f} \circ \varphi^{-1}\|^r_{\varphi(B_j)} < L_j, \text{ for all } \tilde{f} \in \mathcal{M}_j. \tag{3}$$

Hence there exists $\varepsilon'_j > 0$ such that the following holds: If $\tilde{f} \in \mathcal{M}_j$, and if $\tilde{h}\colon N \to P$ is a $\mathrm{C}^\infty$ map, with $\tilde{h}(\overline{V}_j) \subset W$, then

$$\|\omega \circ \tilde{h} \circ (\psi'_j)^{-1} - \omega \circ h \circ (\psi'_j)^{-1}\|^r_{\psi'_j(\overline{V}_j)} < \varepsilon'_j \tag{4}$$

implies that $\|(\omega \circ \tilde{h} \circ (\psi'_j)^{-1} - \omega \circ h \circ (\psi'_j)^{-1}) \circ (\psi_j \circ \tilde{f} \circ \varphi^{-1})\|^r_{\varphi(B_j)} < \varepsilon/2$, i.e., that

$$\|\omega \circ \tilde{h} \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ \tilde{f} \circ \varphi^{-1}\|^r_{\varphi(B_j)} < \varepsilon/2. \tag{5}$$

We define

$$\mathcal{M}'_j = \mathcal{N}^r(h; \overline{V}_j, (V'_j, \psi'_j), (W, \omega), \varepsilon'_j), \quad 1 \le j \le t.$$

If $\tilde{f} \in \mathcal{M}_j$ and $\tilde{h} \in \mathcal{M}'_j$, then $(\tilde{h} \circ \tilde{f})(B_j) \subset \tilde{h}(V_j) \subset W$, and by (2) and (5) we have that

$$\|\omega \circ \tilde{h} \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}\|^r_{\varphi(B_j)} < \varepsilon, \quad 1 \le j \le t.$$

Thus, if $(\tilde{f}, \tilde{h}) \in \bigcap_{j=1}^t \mathcal{M}_j \times \mathcal{M}'_j$, then $(\tilde{h} \circ \tilde{f})(B) \subset W$, and

$$\|\omega \circ \tilde{h} \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}\|^r_{\varphi(B)} < \varepsilon$$

holds, which means that $\tilde{h} \circ \tilde{f} = \Gamma(\tilde{f}, \tilde{h}) \in \mathcal{N}^r(h \circ f; B, (U, \varphi), (W, \omega), \varepsilon) = \mathcal{N}$. □

If we in Lemma 2.1 take $N = P$ and $h = \tilde{h} = \mathrm{id}_N$, we see that the proof of Lemma 2.1 proves the following.

REMARK 2.2. Let $f \in \mathrm{C}^\infty(M, N)$, and let $\mathcal{N} = \mathcal{N}^r(f; B, (U, \varphi), (V, \psi), \varepsilon)$ be an elementary $\mathrm{C}^r$ neighborhood of $f$. Suppose that we are given compact subsets $B_j$ of $M$ and charts $(V_j, \psi_j)$ in $N$, $1 \le j \le t$, such that
(a) $B = \bigcup_{j=1}^t B_j$
(b) $f(B_j) \subset V_j \subset \overline{V}_j \subset V'_j \subset V$, where $(V'_j, \psi'_j)$ are charts in $N$, $\psi_j = \psi'_j \mid V_j$ and $\overline{V}_j$ are compact, $1 \le j \le t$.
Then there exist $\varepsilon_j$, $1 \le j \le t$, so that if we set $\mathcal{N}_j = \mathcal{N}^r(f; B_j, (U, \varphi), (V_j, \psi_j), \varepsilon_j)$, then $\bigcap_{j=1}^t \mathcal{N}_j \subset \mathcal{N}$.

Using Lemma 2.1 we can now easily prove Proposition 2.3 below, see [3], Proposition I.4.5.5.

**Proposition 2.3.** *Let* $(f, h) \in \mathrm{C}^\infty_{\mathrm{vS}}(M, N) \times \mathrm{C}^\infty_{\mathrm{vS}}(N, P)$*, where the map* $f : M \to N$ *is proper. Then the map*

$$\Gamma : \mathrm{C}^\infty_{\mathrm{vS}}(M, N) \times \mathrm{C}^\infty_{\mathrm{vS}}(N, P) \to \mathrm{C}^\infty_{\mathrm{vS}}(M, P), \quad (\tilde{f}, \tilde{h}) \mapsto \tilde{h} \circ \tilde{f},$$

*is continuous at* $(f, h)$*.*

Proof. Let $\mathcal{V} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ be any basic very-strong C$^\infty$ neighborhood of $\Gamma(f, h) = h \circ f$ in $\mathrm{C}^\infty_{\mathrm{vS}}(M, P)$. Here each $\mathcal{N}_i = \mathcal{N}^{r_i}(h \circ f; B_i, (U_i, \varphi_i), (W_i, \omega_i), \varepsilon_i)$, $i \in \Lambda$, is an elementary C$^{r_i}$ neighborhood of $h \circ f$, and $1 \le r_i < \infty$, $i \in \Lambda$, and the family $\{B_i\}_{i\in\Lambda}$ is locally finite in $M$. Since the map $f$ is proper it follows that the family $\{f(B_i)\}_{i\in\Lambda}$ is locally finite in $N$. Each set $f(B_i)$ is compact, and thus there exists, for each $i \in \Lambda$, a compact neighborhood $Q_i$ of $f(B_i)$ in $N$, such that the family $\{Q_i\}_{i\in\Lambda}$ is locally finite in $N$, see [2], Theorem 30.C.10.

