Usami, Y.
Osaka J. Math.
17 (1980), 793-799

A NOTE ON HEIGHT OF EXCEPTIONAL CHARACTER DEGREES

YOKO USAMI

(Received July 13, 1979)

1. Introduction

We assume

(*) *G is a finite group with a Sylow p-group P satisfying*

$$C_G(x) = C_G(P), \qquad all\ x \in P^\sharp .$$

Hypothesis (*) implies that P is an abelian trivial intersection subgroup and $C_G(P)=P\times V$, for some p'-group V. In fact, $S=(P\times V)-V$ is a T.I. set. Furthermore, $N_G(P)/V$ is a Frobenius group with Frobenius kernel PV/V. We set $|P|=q$, $s=|N_G(P): C_G(P)|$ and $st=q-1$, where t is the number of p-classes of G. We set $N=N_G(P)$.

Under this hypothesis (*) R. Brauer and H.S. Leonard, Jr. [1,3] have shown the following results.

(a) There is a one to one correspondence from the p-blocks of G of full defect onto the N-classes of irreducible characters of V. (See (1D) [3] for details.)

(b) A p-block B of G of full defect associated with an N-class Φ of irreducible characters of V contains a family of exceptional characters $\{\Lambda_i\}$ $(1\leqq i\leqq |\Phi|t)$, if $|\Phi|t>1$.

(c) Let $\varphi\in\Phi$, and let $W(\varphi)$ be the inertia group of φ in N. If $f=|\Phi|$ and $|W(\varphi)|=e|C_G(P)|$, then $ef=s$. Let Λ be any member of $\{\Lambda_i\}$. Then

$$\Lambda_{|S} = \delta\lambda + c\sum_{\varphi\in\Phi} 1_P\varphi\,, \tag{1}$$

$$\Lambda(1)\equiv\delta s\varphi(1)+cf\varphi(1)\equiv(\delta e+c)f\varphi(1) \qquad (\text{mod } q), \tag{2}$$

where $\delta=\pm 1$, $\varphi\in\Phi$, $c\in\boldsymbol{Z}$, and λ is an appropriately chosen exceptional character of N. In particular, the p-block b of λ in N lifts to the p-block B of Λ in G. In addition $\lambda=(\mu\varphi)^N$, where μ is some nonprincipal irreducible character of P.

(d) Moreover it follows from (3.10) in [1] that

$$\Lambda_{|V}\equiv\delta\lambda_{|V}+c\sum_{\varphi\in\Phi}\varphi\equiv(\delta e+c)\sum_{\varphi\in\Phi}\varphi \qquad (\text{mod } q) \tag{3}$$

in the ring of all algebraic integers.

Using (a),(b) and (c), D.A. Sibley has proved the following two theorems.

Theorem 1 (Sibley [4]). *Suppose* (*) *holds and G has at least three classes of p-elements. Then* $c=0$. *In particular* $\Lambda(1)\equiv\delta\lambda(1)$ (mod q).

Theorem 2 (Sibley [5]). *Suppose*
(♯) *G is a finite group with a Sylow p-group P satisfying*

$$C_G(x) = P, \qquad all\ x \in P^{\#}.$$

Under this hypothesis (♯), $p \nmid \Lambda(1)$ *if G has at least two classes of p-elements.*

We remark that instead of hypothesis (♯), Sibley has proved Theorem 2 under the following hypothesis:

(♯′) *A Sylow p-group P of G is an abelian* T.I. *set, and* $N_G(P)$ *is a Frobenius group with Frobenius kernel P.*

It is easily seen that hypothesis (♯) is equivalent to hypothesis (♯′).
In this paper we shall prove the following theorem, which has been conjectured by Sibley [5].

Main Theorem. *Suppose* (*) *holds and G has at least two classes of p-elements. Then* $p \nmid \Lambda(1)$.

EXAMPLE. Let $G=SL(2,q)$, where q is a power of an odd prime p and $\frac{q-1}{2}$ is odd. Then G satisfies (*) with $|V|=2$ and has two classes of p-elements. $N_G(P)$ has two families of exceptional characters and both degrees are $\frac{q-1}{2}$. On the other hand G has two families of exceptional characters and their degrees are $\frac{q-1}{2}$ and $\frac{q+1}{2}$, which are prime to p. Moreover if we choose δ appropriately, we can take $c=0$ in both families.

