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A NOTE ON HEIGHT OF EXCEPTIONAL CHARACTER DEGREES

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1. Introduction

We assume

(*) *G is a finite group with a Sylow p-group P satisfying*

$$C_G(x) = C_G(P), \quad \text{all } x \in P^*.$$

Hypothesis (*) implies that P is an abelian trivial intersection subgroup and $C_G(P) = P \times V$, for some p' -group V . In fact, $S = (P \times V) - V$ is a T.I. set. Furthermore, $N_G(P)/V$ is a Frobenius group with Frobenius kernel PV/V . We set $|P| = q$, $s = |N_G(P) : C_G(P)|$ and $st = q - 1$, where t is the number of p -classes of G . We set $N = N_G(P)$.

Under this hypothesis (*) R. Brauer and H.S. Leonard, Jr. [1,3] have shown the following results.

- (a) There is a one to one correspondence from the p -blocks of G of full defect onto the N -classes of irreducible characters of V . (See (1D) [3] for details.)
- (b) A p -block B of G of full defect associated with an N -class Φ of irreducible characters of V contains a family of exceptional characters $\{\Lambda_i\}$ ($1 \leq i \leq |\Phi|t$), if $|\Phi|t > 1$.
- (c) Let $\varphi \in \Phi$, and let $W(\varphi)$ be the inertia group of φ in N . If $f = |\Phi|$ and $|W(\varphi)| = e|C_G(P)|$, then $ef = s$. Let Λ be any member of $\{\Lambda_i\}$. Then

$$\Lambda|_S = \delta\lambda + c \sum_{\varphi \in \Phi} 1_P \varphi, \tag{1}$$

$$\Lambda(1) \equiv \delta s \varphi(1) + c f \varphi(1) \equiv (\delta e + c) f \varphi(1) \pmod{q}, \tag{2}$$

where $\delta = \pm 1$, $\varphi \in \Phi$, $c \in \mathbf{Z}$, and λ is an appropriately chosen exceptional character of N . In particular, the p -block b of λ in N lifts to the p -block B of Λ in G . In addition $\lambda = (\mu\varphi)^N$, where μ is some nonprincipal irreducible character of P .

- (d) Moreover it follows from (3.10) in [1] that

$$\Lambda|_V \equiv \delta \lambda|_V + c \sum_{\varphi \in \Phi} \varphi \equiv (\delta e + c) \sum_{\varphi \in \Phi} \varphi \pmod{q} \tag{3}$$

in the ring of all algebraic integers.

Using (a),(b) and (c), D.A. Sibley has proved the following two theorems.

Theorem 1 (Sibley [4]). *Suppose (*) holds and G has at least three classes of p -elements. Then $c=0$. In particular $\Lambda(1) \equiv \delta\lambda(1) \pmod{q}$.*

Theorem 2 (Sibley [5]). *Suppose*

(#) *G is a finite group with a Sylow p -group P satisfying*

$$C_G(x) = P, \quad \text{all } x \in P^*.$$

Under this hypothesis (#), $p \nmid \Lambda(1)$ if G has at least two classes of p -elements.

We remark that instead of hypothesis (#), Sibley has proved Theorem 2 under the following hypothesis:

(#') *A Sylow p -group P of G is an abelian T.I. set, and $N_G(P)$ is a Frobenius group with Frobenius kernel P .*

It is easily seen that hypothesis (#) is equivalent to hypothesis (#').

In this paper we shall prove the following theorem, which has been conjectured by Sibley [5].

Main Theorem. *Suppose (*) holds and G has at least two classes of p -elements. Then $p \nmid \Lambda(1)$.*

EXAMPLE. Let $G = SL(2, q)$, where q is a power of an odd prime p and $\frac{q-1}{2}$ is odd. Then G satisfies (*) with $|V|=2$ and has two classes of p -elements. $N_G(P)$ has two families of exceptional characters and both degrees are $\frac{q-1}{2}$. On the other hand G has two families of exceptional characters and their degrees are $\frac{q-1}{2}$ and $\frac{q+1}{2}$, which are prime to p . Moreover if we choose δ appropriately, we can take $c=0$ in both families.