By Lemma 2.1 there exist for each $\mathcal{N}_i$ finitely many elementary C$^{r_i}$ neighborhoods

$$\mathcal{M}_{i,j} = \mathcal{N}^{r_i}(f; B_{i,j}, (U_i, \varphi_i), (V_{i,j}, \psi_{i,j}), \varepsilon_{i,j})$$

and

$$\mathcal{M}'_{i,j} = \mathcal{N}^{r_i}(h; \overline{V}_{i,j}, (V'_{i,j}, \psi'_{i,j}), (W_i, \omega_i), \varepsilon'_{i,j}),$$

of $f$ and $h$, respectively, $1 \le j \le t(i)$, such that $\Gamma(\bigcap_{j=1}^{t(i)}(\mathcal{M}_{i,j} \times \mathcal{M}'_{i,j})) \subset \mathcal{N}_i$. Here $\bigcup_{j=1}^{t(i)} B_{i,j} = B_i$ and $\overline{V}_{i,j} \subset Q_i$, $1 \le j \le t(i)$.

Since the family $\{B_{i,j} \mid i \in \Lambda,\ 1 \le j \le t(i)\}$ is locally finite in $M$, we have that $\mathcal{U} = \bigcap_{i\in\Lambda} \bigcap_{j=1}^{t(i)} \mathcal{M}_{i,j}$ is a basic very-strong C$^\infty$ neighborhood of $f$ in $\mathrm{C}^\infty_{\mathrm{vS}}(M, N)$. Moreover, since the family $\{Q_i\}_{i\in\Lambda}$ is locally finite in $N$ it follows that the family $\{\overline{V}_{i,j} \mid i \in \Lambda,\ 1 \le j \le t(i)\}$ is locally finite in $N$. Hence $\mathcal{U}' = \bigcap_{i\in\Lambda} \bigcap_{j=1}^{t(i)} \mathcal{M}'_{i,j}$ is a basic very-strong C$^\infty$ neighborhood of $h$ in $\mathrm{C}^\infty_{\mathrm{vS}}(N, P)$. Furthermore $\Gamma(\mathcal{U} \times \mathcal{U}') \subset \mathcal{V}$. This completes the proof. □

Let $\mathrm{Prop}^\infty(M, N)$ denote the set of all proper C$^\infty$ maps from $M$ to $N$. We give $\mathrm{Prop}^\infty(M, N)$ the relative topology from $\mathrm{C}^\infty_{\mathrm{vS}}(M, N)$, and denote the obtained topological space by $\mathrm{Prop}^\infty_{\mathrm{vS}}(M, N)$. As a corollary of Proposition 2.3 we obtain.

**Corollary 2.4.** *The map* $\Gamma : \mathrm{Prop}^\infty_{\mathrm{vS}}(M, N) \times \mathrm{C}^\infty_{\mathrm{vS}}(N, P) \to \mathrm{C}^\infty_{\mathrm{vS}}(M, P)$, $(f, h) \mapsto h \circ f$*, is continuous.*

As another corollary of Proposition 2.3 we have the following result.

**Proposition 2.5.** *Let $f: M \to N$ be a proper $C^\infty$ map. Then the induced map*

$$f^*: C^\infty_{vS}(N, P) \to C^\infty_{vS}(M, P), \quad h \mapsto h \circ f,$$

*is continuous.*

Using Lemma 2.1 we can also prove the following.

**Proposition 2.6.** *Let $h: N \to P$ be a $C^\infty$ map. Then the induced map*

$$h_*: C^\infty_{vS}(M, N) \to C^\infty_{vS}(M, P), \quad f \mapsto h \circ f,$$

*is continuous.*

Proof. Let $f \in C^\infty_{vS}(M, N)$, and let $\mathcal{V} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ be any basic very-strong $C^\infty$ neighborhood of $h_*(f) = h \circ f$ in $C^\infty_{vS}(M, P)$. Here each $\mathcal{N}_i = \mathcal{N}^{r_i}(h \circ f; B_i, (U_i, \varphi_i), (W_i, \psi_i), \varepsilon_i)$, $i \in \Lambda$, is an elementary $C^{r_i}$ neighborhood of $h \circ f$, and $1 \leq r_i < \infty$, $i \in \Lambda$, and the family $\{B_i\}_{i\in\Lambda}$ is locally finite in $M$. By Lemma 2.1 we find for each $\mathcal{N}_i$ finitely many elementary $C^{r_i}$ neighborhoods $\mathcal{M}_{i,j}$ of $f$, $1 \leq j \leq t(i)$, such that $\Gamma(\bigcap_{j=1}^{t(i)} \mathcal{M}_{i,j} \times \{h\}) = h_*(\bigcap_{j=1}^{t(i)} \mathcal{M}_{i,j}) \subset \mathcal{N}_i$. Here $\mathcal{M}_{i,j} = \mathcal{N}^{r_i}(f; B_{i,j}, (U_i, \varphi_i), (V_{i,j}, \psi_{i,j}), \varepsilon_{i,j})$ and $\bigcup_{j=1}^{t(i)} B_{i,j} = B_i$. Now the family $\{B_{i,j} \mid i \in \Lambda, 1 \leq j \leq t(i)\}$ is locally finite in $M$, and hence $\mathcal{U} = \bigcap_{i\in\Lambda} \bigcap_{j=1}^{t(i)} \mathcal{M}_{i,j}$ is a basic very-strong neighborhood of $f$. Furthermore we have that $h_*(\mathcal{U}) \subset \mathcal{V}$. This completes the proof. □

## 3. The product theorem

By the product theorem we mean the result in Proposition 3.1 below. Using the product theorem and also Proposition 2.5 we prove Corollary 3.2, which will be important for us later on in the paper. Corollary 3.2 plays a key role in the proof of Lemma 6.3. Note that Corollary 3.2 is the very-strong $C^\infty$ topology version of Lemma 2.2 in [6].

