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# A NOTE ON HEIGHT OF EXCEPTIONAL CHARACTER DEGREES

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## 1. Introduction

We assume

(*) *G is a finite group with a Sylow p-group P satisfying*

$$C_G(x) = C_G(P), \qquad \text{all } x \in P^{\#}.$$

Hypothesis (*) implies that $P$ is an abelian trivial intersection subgroup and $C_G(P)=P\times V$, for some $p'$-group $V$. In fact, $S=(P\times V)-V$ is a T.I. set. Furthermore, $N_G(P)/V$ is a Frobenius group with Frobenius kernel $PV/V$. We set $|P|=q$, $s=|N_G(P): C_G(P)|$ and $st=q-1$, where $t$ is the number of $p$-classes of $G$. We set $N=N_G(P)$.

Under this hypothesis (*) R. Brauer and H.S. Leonard, Jr. [1,3] have shown the following results.

(a) There is a one to one correspondence from the $p$-blocks of $G$ of full defect onto the $N$-classes of irreducible characters of $V$. (See (1D) [3] for details.)

(b) A $p$-block $B$ of $G$ of full defect associated with an $N$-class $\Phi$ of irreducible characters of $V$ contains a family of exceptional characters $\{\Lambda_i\}$ $(1\leqq i\leqq |\Phi|t)$, if $|\Phi|t>1$.

(c) Let $\varphi\in\Phi$, and let $W(\varphi)$ be the inertia group of $\varphi$ in $N$. If $f=|\Phi|$ and $|W(\varphi)|=e|C_G(P)|$, then $ef=s$. Let $\Lambda$ be any member of $\{\Lambda_i\}$. Then

$$\Lambda_{|S} = \delta\lambda + c\sum_{\varphi\in\Phi} 1_P\varphi\,, \tag{1}$$

$$\Lambda(1)\equiv\delta s\varphi(1)+cf\varphi(1)\equiv(\delta e+c)f\varphi(1) \qquad (\mathrm{mod}\ q), \tag{2}$$

where $\delta=\pm 1$, $\varphi\in\Phi$, $c\in\boldsymbol{Z}$, and $\lambda$ is an appropriately chosen exceptional character of $N$. In particular, the $p$-block $b$ of $\lambda$ in $N$ lifts to the $p$-block $B$ of $\Lambda$ in $G$. In addition $\lambda=(\mu\varphi)^N$, where $\mu$ is some nonprincipal irreducible character of $P$.

(d) Moreover it follows from (3.10) in [1] that

$$\Lambda_{|V}\equiv\delta\lambda_{|V}+c\sum_{\varphi\in\Phi}\varphi\equiv(\delta e+c)\sum_{\varphi\in\Phi}\varphi \qquad (\mathrm{mod}\ q) \tag{3}$$

in the ring of all algebraic integers.

Using (a), (b) and (c), D.A. Sibley has proved the following two theorems.

**Theorem 1** (Sibley [4]). *Suppose* $(*)$ *holds and* $G$ *has at least three classes of* $p$*-elements. Then* $c=0$. *In particular* $\Lambda(1)\equiv\delta\lambda(1)$ (mod $q$).

**Theorem 2** (Sibley [5]). *Suppose*
($\sharp$) *$G$ is a finite group with a Sylow $p$-group $P$ satisfying*

$$C_G(x) = P, \qquad \text{all } x\in P^{\sharp}.$$

*Under this hypothesis* ($\sharp$), $p\not\mid\Lambda(1)$ *if $G$ has at least two classes of $p$-elements.*

We remark that instead of hypothesis ($\sharp$), Sibley has proved Theorem 2 under the following hypothesis:

($\sharp'$) *A Sylow $p$-group $P$ of $G$ is an abelian* T.I. *set, and $N_G(P)$ is a Frobenius group with Frobenius kernel $P$.*

It is easily seen that hypothesis ($\sharp$) is equivalent to hypothesis ($\sharp'$).

In this paper we shall prove the following theorem, which has been conjectured by Sibley [5].

**Main Theorem.** *Suppose* $(*)$ *holds and* $G$ *has at least two classes of* $p$*-elements. Then* $p\not\mid\Lambda(1)$.