2. Proof of Main Theorem

The main theorem can be proved by similar way to Theorem 2 with the addition of block calculations as the proof of Theorem 1.

Suppose by way of contradiction that $p\,|\,\Lambda(1)$. Then first of all we claim also $p\,|\,\Lambda(x)$ for any p-regular element x which is not conjugate to an element of V in G. Let $g_i \in P^{\#}$ and K_i be the class of G containing g_i and K be the class of G containing x. We define a class function θ_{ix} by

$$\theta_{ix}(a) = |\{(g_i', x')\,|\,g_i' \in K_i, x' \in K, g_i'x' = a\}|\,.$$

We have the well-known formula

$$\theta_{ix} = \frac{|G|}{|C_G(g_i)|\,|C_G(x)|} \sum_{\chi} \frac{\bar{\chi}(g_i)\bar{\chi}(x)}{\chi(1)} \chi ,$$

where the sum is over all irreducible characters χ of G. We now define another class function θ_{ix}' by

$$\theta_{ix}'(a) = \begin{cases} \theta_{ix}(a), & \text{if } a \text{ is } p\text{-singular,} \\ 0, & \text{otherwise.} \end{cases}$$

We may write

$$\theta_{ix}' = \sum_a{}' \frac{\theta_{ix}(a)}{|C_G(a)|} \sum_{\chi} \bar{\chi}(a)\chi ,$$

where the sum is over a complete set of representatives a of the p-singular classes a^G of G. By $\theta_{ix}'{}_{|B}$ we mean

$$\theta_{ix}'{}_{|B} = \sum_a{}' \frac{\theta_{ix}(a)}{|C_G(a)|} \sum_{\chi \in B} \bar{\chi}(a)\chi ,$$

and by $\theta_{ix|B}$ we mean

$$\theta_{ix|B} = \frac{|G|}{|C_G(g_i)|\,|C_G(x)|} \sum_{\chi \in B} \frac{\bar{\chi}(g_i)\bar{\chi}(x)}{\chi(1)} \chi .$$

Lemma 1. $\theta_{ix}(g_k v) \equiv 0 \pmod{q}$ *for* $g_k \in P^\#$ *and* $v \in V$.

Proof. The lemma follows easily, because P acts by conjugation fixed-point-free on the set of the pairs (g_i', x'), where $g_i' \in K_i, x' \in K$ and $g_i'x' = g_k v$. (q.e.d.)

Let $\mathfrak{m}$ be $\{\frac{z}{y} \mid y$ is a rational integer which is prime to p, and z is an algebraic integer$\}$.

Lemma 2. $\theta_{ix|B}(g_k)$ *is in* $\mathfrak{m}$ *and* $\theta_{ix|B}(g_k) \equiv 0 \pmod{q\mathfrak{m}}$.

Proof. Since $\theta_{ix} - \theta_{ix}'$ vanishes on p-singular elements, the "Truncation of Relations" theorem (see [2] (IV.6.3)) shows that $\theta_{ix|B} - \theta_{ix}'{}_{|B}$ vanishes on p-singular elements. In particular

$$\theta_{ix|B}(g_k) = \theta_{ix}'{}_{|B}(g_k) = \sum_a{}' \frac{\theta_{ix}(a)}{|C_G(a)|} \sum_{\chi \in B} \bar{\chi}(a)\chi(g_k) .$$

We can claculate $\sum_{\chi \in B} \bar{\chi}(a)\chi(g_k)$ by (5)[4] and it becomes

$$\theta_{ix}{}'_{|B}(g_k)=\sum_{g_k v}{}' \frac{\theta_{ix}(g_k v)}{|C_G(g_k v)|} q \sum_{\varphi\in\Phi} \bar{\varphi}(v)\varphi(1)\,,$$

where the sum is over a complete set of representatives of p-singular classes in which g_k can be chosen as p-part. Since $\theta_{ix}(g_k v)\equiv 0 \pmod q$ by Lemma 1, the result follows. (q.e.d.)

