<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>A note on height of exceptional character degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Usami, Yoko</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 17(3) P.793-P.799</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1980</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/7091">https://doi.org/10.18910/7091</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/7091</td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
1. Introduction

We assume

\[ (*) \quad G \text{ is a finite group with a Sylow } p\text{-group } P \text{ satisfying} \]

\[ C_G(x) = C_G(P), \quad \text{all } x \in P^t. \]

Hypothesis (*) implies that \( P \) is an abelian trivial intersection subgroup and \( C_G(P) = P \times V \), for some \( p' \)-group \( V \). In fact, \( S = (P \times V) - V \) is a T.I. set. Furthermore, \( N_G(P)/V \) is a Frobenius group with Frobenius kernel \( PV/V \). We set \( |P| = q, s = |N_G(P); C_G(P)| \) and \( st = q - 1 \), where \( t \) is the number of \( p \)-classes of \( G \). We set \( N = N_G(P) \).

Under this hypothesis (*) R. Brauer and H.S. Leonard, Jr. [1,3] have shown the following results.

(a) There is a one to one correspondence from the \( p \)-blocks of \( G \) of full defect onto the \( N \)-classes of irreducible characters of \( V \). (See (1D) [3] for details.)

(b) A \( p \)-block \( B \) of \( G \) of full defect associated with an \( N \)-class \( \Phi \) of irreducible characters of \( V \) contains a family of exceptional characters \( \{ \Lambda_i \} \) \( (1 \leq i \leq |\Phi|) \), if \( |\Phi|f > 1 \).

(c) Let \( \varphi \in \Phi \), and let \( W(\varphi) \) be the inertia group of \( \varphi \) in \( N \). If \( f = |\Phi| \) and \( |W(\varphi)| = e|C_G(P)| \), then \( ef = s \). Let \( \Lambda \) be any member of \( \{ \Lambda_i \} \). Then

\[ \Lambda_{is} = \delta \lambda + c \sum_{\varphi \in \Phi} 1_{\varphi \varphi}, \quad (1) \]

\[ \Lambda(1) = \delta \varphi(1) + ef \varphi(1) \equiv (\delta e + c) f \varphi(1) \pmod{q}, \quad (2) \]

where \( \delta = \pm 1 \), \( \varphi \in \Phi \), \( c \in \mathbb{Z} \), and \( \lambda \) is an appropriately chosen exceptional character of \( N \). In particular, the \( p \)-block \( b \) of \( \lambda \) in \( N \) lifts to the \( p \)-block \( B \) of \( \Lambda \) in \( G \). In addition \( \lambda = (\mu \varphi)^N \), where \( \mu \) is some nonprincipal irreducible character of \( P \).

(d) Moreover it follows from (3.10) in [1] that

\[ \Lambda_{i\varphi} = \delta \lambda_{i\varphi} + c \sum_{\varphi \in \Phi} \varphi \equiv (\delta e + c) \sum_{\varphi} \varphi \pmod{q} \quad (3) \]
in the ring of all algebraic integers.

Using (a), (b) and (c), D.A. Sibley has proved the following two theorems.

**Theorem 1** (Sibley [4]). *Suppose (*) holds and G has at least three classes of p-elements. Then c=0. In particular \( \Lambda(1) \equiv 8\lambda(1) \mod q \).*

**Theorem 2** (Sibley [5]). *Suppose

(\#) G is a finite group with a Sylow p-group \( P \) satisfying

\[
C_G(x) = P, \quad \text{all } x \in P^i.
\]

Under this hypothesis (\#), \( p \not| \Lambda(1) \) if G has at least two classes of p-elements.

We remark that instead of hypothesis (\#), Sibley has proved Theorem 2 under the following hypothesis:

(\#'') A Sylow p-group \( P \) of G is an abelian T.I. set, and \( N_G(P) \) is a Frobenius group with Frobenius kernel \( P \).

It is easily seen that hypothesis (\#) is equivalent to hypothesis (\#').

In this paper we shall prove the following theorem, which has been conjectured by Sibley [5].