EXAMPLE. Let $G=SL(2,q)$, where $q$ is a power of an odd prime $p$ and $\frac{q-1}{2}$ is odd. Then $G$ satisfies $(*)$ with $|V|=2$ and has two classes of $p$-elements. $N_G(P)$ has two families of exceptional characters and both degrees are $\frac{q-1}{2}$. On the other hand $G$ has two families of exceptional characters and their degrees are $\frac{q-1}{2}$ and $\frac{q+1}{2}$, which are prime to $p$. Moreover if we choose $\delta$ appropriately, we can take $c=0$ in both families.

## 2. Proof of Main Theorem

The main theorem can be proved by similar way to Theorem 2 with the addition of block calculations as the proof of Theorem 1.

Suppose by way of contradiction that $p\,|\,\Lambda(1)$. Then first of all we claim also $p\,|\,\Lambda(x)$ for any $p$-regular element $x$ which is not conjugate to an element of $V$ in $G$. Let $g_i\in P^{\sharp}$ and $K_i$ be the class of $G$ containing $g_i$ and $K$ be the class of $G$ containing $x$. We define a class function $\theta_{ix}$ by

$$\theta_{ix}(a) = |\{(g_i', x')\,|\,g_i'\in K_i, x'\in K, g_i'x' = a\}|\,.$$

We have the well-known formula

$$\theta_{ix} = \frac{|G|}{|C_G(g_i)|\,|C_G(x)|}\sum_{\chi}\frac{\bar{\chi}(g_i)\bar{\chi}(x)}{\chi(1)}\chi\,,$$

where the sum is over all irreducible characters $\chi$ of $G$. We now define another class function $\theta_{ix}'$ by

$$\theta_{ix}'(a) = \begin{cases}\theta_{ix}(a), & \text{if } a \text{ is } p\text{-singular},\\ 0, & \text{otherwise.}\end{cases}$$

We may write

$$\theta_{ix}' = \sum_a{}' \frac{\theta_{ix}(a)}{|C_G(a)|}\sum_{\chi}\bar{\chi}(a)\chi\,,$$

where the sum is over a complete set of representatives $a$ of the $p$-singular classes $a^G$ of $G$. By $\theta_{ix}'{}_{|B}$ we mean

$$\theta_{ix}'{}_{|B} = \sum_a{}' \frac{\theta_{ix}(a)}{|C_G(a)|}\sum_{\chi\in B}\bar{\chi}(a)\chi\,,$$

and by $\theta_{ix|B}$ we mean

$$\theta_{ix|B} = \frac{|G|}{|C_G(g_i)|\,|C_G(x)|}\sum_{\chi\in B}\frac{\bar{\chi}(g_i)\bar{\chi}(x)}{\chi(1)}\chi\,.$$

**Lemma 1.** $\theta_{ix}(g_k v)\equiv 0 \pmod q$ *for* $g_k\in P^\#$ *and* $v\in V$.

Proof. The lemma follows easily, because $P$ acts by conjugation fixed-point-free on the set of the pairs $(g_i', x')$, where $g_i'\in K_i$, $x'\in K$ and $g_i'x'=g_k v$. (q.e.d.)

Let $\mathfrak{m}$ be $\{\frac{z}{y}\,|\,y$ is a rational integer which is prime to $p$, and $z$ is an algebraic integer$\}$.

**Lemma 2.** $\theta_{ix|B}(g_k)$ *is in* $\mathfrak{m}$ *and* $\theta_{ix|B}(g_k)\equiv 0 \pmod{q\mathfrak{m}}$.

Proof. Since $\theta_{ix}-\theta_{ix}'$ vanishes on $p$-singular elements, the "Truncation of Relations" theorem (see [2] (IV.6.3)) shows that $\theta_{ix|B}-\theta_{ix}'{}_{|B}$ vanishes on $p$-singular elements. In particular

$$\theta_{ix|B}(g_k) = \theta_{ix}'{}_{|B}(g_k) = \sum_a{}' \frac{\theta_{ix}(a)}{|C_G(a)|}\sum_{\chi\in B}\bar{\chi}(a)\chi(g_k)\,.$$

