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$\mathbb{Z}_n$-EQUIVARIANT GOERITZ MATRICES FOR PERIODIC LINKS

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1. Introduction

A link $l^{(n)}$ in S^3 is said to have *period* $n (n \geq 2)$ if there is an n-periodic homeomorphism ϕ from S^3 onto itself such that $l^{(n)}$ is invariant under ϕ and the fixed point set $\tilde{f}$ of the $\mathbb{Z}_n$-action induced by ϕ is homeomorphic to a 1-sphere in S^3 disjoint from $l^{(n)}$. By the positive solution of the Smith conjecture [10], $\tilde{f}$ is unknotted and so the homeomorphism ϕ is conjugate to one point compactification of the $(2\pi/n)$-rotation about the z-axis in $\mathbb{R}^3$. Hence the quotient map $\pi\colon S^3 \to S^3/\mathbb{Z}_n$ is an n-fold branched cyclic cover branched along $\pi(\tilde{f}) = f$, and $l = \pi(l^{(n)})$ is also a link in the orbit space $S^3/\mathbb{Z}_n \cong S^3$, which is called the *factor link* of $l^{(n)}$.

There are several studies about the relationship between polynomial invariants of $l^{(n)}$ and those of l [5, 11, 14, 15, 16], and also some numerical invariants [3, 4, 9, 13] (see also references therein). In particular, Gordon-Litherland-Murasugi [4] gave a necessary congruence condition mod 4 on the signature of a knot in S^3 for it to have odd prime power period n, by using a $\mathbb{Z}_n$-invariant Hermitian form.

Now let $l = k_1 \cup \cdots \cup k_\mu$ be an oriented link in S^3 of μ components and let f be the oriented trivial knot such that $l \cap f = \emptyset$. For any integer $n \geq 2$, let $\pi\colon S^3 \to S^3$ be the n-fold branched cyclic cover branched along f. We denote the preimage $\pi^{-1}(l)$ and $\pi^{-1}(k_i)$ by $l^{(n)}$ and $k_i^{(n)}$, respectively. Then $k_i^{(n)} = k_{i1} \cup \cdots \cup k_{i\nu_i}$ is a link of ν_i components, where ν_i is the greatest common divisor of n and $\lambda_i = Lk(k_i, f)$, the *linking number* of k_i and f. We give an orientation to $k_i^{(n)}$ inherited from k_i. Then $l^{(n)} = k_1^{(n)} \cup \cdots \cup k_\mu^{(n)} = k_{11} \cup \cdots \cup k_{1\nu_1} \cup \cdots \cup k_{\mu 1} \cup \cdots \cup k_{\mu\nu_\mu}$ is an oriented n-periodic link in S^3 with l as its factor link. Throughout this paper we call such an oriented link $l^{(n)}$ the *n-periodic covering link over $l_1 = l \cup f$*. Notice that every link in S^3 with cyclic period arises in this manner.

Section 2 of the present paper reviews the definitions of Goeritz matrix for a link and its invariants. In Section 3, we characterize a $\mathbb{Z}_n$-equivariant Goeritz matrix for an n-periodic covering link $l^{(n)}$ in terms of its factor link $l \cup f$. In Section 4, we derive a necessary congruence condition mod 4 on the signature of a link for it to be an n-periodic covering link over a certain link. In Section 5, we give a congruence mod p between the reduced Alexander polynomial of an n-periodic covering link $l^{(n)}$ with

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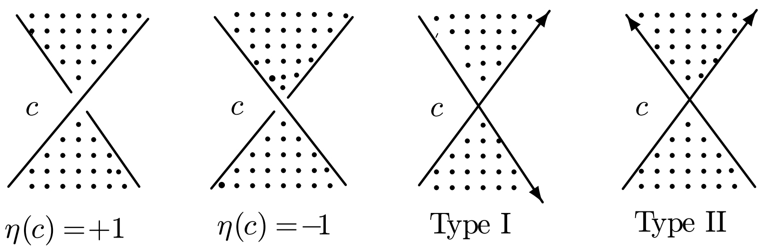

Fig. 1.

odd prime power period $n = p^r (r > 0)$ and that of its factor link l, which is a natural generalization of Murasugi's congruence on periodic knots [14, Theorem 2]. Using this generalized congruence, we also generalize Theorem 1.1 in Gordon-Litherland-Murasugi [4]. In Section 6, we show that the $\mathbb{Z}_n$-equivariant Goeritz matrix gives a more practical way to calculate the signature invariant $\tau_n(l,\pi)$ [4, 19] of a link l in the 2-fold branched cyclic cover $\mathcal{M}$ of S^3 branched along a certain link, where $\pi\colon N_n \to N$ is an n-fold branched cyclic cover of a 4-manifold N branched over a surface F such that $\partial(N, F) = (\mathcal{M}, l)$.

2. Goeritz matrix

Let l be an oriented link in S^3 and let L be its link diagram in the plane $\mathbb{R}^2 \subset \mathbb{R}^3 \cup \{\infty\}$. Color the regions of $\mathbb{R}^2 - L$ alternately black and white. Denote the white regions by $X_0, X_1, \dots, X_u$. (We always take the unbounded region to be white and denote it by X_0.) Let $C(L)$ denote the set of all crossings of L. Assign an *incidence number* $\eta(c) = \pm 1$ to each crossing $c \in C(L)$ and define a crossing $c \in C(L)$ to be of *type* I or *type* II as indicated in Fig. 1.

Let $g_{ij} = -\sum_{c\in C_L(X_i,X_j)} \eta(c)$ for $i \neq j$ and $g_{ii} = -\sum_{j\neq i} g_{ij}$, where $C_L(X_i, X_j) = \{c \in C(L) \mid c \text{ is incident to both } X_i \text{ and } X_j\}$. Let $G'(L) = (g_{ij})_{0\le i,j\le u}$. The principal minor $G(L) = (g_{ij})_{1\le i,j\le u}$ of $G'(L)$ is called the *Goeritz matrix* of l associated to L [1, 2]. Let L_1 and L_2 be two diagrams of l. Then Kyle [8] showed that $G(L_1)$ and $G(L_2)$ are *equivalent*, i.e., they can be transformed into each other by a finite number of transformations of the following types and their inverses:

(I) $G \to UGU^t$, where U is a unimodular matrix of integers,

(II) $G \to \begin{pmatrix} G & 0 \\ 0 & \pm 1 \end{pmatrix}$,

(III) $G \to \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$.

In that paper he also showed that a non-singular matrix B equivalent to the Goeritz matrix $G(L)$ associated to any diagram L of a link l is a relation matrix for the torsion group of $H_1(\mathcal{M}_2(l);\mathbb{Z})$ of the 2-fold branched cyclic cover $\mathcal{M}_2(l)$ of S^3 branched along the link l, and that $B^{-1}(\text{mod } 1)$ is the linking matrix of $\mathcal{M}_2(l)$.