**Main Theorem.** *Suppose (*) holds and G has at least two classes of p-elements. Then \( p \not| \Lambda(1) \).*

**Example.** Let \( G=SL(2,q) \), where \( q \) is a power of an odd prime \( p \) and \( \frac{q-1}{2} \) is odd. Then \( G \) satisfies (*) with \( |V|=2 \) and has two classes of p-elements. \( N_G(P) \) has two families of exceptional characters and both degrees are \( \frac{q-1}{2} \).

On the other hand \( G \) has two families of exceptional characters and their degrees are \( \frac{q-1}{2} \) and \( \frac{q+1}{2} \), which are prime to \( p \). Moreover if we choose \( \delta \) appropriately, we can take \( c=0 \) in both families.

**2. Proof of Main Theorem**

The main theorem can be proved by similar way to Theorem 2 with the addition of block calculations as the proof of Theorem 1.

Suppose by way of contradiction that \( p \not| \Lambda(1) \). Then first of all we claim also \( p \not| \Lambda(x) \) for any p-regular element \( x \) which is not conjugate to an element of \( V \) in \( G \). Let \( g_i \in P^i \) and \( K_i \) be the class of \( G \) containing \( g_i \) and \( K \) be the class of \( G \) containing \( x \). We define a class function \( \theta_{ix} \) by

\[
\theta_{ix}(a) = | \{(g_i', x') | g_i' \in K_i, x' \in K, g_i'x' = a \} | .
\]
We have the well-known formula

$$\theta_{ix} = \frac{|G|}{|C_{G}(g)|} \frac{\sum x \chi(g) \chi(x)}{\chi(1)}$$

where the sum is over all irreducible characters $\chi$ of $G$. We now define another class function $\theta_{ix}'$ by

$$\theta_{ix}'(a) = \begin{cases} \theta_{ix}(a), & \text{if } a \text{ is } p\text{-singular}, \\ 0, & \text{otherwise}. \end{cases}$$

We may write

$$\theta_{ix}' = \frac{\sum a \theta_{ix}(a)}{|C_{G}(a)|} \frac{\sum \chi(a) \chi}{\chi(1)}$$

where the sum is over a complete set of representatives $a$ of the $p$-singular classes $a^{G}$ of $G$. By $\theta_{ix}'_{IB}$ we mean

$$\theta_{ix}'_{IB} = \frac{\sum a \theta_{ix}(a)}{|C_{G}(a)|} \frac{\sum \chi(a) \chi}{\chi(1)}$$

and by $\theta_{ixlB}$ we mean

$$\theta_{ixlB} = \frac{|G|}{|C_{G}(g)|} \frac{\sum x \chi(g) \chi(x)}{\chi(1)}$$

**Lemma 1.** $\theta_{ix}(g_{x}v) \equiv 0 \pmod{q}$ for $g_{x} \in P^{t}$ and $v \in V$.

**Proof.** The lemma follows easily, because $P$ acts by conjugation fixed-point-free on the set of the pairs $(g', x')$, where $g' \subseteq K, x' \subseteq K$ and $g'x' = g_{x}v$.

Let $m$ be $\{ \frac{x}{y} | y \text{ is a rational integer which is prime to } p, \text{ and } z \text{ is an algebraic integer} \}$.

**Lemma 2.** $\theta_{ixlB}(g_{x})$ is in $m$ and $\theta_{ixlB}(g_{x}) \equiv 0 \pmod{qm}$.

**Proof.** Since $\theta_{ix} - \theta_{ix}'$ vanishes on $p$-singular elements, the "Truncation of Relations" theorem (see [2] (IV.6.3)) shows that $\theta_{ixlB} - \theta_{ixlB}'$ vanishes on $p$-singular elements. In particular

$$\theta_{ixlB}(g_{x}) = \theta_{ixlB}'(g_{x}) = \frac{\sum a \theta_{ix}(a)}{|C_{G}(a)|} \frac{\sum \chi(a) \chi(g_{x})}{\chi(1)}$$

We can calculate $\sum \chi(a) \chi(g_{x})$ by (5)[4] and it becomes
\[ \theta_{\iota} (g_k) = \sum_{v \in \mathcal{V}} \theta_{\iota}(g_kv) \frac{q \sum \varphi(v) \varphi(1)}{|C_0(g_kv)|}, \]

where the sum is over a complete set of representatives of \( p \)-singular classes in which \( g_k \) can be chosen as \( p \)-part. Since \( \theta_{\iota}(g_kv) \equiv 0 \pmod{q} \) by Lemma 1, the result follows.