We can claculate $\sum_{\chi\in B}\bar{\chi}(a)\chi(g_k)$ by (5)[4] and it becomes

$$\theta_{ix}{}'_{|B}(g_k)=\sum_{g_kv}{}'\frac{\theta_{ix}(g_kv)}{|C_G(g_kv)|}q\sum_{\varphi\in\Phi}\bar{\varphi}(v)\varphi(1)\,,$$

where the sum is over a complete set of representatives of $p$-singular classes in which $g_k$ can be chosen as $p$-part. Since $\theta_{ix}(g_kv)\equiv 0 \pmod q$ by Lemma 1, the result follows. (q.e.d.)

**Lemma 3.** *If* $p\,|\,\Lambda(1)$, *then* $p\,|\,\Lambda(x)$ *in* $\mathfrak{m}$ *for any p-regular element* $x\notin V^G$.

Proof. We can compute the difference between $\theta_{ix|B}$ and $\theta_{jx|B}$ for $g_i, g_j\in P^\sharp$, as Sibley did in [5]:

$$\begin{aligned}\theta_{ix|B}(g_k)-\theta_{jx|B}(g_k)&=\frac{|G|}{|PV|\,|C_G(x)|}\sum_{\Lambda}\left\{\frac{\bar\Lambda(g_i)\bar\Lambda(x)\Lambda(g_k)}{\Lambda(1)}-\frac{\bar\Lambda(g_j)\bar\Lambda(x)\Lambda(g_k)}{\Lambda(1)}\right\}\\&=\frac{|G|\bar\Lambda(x)}{q|V|\,|C_G(x)|\Lambda(1)}\sum_{\Lambda}\{\bar\Lambda(g_i)\Lambda(g_k)-\bar\Lambda(g_j)\Lambda(g_k)\}\,,\end{aligned}$$

where $\{\Lambda\}$ are the exceptional characters in $B$. (These equalities follow from the facts that $X(g_i)=X(g_j)$ for any nonexceptional character $X$ in $B$ ((1$D$) (v) [3]), and that $\Lambda(x)$ and $\Lambda(1)$ are independent of the choice of $\Lambda$((2B) [1]).) On the other hand,

$$\begin{aligned}\sum_{\Lambda}\{\bar\Lambda(g_i)\Lambda(g_k)-\bar\Lambda(g_j)\Lambda(g_k)\}&=\sum_{\chi\in B}\{\bar\chi(g_i)\chi(g_k)-\bar\chi(g_j)\chi(g_k)\}\\&=qf\varphi(1)^2(\delta_{g_ig_k}-\delta_{g_jg_k})\,,\end{aligned}$$

where $\delta_{gh}$ is defined for $g, h\in P^\sharp$ by

$$\delta_{gh}=\begin{cases}1 & g\sim h\,,\\ 0 & \text{otherwise.}\end{cases}$$

The last equality holds by (5) [4]. As $G$ has at least two classes of $p$-elements, we can choose $g_i=g_k$ and $g_j\not\sim g_k$. Then by Lemma 2

$$0\equiv\theta_{ix|B}(g_k)-\theta_{jx|B}(g_k)=\frac{|G|\bar\Lambda(x)f\varphi(1)^2}{|V|\,|C_G(x)|\Lambda(1)}\pmod{q\mathfrak{m}}\,.$$

Then $p\,|\,\Lambda(x)$. (q.e.d.)

We now calculate $||\Lambda||^2$. This gives

$$\begin{aligned}1=||\Lambda||^2=\frac{1}{|G|}\sum_{g\in S}{}'\frac{|G|\Lambda(g)\bar\Lambda(g)}{|C_G(g)|}+\frac{1}{|G|}\sum_{v\in V}{}'\frac{|G|\Lambda(v)\bar\Lambda(v)}{|C_G(v)|}\\+\frac{1}{|G|}\sum_{x\notin V^G}{}'\frac{|G|\Lambda(x)\bar\Lambda(x)}{|C_G(x)|}\,,\end{aligned}\tag{4}$$

where the first and the second sums are over complete sets of representatives of

$G$-conjugacy classes and the third sum is over that of $G$-conjugacy classes of $p$-regular elements which are not in $V^G$. Then by Lemma 3 we may write the third sum as $p^2R$ where $R=\frac{z}{y}$ for some algebraic integer $z$ and some rational integer $y$ which is prime to $p$.