**Proposition 3.1.** *Let $M$, $N_1$ and $N_2$ be $C^\infty$ manifolds, and let $q_i: N_1 \times N_2 \to N_i$, $i = 1, 2$, denote the projection maps. Then the natural bijection*

$$\iota: C^\infty_{vS}(M, N_1 \times N_2) \to C^\infty_{vS}(M, N_1) \times C^\infty_{vS}(M, N_2), \quad f \mapsto (q_1 \circ f, q_2 \circ f),$$

*is a homeomorphism.*

Proof. It follows by Proposition 2.6 that $\iota$ is continuous. The fact that $\iota^{-1}$ is continuous is seen as follows. Let $(f_1, f_2) \in C^\infty_{vS}(M, N_1) \times C^\infty_{vS}(M, N_2)$, and denote $\iota^{-1}(f_1, f_2) = f$. Let $\mathcal{V} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ be any basic very-strong $C^\infty$ neighborhood of $f$ in

$C^\infty_{vS}(M, N_1 \times N_2)$. Here each $\mathcal{N}_i = \mathcal{N}^{r_i}(f; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i)$, $i \in \Lambda$, is an elementary $C^{r_i}$ neighborhood of $f$ in $C^\infty(M, N_1 \times N_2)$, and $1 \le r_i < \infty$, $i \in \Lambda$, and the family $\{B_i\}_{i\in\Lambda}$ is locally finite in $M$. It follows by Remark 2.2 that we may assume that each of the elementary $C^{r_i}$ neighborhoods $\mathcal{N}_i$, $i \in \Lambda$, is of the type where the chart in $N_1 \times N_2$ is a product chart, i.e., we can assume that each $\mathcal{N}_i$ is of the form

$$\mathcal{N}_i = \mathcal{N}^{r_i}(f; B_i, (U_i, \varphi_i), (V_i^{(1)} \times V_i^{(2)}, \psi_i^{(1)} \times \psi_i^{(2)}), \varepsilon_i), \quad i \in \Lambda. \tag{1}$$

It is readily seen that if $\mathcal{N}_i$ is as in (1), and we set

$$\mathcal{N}_i^{(k)} = \mathcal{N}^{r_i}(f_k; B_i, (U_i, \varphi_i), (V_i^{(k)}, \psi_i^{(k)}), \varepsilon_i), \quad k = 1,\ 2,$$

then $\iota^{-1}(\mathcal{N}_i^{(1)} \times \mathcal{N}_i^{(2)}) \subset \mathcal{N}_i$. Now $\mathcal{U}^{(k)} = \bigcap_{i\in\Lambda} \mathcal{N}_i^{(k)}$ is a basic very-strong $C^\infty$ neighborhood of $f_k = q_k \circ f$ in $C^\infty(M, N_k)$, $k = 1, 2$. Furthermore $\iota^{-1}(\mathcal{U}^{(1)} \times \mathcal{U}^{(2)}) \subset \bigcap_{i\in\Lambda} \mathcal{N}_i = \mathcal{V}$. This proves that $\iota^{-1}$ is continuous. □

**Corollary 3.2.** *Let $M$ and $N$ be $C^\infty$ manifolds, and let $Q$ be a compact $C^\infty$ manifold. Then the map*

$$\chi\colon C^\infty_{vS}(M, N) \to C^\infty_{vS}(Q \times M, Q \times N), \quad f \mapsto \mathrm{id} \times f,$$

*is continuous.*

Proof. It follows by Proposition 3.1 that it is enough to prove that the maps

$$C^\infty_{vS}(M, N) \to C^\infty_{vS}(Q \times M, Q), \quad f \mapsto q_1 \circ (\mathrm{id} \times f) \tag{i}$$

and

$$C^\infty_{vS}(M, N) \to C^\infty_{vS}(Q \times M, N), \quad f \mapsto q_2 \circ (\mathrm{id} \times f) \tag{ii}$$

are continuous. Here $q_1\colon Q \times N \to Q$ and $q_2\colon Q \times N \to N$ denote the projection maps. The map in (i) is the constant map from $C^\infty_{vS}(M, N)$ onto the element $p_1 \in C^\infty(Q \times M, Q)$, where $p_1\colon Q \times M \to Q$ is the projection, and hence (i) is continuous.

Observe that $q_2 \circ (\mathrm{id} \times f) = f \circ p_2$, where $p_2\colon Q \times M \to M$ is the projection. Thus the map in (ii) equals the map $p_2^*\colon C^\infty_{vS}(M, N) \to C^\infty_{vS}(Q \times M, N)$, $f \mapsto f \circ p_2$. Since $Q$ is compact, the projection $p_2$ is a proper map, and hence $p_2^*$ is continuous by Proposition 2.5. □

## 4. Whitney approximation

In this section we prove Theorem A. The following basic result was proved by H. Whitney in 1932–33.

**Lemma 4.1** (H. Whitney). *Let $U$ be an open subset of $\mathbb{R}^m$, and let $U_1, U_2, \dots$ be bounded open sets* (*some of which may be empty*) *such that $\bigcup_{q=1}^{\infty} U_q = U$ and $\overline{U}_q \subset U_{q+1}$ for all $q \geq 1$. Then if $f\colon U \to \mathbb{R}^n$ is a $\mathrm{C}^\infty$ map, and $\varepsilon_1 \geq \varepsilon_2 \geq \cdots$ are given positive numbers, and $k_1 \leq k_2 \leq \cdots$ are given positive integers, there is real analytic map $h\colon U \to \mathbb{R}^n$ such that, for each $1 \leq j \leq n$, we have that*

$$|\mathrm{D}^\alpha (h-f)_j(x)| < \varepsilon_q, \quad \textit{for all} \quad x \in U - U_q,$$

*and all $\alpha = (\alpha_1, \dots, \alpha_m)$ with $|\alpha| \leq k_q$, $q = 1, 2, \dots$.*

Proof. See [16], Lemma 6. In [16] the formulation of this result is given in the case when $k_q = q$, for $q \geq 1$. The above form of the result is an immediate consequence of this one. □

We begin by showing that Lemma 4.1 gives us the following result.