(q.e.d.)

**Lemma 3.** If \( p \mid \Lambda(1) \), then \( p \mid \Lambda(x) \) in \( m \) for any \( p \)-regular element \( x \in V^c \).

Proof. We can compute the difference between \( \theta_{\iota} \) and \( \theta_{\iota} \) for \( g_i, g_j \in P^\ast \), as Sibley did in [5]:

\[ \theta_{\iota}(g_k) - \theta_{\iota}(g_k) = \frac{|G|}{|PV||C_0(x)|} \sum_{\Lambda} \left\{ \frac{\bar{\Lambda}(g_i)\bar{\Lambda}(x)\Lambda(g_k) - \bar{\Lambda}(g_j)\bar{\Lambda}(x)\Lambda(g_k)}{\Lambda(1)} \right\}, \]

where \( \{\Lambda\} \) are the exceptional characters in \( B \). (These equalities follow from the facts that \( X(g_i) = X(g_j) \) for any nonexceptional character \( X \) in \( B \) ((1D) \( v \) [3]), and that \( \Lambda(x) \) and \( \Lambda(1) \) are independent of the choice of \( \Lambda((2B) \) [1]).)

On the other hand,

\[ \sum_{\Lambda} \{\bar{\Lambda}(g_i)\Lambda(g_k) - \bar{\Lambda}(g_j)\Lambda(g_k)\} = \sum_{x \in B} \{\bar{\Lambda}(g_i)\Lambda(x) - \bar{\Lambda}(g_j)\Lambda(x)\}, \]

\[ = qf \varphi(1)^2 (\delta_{\varphi h} - \delta_{\varphi h}), \]

where \( \delta_{gh} \) is defined for \( g, h \in P^\ast \) by

\[ \delta_{gh} = \begin{cases} 1 & \text{if } g \sim h, \\ 0 & \text{otherwise}. \end{cases} \]

The last equality holds by (5) [4]. As \( G \) has at least two classes of \( p \)-elements, we can choose \( g_i = g_k \) and \( g_j \sim g_k \). Then by Lemma 2

\[ 0 \equiv \theta_{\iota}(g_k) - \theta_{\iota}(g_k) = \frac{|G|\bar{\Lambda}(x)f \varphi(1)^2}{|V||C_0(x)|\Lambda(1)} \pmod{qm}. \]

Then \( p \mid \Lambda(x) \).

(q.e.d.)

We now calculate \( ||\Lambda||^2 \). This gives

\[ 1 = ||\Lambda||^2 = \frac{1}{|G|} \sum_{g \in B} \frac{|G|\Lambda(g)\bar{\Lambda}(g)}{|C_0(g)|} + \frac{1}{|G|} \sum_{v \in \mathcal{V}} \frac{|G|\Lambda(v)\bar{\Lambda}(v)}{|C_0(v)|}, \]

\[ + \frac{1}{|G|} \sum_{x \in V^c \setminus V^0} \frac{|G|\Lambda(x)\bar{\Lambda}(x)}{|C_0(x)|}, \]

where the first and the second sums are over complete sets of representatives of
G-conjugacy classes and the third sum is over that of G-conjugacy classes of p-regular elements which are not in $V^C$. Then by Lemma 3 we may write the third sum as $p^2R$ where $R = \frac{z}{y}$ for some algebraic integer $z$ and some rational integer $y$ which is prime to $p$.