Lemma 3. *If $p\,|\,\Lambda(1)$, then $p\,|\,\Lambda(x)$ in $\mathfrak{m}$ for any p-regular element $x\notin V^G$.*

Proof. We can compute the difference between $\theta_{ix|B}$ and $\theta_{jx|B}$ for $g_i, g_j\in P^\sharp$, as Sibley did in [5]:

$$\begin{aligned}\theta_{ix|B}(g_k)-\theta_{jx|B}(g_k) &= \frac{|G|}{|PV|\,|C_G(x)|}\sum_{\Lambda}\left\{\frac{\bar{\Lambda}(g_i)\bar{\Lambda}(x)\Lambda(g_k)}{\Lambda(1)}-\frac{\bar{\Lambda}(g_j)\bar{\Lambda}(x)\Lambda(g_k)}{\Lambda(1)}\right\}\\ &= \frac{|G|\bar{\Lambda}(x)}{q|V|\,|C_G(x)|\Lambda(1)}\sum_{\Lambda}\{\bar{\Lambda}(g_i)\Lambda(g_k)-\bar{\Lambda}(g_j)\Lambda(g_k)\}\,,\end{aligned}$$

where $\{\Lambda\}$ are the exceptional characters in B. (These equalities follow from the facts that $X(g_i)=X(g_j)$ for any nonexceptional character X in B ((1D) (v) [3]), and that $\Lambda(x)$ and $\Lambda(1)$ are independent of the choice of Λ((2B) [1]).) On the other hand,

$$\begin{aligned}\sum_{\Lambda}\{\bar{\Lambda}(g_i)\Lambda(g_k)-\bar{\Lambda}(g_j)\Lambda(g_k)\} &= \sum_{\chi\in B}\{\bar{\chi}(g_i)\chi(g_k)-\bar{\chi}(g_j)\chi(g_k)\}\\ &= qf\varphi(1)^2(\delta_{g_i g_k}-\delta_{g_j g_k})\,,\end{aligned}$$

where δ_{gh} is defined for $g, h\in P^\sharp$ by

$$\delta_{gh}=\begin{cases}1 & g\sim h\,,\\ 0 & \text{otherwise.}\end{cases}$$

The last equality holds by (5) [4]. As G has at least two classes of p-elements, we can choose $g_i=g_k$ and $g_j\not\sim g_k$. Then by Lemma 2

$$0\equiv\theta_{ix|B}(g_k)-\theta_{jx|B}(g_k)=\frac{|G|\bar{\Lambda}(x)f\varphi(1)^2}{|V|\,|C_G(x)|\Lambda(1)} \qquad (\text{mod } q\mathfrak{m})\,.$$

Then $p\,|\,\Lambda(x)$. (q.e.d.)

We now calculate $||\Lambda||^2$. This gives

$$\begin{aligned}1=||\Lambda||^2 &= \frac{1}{|G|}\sum_{g\in S}{}'\frac{|G|\Lambda(g)\bar{\Lambda}(g)}{|C_G(g)|}+\frac{1}{|G|}\sum_{v\in V}{}'\frac{|G|\Lambda(v)\bar{\Lambda}(v)}{|C_G(v)|}\\ &+\frac{1}{|G|}\sum_{x\notin V^G}{}'\frac{|G|\Lambda(x)\bar{\Lambda}(x)}{|C_G(x)|}\,,\end{aligned} \tag{4}$$

where the first and the second sums are over complete sets of representatives of

G-conjugacy classes and the third sum is over that of G-conjugacy classes of p-regular elements which are not in V^G. Then by Lemma 3 we may write the third sum as p^2R where $R=\frac{z}{y}$ for some algebraic integer z and some rational integer y which is prime to p.