**Proposition 4.2.** *Let $U$ be an open subset of $\mathbb{R}^m$. Then the set $\mathrm{C}^\omega(U, \mathbb{R}^n)$ is dense in $\mathrm{C}^\infty_{\mathrm{vS}}(U, \mathbb{R}^n)$.*

Proof. Suppose $f \in \mathrm{C}^\infty_{\mathrm{vS}}(U, \mathbb{R}^n)$ and let $\mathcal{U} = \bigcap_{i \in \Lambda} \mathcal{N}^{r_i}(f; B_i, \varepsilon_i)$ be any basic very-strong $\mathrm{C}^\infty$ neighborhood of $f$. Here $1 \leq r_i < \infty$, and $\varepsilon_i > 0$, $i \in \Lambda$, and each $B_i$, $i \in \Lambda$, is a compact subset of $U$, and the family $\{B_i\}_{i \in \Lambda}$ is locally finite in $U$.

First we choose bounded open subsets $\emptyset = U_1, U_2, \dots$ of $U$ such that

a) $\bigcup_{q=1}^{\infty} U_q = U$,

b) $\overline{U}_q \subset U_{q+1}$, $q = 1, 2, \dots$ .

Next we define subsets $\Lambda_q$ of $\Lambda$, $q \geq 1$, in the following way. We set

$$\Lambda_q = \{ i \in \Lambda \mid B_i \cap U_{q+1} \neq \emptyset \}, \quad q = 1, 2, \dots .$$

Clearly $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_q \subset \Lambda_{q+1} \subset \cdots$, and $\bigcup_{q=1}^{\infty} \Lambda_q = \Lambda$. Since $\overline{U}_{q+1}$ is a compact subset of $U$ and the family $\{B_i\}_{i \in \Lambda}$ is locally finite in $U$, it follows that $B_i \cap \overline{U}_{q+1} \neq \emptyset$, and hence also that $B_i \cap U_{q+1} \neq \emptyset$, for only finitely many $i \in \Lambda$. Thus each $\Lambda_q$ is a finite set. We define

$$\begin{aligned} \overline{r}_q &= \max \{ r_i \mid i \in \Lambda_q \}, \quad q = 1, 2, \dots, \\ \overline{\varepsilon}_q &= \min \{ \varepsilon_i \mid i \in \Lambda_q \}, \quad q = 1, 2, \dots . \end{aligned} \tag{1}$$

Then $\overline{r}_1 \leq \overline{r}_2 \leq \dots$, and $\overline{\varepsilon}_1 \geq \overline{\varepsilon}_2 \geq \dots$ .

By Lemma 4.1 there exists a real analytic map $h\colon U \to \mathbb{R}^n$ such that, for each $1 \leq j \leq n$,

$$|\mathrm{D}^\alpha (h-f)_j(x)| < \overline{\varepsilon}_q, \quad \text{for all} \quad x \in U - U_q, \tag{2}$$

and all $\alpha = (\alpha_1, \dots, \alpha_m)$ with $|\alpha| \le \overline{r}_q$, $q = 1, 2, \dots$.

Now consider a fixed compact set $B_i$, $i \in \Lambda$. We let $q(i)$ be the least integer for which $i \in \Lambda_{q(i)}$. Thus

$$i \in \Lambda_{q(i)} - \Lambda_{q(i)-1},$$

where $\Lambda_0 = \emptyset$. Since $i \notin \Lambda_{q(i)-1}$ we have that

$$B_i \subset U - U_{q(i)}. \tag{3}$$

Since we for each $i \in \Lambda$ have that $i \in \Lambda_{q(i)}$, it follows by (1) that

$$\overline{r}_{q(i)} \ge r_i, \quad \text{and} \quad \overline{\varepsilon}_{q(i)} \le \varepsilon_i, \quad \text{for each} \quad i \in \Lambda.$$

It now follows by (2) and (3) that, for each $1 \le j \le n$,

$$|D^\alpha (h - f)_j(x)| < \overline{\varepsilon}_{q(i)} \le \varepsilon_i, \quad \text{for all} \quad x \in B_i,$$

and all $\alpha = (\alpha_1, \dots, \alpha_m)$ with $|\alpha| \le \overline{r}_{q(i)}$, and hence in particular for all $\alpha$ with $|\alpha| \le r_i$, $i \in \Lambda$. Thus we have that

$$\|h - f\|_{B_i}^{r_i} < \varepsilon_i, \quad i \in \Lambda.$$

Hence $h \in \bigcap_{i \in \Lambda} \mathcal{N}^{r_i}(f; B_i, \varepsilon_i) = \mathcal{U}$. Now $h \in \mathcal{U} \cap C^\omega(U, \mathbb{R}^n)$, and this proves that the set $C^\omega(U, \mathbb{R}^n)$ is dense in $C^\infty_{vS}(U, \mathbb{R}^n)$. □

Using Proposition 4.2 we prove the following.