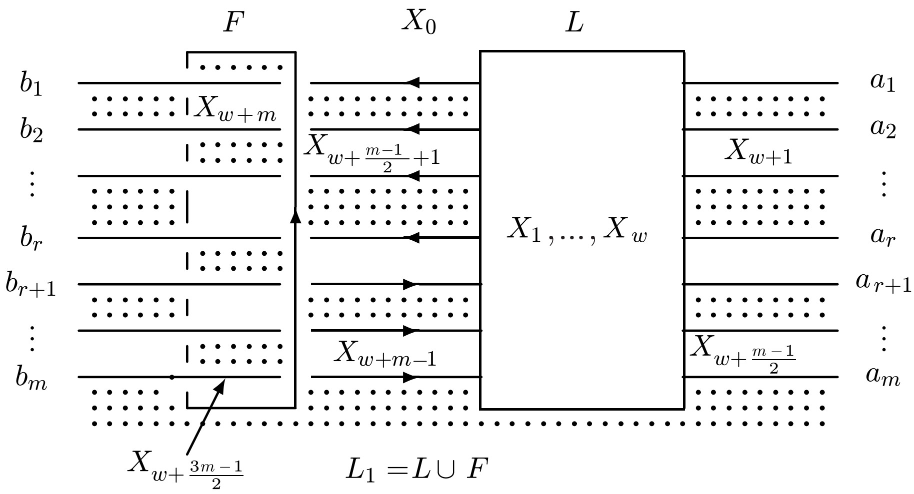

Fig. 2.

On the other hand, let $S(L)$ denote the compact surface with boundary L, which is built up out of disks and bands. Each disk lies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and is a closed black region less a small neighborhood of each crossing. Each crossing gives a small half-twisted band. Let $\beta_0(L)$ denote the number of connected components of the surface $S(L)$. Let $C_{\mathrm{II}}(L) = \{c_1, c_2, \ldots, c_s\}$ denote the set of all crossings of type II in L and let $A(L) = \mathrm{diag}(-\eta(c_1), -\eta(c_2), \ldots, -\eta(c_s))$, an $s \times s$ diagonal matrix. Then Traldi [18] defined the *modified Goeritz matrix* $H(L)$ of l associated to L by $H(L) = G(L) \oplus A(L) \oplus B(L)$, where $B(L)$ denotes the $(\beta_0(L) - 1) \times (\beta_0(L) - 1)$ zero matrix, and showed that the signature $\sigma(l)$ and the Murasugi nullity $\mathcal{N}(l)$ [12] of an oriented link l in S^3 are given by the formula: $\sigma(l) = \sigma(H(L))$ and $\mathcal{N}(l) = \mathcal{N}(H(L)) + 1$, where $\sigma(H(L))$ and $\mathcal{N}(H(L))$ denote the signature and the nullity of the symmetric matrix $H(L)$, respectively.

3. $\mathbb{Z}_n$-equivariant Goeritz matrix

Let $l_1 = l \cup f$ be an oriented link in S^3 with an unknotted component f such that $\lambda = Lk(l, f)$ is an odd integer. Applying an isotopy deformation if necessary, we can choose an oriented link diagram $L_1 = L \cup F$ in $\mathbb{R}^2 \subset \mathbb{R}^3 \cup \{\infty\}$ which has the form shown in Fig. 2, in which L and F represent the diagrams of l and f, respectively, in the link l_1 and a_i is identified with b_i for each $i = 1, 2, \ldots, m$. Note that $\lambda = Lk(l, f) = 2r - m$ and m is an odd integer.

Color the regions of $\mathbb{R}^2 - L_1$ alternately black and white. Without loss of generality we may assume that the surfaces $S(L_1)$ and $S(L) = S(L_1 - F)$ are connected and the orientations of l and f are as indicated in Fig. 2. (If not, by applying Reidemeister moves to L_1, deform L_1 to $L_1' = L' \cup F$ so that L_1' is equivalent to the diagram L_1, which has the required orientation and $S(L'_1)$ and $S(L') = S(L'_1 - F)$ are connected.) We denote the unbounded white region by X_0 and denote the other white regions as follows. Let $X_1, X_2, \ldots, X_w$ denote the white regions of L_1 each of which does not meet the component F. The white regions of L_1 each of which meets the component F are denoted by $X_{w+1}, X_{w+2}, \ldots, X_{w+(m-1)/2}, X_{w+(m-1)/2+1}, X_{w+(m-1)/2+2}, \ldots, X_{w+m-1}$, and $X_{w+m}, X_{w+m+1}, \ldots, X_{w+(3m-1)/2}$ as indicated in Fig. 2.

Let $G(L_1) = (g_{ij})_{1\leq i,j\leq w+(3m-1)/2}$ be the Goeritz matrix of l_1 associated to L_1 and we denote submatrices of $G(L_1)$ as follows: $M = (g_{ij})_{1\leq i,j\leq w}$, $N_1 = (g_{ij})_{w+1\leq i,j\leq w+(m-1)/2}$, $N_2 = (g_{ij})_{w+(m-1)/2+1\leq i,j\leq w+m-1}$, $P = (g_{ij})_{1\leq i\leq w, w+1\leq j\leq w+(m-1)/2}$, $Q = (g_{ij})_{1\leq i\leq w, w+(m-1)/2+1\leq j\leq w+m-1}$, and $R_1 = (g_{ij})_{w+1\leq i\leq w+(m-1)/2, w+(m-1)/2+1\leq j\leq w+m-1}$. In this situation, we obtain the following two lemmas:

Lemma 3.1 ([9]).

$$VH(L_1)V^t = \begin{pmatrix} M & P-Q \\ P^t-Q^t & N-R \end{pmatrix} \oplus A(L) \oplus Y \begin{pmatrix} I_a & O & O \\ O & -I_b & O \\ O & O & 2 \end{pmatrix} Y^{-1},$$

where V is a unimodular integral matrix, $N = N_1+N_2$, $R = R_1+R_1^t$, $a-b = -Lk(l, f)-1$, and Y is an invertible rational matrix.

Lemma 3.2. *The Goeritz matrix of l associated to the link diagram $L(= L_1 - F)$ is equivalent to the matrix*:

$$G(L) = \begin{pmatrix} M & P+Q \\ P^t+Q^t & N+R \end{pmatrix}.$$

Proof. Let $L = L_1 - F$, the diagram in $\mathbb{R}^2$ obtained from L_1 in Fig. 2 by deleting the unknotted component F. Then L is a diagram of the link l. The coloring of L_1 then induces a coloring of L such that the white regions X_{w+i} and $X_{w+(m-1)/2+i}$ in L_1 become the same region in L, denoted by X_{w+i}, for each $i = 1, 2, \ldots, (m-1)/2$. Then it is not difficult to see that the matrix $G(L) = (g_{ij})_{1\leq i,j\leq w+(m-1)/2}$ is of the required form. □

Theorem 3.3. *Let n be any integer greater than or equal to* 3. *Then the Goeritz matrix of the n-periodic covering link $l^{(n)}$ over $l_1 = l \cup f$ is equivalent to the symmetric* *block-wise circulant matrix of the form*:

$$G(L^{(n)}) = \begin{pmatrix} M & P & O & Q & O & O & \cdots & O & O & O & O \\ P^t & N & O & R_1 & O & O & \cdots & O & O & Q^t & R_1^t \\ O & O & M & P & O & Q & \cdots & O & O & O & O \\ Q^t & R_1^t & P^t & N & O & R_1 & \cdots & O & O & O & O \\ O & O & O & O & M & P & \cdots & O & O & O & O \\ O & O & Q^t & R_1^t & P^t & N & \cdots & O & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ O & Q & O & O & O & O & \cdots & O & O & M & P \\ O & R_1 & O & O & O & O & \cdots & Q^t & R_1^t & P^t & N \end{pmatrix}$$

Consequently, $G(L^{(n)})$ *is a relation matrix for* $H_1(\mathcal{M}_2(l^{(n)}); \mathbb{Z})$.