**Lemma 4.** Let $T_1$ be the first term of (4). Then

$$T_1 = 1 + \frac{|V|f}{|N|} \left\{ - (\delta e + c)^2 + c^2 \right\} .$$

**Proof.** By (1),

$$T_1 = \frac{1}{|G|} \sum_{\phi \in \mathcal{C}} \frac{|G|}{|C_G(g)|} (\delta \lambda(g) + c \sum_{\phi \in \mathcal{C}} 1_p \varphi(g)) (\delta \overline{\lambda}(g) + c \sum_{\phi \in \mathcal{C}} 1_p \overline{\varphi}(g)) ,$$

where the sum is over a complete set of representatives of G-conjugacy classes. Since $S$ is a T.I. set, the representatives of G-conjugacy classes of $S$ coincide with those of $N$-conjugacy classes of $S$ and $|C_G(g)| = |C_N(g)|$. Then

$$T_1 = \frac{|G:N|}{|G|} \sum_{\phi \in \mathcal{C}} \frac{|N|}{|C_N(g)|} (\delta \lambda(g) + c \sum_{\phi \in \mathcal{C}} 1_p \varphi(g)) (\delta \overline{\lambda}(g) + c \sum_{\phi \in \mathcal{C}} 1_p \overline{\varphi}(g)) ,$$

where the sum is over a complete set of representatives of $N$-conjugacy classes. Since $\lambda$ is a character of $N$ and $\sum_{\phi \in \mathcal{C}} 1_p \varphi$ is an $N$-invariant character of $PV$,

$$T_1 = \frac{1}{|N|} \left\{ \sum_{\phi \in \mathcal{C}} \lambda(g) \overline{\lambda}(g) + \delta e \sum_{\phi \in \mathcal{C}} [\lambda(g) (\sum_{\phi \in \mathcal{C}} 1_p \varphi(g)) + \overline{\lambda}(g) (\sum_{\phi \in \mathcal{C}} 1_p \varphi(g))] \right. \\
+ \left. c^2 \sum_{\phi \in \mathcal{C}} (\sum_{\phi \in \mathcal{C}} 1_p \varphi(g)) (\sum_{\phi \in \mathcal{C}} 1_p \overline{\varphi}(g)) \right\} ,$$

where the sums $\sum_{\phi \in \mathcal{C}}$ are over all elements of $S$. We can express $\lambda_{1\times \mathcal{C}}$ as follows:

$$\lambda_{1\times \mathcal{C}} = \sum_{\phi \in \mathcal{C}} (\mu_1 + \mu_2 + \cdots + \mu_e)^n \varphi^n ,$$

where $n$ ranges over a cross section of $W(\phi)$ in $N$, and $\mu_1, \mu_2, \cdots, \mu_e$ are distinct irreducible nonprincipal characters of $P$. Note that

$$\sum_{\phi \in \mathcal{C}} \varphi^n = \sum_{\phi \in \mathcal{C}} \varphi .$$

From the orthogonality relations we get

$$\sum_{\phi \in \mathcal{C}} \lambda(g) \overline{\lambda}(g) = |N| - \sum_{\phi \in \mathcal{C}} |\lambda(\phi)|^2 \\
= |N| - \sum_{\phi \in \mathcal{C}} e \sum_{\phi \in \mathcal{C}} |\varphi(\phi)|^2 \\
= |N| - e^2 |V| f ,$$
\[
\sum_{g \in G} \lambda (g) \left( \sum_{\varphi \in \Phi} 1_{\varphi} \varphi (g) \right) = \sum_{g \in G} \left( \sum_{\varphi \in \Phi} (\mu_1 + \mu_2 + \cdots + \mu_s)^n \varphi^n (g) \right) \left( \sum_{\varphi \in \Phi} 1_{\varphi} \varphi (g) \right) \\
= - \sum_{\varphi \in \Phi} \left( \sum_{s \in S} e \varphi^s (v) \right) \left( \sum_{\varphi \in \Phi} \varphi (v) \right) \\
= -e \sum_{\varphi \in \Phi} \varphi (v) \left( \sum_{\varphi \in \Phi} \varphi (v) \right) \\
= -e |V| f ,
\]
and
\[
\sum_{g \in G} \lambda (g) \left( \sum_{\varphi \in \Phi} 1_{\varphi} \varphi (g) \right) = -e |V| f ,
\]
and
\[
\sum_{g \in G} \left| \sum_{\varphi \in \Phi} 1_{\varphi} \varphi (g) \right|^2 = (q-1) \sum_{v \in V} \left| \sum_{\varphi \in \Phi} \varphi (v) \right|^2 = (q-1) |V| f .
\]
Then
\[
T_1 = \frac{1}{|N|} \left\{ |N| - e^2 |V| f - 2de |V| f + c^2 (q-1) |V| f \right\} \\
= 1 + \frac{|V| f}{|N|} \left\{ -(d+e)^2 + c^2 q \right\} .
\]
(q.c.d.)