Lemma 4. *Let T_1 be the first term of* (4). *Then*

$$T_1 = 1+\frac{|V|f}{|N|}\{-(\delta e+c)^2+c^2j\} .$$

Proof. By (1),

$$T_1 = \frac{1}{|G|}\sum_{g\in S}{}' \frac{|G|}{|C_G(g)|}(\delta\lambda(g)+c\sum_{\varphi\in\Phi}1_P\varphi(g))\,(\delta\bar{\lambda}(g)+c\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g)) ,$$

where the sum is over a complete set of representatives of G-conjugacy classes. Since S is a T.I. set, the representatives of G-conjugacy classes of S coincide with those of N-conjugacy classes of S and $|C_G(g)|=|C_N(g)|$. Then

$$T_1 = \frac{|G:N|}{|G|}\sum_{g\in S}{}' \frac{|N|}{|C_N(g)|}(\delta\lambda(g)+c\sum_{\varphi\in\Phi}1_P\varphi(g))\,(\delta\bar{\lambda}(g)+c\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g)) ,$$

where the sum is over a complete set of representatives of N-conjugacy classes. Since λ is a character of N and $\sum_{\varphi\in\Phi}1_P\varphi$ is an N-invariant character of PV,

$$T_1 = \frac{1}{|N|}\{\sum_{g\in S}\lambda(g)\bar{\lambda}(g)+\delta c\sum_{g\in S}[\lambda(g)\,(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))+\bar{\lambda}(g)\,(\sum_{\varphi\in\Phi}1_P\varphi(g))]$$
$$+c^2\sum_{g\in S}(\sum_{\varphi\in\Phi}1_P\varphi(g))\,(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))\} ,$$

where the sums $\sum_{g\in S}$ are over all elements of S. We can express $\lambda_{|P\times V}$ as follows:

$$\lambda_{|P\times V} = \sum_n(\mu_1+\mu_2+\cdots+\mu_e)^n\varphi^n ,$$

where n ranges over a cross section of $W(\varphi)$ in N, and $\mu_1, \mu_2, \cdots, \mu_e$ are distinct irreducible nonprincipal characters of P. Note that

$$\sum_n\varphi^n = \sum_{\varphi\in\Phi}\varphi .$$

From the orthogonality relations we get

$$\begin{aligned}\sum_{g\in S}\lambda(g)\bar{\lambda}(g) &= |N|-\sum_{v\in V}|\lambda(v)|^2\\ &= |N|-\sum_{v\in V}|e\sum_{\varphi\in\Phi}\varphi(v)|^2\\ &= |N|-e^2|V|f ,\end{aligned}$$

$$\sum_{g\in S}\lambda(g)\left(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g)\right)=\sum_{g\in S}\left(\sum_n(\mu_1+\mu_2+\cdots+\mu_e)^n\varphi^n(g)\right)\left(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g)\right)$$
$$=-\sum_{v\in V}\left(\sum_n e\varphi^n(v)\right)\left(\sum_{\varphi\in\Phi}\bar{\varphi}(v)\right)$$
$$=-e\sum_{v\in V}\left(\sum_{\varphi\in\Phi}\varphi(v)\sum_{\varphi\in\Phi}\bar{\varphi}(v)\right)$$
$$=-e|V|f,$$
$$\sum_{g\in S}\bar{\lambda}(g)\left(\sum_{\varphi\in\Phi}1_P\varphi(g)\right)=-e|V|f,$$

and

$$\sum_{g\in S}\left|\sum_{\varphi\in\Phi}1_P\varphi(g)\right|^2=(q-1)\sum_{v\in V}\left|\sum_{\varphi\in\Phi}\varphi(v)\right|^2=(q-1)|V|f.$$

Then

$$T_1=\frac{1}{|N|}\{|N|-e^2|V|f-2\delta ce|V|f+c^2(q-1)|V|f\}$$
$$=1+\frac{|V|f}{|N|}\{-(\delta e+c)^2+c^2q\}. \qquad \text{(q.e.d.)}$$

Multiplying (4) by $q|V|$ we get

$$\frac{|V|f}{s}\{-(\delta e+c)^2+c^2q\}+T_2+p^2qR|V|=0,$$

where

$$T_2=\frac{1}{|G:PV|}\sum_{v\in V}{}'\frac{|G|\Lambda(v)\bar{\Lambda}(v)}{|C_G(v)|}.$$

Then

$$\frac{|V|f}{s}\{-(\delta e+c)^2+c^2q\}+T_2\equiv 0 \qquad (\mathrm{mod}\ pq\mathfrak{m}). \qquad (5)$$