Proof. Let $L_1 = L \cup F$ be a diagram of l_1 given by Fig. 2. We may assume that F represents the z-axis $\cup\{\infty\}$ and L lies in an annulus $A \subset \mathbb{R}^2$. Let $\varphi_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the n-fold branched cyclic cover branched at the origin and let $L^{(n)} = \varphi_n^{-1}(L)$. Then $L^{(n)}$ is an n-periodic diagram in an annulus $A \subset \mathbb{R}^2$ of the n-periodic covering link $l^{(n)}$. Also the coloring of $\mathbb{R}^2 - L$ induces a coloring of $\mathbb{R}^2 - L^{(n)}$. Let $\tilde{\varphi}_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the $(2\pi/n)$-rotation of $\mathbb{R}^2$ about the origin. We denote the white regions of $\mathbb{R}^2 - L^{(n)}$ as follows.

For each $i = 1, 2, \ldots, (m-1)/2$, let X^1_{w+i} denote the white region in $\mathbb{R}^2 - L^{(n)}$ which meets the line $\theta = 0$ in the polar coordinate system of $\mathbb{R}^2$ such that $\varphi_n(X^1_{w+i}) = X_{w+i}$. Now let D be the closed domain in A bounded by two half lines $\theta = 0$ and $\theta = 2\pi/n$. Then for the white region $X_i (i = 1, 2, \ldots, w)$ of $\mathbb{R}^2 - L$, we denote the white region $\varphi_n^{-1}(X_i) \cap D$ in $\mathbb{R}^2 - L^{(n)}$ by X^1_i. Finally, for each $p = 2, \ldots, n$ and $i = 1, 2, \ldots, w + (m-1)/2$, $X^p_i = \tilde{\varphi}_n^{p-1}(X^1_i)$ and $X^0_0 = \varphi_n^{-1}(X_0)$.

For $p, q = 0, 1, \ldots, n$, let $G_{pq} = (g^{pq}_{ij})_{1 \le i,j \le w+(m-1)/2}$ be the matrix defined as follows. If $p \ne q$ or $i \ne j$, then $g^{pq}_{ij} = -\sum_{c \in C_{L^{(n)}}(X^p_i, X^q_j)} \eta(c)$. If $p = q$ and $i = j$, then $g^{pp}_{ii} = -\sum_{q \ne p \text{ or } j \ne i} g^{pq}_{ij}$. Then it is not difficult to see that the Goeritz matrix of $l^{(n)}$ associated to $L^{(n)}$ is equivalent to the symmetric block matrix given by $G(L^{(n)}) = (G_{pq})_{1 \le p,q \le n}$, where

$$G_{11} = \begin{pmatrix} M & P \\ P^t & N \end{pmatrix}, G_{12} = \begin{pmatrix} O & Q \\ O & R_1 \end{pmatrix}, G_{1n} = \begin{pmatrix} O & O \\ Q^t & R_1^t \end{pmatrix},$$

$$G_{1q} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} (3 \le q \le n-1), \; G_{pq} = G^t_{qp}, \; G_{pq} = G_{p+1q+1} (1 \le p, \; q \le n),$$

and $G_{1n-i} = G^t_{1i+2} (i = 1, \ldots, n-2)$. This completes the proof. □

4. Signature and nullity

Lemma 4.1. *Let $l_1 = l \cup f$ be an oriented link in S^3 with an unknotted component f such that $\lambda = Lk(l, f)$ is an odd integer. For any integer $n \geq 2$, let $l^{(n)}$ be the n-periodic covering link over l_1. Let $\xi = e^{2\pi i/n}$, where $i = \sqrt{-1}$, and define the Hermitian matrix*

$$\Lambda_{L_1}(n : \xi^j) = \begin{pmatrix} M & P + \xi^j Q \\ P^t + \xi^{-j} Q^t & N + \xi^j R_1 + \xi^{-j} R_1^t \end{pmatrix},$$

where M, N, P, Q, and R_1 are matrices as in Section 3.
(1) *If n is an odd integer, then*

$$\sigma(l^{(n)}) = n\sigma(l) + 2 \sum_{j=1}^{(n-1)/2} \sigma(\Lambda_{L_1}(n; \xi^j) \oplus -G(L)), \tag{4.1}$$

$$\mathcal{N}(l^{(n)}) = \mathcal{N}(l) + 2 \sum_{j=1}^{(n-1)/2} \mathcal{N}(\Lambda_{L_1}(n; \xi^j)). \tag{4.2}$$

(2) *If n is an even integer, then*

$$\sigma(l^{(n)}) = (n-1)\sigma(l) + \sigma(l \cup f) + Lk(l, f) + 2 \sum_{j=1}^{(n-2)/2} \sigma(\Lambda_{L_1}(n; \xi^j) \oplus -G(L)) \tag{4.3}$$

$$\mathcal{N}(l^{(n)}) = \mathcal{N}(l) + \mathcal{N}(l \cup f) + 2 \sum_{j=1}^{(n-2)/2} \mathcal{N}(\Lambda_{L_1}(n; \xi^j)) - 1. \tag{4.4}$$

Proof. Let I_n denote the $n \times n$ identity matrix and let T and U be the $n \times n$ matrices:

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, U = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{2(n-1)} & \cdots & \xi^{(n-1)^2} \end{pmatrix}.$$

Let $G(L^{(n)})$ be the Goeritz matrix of $l^{(n)}$ given by Theorem 3.3. Then $G(L^{(n)}) = G_{11} \otimes I_n + G_{12} \otimes T + \cdots + G_{1n} \otimes T^{n-1}$ and $U^{-1} T^j U = \text{diag}(1, \xi^j, \xi^{2j}, \ldots, \xi^{(n-1)j})$ ($j = 1,$

$2, \dots, n-1$). Hence, combining Lemma 3.2, we obtain that

$$
\begin{aligned}
&(I_n \otimes U)^{-1} G(L^{(n)})(I_n \otimes U) \\
&\quad = G_{11} \otimes U^{-1} I_n U + G_{12} \otimes U^{-1} T U + \cdots + G_{1n} \otimes U^{-1} T^{n-1} U \\
&\quad = \operatorname{diag}(G_{11} + G_{12} + \cdots + G_{1n}, G_{11} + \xi G_{12} + \cdots + \xi^{n-1} G_{1n}, \\
&\qquad\qquad \cdots, G_{11} + \xi^{n-1} G_{12} + \cdots + \xi^{(n-1)^2} G_{1n}) \\
&\quad = G(L) \oplus \left(\bigoplus_{j=1}^{n-1} \Lambda_{L_1}(n : \xi^j) \right).
\end{aligned} \tag{4.5}
$$