2. Proof of Main Theorem

The main theorem can be proved by similar way to Theorem 2 with the addition of block calculations as the proof of Theorem 1.

Suppose by way of contradiction that $p \mid \Lambda(1)$. Then first of all we claim also $p \mid \Lambda(x)$ for any p -regular element x which is not conjugate to an element of V in G . Let $g_i \in P^*$ and K_i be the class of G containing g_i and K be the class of G containing x . We define a class function θ_{ix} by

$$\theta_{ix}(a) = |\{(g_i', x') \mid g_i' \in K_i, x' \in K, g_i' x' = a\}|.$$

We have the well-known formula

$$\theta_{ix} = \frac{|G|}{|C_G(g_i)||C_G(x)|} \sum_{\chi} \frac{\bar{\chi}(g_i)\bar{\chi}(x)}{\chi(1)} \chi,$$

where the sum is over all irreducible characters χ of G . We now define another class function θ_{ix}' by

$$\theta_{ix}'(a) = \begin{cases} \theta_{ix}(a), & \text{if } a \text{ is } p\text{-singular,} \\ 0, & \text{otherwise.} \end{cases}$$

We may write

$$\theta_{ix}' = \sum_a' \frac{\theta_{ix}(a)}{|C_G(a)|} \sum_{\chi} \bar{\chi}(a)\chi,$$

where the sum is over a complete set of representatives a of the p -singular classes a^G of G . By $\theta_{ix}'|_B$ we mean

$$\theta_{ix}'|_B = \sum_a' \frac{\theta_{ix}(a)}{|C_G(a)|} \sum_{\chi \in B} \bar{\chi}(a)\chi,$$

and by $\theta_{ix}|_B$ we mean

$$\theta_{ix}|_B = \frac{|G|}{|C_G(g_i)||C_G(x)|} \sum_{\chi \in B} \frac{\bar{\chi}(g_i)\bar{\chi}(x)}{\chi(1)} \chi.$$

Lemma 1. $\theta_{ix}(g_kv) \equiv 0 \pmod{q}$ for $g_k \in P^*$ and $v \in V$.

Proof. The lemma follows easily, because P acts by conjugation fixed-point-free on the set of the pairs (g_i', x') , where $g_i' \in K_i, x' \in K$ and $g_i'x' = g_kv$. (q.e.d.)

Let m be $\{\frac{z}{y} | y \text{ is a rational integer which is prime to } p, \text{ and } z \text{ is an algebraic integer}\}$.

Lemma 2. $\theta_{ix|_B}(g_k)$ is in m and $\theta_{ix|_B}(g_k) \equiv 0 \pmod{qm}$.

Proof. Since $\theta_{ix} - \theta_{ix}'$ vanishes on p -singular elements, the "Truncation of Relations" theorem (see [2] (IV.6.3)) shows that $\theta_{ix|_B} - \theta_{ix}'|_B$ vanishes on p -singular elements. In particular

$$\theta_{ix|_B}(g_k) = \theta_{ix}'|_B(g_k) = \sum_a' \frac{\theta_{ix}(a)}{|C_G(a)|} \sum_{\chi \in B} \bar{\chi}(a)\chi(g_k).$$

We can calculate $\sum_{\chi \in B} \bar{\chi}(a)\chi(g_k)$ by (5)[4] and it becomes

$$\theta_{ix' | B}(g_k) = \sum_{g_k v} \frac{\theta_{ix}(g_k v)}{|C_G(g_k v)|} q \sum_{\varphi \in \Phi} \bar{\varphi}(v) \varphi(1),$$

where the sum is over a complete set of representatives of p -singular classes in which g_k can be chosen as p -part. Since $\theta_{ix}(g_k v) \equiv 0 \pmod{q}$ by Lemma 1, the result follows. (q.e.d.)