Multiplying (4) by \( q | V | \) we get
\[
\frac{|V| f}{s} \left\{ -(d+e)^2 + c^2 q \right\} + T_2 + p^2 q R | V | = 0 ,
\]
where
\[
T_2 = \frac{1}{|G : PV|} \sum_{v \in V} \frac{|G| \Lambda (v) \overline{\Lambda} (v)}{|C_G (v)|} .
\]
Then
\[
\frac{|V| f}{s} \left\{ -(d+e)^2 + c^2 q \right\} + T_2 \equiv 0 \pmod{pqm} . \tag{5}
\]

**Lemma 5.** \( T_2 \equiv \frac{|V| f (d+e)^2}{s} \pmod{pqm} . \)

Proof. Let \( \{ v_j \} (1 \leq j \leq u) \) be the representatives of \( G \)-conjugacy classes of \( V \). Then these are also the representatives of \( N \)-conjugacy classes, because \( N_G (P) \) controls fusion of \( C_G (P) \). Note that \( p | (d+e) \) from (2), because we have assumed that \( p | \Lambda (1) \). By (3),
\[
T_2 = \frac{1}{|G : PV| \sum_{j=1}^u \frac{|G|}{|C_G (v_j)|} \Lambda (v_j) \overline{\Lambda} (v_j) \\
= \frac{1}{|G : PV| \sum_{j=1}^u \frac{|G|}{|C_G (v_j)|} \delta (d+e)^2 \left\{ \sum_{\varphi \in \Phi} \varphi (v_j) \right\} \left( \sum_{\varphi \in \Phi} \varphi (v_j) \right) \} \pmod{pqm} .
\]

We now set \( \zeta = \sum_{\varphi \in \Phi} \varphi \). Since \( \zeta \) is an \( N \)-invariant character of \( V \) and \( \{ v_j \} \) are also the representatives of \( N \)-conjugacy classes,
Then

\[ T_2 \equiv \frac{(\delta e+c)^2}{s} \sum_{v \in V} |\xi(v)|^2 \equiv \frac{|V|f(\delta e+c)^2}{s} \quad (\text{mod } pqm). \]

Since \( P \subseteq C_G(v) \) and \( P \) is a T.I. Sylow \( p \)-group of \( C_G(v) \),

\[ |C_G(v): N_G(P)\cap C_G(v)| \equiv 1 \quad (\text{mod } q). \]

Thus

\[ T_2 \equiv \frac{(\delta e+c)^2}{s} \sum_{v \in V} |\xi(v)|^2 \equiv \frac{|V|f(\delta e+c)^2}{s} \quad (\text{mod } pqm). \quad (\text{q.e.d.}) \]

Then by (5) we get the congruence

\[ \frac{|V|f^2q}{s} \equiv 0 \quad (\text{mod } pqm). \]

Hence we get \( p \mid c^2 \). This contradicts \( p \mid \Lambda(1) \). This completes the proof of the main theorem.

Acknowledgment. The author would like to express her hearty thanks to Professor Yukio Tsushima for his helpful advices.

References


Department of Mathematics
Ochanomizu University
Ohtsuka, Bunkyo-ku, Tokyo 112
Japan