**Lemma 4.** *Let $T_1$ be the first term of* (4). *Then*

$$T_1 = 1+\frac{|V|f}{|N|}\{-(\delta e+c)^2+c^2j\} .$$

Proof. By (1),

$$T_1 = \frac{1}{|G|}\sum_{g\in S}{}'\frac{|G|}{|C_G(g)|}(\delta\lambda(g)+c\sum_{\varphi\in\Phi}1_P\varphi(g))\,(\delta\bar{\lambda}(g)+c\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))\,,$$

where the sum is over a complete set of representatives of $G$-conjugacy classes. Since $S$ is a T.I. set, the representatives of $G$-conjugacy classes of $S$ coincide with those of $N$-conjugacy classes of $S$ and $|C_G(g)|=|C_N(g)|$. Then

$$T_1 = \frac{|G:N|}{|G|}\sum_{g\in S}{}'\frac{|N|}{|C_N(g)|}(\delta\lambda(g)+c\sum_{\varphi\in\Phi}1_P\varphi(g))\,(\delta\bar{\lambda}(g)+c\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))\,,$$

where the sum is over a complete set of representatives of $N$-conjugacy classes. Since $\lambda$ is a character of $N$ and $\sum_{\varphi\in\Phi}1_P\varphi$ is an $N$-invariant character of $PV$,

$$\begin{aligned}T_1 = \frac{1}{|N|}\ \{&\sum_{g\in S}\lambda(g)\bar{\lambda}(g)+\delta c\sum_{g\in S}[\lambda(g)\,(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))+\bar{\lambda}(g)\,(\sum_{\varphi\in\Phi}1_P\varphi(g))]\\ &+c^2\sum_{g\in S}(\sum_{\varphi\in\Phi}1_P\varphi(g))\,(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))\}\,,\end{aligned}$$

where the sums $\sum_{g\in S}$ are over all elements of $S$. We can express $\lambda_{|P\times V}$ as follows:

$$\lambda_{|P\times V} = \sum_n(\mu_1+\mu_2+\cdots+\mu_e)^n\varphi^n\,,$$

where $n$ ranges over a cross section of $W(\varphi)$ in $N$, and $\mu_1,\mu_2,\cdots,\mu_e$ are distinct irreducible nonprincipal characters of $P$. Note that

$$\sum_n\varphi^n = \sum_{\varphi\in\Phi}\varphi\,.$$

From the orthogonality relations we get

$$\begin{aligned}\sum_{g\in S}\lambda(g)\bar{\lambda}(g) &= |N|-\sum_{v\in V}|\lambda(v)|^2\\ &= |N|-\sum_{v\in V}|e\sum_{\varphi\in\Phi}\varphi(v)|^2\\ &= |N|-e^2|V|f,\end{aligned}$$

$$\sum_{g\in S}\lambda(g)\,(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))=\sum_{g\in S}(\sum_n(\mu_1+\mu_2+\cdots+\mu_e)^n\varphi^n(g))\,(\sum_{\varphi\in\Phi}1_P\bar{\varphi}(g))$$
$$=-\sum_{v\in V}(\sum_n e\varphi^n(v))\,(\sum_{\varphi\in\Phi}\bar{\varphi}(v))$$
$$=-e\sum_{v\in V}(\sum_{\varphi\in\Phi}\varphi(v)\sum_{\varphi\in\Phi}\bar{\varphi}(v))$$
$$=-e|V|f,$$
$$\sum_{g\in S}\bar{\lambda}(g)\,(\sum_{\varphi\in\Phi}1_P\varphi(g))=-e|V|f,$$

and

$$\sum_{g\in S}|\sum_{\varphi\in\Phi}1_P\varphi(g)|^2=(q-1)\sum_{v\in V}|\sum_{\varphi\in\Phi}\varphi(v)|^2=(q-1)|V|f.$$