**Proposition 4.3.** *Let $M$ be a real analytic manifold. Then the set $C^\omega(M, \mathbb{R}^n)$ is dense in $C^\infty_{vS}(M, \mathbb{R}^n)$.*

Proof. By the Grauert-Morrey imbedding theorem, see [4], Theorem 3, we may consider $M$ as a real analytic closed submanifold of some euclidean space $\mathbb{R}^m$. Let $i: M \hookrightarrow \mathbb{R}^m$ denote the inclusion. By Proposition 2.5 the induced map

$$i^*: C^\infty_{vS}(\mathbb{R}^m, \mathbb{R}^n) \to C^\infty_{vS}(M, \mathbb{R}^n), \quad f \mapsto f \mid M,$$

is continuous. Furthermore $i^*$ is surjective, since each $C^\infty$ map $f': M \to \mathbb{R}^n$ can be extended to a $C^\infty$ map $f: \mathbb{R}^m \to \mathbb{R}^n$, see e.g. [13], Proposition 2.5.14.

Let $\mathcal{U}$ be a non-empty open subset of $C^\infty_{vS}(M, \mathbb{R}^n)$. Then $(i^*)^{-1}(\mathcal{U})$ is a non-empty open subset of $C^\infty_{vS}(\mathbb{R}^m, \mathbb{R}^n)$, and hence we have by Proposition 4.2 that there exists a real analytic map $h: \mathbb{R}^m \to \mathbb{R}^n$ such that $h \in (i^*)^{-1}(\mathcal{U})$. Then $h \circ i = h|: M \to \mathbb{R}^n$ is real analytic, and thus $h \circ i = i^*(h) \in \mathcal{U} \cap C^\omega(M, \mathbb{R}^n)$. □

We can now prove Theorem A, given in the introduction.

**Theorem 4.4.** *Let $M$ and $N$ be real analytic manifolds. Then $\mathrm{C}^{\omega}(M,N)$ is dense in $\mathrm{C}^{\infty}_{\mathrm{vS}}(M,N)$.*

Proof. By the Grauert-Morrey imbedding theorem, [4], Theorem 3, we can consider $N$ as a real analytic closed submanifold of some $\mathbb{R}^n$. It is well known that there then exist an open neighborhood $W$ of $N$ in $\mathbb{R}^n$ and a real analytic retraction $r\colon W \to N$, see the corresponding step in the proof of Theorem 7.2. By Proposition 2.6 the induced map

$$r_*\colon \mathrm{C}^{\infty}_{\mathrm{vS}}(M,W) \to \mathrm{C}^{\infty}_{\mathrm{vS}}(M,N), \quad f \mapsto r \circ f,$$

is continuous, and clearly $r_*$ is surjective.

Suppose $\mathcal{U}$ is a non-empty open subset of $\mathrm{C}^{\infty}_{\mathrm{vS}}(M,N)$. Then $r_*^{-1}(\mathcal{U})$ is a non-empty open subset of $\mathrm{C}^{\infty}_{\mathrm{vS}}(M,W)$, and since $\mathrm{C}^{\infty}(M,W)$ is open in $\mathrm{C}^{\infty}_{\mathrm{vS}}(M,\mathbb{R}^n)$, see Lemma 1.4, it follows that $r_*^{-1}(\mathcal{U})$ is open in $\mathrm{C}^{\infty}_{\mathrm{vS}}(M,\mathbb{R}^n)$. By Proposition 4.3 we now know that there exists a real analytic map $h\colon M \to \mathbb{R}^n$ such that $h \in r_*^{-1}(\mathcal{U})$. Then $h\colon M \to W \subset \mathbb{R}^n$, and $r \circ h\colon M \to N$ is a real analytic map. Thus $r \circ h = r_*(h) \in \mathcal{U} \cap \mathrm{C}^{\omega}(M,N)$, and this completes the proof. □

## 5. The very-strong topology $\mathrm{C}^{\infty}$ on $\mathrm{C}^{\infty,K}(M,N)$

Let $K$ be a compact Lie group. If $M$ and $N$ are $\mathrm{C}^{\infty}$ $K$-manifolds, we let $\mathrm{C}^{\infty,K}(M,N)$ denote the set of all $K$-equivariant $\mathrm{C}^{\infty}$ maps from $M$ to $N$. We give $\mathrm{C}^{\infty,K}(M,N)$ the relative topology from $\mathrm{C}^{\infty}_{\mathrm{vS}}(M,N)$, and denote the set $\mathrm{C}^{\infty,K}(M,N)$ with this topology by $\mathrm{C}^{\infty,K}_{\mathrm{vS}}(M,N)$. As immediate consequences of Propositions 2.5 and 2.6, respectively, we now obtain.

**Proposition 5.1.** *Let $M$, $N$ and $P$ be $\mathrm{C}^{\infty}$ $K$-manifolds and let $f\colon M \to N$ be a $K$-equivariant proper $\mathrm{C}^{\infty}$ map. Then the induced map*

$$f^*\colon \mathrm{C}^{\infty,K}_{\mathrm{vS}}(N,P) \to \mathrm{C}^{\infty,K}_{\mathrm{vS}}(M,P), \quad h \mapsto h \circ f,$$

*is continuous.*

**Proposition 5.2.** *Let $M$, $N$ and $P$ be $\mathrm{C}^{\infty}$ $K$-manifolds, and let $h\colon N \to P$ be a $K$-equivariant $\mathrm{C}^{\infty}$ map. Then the induced map*

$$h_*\colon \mathrm{C}^{\infty,K}_{\mathrm{vS}}(M,N) \to \mathrm{C}^{\infty,K}_{\mathrm{vS}}(M,P), \quad f \mapsto h \circ f,$$