Lemma 5. $T_2\equiv\dfrac{|V|f(\delta e+c)^2}{s} \quad (\mathrm{mod}\ pq\mathfrak{m}).$

Proof. Let $\{v_j\}$ $(1\leqq j\leqq u)$ be the representatives of G-conjugacy classes of V. Then these are also the representatives of N-conjugacy classes, because $N_G(P)$ controls fusion of $C_G(P)$. Note that $p|(\delta e+c)$ from (2), because we have assumed that $p|\Lambda(1)$. By (3),

$$T_2=\frac{1}{|G:PV|}\sum_{j=1}^{u}\frac{|G|}{|C_G(v_j)|}\Lambda(v_j)\bar{\Lambda}(v_j)$$
$$\equiv\frac{1}{|G:PV|}\sum_{j=1}^{u}\frac{|G|}{|C_G(v_j)|}(\delta e+c)^2\left\{\left(\sum_{\varphi\in\Phi}\varphi(v_j)\right)\left(\sum_{\varphi\in\Phi}\bar{\varphi}(v_j)\right)\right\} \quad (\mathrm{mod}\ pq\mathfrak{m}).$$

We now set $\zeta=\sum_{\varphi\in\Phi}\varphi$. Since ζ is an N-invariant character of V and $\{v_j\}$ are also the representatives of N-conjugacy classes,

$$|\zeta(v_j)|^2=\frac{|C_N(v_j)|}{|N|}\sum_{v\underset{N}{\sim}v_j}|\zeta(v)|^2 .$$

Then

$$\begin{aligned}T_2&\equiv\frac{(\delta e+c)^2}{|G:PV|}\sum_{j=1}^{u}\frac{|G|}{|C_G(v_j)|}|\zeta(v_j)|^2\\&\equiv\frac{(\delta e+c)^2}{|G:PV|}\sum_{j=1}^{u}\left\{\frac{|G|\,|C_N(v_j)|}{|C_G(v_j)|\,|N|}\sum_{v\underset{N}{\sim}v_j}|\zeta(v)|^2\right\}\\&\equiv\frac{(\delta e+c)^2}{s}\sum_{v\in V}\frac{|\zeta(v)|^2}{|C_G(v):C_N(v)|}\qquad(\mathrm{mod}\ pq\mathfrak{m}) .\end{aligned}$$

Since $P\subseteqq C_G(v)$ and P is a T.I. Sylow p-group of $C_G(v)$,

$$|C_G(v): C_N(v)|\equiv|C_G(v): N_G(P)\cap C_G(v)|\equiv 1\qquad(\mathrm{mod}\ q) .$$

Thus

$$T_2\equiv\frac{(\delta e+c)^2}{s}\sum_{v\in V}|\zeta(v)|^2\equiv\frac{|V|f(\delta e+c)^2}{s}\qquad(\mathrm{mod}\ pq\mathfrak{m}) .\qquad\text{(q.e.d.)}$$

Then by (5) we get the congruence

$$\frac{|V|fc^2q}{s}\equiv 0\qquad(\mathrm{mod}\ pq\mathfrak{m}) .$$

Hence we get $p|c^2$. This contradicts $p|\Lambda(1)$. This completes the proof of the main theorem.

Acknowledgment. The author would like to express her hearty thanks to Professor Yukio Tsushima for his helpful advices.

References

[1] R. Brauer and H.S. Leonard, Jr.: *On finite groups with an abelian Sylow group,* Canad. J. Math. **14** (1962), 436–450.

[2] W. Feit: Representations of finite groups (Lecture note), Yale University, New Haven, Conn., 1969.

[3] H.S. Leonard, Jr.: *Finite linear groups having an abelian Sylow subgroup,* J. Algebra **20** (1972), 57–69.

[4] D.A. Sibley: *Finite linear groups with a strongly self-centralizing Sylow subgroup* II, J. Algebra **36** (1975), 319–332.

[5] D.A. Sibley: *Height of exceptional character degrees,* Proc. Amer. Math. Soc. **69** (1978), 16–18.

Department of Mathematics
Ochanomizu University
Ohtsuka, Bunkyo-ku, Tokyo 112
Japan