Then

$$T_1=\frac{1}{|N|}\{|N|-e^2|V|f-2\delta ce|V|f+c^2(q-1)|V|f\}$$
$$=1+\frac{|V|f}{|N|}\{-(\delta e+c)^2+c^2q\}. \qquad \text{(q.e.d.)}$$

Multiplying (4) by $q|V|$ we get

$$\frac{|V|f}{s}\{-(\delta e+c)^2+c^2q\}+T_2+p^2qR|V|=0,$$

where

$$T_2=\frac{1}{|G:PV|}\sum_{v\in V}{}'\frac{|G|\Lambda(v)\bar{\Lambda}(v)}{|C_G(v)|}.$$

Then

$$\frac{|V|f}{s}\{-(\delta e+c)^2+c^2q\}+T_2\equiv 0 \qquad (\mathrm{mod}\ pq\mathfrak{m}). \tag{5}$$

**Lemma 5.** $\quad T_2\equiv\dfrac{|V|f(\delta e+c)^2}{s} \qquad (\mathrm{mod}\ pq\mathfrak{m}).$

Proof. Let $\{v_j\}$ $(1\leqq j\leqq u)$ be the representatives of $G$-conjugacy classes of $V$. Then these are also the representatives of $N$-conjugacy classes, because $N_G(P)$ controls fusion of $C_G(P)$. Note that $p\,|\,(\delta e+c)$ from (2), because we have assumed that $p\,|\,\Lambda(1)$. By (3),

$$T_2=\frac{1}{|G:PV|}\sum_{j=1}^{u}\frac{|G|}{|C_G(v_j)|}\Lambda(v_j)\bar{\Lambda}(v_j)$$
$$\equiv\frac{1}{|G:PV|}\sum_{j=1}^{u}\frac{|G|}{|C_G(v_j)|}(\delta e+c)^2\{(\sum_{\varphi\in\Phi}\varphi(v_j))\,(\sum_{\varphi\in\Phi}\bar{\varphi}(v_j))\} \quad (\mathrm{mod}\ pq\mathfrak{m}).$$

We now set $\zeta=\sum_{\varphi\in\Phi}\varphi$. Since $\zeta$ is an $N$-invariant character of $V$ and $\{v_j\}$ are also the representatives of $N$-conjugacy classes,

$$|\zeta(v_j)|^2 = \frac{|C_N(v_j)|}{|N|} \sum_{v \underset{N}{\sim} v_j} |\zeta(v)|^2 .$$

Then

$$\begin{aligned} T_2 &\equiv \frac{(\delta e+c)^2}{|G:PV|} \sum_{j=1}^{u} \frac{|G|}{|C_G(v_j)|} |\zeta(v_j)|^2 \\ &\equiv \frac{(\delta e+c)^2}{|G:PV|} \sum_{j=1}^{u} \left\{ \frac{|G|\,|C_N(v_j)|}{|C_G(v_j)|\,|N|} \sum_{v \underset{N}{\sim} v_j} |\zeta(v)|^2 \right\} \\ &\equiv \frac{(\delta e+c)^2}{s} \sum_{v \in V} \frac{|\zeta(v)|^2}{|C_G(v):C_N(v)|} \qquad (\mathrm{mod}\ pq\mathfrak{m}) . \end{aligned}$$

Since $P \subseteq C_G(v)$ and $P$ is a T.I. Sylow $p$-group of $C_G(v)$,

$$|C_G(v) : C_N(v)| \equiv |C_G(v) : N_G(P) \cap C_G(v)| \equiv 1 \qquad (\mathrm{mod}\ q) .$$

Thus

$$T_2 \equiv \frac{(\delta e+c)^2}{s} \sum_{v \in V} |\zeta(v)|^2 \equiv \frac{|V|\, f(\delta e+c)^2}{s} \qquad (\mathrm{mod}\ pq\mathfrak{m}) . \qquad \text{(q.e.d.)}$$

Then by (5) we get the congruence

$$\frac{|V|\, f c^2 q}{s} \equiv 0 \qquad (\mathrm{mod}\ pq\mathfrak{m}) .$$

Hence we get $p \mid c^2$. This contradicts $p \mid \Lambda(1)$. This completes the proof of the main theorem.

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