*is continuous.*

## 6. The averaging map and the very-strong $C^\infty$ topology

Let $K$ be a compact Lie group and assume to begin with that $M$ is any $C^\infty$ manifold. Suppose that we are given a $C^\infty$ map

$$f\colon K\times M\to\mathbb{R}^n.$$

We then define

$$\hat{A}(f)\colon M\to\mathbb{R}^n$$

by

$$\hat{A}(f)(x)=\int_K f(g,x)\,dg.$$

Here the integral is the Haar integral, obtained by integrating each coordinate function of $f$. That is if $f(g,x)=(f_1(g,x),\dots,f_n(g,x))$, then $\hat{A}(f)(x)=(v_1(x),\dots,v_n(x))$, where $v_j(x)=\int_K f_j(g,x)\,dg$, $1\le j\le n$. It is a well-known fact that $\hat{A}(f)\colon M\to\mathbb{R}^n$ is a $C^\infty$ map, see e.g., [1], Theorem 0.3.3. Hence we have that

$$\hat{A}\colon C^\infty(K\times M,\mathbb{R}^n)\to C^\infty(M,\mathbb{R}^n),\quad f\mapsto\hat{A}(f). \tag{1}$$

We show in Proposition 6.2 that $\hat{A}$ is continuous when we give both function spaces the very-strong $C^\infty$ topology. First we prove the following.

**Lemma 6.1.** *Suppose $f\in C^\infty(K\times M,\mathbb{R}^n)$, and let $\mathcal{N}=\mathcal{N}^r(\hat{A}(f);B,(U,\varphi),\varepsilon)$ be any elementary $C^r$ neighborhood of $\hat{A}(f)$ in $C^\infty(M,\mathbb{R}^n)$, where $1\le r<\infty$. Then there exist finitely many elementary $C^r$ neighborhoods $\mathcal{M}_p$ of $f$, of the form $\mathcal{M}_p=\mathcal{N}^r(f;Q_p\times B,(W_p\times U,\omega_p\times\varphi),\varepsilon)$, $1\le p\le s$, such that $\hat{A}(\bigcap_{p=1}^s\mathcal{M}_p)\subset\mathcal{N}$.*

Proof. Since $K$ is compact we can find finitely many charts $(W_p,\omega_p)$ in $K$ and compact subsets $Q_p$ of $W_p$, $1\le p\le s$, such that $\bigcup_{p=1}^s Q_p=K$. We set

$$\mathcal{M}_p=\mathcal{N}^r(f;Q_p\times B,\ (W_p\times U,\omega_p\times\varphi),\ \varepsilon),\quad 1\le p\le s.$$

If $h\in\mathcal{M}_p$, then we have for each $1\le j\le n$ that

$$\left|D^\gamma(h\circ(\omega_p\times\varphi)^{-1}-f\circ(\omega_p\times\varphi)^{-1})_j(v)\right|<\varepsilon,$$

for all $v=(t_1,\dots,t_k,u_1,\dots,u_m)\in\omega_p(Q_p)\times\varphi(B)\subset\mathbb{R}^k\times\mathbb{R}^m$, and every $(k+m)$-tuple $\gamma=(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_m)$ with $|\gamma|\le r$. Here $k=\dim K$ and $m=\dim M$. In particular we have that

$$\left|D^\beta(h_g\circ\varphi^{-1}-f_g\circ\varphi^{-1})_j(u)\right|<\varepsilon,\quad 1\le j\le n, \tag{2}$$

for all $g \in Q_p$ and all $u = (u_1, \dots, u_m) \in \varphi(B) \subset \mathbb{R}^m$, and every $\beta = (\beta_1, \dots, \beta_m)$ with $|\beta| \le r$. Here $h_g(x) = h(g,x)$ and $f_g(x) = f(g,x)$, for all $g \in K$ and all $x \in M$.

We have that

$$(\hat{A}(h) \circ \varphi^{-1} - \hat{A}(f) \circ \varphi^{-1})_j(u) = \int_K (h_g \circ \varphi^{-1} - f_g \circ \varphi^{-1})_j(u)\, dg, \quad 1 \le j \le n,$$

for all $u \in \varphi(U)$. Hence

$$\mathrm{D}^\beta(\hat{A}(h) \circ \varphi^{-1} - \hat{A}(f) \circ \varphi^{-1})_j(u) = \int_K \mathrm{D}^\beta(h_g \circ \varphi^{-1} - f_g \circ \varphi^{-1})_j(u)\, dg, \quad 1 \le j \le n,$$

for all $u \in \varphi(U)$.

By (2), and since $\bigcup_{p=1}^s Q_p = K$, it now follows that if $h \in \bigcap_{p=1}^s \mathcal{M}_p$, then

$$\left|\mathrm{D}^\beta(\hat{A}(h) \circ \varphi^{-1} - \hat{A}(f) \circ \varphi^{-1})_j(u)\right| \le \int_K \left|\mathrm{D}^\beta(h_g \circ \varphi^{-1} - f_g \circ \varphi^{-1})_j(u)\right| dg < \varepsilon,$$

for $1 \le j \le n$, and for all $u \in \varphi(B)$, and every $\beta$ with $|\beta| \le r$. Thus $\hat{A}(h) \in \mathcal{N}^r(\hat{A}(f); B, (U, \varphi), \varepsilon) = \mathcal{N}$. □

**Proposition 6.2.** *The map* $\hat{A}\colon \mathrm{C}^\infty_{\mathrm{vS}}(K \times M, \mathbb{R}^n) \to \mathrm{C}^\infty_{\mathrm{vS}}(M, \mathbb{R}^n)$ *is continuous.*