Lemma 3. *If $p \mid \Lambda(1)$, then $p \mid \Lambda(x)$ in m for any p -regular element $x \in V^G$.*

Proof. We can compute the difference between $\theta_{ix | B}$ and $\theta_{jx | B}$ for $g_i, g_j \in P^*$, as Sibley did in [5]:

$$\begin{aligned} \theta_{ix | B}(g_k) - \theta_{jx | B}(g_k) &= \frac{|G|}{|PV| |C_G(x)|} \sum_{\Lambda} \left\{ \frac{\bar{\Lambda}(g_i) \bar{\Lambda}(x) \Lambda(g_k)}{\Lambda(1)} - \frac{\bar{\Lambda}(g_j) \bar{\Lambda}(x) \Lambda(g_k)}{\Lambda(1)} \right\} \\ &= \frac{|G| \bar{\Lambda}(x)}{q |V| |C_G(x)| \Lambda(1)} \sum_{\Lambda} \{ \bar{\Lambda}(g_i) \Lambda(g_k) - \bar{\Lambda}(g_j) \Lambda(g_k) \}, \end{aligned}$$

where $\{\Lambda\}$ are the exceptional characters in B . (These equalities follow from the facts that $X(g_i) = X(g_j)$ for any nonexceptional character X in B ((1D) (v) [3]), and that $\Lambda(x)$ and $\Lambda(1)$ are independent of the choice of Λ ((2B) [1]).) On the other hand,

$$\begin{aligned} \sum_{\Lambda} \{ \bar{\Lambda}(g_i) \Lambda(g_k) - \bar{\Lambda}(g_j) \Lambda(g_k) \} &= \sum_{x \in B} \{ \bar{\chi}(g_i) \chi(g_k) - \bar{\chi}(g_j) \chi(g_k) \} \\ &= q f \varphi(1)^2 (\delta_{g_i g_k} - \delta_{g_j g_k}), \end{aligned}$$

where δ_{gh} is defined for $g, h \in P^*$ by

$$\delta_{gh} = \begin{cases} 1 & g \sim h, \\ 0 & \text{otherwise.} \end{cases}$$

The last equality holds by (5) [4]. As G has at least two classes of p -elements, we can choose $g_i = g_k$ and $g_j \not\sim g_k$. Then by Lemma 2

$$0 \equiv \theta_{ix | B}(g_k) - \theta_{jx | B}(g_k) = \frac{|G| \bar{\Lambda}(x) f \varphi(1)^2}{|V| |C_G(x)| \Lambda(1)} \pmod{qm}.$$

Then $p \mid \Lambda(x)$. (q.e.d.)

We now calculate $\|\Lambda\|^2$. This gives

$$\begin{aligned} 1 = \|\Lambda\|^2 &= \frac{1}{|G|} \sum_{g \in B} \frac{|G| \Lambda(g) \bar{\Lambda}(g)}{|C_G(g)|} + \frac{1}{|G|} \sum_{v \in V} \frac{|G| \Lambda(v) \bar{\Lambda}(v)}{|C_G(v)|} \\ &\quad + \frac{1}{|G|} \sum_{x \in V^G} \frac{|G| \Lambda(x) \bar{\Lambda}(x)}{|C_G(x)|}, \end{aligned} \tag{4}$$

where the first and the second sums are over complete sets of representatives of

G -conjugacy classes and the third sum is over that of G -conjugacy classes of p -regular elements which are not in V^G . Then by Lemma 3 we may write the third sum as p^2R where $R = \frac{z}{y}$ for some algebraic integer z and some rational integer y which is prime to p .