It is clear that $A(L^{(n)}) = \bigoplus_{j=1}^{n} A(L)$ and $A(L^{(n)})$ is nonsingular. Since the surface $S(L^{(n)})$ is connected, $B(L^{(n)})$ is the empty matrix. It thus follows from (4.5) that $\mathcal{N}(l^{(n)}) = \mathcal{N}(G(L^{(n)})) + 1 = \mathcal{N}(G(L) \oplus (\bigoplus_{j=1}^{n-1} \Lambda_{L_1}(n; \xi^j))) + 1 = \mathcal{N}(l) + \sum_{j=1}^{n-1} \mathcal{N}(\Lambda_{L_1}(n; \xi^j))$ and

$$
\begin{aligned}
\sigma(l^{(n)}) &= \sigma(H(L^{(n)})) = \sigma(G(L^{(n)}) \oplus A(L^{(n)})) \\
&= \sigma\left(\bigoplus_{j=1}^{n} (G(L) \oplus A(L)) \oplus \left(\bigoplus_{j=1}^{n-1} (\Lambda_{L_1}(n; \xi^j) \oplus -G(L)) \right) \right).
\end{aligned}
$$

Therefore we obtain

$$
\sigma(l^{(n)}) = n\sigma(l) + \sum_{j=1}^{n-1} \sigma(\Lambda_{L_1}(n; \xi^j) \oplus -G(L)). \tag{4.6}
$$

Since $\xi^{n-j} = \overline{\xi^j}$, $\Lambda_{L_1}(n; \xi^j) = \Lambda_{L_1}(n; \overline{\xi^j})^t$, i.e., $\Lambda_{L_1}(n; \xi^j)$ is a Hermitian matrix, and hence $\mathcal{N}(\Lambda_{L_1}(n; \xi^j)) = \mathcal{N}(\Lambda_{L_1}(n; \xi^{n-j}))$ and $\sigma(\Lambda_{L_1}(n; \xi^j)) = \sigma(\Lambda_{L_1}(n; \xi^{n-j}))$. Thus assertion (1) follows.

If $n = 2$, then it follows from [9] that $\sigma(l^{(2)}) = \sigma(l) + \sigma(l \cup f) + Lk(l, f)$ and $\mathcal{N}(l^{(2)}) = \mathcal{N}(l) + \mathcal{N}(l \cup f) - 1$. If n is an even integer with $n > 2$, then, by Lemma 3.1, $\mathcal{N}(\Lambda_{L_1}(n; \xi^{n/2})) = \mathcal{N}(H(L \cup F)) = \mathcal{N}(l \cup f) - 1$ and $\sigma(\Lambda_{L_1}(n; \xi^{n/2}) \oplus A(L)) = \sigma(l \cup f) + Lk(l, f)$. This implies assertion (2). □

Theorem 4.2. *Let $l_1 = l \cup f$ be an oriented link in S^3 of $\mu + 1$ components such that f is unknotted. For any integer $n \geq 2$, let $l^{(n)}$ be the n-periodic covering link over l_1. We assume that $\mathcal{N}(l^{(n)}) = \mathcal{N}(l)$.*

(1) *If either $Lk(l, f)$ and $\mathcal{N}(l)$ are odd or $Lk(l, f)$ and $\mathcal{N}(l)$ are even, then*

$$
\sigma(l^{(n)}) \equiv \begin{cases} n\sigma(l) & (\text{mod } 4) \text{ if } n \text{ is odd}, \\ (n-1)\sigma(l) + \sigma(l \cup f) + Lk(l, f) & (\text{mod } 4) \text{ if } n \text{ is even}. \end{cases}
$$

(2) *If either $Lk(l, f)$ is odd and $\mathcal{N}(l)$ is even or $Lk(l, f)$ is even and $\mathcal{N}(l)$ is odd,*

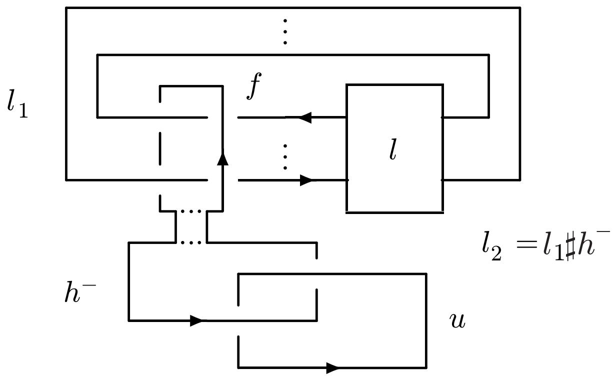

Fig. 3.

then

$$\sigma(l^{(n)}) \equiv \begin{cases} n\sigma(l) + n - 1 & \text{(mod 4) } \textit{if } n \textit{ is odd}, \\ (n-1)\sigma(l) + \sigma(l \cup f) + Lk(l, f) + n - 2 & \text{(mod 4) } \textit{if } n \textit{ is even}. \end{cases}$$

Proof.

CASE I. $Lk(l, f) \equiv 1 \pmod 2$:

(i) If $\mathcal{N}(l^{(n)}) = \mathcal{N}(l)$ is odd, then, from (4.2) and (4.4), $\mathcal{N}(\Lambda_{L_1}(n;\xi^j)) = 0$ for each j. Notice that $\mathcal{N}(G(L)) = \mathcal{N}(l)-1$ is even and $\Lambda_{L_1}(n;\xi^j)\oplus -G(L)$ is a $(2w+m-1)\times(2w+m-1)$ square matrix for each j. Since $2w+m-1$ is also even, $\sigma(\Lambda_{L_1}(n;\xi^j)\oplus -G(L))$ must be even. By (4.1) and (4.3), the desired result follows.

(ii) If $\mathcal{N}(l^{(n)}) = \mathcal{N}(l)$ is even, then, from (4.2) and (4.4), $\mathcal{N}(\Lambda_{L_1}(n;\xi^j)) = 0$ for each j. In this case, $\mathcal{N}(G(L)) = \mathcal{N}(l) - 1$ is odd and so $\sigma(\Lambda_{L_1}(n;\xi^j) \oplus -G(L))$ must be odd, say $2k_j + 1$ for some $k_j \in \mathbb{Z}$. From (4.1) and (4.3), we obtain that

$$\sigma(l^{(n)}) = \begin{cases} n\sigma(l) + 2\sum_{j=1}^{(n-1)/2}(2k_j + 1) & \text{if } n \text{ is odd}, \\ (n-1)\sigma(l) + \sigma(l \cup f) + Lk(l, f) + 2\sum_{j=1}^{(n-2)/2}(2k_j + 1) & \text{if } n \text{ is even}. \end{cases}$$

This implies the desired result.

CASE II. $Lk(l, f) \equiv 0 \pmod 2$: Let $l_2 = l_1\sharp h^-$ denote the connected sum of $l_1 = l \cup f$ and the left handed Hopf link h^- as shown in Fig. 3. It is easy to see that $l_2 = l_1\sharp h^- = (l \cup f)\sharp h^-$ is ambient isotopic to the link $(l \circ u) \cup f$. The link $(l \circ u)^{(n)}$ is also ambient isotopic to the link $l^{(n)} \circ u$. Note that $Lk(l \circ u, f) = Lk(l, f) - 1$ is odd, where $l \circ u$ denotes the split link consisting of l and the unknot u which is one of the components of h^-. Observe that $\mathcal{N}(l \circ u) = \mathcal{N}(l)+1$, $\mathcal{N}((l \circ u)^{(n)}) = \mathcal{N}(l^{(n)} \circ u) = \mathcal{N}(l^{(n)})+1$ [12, Lemma 6.4].