Proof. Let $f \in \mathrm{C}^\infty_{\mathrm{vS}}(K \times M, \mathbb{R}^n)$ and let $\mathcal{V} = \bigcap_{i \in \Lambda} \mathcal{N}_i$ be any basic very-strong $\mathrm{C}^\infty$ neighborhood of $\hat{A}(f)$ in $\mathrm{C}^\infty_{\mathrm{vS}}(M, \mathbb{R}^n)$. Here each $\mathcal{N}_i = \mathcal{N}^{r_i}(\hat{A}(f); B_i, (U_i, \varphi_i), \varepsilon_i)$, $i \in \Lambda$, is an elementary $\mathrm{C}^{r_i}$ neighborhood of $\hat{A}(f)$ in $\mathrm{C}^\infty(M, \mathbb{R}^n)$, where $1 \le r_i < \infty$, $i \in \Lambda$, and the family $\{B_i\}_{i \in \Lambda}$ is locally finite in $M$. By Lemma 6.1 we find for each $\mathcal{N}_i$, $i \in \Lambda$, finitely many elementary $\mathrm{C}^{r_i}$ neighborhoods $\mathcal{M}_{i,p}$ of $f$, $1 \le p \le s(i)$, such that $\hat{A}(\bigcap_{p=1}^{s(i)} \mathcal{M}_{i,p}) \subset \mathcal{N}_i$. Here

$$\mathcal{M}_{i,p} = \mathcal{N}^{r_i}(f; Q_p \times B_i, (W_p \times U_i, \omega_p \times \varphi_i), \varepsilon_i), \quad i \in \Lambda,\ 1 \le p \le s(i),$$

and $\bigcup_{p=1}^{s(i)} Q_p = K$. Now the family $\{Q_p \times B_i \mid 1 \le p \le s(i),\ i \in \Lambda\}$ is locally finite in $K \times M$, and hence $\mathcal{U} = \bigcap_{i \in \Lambda} \bigcap_{p=1}^{s(i)} \mathcal{M}_{i,p}$ is a basic very-strong $\mathrm{C}^\infty$ neighborhood of $f$ in $\mathrm{C}^\infty_{\mathrm{vS}}(K \times M, \mathbb{R})$. Furthermore we have that $\hat{A}(\mathcal{U}) \subset \mathcal{V}$. □

Let us now turn our attention to the case where $M$ is a $\mathrm{C}^\infty$ $K$-manifold. We let

$$\Phi\colon K \times M \to M,\ (g, x) \mapsto gx,$$

denote the given action of $K$ on $M$. Suppose $\mathbb{R}^n(\theta)$ is a linear representation space for $K$, and let

$$\Theta\colon K \times \mathbb{R}^n(\theta) \to \mathbb{R}^n(\theta),\ (g, y) \mapsto \theta(g)y = gy,$$

denote the corresponding action of $K$ on $\mathbb{R}^n(\theta)$. If $f\colon M \to \mathbb{R}^n(\theta)$ is any C$^\infty$ map, we define

$$A(f)\colon M \to \mathbb{R}^n(\theta) \tag{3}$$

by

$$A(f)(x) = \int_K g^{-1} f(gx)\, dg, \quad x \in M.$$

Then $A(f)$ is a $K$-equivariant C$^\infty$ map, and $A(f) = f$, if $f$ is $K$-equivariant. Thus we obtain a surjective map, in fact a retraction,

$$A\colon \mathrm{C}^\infty(M, \mathbb{R}^n(\theta)) \to \mathrm{C}^{\infty,K}(M, \mathbb{R}^n(\theta)), \quad f \mapsto A(f). \tag{4}$$

We prove in Theorem 6.4 that $A$ is continuous in the very-strong C$^\infty$ topology. In order to do this we first establish the following key lemma.

**Lemma 6.3.** *Let $M$ be a C$^\infty$ $K$-manifold, where $K$ is a compact Lie group, and let $\mathbb{R}^n(\theta)$ be a linear representation space for $K$. For each C$^\infty$ map $f\colon M \to \mathbb{R}^n(\theta)$ we define $f_{(\Phi,\Theta)}\colon K \times M \to \mathbb{R}^n(\theta)$, $(g, x) \mapsto g^{-1} f(gx)$. Then the map*

$$\Omega\colon \mathrm{C}^\infty_{\mathrm{vS}}(M, \mathbb{R}^n(\theta)) \to \mathrm{C}^\infty_{\mathrm{vS}}(K \times M, \mathbb{R}^n(\theta)),\ \ f \mapsto f_{(\Phi,\Theta)},$$

*is continuous.*

Proof. We have that $f_{(\Phi,\Theta)} = \Theta' \circ (\mathrm{id} \times f) \circ \Phi_\triangle\colon K \times M \to \mathbb{R}^n(\theta)$, where

$$\Phi_\triangle\colon K \times M \to K \times M,\ \ (g, x) \mapsto (g, gx),$$

and

$$\Theta'\colon K \times \mathbb{R}^n(\theta) \to \mathbb{R}^n(\theta), \quad (g, y) \mapsto g^{-1} y,$$

Therefore $\Omega$ equals the composite map

$$\mathrm{C}^\infty_{\mathrm{vS}}(M, \mathbb{R}^n(\theta)) \xrightarrow{\chi} \mathrm{C}^\infty_{\mathrm{vS}}(K \times M, K \times \mathbb{R}^n(\theta)) \xrightarrow{\Phi^*_\triangle} \mathrm{C}^\infty_{\mathrm{vS}}(K \times M, K \times \mathbb{R}^n(\theta))$$
$$\xrightarrow{\Theta'_*} \mathrm{C}^\infty_{\mathrm{vS}}(K \times M, \mathbb{R}^n(\theta)).$$

Here $\chi(f) = \mathrm{id} \times f$. The map $\chi$ is continuous by Corollary 3.2. The map $\Phi_\triangle$ is easily seen to be proper, and hence $\Phi^*_\triangle$ is continuous by Proposition 2.5. The fact that $\Theta'_*$ is continuous follows by Proposition 2.6. □

**Theorem 6.4.** *Let $M$ be a $C^\infty$ $K$-manifold, where $K$ is a compact Lie group, and let $\mathbb{R}^n(\theta)$ be a linear representation space for $K$. Then*

$$A\colon C^\infty_{vS}(M, \mathbb{R}^n(\theta)) \to C^{\infty,K}_{vS}(M, \mathbb{R}^n(\theta)), \quad f \mapsto A(f),$$