Lemma 4. *Let T_1 be the first term of (4). Then*

$$T_1 = 1 + \frac{|V|f}{|N|} \{ -(\delta e + c)^2 + c^2 \} .$$

Proof. By (1),

$$T_1 = \frac{1}{|G|} \sum_{g \in S} \frac{|G|}{|C_G(g)|} (\delta \lambda(g) + c \sum_{\varphi \in \Phi} 1_P \varphi(g)) (\delta \bar{\lambda}(g) + c \sum_{\varphi \in \Phi} 1_P \bar{\varphi}(g)) ,$$

where the sum is over a complete set of representatives of G -conjugacy classes. Since S is a T.I. set, the representatives of G -conjugacy classes of S coincide with those of N -conjugacy classes of S and $|C_G(g)| = |C_N(g)|$. Then

$$T_1 = \frac{|G:N|}{|G|} \sum_{g \in S} \frac{|N|}{|C_N(g)|} (\delta \lambda(g) + c \sum_{\varphi \in \Phi} 1_P \varphi(g)) (\delta \bar{\lambda}(g) + c \sum_{\varphi \in \Phi} 1_P \bar{\varphi}(g)) ,$$

where the sum is over a complete set of representatives of N -conjugacy classes. Since λ is a character of N and $\sum_{\varphi \in \Phi} 1_P \varphi$ is an N -invariant character of PV ,

$$T_1 = \frac{1}{|N|} \left\{ \sum_{g \in S} \lambda(g) \bar{\lambda}(g) + \delta c \sum_{g \in S} [\lambda(g) (\sum_{\varphi \in \Phi} 1_P \bar{\varphi}(g)) + \bar{\lambda}(g) (\sum_{\varphi \in \Phi} 1_P \varphi(g))] + c^2 \sum_{g \in S} (\sum_{\varphi \in \Phi} 1_P \varphi(g)) (\sum_{\varphi \in \Phi} 1_P \bar{\varphi}(g)) \right\} ,$$

where the sums $\sum_{g \in S}$ are over all elements of S . We can express $\lambda_{|P \times V}$ as follows:

$$\lambda_{|P \times V} = \sum_n (\mu_1 + \mu_2 + \dots + \mu_e)^n \varphi^n ,$$

where n ranges over a cross section of $W(\varphi)$ in N , and $\mu_1, \mu_2, \dots, \mu_e$ are distinct irreducible nonprincipal characters of P . Note that

$$\sum_n \varphi^n = \sum_{\varphi \in \Phi} \varphi .$$

From the orthogonality relations we get

$$\begin{aligned} \sum_{g \in S} \lambda(g) \bar{\lambda}(g) &= |N| - \sum_{v \in V} |\lambda(v)|^2 \\ &= |N| - \sum_{v \in V} |e \sum_{\varphi \in \Phi} \varphi(v)|^2 \\ &= |N| - e^2 |V| f , \end{aligned}$$

$$\begin{aligned} \sum_{g \in S} \lambda(g) \left(\sum_{\varphi \in \Phi} 1_P \bar{\varphi}(g) \right) &= \sum_{g \in S} \left(\sum_{\pi} (\mu_1 + \mu_2 + \dots + \mu_e)^\pi \varphi^\pi(g) \right) \left(\sum_{\varphi \in \Phi} 1_P \bar{\varphi}(g) \right) \\ &= - \sum_{v \in V} \left(\sum_{\pi} e \varphi^\pi(v) \right) \left(\sum_{\varphi \in \Phi} \bar{\varphi}(v) \right) \\ &= -e \sum_{v \in V} \left(\sum_{\varphi \in \Phi} \varphi(v) \right) \sum_{\varphi \in \Phi} \bar{\varphi}(v) \\ &= -e |V| f, \\ \sum_{g \in S} \bar{\lambda}(g) \left(\sum_{\varphi \in \Phi} 1_P \varphi(g) \right) &= -e |V| f, \end{aligned}$$

and

$$\sum_{g \in S} \left| \sum_{\varphi \in \Phi} 1_P \varphi(g) \right|^2 = (q-1) \sum_{v \in V} \left| \sum_{\varphi \in \Phi} \varphi(v) \right|^2 = (q-1) |V| f.$$