If $\mathcal{N}(l^{(n)}) = \mathcal{N}(l)$ is even, then, from (4.2) and (4.4), $\mathcal{N}(\Lambda_{L_1}(n;\xi^j)) = 0$ for each j and $\mathcal{N}((l \circ u)^{(n)}) = \mathcal{N}(l \circ u)$ is odd. By the argument in (i) above, we have that

$\sigma((l \circ u)^{(n)}) \equiv$

$$\begin{cases} n\sigma(l \circ u) & (\text{mod } 4) \text{ if } n \text{ is odd,} \\ (n-1)\sigma(l \circ u) + \sigma((l \circ u) \cup f) + Lk(l \circ u, f) & (\text{mod } 4) \text{ if } n \text{ is even.} \end{cases}$$

Finally, if $\mathcal{N}(l^{(n)}) = \mathcal{N}(l)$ is odd, then $\mathcal{N}((l \circ u)^{(n)}) = \mathcal{N}(l \circ u)$ is even. By the argument in (ii) above, we obtain that $\sigma((l \circ u)^{(n)}) \equiv$

$$\begin{cases} n\sigma(l \circ u) + n - 1 & (\text{mod } 4) \text{ if } n \text{ is odd,} \\ (n-1)\sigma(l \circ u) + \sigma((l \circ u) \cup f) + Lk(l \circ u, f) + n - 2 & (\text{mod } 4) \text{ if } n \text{ is even.} \end{cases}$$

Note that $\sigma(l \circ u) = \sigma(l)$, $\sigma((l \circ u) \cup f) = \sigma((l \cup f)\sharp h^-) = \sigma(l \cup f) + \sigma(h^-) = \sigma(l \cup f) + 1$ [12, Lemma 7.2, 7.4] and $Lk(l \circ u, f) = Lk(l, f) - 1$. This completes the proof. □

Theorem 4.3. *Let $l = k_1 \cup \cdots \cup k_\mu$ be an oriented link in S^3 of μ components and let $l_1 = l \cup f$ be an oriented link in S^3 of $\mu + 1$ components such that f is unknotted. For any integer $n \geq 2$, let $l^{(n)}$ be the n-periodic covering link over l_1. Then*

$$\mathcal{N}(l^{(n)}) = \begin{cases} \mathcal{N}(l) + 2\rho_1 & \text{if } n \text{ is odd,} \\ \mathcal{N}(l) + \mathcal{N}(l \cup f) + 2\rho_2 - 1 & \text{if } n \text{ is even,} \end{cases}$$

where ρ_1 and ρ_2 are some integers with the following properties:

$$\frac{(1 - \mathcal{N}(l))}{2} \leq \rho_1 \leq \frac{1}{2}\left(\sum_{i=1}^{\mu} \nu_i - \mathcal{N}(l)\right),$$

$$1 - \frac{\mathcal{N}(l) + \mathcal{N}(l \cup f)}{2} \leq \rho_2 \leq \frac{1}{2}\left(\sum_{i=1}^{\mu} \nu_i - \mathcal{N}(l) - \mathcal{N}(l \cup f) + 1\right),$$

where ν_i denotes the greatest common divisor of n and $Lk(k_i, f)$.

Proof.

CASE I. $Lk(l, f) \equiv 1$ (mod 2). By (4.2) and (4.4), if n is odd, then $\mathcal{N}(l^{(n)}) = \mathcal{N}(l) + 2\rho_1$ for some integer ρ_1, and if n is even, then $\mathcal{N}(l^{(n)}) = \mathcal{N}(l) + \mathcal{N}(l \cup f) + 2\rho_2 - 1$ for some integer ρ_2.

CASE II. $Lk(l, f) \equiv 0$ (mod 2). Let $l_2 = l_1 \sharp h^-$ denote the connected sum of $l_1 = l \cup f$ and the left handed Hopf link h^- as shown in Fig. 3. As Case I, if n is odd, then $\mathcal{N}((l \circ u)^{(n)}) = \mathcal{N}(l \circ u) + 2\rho_1$, and if n is even, then $\mathcal{N}((l \circ u)^{(n)}) = \mathcal{N}(l \circ u) + \mathcal{N}((l \circ u) \cup f) + 2\rho_2 - 1$. Note that $\mathcal{N}(l \circ u) = \mathcal{N}(l) + 1$, $\mathcal{N}((l \circ u)^{(n)}) = \mathcal{N}(l^{(n)} \circ u) = \mathcal{N}(l^{(n)}) + 1$, and $\mathcal{N}((l \circ u) \cup f) = \mathcal{N}((l \cup f)\sharp h^-) = \mathcal{N}(l \cup f) + \mathcal{N}(h^-) - 1 = \mathcal{N}(l \cup f)$ [12, Lemmas 6.3, 6.4]. Since the number of components of $l^{(n)}$ is equal to $\sum_{i=1}^{\mu} \nu_i$, $1 \leq \mathcal{N}(l^{(n)}) \leq \sum_{i=1}^{\mu} \nu_i$. So $1 \leq \mathcal{N}(l) + 2\rho_1 \leq \sum_{i=1}^{\mu} \nu_i (n\text{: odd})$ and $1 \leq \mathcal{N}(l) + \mathcal{N}(l \cup f) + 2\rho_2 - 1 \leq \sum_{i=1}^{\mu} \nu_i (n\text{: even})$. This implies the results. □

5. The reduced Alexander polynomial

Let $l = k_1 \cup \cdots \cup k_\mu$ be an oriented link in S^3 of μ components, let E be the exterior of l, and let $\pi_1(E)$ be the link group of l. Let t_i be the homology class in $H_1(E)$ represented by a meridian of $k_i (1 \leq i \leq \mu)$. Then $H_1(E)$ is a free abelian group of rank μ generated by $t_1, \ldots, t_\mu$. Let $\gamma\colon \pi_1(E) \to H_1(E)$ be the Hurewicz epimorphism and let E_γ be the universal abelian covering space of E corresponding to the kernel of γ. Then $H_1(E)$ acts on E_γ as the covering transformation group and so $H_1(E_\gamma)$ can be regarded as a module over the integral group ring $\mathbb{Z}H_1(E)$ of $H_1(E)$. By regarding $H_1(E)$ as the multiplicative free abelian group F_μ with basis $t_1, \ldots, t_\mu$, we can identify $\mathbb{Z}H_1(E)$ with the Laurent polynomial ring Λ in the variables $t_1, \ldots, t_\mu$, so that we can regard $H_1(E_\gamma)$ as a Λ-module. The 0-th characteristic polynomial of $H_1(E_\gamma)$, i.e., the greatest common divisor of the elements of the 0-th elementary ideal of $H_1(E_\gamma)$, is called the *Alexander polynomial of l on μ variables*, and denoted by $\Delta_l(t_1, \ldots, t_\mu)$.