*is continuous.*

Proof. Note that for every $f \in C^\infty_{vS}(M, \mathbb{R}^n(\theta))$ we have that $A(f) = \hat{A}(f_{(\Phi,\Theta)})$, that is, $A(f) = (\hat{A} \circ \Omega)(f)$. Hence Theorem 6.4 follows by Proposition 6.2 and Lemma 6.3. □

## 7. Proof of Theorem C

**Proposition 7.1.** *Let $M$ be a real analytic $K$-manifold, where $K$ is a compact Lie group, and let $\mathbb{R}^n(\theta)$ be a linear representation space for $K$. Then $C^{\omega,K}(M, \mathbb{R}^n(\theta))$ is dense in $C^{\infty,K}_{vS}(M, \mathbb{R}^n(\theta))$.*

Proof. Let $\mathcal{U}$ be a non-empty, open subset of $C^{\infty,K}_{vS}(M, \mathbb{R}^n(\theta))$. By Theorem 6.4 we know that $A^{-1}(\mathcal{U})$ is an open subset of $C^\infty_{vS}(M, \mathbb{R}^n(\theta))$, and since $A$ is surjective $A^{-1}(\mathcal{U})$ is non-empty. Hence we have by Proposition 4.3 that there exists a real analytic map $f\colon M \to \mathbb{R}^n(\theta)$, such that $f \in A^{-1}(\mathcal{U})$. Since $f\colon M \to \mathbb{R}^n(\theta)$ is real analytic, it follows that also the $K$-equivariant map $A(f)\colon M \to \mathbb{R}^n(\theta)$ is real analytic, see [9], Theorem 1.16. Thus $A(f) \in \mathcal{U} \cap C^{\omega,K}(M, \mathbb{R}^n(\theta))$, and this completes the proof. □

**Theorem 7.2.** *Let $M$ and $N$ be real analytic $K$-manifolds, where $K$ is a compact Lie group, and assume that the number of $K$-isotropy types in $N$ is finite. Then $C^{\omega,K}(M, N)$ is dense in $C^{\infty,K}_{vS}(M, N)$.*

By [12], Theorem 1.1, we may consider $N$ as a $K$-invariant real analytic closed submanifold of some linear representation space $\mathbb{R}^n(\rho)$ for $K$. Then there exist a $K$-invariant open neighborhood $W$ of $N$ in $\mathbb{R}^n(\rho)$ and a $K$-equivariant real analytic retraction $r\colon W \to N$, see [8], Theorem I. Furthermore we have by Proposition 5.2 that the induced map

$$r_*\colon C^{\infty,K}_{vS}(M, W) \to C^{\infty,K}_{vS}(M, N), \quad f \mapsto r \circ f,$$

is continuous, and clearly $r_*$ is surjective.

Let $\mathcal{U}$ be a non-empty open subset of $C^{\infty,K}_{vS}(M, N)$. Then $r_*^{-1}(\mathcal{U})$ is a non-empty open subset of $C^{\infty,K}_{vS}(M, W)$. Since $C^{\infty,K}(M, W)$ is an open subset of $C^{\infty,K}_{vS}(M, \mathbb{R}^n(\rho))$, see Lemma 1.4, it follows that $r_*^{-1}(\mathcal{U})$ is open in $C^{\infty,K}_{vS}(M, \mathbb{R}^n(\rho))$. By Proposition 7.1 there exists a real analytic $K$-equivariant map $f\colon M \to \mathbb{R}^n(\rho)$ such that $f \in r_*^{-1}(\mathcal{U})$.

Then $f: M \to W \subset \mathbb{R}^n(\rho)$, and $r \circ f: M \to N$ is a $K$-equivariant real analytic map. Now $r \circ f = r_*(f) \in \mathcal{U} \cap C^{\omega,K}(M, N)$, and this completes the proof.

## 8. The equivariant glueing lemma

The purpose of this section is simply to record the following straightforward equivariant version of Lemma B in the introduction.

**Lemma 8.1.** *Let $f: M \to N$ be a $K$-equivariant $C^\infty$ map between $C^\infty$ $K$-manifolds, and let $U$ be a $K$-invariant open subset of $M$. Then there exists an open neighborhood $\mathcal{N}$ of $f|U$ in $C^{\infty,K}_{vS}(U, N)$ such that the following holds: If $h \in \mathcal{N}$ and we define $E(h): M \to N$ by*

$$E(h)(x) = \begin{cases} h(x), & x \in U \\ f(x), & x \in M - U, \end{cases}$$

*then $E(h)$ is a $K$-equivariant $C^\infty$ map. Furthermore $E: \mathcal{N} \to C^{\infty,K}_{vS}(M, N)$, $h \mapsto E(h)$, is continuous.*

Proof. It is clear that it is enough to prove the lemma in the case when $K = \{e\}$, and this case is given in [3], I.4.3.4.4. ☐

## 9. A correction to the paper [6]

Lemma 2.3 in [6], given for the strong $C^\infty$ topology, is not correct as stated. This mistake was pointed out to me by Sarah Packman (a graduate student at Berkeley) [14]. The best way to correct this mistake is to simply replace Lemma 2.3 in [6] by its valid very-strong $C^\infty$ topology version, i.e., by Lemma 8.1 above. As a consequence of this change one should also replace [6], Theorem 2.1 (this is the result of T. Matumoto and M. Shiota [12], Theorem 1.2) by the corresponding result for the very-strong $C^\infty$ topology, i.e., by Theorem 7.2 of the present paper. When these two replacements have been done, the proof of the main result in [6] requires no other changes and is correct as it stands.

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