Then

$$\begin{aligned} T_1 &= \frac{1}{|N|} \{ |N| - e^2 |V| f - 2\delta c e |V| f + c^2 (q-1) |V| f \} \\ &= 1 + \frac{|V| f}{|N|} \{ -(\delta e + c)^2 + c^2 q \}. \end{aligned} \tag{q.e.d.}$$

Multiplying (4) by $q|V|$ we get

$$\frac{|V| f}{s} \{ -(\delta e + c)^2 + c^2 q \} + T_2 + p^2 q R |V| = 0,$$

where

$$T_2 = \frac{1}{|G:PV|} \sum_{v \in V} \frac{|G| |\Lambda(v) \bar{\Lambda}(v)|}{|C_G(v)|}.$$

Then

$$\frac{|V| f}{s} \{ -(\delta e + c)^2 + c^2 q \} + T_2 \equiv 0 \pmod{pqm}. \tag{5}$$

Lemma 5. $T_2 \equiv \frac{|V| f (\delta e + c)^2}{s} \pmod{pqm}.$

Proof. Let $\{v_j\} (1 \leq j \leq u)$ be the representatives of G -conjugacy classes of V . Then these are also the representatives of N -conjugacy classes, because $N_G(P)$ controls fusion of $C_G(P)$. Note that $p | (\delta e + c)$ from (2), because we have assumed that $p | \Lambda(1)$. By (3),

$$\begin{aligned} T_2 &= \frac{1}{|G:PV|} \sum_{j=1}^u \frac{|G|}{|C_G(v_j)|} \Lambda(v_j) \bar{\Lambda}(v_j) \\ &\equiv \frac{1}{|G:PV|} \sum_{j=1}^u \frac{|G|}{|C_G(v_j)|} (\delta e + c)^2 \left\{ \left(\sum_{\varphi \in \Phi} \varphi(v_j) \right) \left(\sum_{\varphi \in \Phi} \bar{\varphi}(v_j) \right) \right\} \pmod{pqm}. \end{aligned}$$

We now set $\zeta = \sum_{\varphi \in \Phi} \varphi$. Since ζ is an N -invariant character of V and $\{v_j\}$ are also the representatives of N -conjugacy classes,

$$|\zeta(v_j)|^2 = \frac{|C_N(v_j)|}{|N|} \sum_{\substack{v \sim v_j \\ N}} |\zeta(v)|^2.$$

Then

$$\begin{aligned} T_2 &\equiv \frac{(\delta e + c)^2}{|G:PV|} \sum_{j=1}^u \frac{|G|}{|C_G(v_j)|} |\zeta(v_j)|^2 \\ &\equiv \frac{(\delta e + c)^2}{|G:PV|} \sum_{j=1}^u \left\{ \frac{|G| |C_N(v_j)|}{|C_G(v_j)| |N|} \sum_{\substack{v \sim v_j \\ N}} |\zeta(v)|^2 \right\} \\ &\equiv \frac{(\delta e + c)^2}{s} \sum_{v \in V} \frac{|\zeta(v)|^2}{|C_G(v):C_N(v)|} \pmod{pqm}. \end{aligned}$$

Since $P \subseteq C_G(v)$ and P is a T.I. Sylow p -group of $C_G(v)$,

$$|C_G(v):C_N(v)| \equiv |C_G(v):N_G(P) \cap C_G(v)| \equiv 1 \pmod{q}.$$

Thus

$$T_2 \equiv \frac{(\delta e + c)^2}{s} \sum_{v \in V} |\zeta(v)|^2 \equiv \frac{|V| f(\delta e + c)^2}{s} \pmod{pqm}. \quad (\text{q.e.d.})$$

Then by (5) we get the congruence

$$\frac{|V| f c^2 q}{s} \equiv 0 \pmod{pqm}.$$

Hence we get $p|c^2$. This contradicts $p \nmid \Lambda(1)$. This completes the proof of the main theorem.

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