Now let $\nu\colon H_1(E) \to F_r$ be an epimorphism from $H_1(E)$ to the free abelian group F_r of rank r with basis $t_1, \ldots, t_r$ and let E_ν be the covering space over E corresponding to the kernel of the composite homomorphism $\nu\gamma\colon \pi_1(E) \to F_r$. Then $H_1(E_\nu)$ can be regarded as a $\mathbb{Z}F_r$-module. The *reduced Alexander polynomial of l on r variables associated to ν* is defined to be the 0-th characteristic polynomial of the $\mathbb{Z}F_r$-module $H_1(E_\nu)$ and denoted by $\tilde{\Delta}_l(t_1, \ldots, t_r)$. If l is a knot, we have $\tilde{\Delta}_l(t) \doteq \Delta_l(t)$. For $\mu \geq 2$, the relationship between the Alexander polynomial $\Delta_l(t_1, \ldots, t_\mu)$ and the reduced one $\tilde{\Delta}_l(t_1, \ldots, t_r)$ is as follow[7, Proposition 7.3.10]:

$$\begin{cases} \tilde{\Delta}_l(t_1) \doteq (t_1 - 1)\Delta_l(\nu(t_1), \ldots, \nu(t_\mu)) & \text{if } r = 1, \\ \tilde{\Delta}_l(t_1, \ldots, t_r) \doteq \Delta_l(\nu(t_1), \ldots, \nu(t_\mu)) & \text{if } r \geq 2. \end{cases} \tag{5.1}$$

Now let $l_1 = k \cup f$ be a two component link in S^3, where f is unknotted and $Lk(k, f) = \lambda$. In [14, Theorem 2], Murasugi showed that the Alexander polynomial $\Delta_{k^{(n)}}(t)$ of the n-periodic covering knot $k^{(n)}$ over $l_1 = k \cup f$, where $n = p^r (r \geq 1)$ and p is an odd prime with $(\lambda, p) = 1$, satisfies the congruence:

$$\Delta_{k^{(n)}}(t) \equiv (1 + t + \cdots + t^{\lambda - 1})^{n-1} \Delta_k(t)^n \pmod{p}. \tag{5.2}$$

The following theorem is a natural generalization of Murasugi's congruence (5.2) on periodic knots.

Theorem 5.1. *Let l be an oriented link in S^3 of μ components, let $l_1 = l \cup f$, where f is unknotted, and let $\lambda = Lk(l, f)$. Let $l^{(n)}$ be the oriented n-periodic covering link in S^3 over l_1 of period $n = p^r (r \geq 1)$, where p is an odd prime. Then the reduced Alexander polynomials $\tilde{\Delta}_{l^{(n)}}(t)$ and $\tilde{\Delta}_l(t)$, where a meridian of each component of $l^{(n)}$ and l corresponds to t, satisfy the congruence*:

$$\tilde{\Delta}_{l^{(n)}}(t) \equiv (1 + t + \cdots + t^{\lambda - 1})^{n-1} \tilde{\Delta}_l(t)^n \pmod{p}. \tag{5.3}$$

Proof. Let $l = k_1 \cup \cdots \cup k_\mu$ be an oriented link in S^3 of μ components and let $l^{(n)} = k_1^{(n)} \cup \cdots \cup k_\mu^{(n)} = k_{11} \cup \cdots \cup k_{1\nu_1} \cup \cdots \cup k_{\mu 1} \cup \cdots \cup k_{\mu\nu_\mu}$ be the oriented n-periodic covering link in S^3 over l_1. If $\mu\nu_\mu = 1$, then the congruence (5.3) is just the Murasugi's congruence (5.2). Assume that $\mu\nu_\mu \geq 2$. Let $\tilde{\Delta}_{l^{(n)}}(t_1, \ldots, t_\mu)$ be the reduced Alexander polynomial of n-periodic covering link $l^{(n)}$ such that for $1 \leq i \leq \mu$, a meridian of each component of $k_i^{(n)} = \pi^{-1}(k_i)$ corresponds to t_i. By [16], the following formula holds:

$$\tilde{\Delta}_{l^{(n)}}(t_1, \ldots, t_\mu) \doteq \Delta_l(t_1, \ldots, t_\mu) \prod_{j=1}^{n-1} \Delta_{l \cup f}(t_1, \ldots, t_\mu, \xi^j), \tag{5.4}$$

where ξ is a primitive n-th root of 1. From (5.1),

$$\begin{cases} \tilde{\Delta}_{l^{(n)}}(t_1) \doteq (t_1 - 1)\Delta_{l^{(n)}}(t_1, \ldots, t_1) & \text{if } \mu = 1 \text{ and } \mu\nu_\mu \geq 2, \\ \tilde{\Delta}_{l^{(n)}}(t_1, \ldots, t_\mu) \doteq \Delta_{l^{(n)}}(t_1, \ldots, t_1, \ldots, t_\mu, \ldots, t_\mu) & \text{if } \mu \geq 2. \end{cases} \tag{5.5}$$

(1) Let $\mu = 1$ and $\mu\nu_\mu \geq 2$, i.e., $l_1 = k_1 \cup f$ and $l^{(n)} = k_{11} \cup \cdots \cup k_{1\nu_1}$. By Torres condition [17], $\Delta_{k_1 \cup f}(t_1, 1) = (t_1^\lambda - 1)(t_1 - 1)^{-1}\Delta_{k_1}(t_1)$. From (5.4) and (5.5), we obtain that

$$(t_1^\lambda - 1)\Delta_{k_1}(t_1)\Delta_{l^{(n)}}(t_1, \ldots, t_1) \doteq \Delta_{k_1}(t_1) \prod_{j=0}^{n-1} \Delta_{k_1 \cup f}(t_1, \xi^j).$$

From [14, Proposition 4.2] and the fact that $\Delta_{k_1}(t) \not\equiv 0 \pmod{p}$, we have that

$$\Delta_{l^{(n)}}(t_1, \ldots, t_1) \equiv (t_1^\lambda - 1)^{n-1}(t_1 - 1)^{-n}\Delta_{k_1}(t_1)^n \pmod{p}.$$

Therefore, by (5.1), we obtain the congruence:

$$\tilde{\Delta}_{l^{(n)}}(t) \equiv (1 + t + \cdots + t^{\lambda-1})^{n-1}\tilde{\Delta}_{k_1}(t)^n \pmod{p}.$$

(2) Let $\mu \geq 2$ and denote $\lambda_i = Lk(k_i, f)$ and so $\lambda = Lk(l, f) = \sum_{i=1}^{\mu} \lambda_i$. By Torres condition [17], $\Delta_{l \cup f}(t_1, \ldots, t_\mu, 1) \doteq (t_1^{\lambda_1} \cdots t_\mu^{\lambda_\mu} - 1)\Delta_l(t_1, \ldots, t_\mu)$. From (5.4) and (5.5), we obtain that

$$\begin{aligned}&(t^\lambda - 1)\Delta_l(t, \ldots, t)\Delta_{l^{(n)}}(t, \ldots, t, \ldots, t, \ldots, t) \\ &\doteq \Delta_l(t, \ldots, t) \prod_{j=0}^{n-1} \Delta_{l \cup f}(t, \ldots, t, \xi^j).\end{aligned}$$

From [14, Proposition 4.2], we obtain the congruence:

$$\begin{aligned}&\Delta_l(t, \ldots, t)\Delta_{l^{(n)}}(t, \ldots, t, \ldots, t, \ldots, t) \\ &\equiv \Delta_l(t, \ldots, t)(t^\lambda - 1)^{n-1}\Delta_l(t, \ldots, t)^n \pmod{p}.\end{aligned}$$

By (5.1),

$$\tilde{\Delta}_l(t)\tilde{\Delta}_{l^{(n)}}(t) \equiv \tilde{\Delta}_l(t)(1+t+\cdots+t^{\lambda-1})^{n-1}\tilde{\Delta}_l(t)^n \pmod p.$$

Therefore we have that either $\tilde{\Delta}_l(t) \equiv 0 \pmod p$ or

$$\tilde{\Delta}_{l^{(n)}}(t) \equiv (1+t+\cdots+t^{\lambda-1})^{n-1}\tilde{\Delta}_l(t)^n \pmod p.$$

If $\tilde{\Delta}_l(t) \equiv 0 \pmod p$, then by (5.4) it is obvious that $\tilde{\Delta}_{l^{(n)}}(t) \equiv 0 \pmod p$. This completes the proof. □

Theorem 5.2. *Let l be an oriented link in S^3 of μ components, let $l_1 = l \cup f$, where f is unknotted, and let $\lambda = Lk(l, f)$. Let $l^{(n)}$ be the oriented n-periodic covering link in S^3 over l_1 of period $n = p^r (r \geq 1)$, where p is an odd prime. Suppose that the reduced Alexander polynomial $\tilde{\Delta}_{l^{(n)}}(t)$ of $l^{(n)}$ satisfies that*
(i) $\tilde{\Delta}_{l^{(n)}}(t)$ *is not a product of non-trivial link polynomials,*
(ii) $\tilde{\Delta}_{l^{(n)}}(t) \not\equiv 0, \pm 1 \pmod p$.
Then
(1) $\tilde{\Delta}_{l^{(n)}}(t) \equiv (1+t+\cdots+t^{\lambda-1})^{n-1} \pmod p$.
(2) *If* $\tilde{\Delta}_{l^{(n)}}(-1) \neq 0$, *then*

$$\sigma(l^{(n)}) \equiv \begin{cases} 0 & \pmod 4 \text{ if } \lambda \text{ is odd,} \\ n-1 & \pmod 4 \text{ if } \lambda \text{ is even.} \end{cases}$$

Proof. (1) From (5.1), (5.4) and (5.5), we obtain that

$$\tilde{\Delta}_{l^{(n)}}(t) \doteq \tilde{\Delta}_l(t) \prod_{j=1}^{n-1} \Delta_{l\cup f}(t, \ldots, t, \xi^j). \tag{5.6}$$

By condition (i), either $\tilde{\Delta}_{l^{(n)}}(t) \doteq \tilde{\Delta}_l(t)$ or $\tilde{\Delta}_l(t) \doteq 1$.
If $\tilde{\Delta}_{l^{(n)}}(t) \doteq \tilde{\Delta}_l(t)$, then, by Theorem 5.1,

$$\tilde{\Delta}_l(t) \equiv (1+t+\cdots+t^{\lambda-1})^{n-1}\tilde{\Delta}_l(t)^n \pmod p.$$

From condition (ii), we obtain that $\tilde{\Delta}_l(t) \not\equiv 0 \pmod p$. So

$$1 \equiv (1+t+\cdots+t^{\lambda-1})^{n-1}\tilde{\Delta}_l(t)^{n-1} \pmod p.$$

Hence all the polynomials $1+t+\cdots+t^{\lambda-1}$ and $\tilde{\Delta}_l(t)$ are congruent to ± 1 modulo p. Hence $\tilde{\Delta}_{l^{(n)}}(t) \equiv \pm 1 \pmod p$. This contradicts to condition (ii). Therefore $\tilde{\Delta}_l(t) \doteq 1$. By Theorem 5.1, the result follows.
(2) Since $\tilde{\Delta}_{l^{(n)}}(-1) \neq 0$, it follows from (5.6) that $\Delta_{l\cup f}(-1, \ldots, -1, \xi^j) \neq 0$ for each $j = 0, 1, \ldots, n-1$. So $\mathcal{N}(l) = 1 = \mathcal{N}(l^{(n)})$. Since $\tilde{\Delta}_l(t) \doteq 1$, $\sigma(l) = 0$. By Theorem 4.2, the result follows. □

6. Applications

Let l be a null-homologous oriented link in a closed oriented 3-manifold $\mathcal{M}$ and let $\pi\colon \mathcal{M}_n \to \mathcal{M}$ be an n-fold branched cyclic cover of $\mathcal{M}$ branched along l. We shall always assume that each oriented meridian of l corresponds to a fixed generator of the group of covering transformations. Let F be a surface properly embedded in a 4-manifold N with $\partial(N, F) = (\mathcal{M}, l)$ and suppose π extends to a covering $N_n \to N$ branched over F. Then the integer

$$\tau_n(l, \pi) = \sigma(N_n) - n\sigma(N) + \frac{(n^2-1)}{3n}[F, \partial F] \cdot [F, \partial F]$$

is an invariant of l and π, where $[F, \partial F] \cdot [F, \partial F]$ denotes algebraic intersection number of homology class $[F, \partial F]$ in $H_2(N, \partial N)$.

If $\mathcal{M}$ is a homology 3-sphere, then Viro [19] shows that $\tau_n(l, \pi)$ can be calculated from a Seifert matrix for l. In general, let l be a null-homologous oriented link in a closed oriented 3-manifold $\mathcal{M}$ and let π_∞ be an infinite cyclic cover of $\mathcal{M} - l$ such that each oriented meridian of l corresponds to a fixed generator of the group of covering transformations. Then, in [4], the authors observed that this invariant $\tau_n(l, \pi)$ can be calculated from a surface $F \subset \mathcal{M}$, called *a spanning surface for* (l, π_∞), such that $\partial F = l$ and the epimorphism $H_1(\mathcal{M} - l) \to \mathbb{Z}$ which determines π_∞ is given by intersection number with F.

Now let $l_1 = l \cup f$ be an oriented link in S^3 such that f is unknotted and $\lambda = Lk(l, f)$ is an odd integer. Let $L_1 = L \cup F$ be a diagram of $l_1 = l \cup f$ which has the form as shown in Fig. 2 and let M, N, P, Q, and R_1 be the matrices defined in the Section 3. For any given integer $n \geq 2$ and $\xi = e^{2\pi i/n} (i = \sqrt{-1})$, define $\mathcal{S}_{L_1}(n; \xi^j)$ to be the Hermitian matrix given by

$$\mathcal{S}_{L_1}(n; \xi^j) = \begin{pmatrix} M & P + \xi^j Q \\ P^t + \xi^{-j} Q^t & N + \xi^j R_1 + \xi^{-j} R_1^t \end{pmatrix} \oplus \begin{pmatrix} -M & -P - Q \\ -P^t - Q^t & -N - R_1 - R_1^t \end{pmatrix}.$$

Theorem 6.1. *Let $l_1 = l \cup f$ be an oriented link in S^3 such that f is unknotted and $\lambda = Lk(l, f)$ is an odd integer. Let $\pi_2\colon \mathcal{M}_2(l) \to S^3$ be the 2-fold branched cyclic cover branched along l and let $f^{(2)} = \pi_2^{-1}(f) \subset \mathcal{M}_2(l)$. Then for any integer $n \geq 2$,*

$$\tau_n(f^{(2)}, \pi) = \sum_{j=1}^{n-1} \sigma(\mathcal{S}_{L_1}(n; \xi^j)).$$

Proof. By Theorem 3.1 in [4], $\tau_n(f^{(2)}, \pi) = \sigma(l^{(n)}) - n\sigma(l) + 2\sigma(f)$. Note that $\sigma(f) = 0$ since f is unknotted. By (4.6), we have that

$$\tau_n(f^{(2)}, \pi) = \sigma(l^{(n)}) - n\sigma(l) = \sum_{j=1}^{n-1} \sigma(\mathcal{S}_{L_1}(n; \xi^j)). \tag{6.1}$$

This completes the proof. □

EXAMPLE 6.2. Let $\beta = \sigma_1\sigma_1\sigma_2\sigma_2\sigma_1^{-1}\sigma_2 \in B_3$ be a braid of 3-strings and let $l_1 = l \cup f$, where $l = \beta^\wedge$ denotes the closed braid with braid axis f(cf. $l = \beta^\wedge$ is the prime knot 5_2). Then $\lambda = Lk(\beta^\wedge, f) = 3$ and $M = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix}$, $P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $R_1 = (0)$, and $N = N_1 + N_2 = (1) + (-1) = (0)$. For any integer $n \geq 2$,

$$\mathcal{S}_{L_1}(n; \xi^j) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & \xi^j \\ 1 & \xi^{-j} & 0 \end{pmatrix} \oplus \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

So $\tau_n(f^{(2)}, \pi) = \sum_{j=1}^{n-1} \sigma(\mathcal{S}_{L_1}(n; \xi^j)) = 0$ for any integer $n \geq 2$. Since $\sigma(\beta^\wedge) = 2$, it follows from (6.1) that $\sigma((\beta^n)^\wedge) = n\sigma(\beta^\wedge) = 2n$ for any integer $n \geq 2$. On the other hand, $\mathcal{N}((\beta^n)^\wedge) = 1$ for any integer $n \geq 2$.

EXAMPLE 6.3. Let $\beta = \sigma_1\sigma_2\sigma_2\sigma_2 \in B_3$ be a braid of 3-strings and let $l_1 = l \cup f$, where $l = \beta^\wedge$(cf. $l = \beta^\wedge$ is the right handed trefoil knot). Then $\lambda = Lk(\beta^\wedge, f) = 3$ and $M = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, $P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $R_1 = (0)$, and $N = N_1 + N_2 = (0) + (-1) = (-1)$. For any integer $n \geq 2$,

$$\mathcal{S}_{L_1}(n; \xi^j) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & \xi^j \\ 1 & \xi^{-j} & -1 \end{pmatrix} \oplus \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

So $\tau_n(f^{(2)}, \pi) = -a_n(j) - 2b_n(j)$, where $a_n(j)$ and $b_n(j)$ denote the numbers of the integers $j (1 \leq j \leq n-1)$ such that $-1 - 2\cos 2\pi j/n = 0$ and $-1 - 2\cos 2\pi j/n > 0$, respectively. Since $\sigma(\beta^\wedge) = -2$, it follows from (6.1) that $\sigma((\beta^n)^\wedge) = -2n - a_n(j) - 2b_n(j)$. On the other hand, we obtain from (4.2) and (4.4) of Lemma 4.1 that $\mathcal{N}((\beta^n)^\wedge) = 3$ or 1 according as n is a multiple of 3 or not.

EXAMPLE 6.4. Let $\beta = \sigma_1\sigma_1\sigma_2 \in B_3$ be a braid of 3-strings and let $l_1 = l \cup f$, where $l = \beta^\wedge$(cf. $l = \beta^\wedge$ is the right handed Hopf link). Then $\lambda = Lk(\beta^\wedge, f) = 3$ and $M = P = Q = (0)$, $R_1 = (1)$, and $N = N_1 + N_2 = (1) + (-1) = (0)$. For any integer $n \geq 2$, $\mathcal{S}_{L_1}(n; \xi^j) = (2\cos(2\pi j/n)) \oplus (-2)$. So $\tau_n(f^{(2)}, \pi) = \sum_{j=1}^{n-1} \sigma(\mathcal{S}_{L_1}(n; \xi^j)) = \sum_{j=1}^{n-1} \epsilon_j - n + 1$ and consequently, $\sigma((\beta^n)^\wedge) = \sum_{j=1}^{n-1} \epsilon_j - 2n + 1$ for any integer $n \geq 2$, where ϵ_j is the sign of the real number $\cos(2\pi j/n)$ ($\epsilon_j = 0$ if $\cos(2\pi j/n) = 0$). On the other hand, we obtain from (4.2) and (4.4) of Lemma 4.1 that $\mathcal{N}((\beta^n)^\wedge) = 3$ or 1 according as n is a multiple of 4 or not.

REMARKS 6.5. (1) Example 6.2, 6.3, and 6.4 show that Theorem 6.1 gives a method to calculate the signature and the nullity of a closed n-periodic braid $(\beta^n)^\wedge$ $(n \geq 2)$ from the braid $\beta \in B_{2m+1} (m \geq 0)$.

(2) Let $l = k_1 \cup \cdots \cup k_\mu$ be an oriented link of μ components and let $l_1 = l \cup f$ be an oriented link in S^3 such that f is unknotted and $\lambda = Lk(l, f)$ is an odd integer. For any integer $n \geq 2$, let $l^{(n)}$ be the n-periodic covering link over l_1. Suppose that $\Delta_{l\cup f}(-1, \ldots, -1, \xi^j) \neq 0$ for each $j = 0, 1, \ldots, n-1$. Let $\Lambda_{L_1}(n; \xi^j)$ be the matrix in Lemma 4.1. Then, from (4.5), we obtain the followings:
(i) Let $\mathcal{O}[H_1(\mathcal{M})]$ denote the order of $H_1(\mathcal{M})$ with integral coefficients. Then

$$\mathcal{O}[H_1(\mathcal{M}_2(l^{(n)}))] = \mathcal{O}[H_1(\mathcal{M}_2(l))] \prod_{j=1}^{n-1} |\det(\Lambda_{L_1}(n; \xi^j))|. \tag{6.2}$$

More precisely, if n is odd, then

$$\mathcal{O}[H_1(\mathcal{M}_2(l^{(n)}))] = \mathcal{O}[H_1(\mathcal{M}_2(l))] \prod_{j=1}^{(n-1)/2} |\det(\Lambda_{L_1}(n; \xi^j))|^2.$$

If n is even, then

$$\mathcal{O}[H_1(\mathcal{M}_2(l^{(n)}))] = \frac{1}{2}\mathcal{O}[H_1(\mathcal{M}_2(l))]\mathcal{O}[H_1(\mathcal{M}_2(l \cup f))] \prod_{j=1}^{(n-2)/2} |\det(\Lambda_{L_1}(n; \xi^j))|^2,$$

where $|\det(\Lambda_{L_1}(n; \xi^j))|$ is the absolute value of the determinant of the Hermitian matrix $\Lambda_{L_1}(n; \xi^j)$.
(ii) It follows from (6.2), [6, Theorem 1] and [16, Theorem 2] that

$$\prod_{j=1}^{n-1} |\det(\Lambda_{L_1}(n : \xi^j))| = \prod_{j=1}^{n-1} |\Delta_{l\cup f}(-1, \ldots, -1, \xi^j)|.$$

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