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\textbf{\textit{Z}}_n^*\text{EQUIVARIANT GOERITZ MATRICES FOR PERIODIC LINKS}

SANG YOUL LEE

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1. Introduction

A link $l^{(n)}$ in $S^3$ is said to have \textit{period} $n(n \geq 2)$ if there is an $n$-periodic homeomorphism $\phi$ from $S^3$ onto itself such that $l^{(n)}$ is invariant under $\phi$ and the fixed point set $\tilde{f}$ of the $\mathbb{Z}_n$-action induced by $\phi$ is homeomorphic to a 1-sphere in $S^3$ disjoint from $l^{(n)}$. By the positive solution of the Smith conjecture [10], $\tilde{f}$ is unknotted and so the homeomorphism $\phi$ is conjugate to one point compactification of the $(2\pi/n)$-rotation about the $z$-axis in $\mathbb{R}^3$. Hence the quotient map $\pi: S^3 \rightarrow S^3/\mathbb{Z}_n$ is an $n$-fold branched cyclic cover branched along $\pi(\tilde{f}) = f$, and $l = \pi(l^{(n)})$ is also a link in the orbit space $S^3/\mathbb{Z}_n \cong S^3$, which is called the \textit{factor link} of $l^{(n)}$.

There are several studies about the relationship between polynomial invariants of $l^{(n)}$ and those of $l$ [5, 11, 14, 15, 16], and also some numerical invariants [3, 4, 9, 13] (see also references therein). In particular, Gordon-Litherland-Murasugi [4] gave a necessary congruence condition mod 4 on the signature of a knot in $S^3$ for it to have odd prime power period $n$, by using a $\mathbb{Z}_n$-invariant Hermitian form.

Now let $l = k_1 \cup \cdots \cup k_\mu$ be an oriented link in $S^3$ of $\mu$ components and let $f$ be the oriented trivial knot such that $l \cap f = \emptyset$. For any integer $n \geq 2$, let $\pi: S^3 \rightarrow S^3$ be the $n$-fold branched cyclic cover branched along $f$. We denote the preimage $\pi^{-1}(l)$ and $\pi^{-1}(k_i)$ by $l^{(n)}$ and $k_i^{(n)}$, respectively. Then $k_i^{(n)} = k_{i1} \cup \cdots \cup k_{i\nu_i}$ is a link of $\nu_i$ components, where $\nu_i$ is the greatest common divisor of $n$ and $\lambda_i = Lk(k_i, f)$, the \textit{linking number} of $k_i$ and $f$. We give an orientation to $k_i^{(n)}$ inherited from $k_i$. Then $l^{(n)} = k_1^{(n)} \cup \cdots \cup k_\mu^{(n)} = k_{11} \cup \cdots \cup k_{1\nu_1} \cup \cdots \cup k_{\mu1} \cup \cdots \cup k_{\mu\nu_\mu}$ is an oriented $\mu$-periodic link in $S^3$ with $l$ as its factor link. Throughout this paper we call such an oriented link $l^{(n)}$ the \textit{n-periodic covering link over $l$} = $l \cup f$. Notice that every link in $S^3$ with cyclic period arises in this manner.

Section 2 of the present paper reviews the definitions of Goeritz matrix for a link and its invariants. In Section 3, we characterize a $\mathbb{Z}_n$-equivariant Goeritz matrix for an $\mu$-periodic covering link $l^{(n)}$ in terms of its factor link $l \cup f$. In Section 4, we derive a necessary congruence condition mod 4 on the signature of a link for it to be an $\mu$-periodic covering link over a certain link. In Section 5, we give a congruence mod $p$ between the reduced Alexander polynomial of an $\mu$-periodic covering link $l^{(n)}$ with

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odd prime power period \( n = p^r (r > 0) \) and that of its factor link \( l \), which is a natural generalization of Murasugi’s congruence on periodic knots [14, Theorem 2]. Using this generalized congruence, we also generalize Theorem 1.1 in Gordon-Litherland-Murasugi [4]. In Section 6, we show that the \( \mathbb{Z}_n \)-equivariant Goeritz matrix gives a more practical way to calculate the signature invariant \( \tau_\eta(l, \pi) \) [4, 19] of a link \( l \) in the 2-fold branched cyclic cover \( \mathcal{M} \) of \( S^3 \) branched along a certain link, where \( \pi: \mathcal{N}_n \to N \) is an \( n \)-fold branched cyclic cover of a 4-manifold \( N \) branched over a surface \( F \) such that \( \partial(N, F) = (\mathcal{M}, l) \).

2. Goeritz matrix

Let \( l \) be an oriented link in \( S^3 \) and let \( L \) be its link diagram in the plane \( \mathbb{R}^2 \subset \mathbb{R}^3 \cup \{\infty\} \). Color the regions of \( \mathbb{R}^2 - L \) alternately black and white. Denote the white regions by \( X_0, X_1, \ldots, X_{lt} \). (We always take the unbounded region to be white and denote it by \( X_0 \).) Let \( C(L) \) denote the set of all crossings of \( L \). Assign an incidence number \( \eta(c) = \pm 1 \) to each crossing \( c \in C(L) \) and define a crossing \( c \in C(L) \) to be of type I or type II as indicated in Fig. 1.

![Fig. 1.](image)

Let \( g_{ij} = -\sum_{c \in C_l(X_i, X_j)} \eta(c) \) for \( i \neq j \) and \( g_{ii} = -\sum_{j \neq i} g_{ij} \), where \( C_l(X_i, X_j) = \{ c \in C(L) \mid c \text{ is incident to both } X_i \text{ and } X_j \} \). Let \( G'(L) = (g_{ij})_{0 \leq i, j \leq lt} \). The principal minor \( G(L) = (g_{ij})_{1 \leq i, j \leq lt} \) of \( G'(L) \) is called the Goeritz matrix of \( L \) associated to \( L \) [1, 2]. Let \( L_1 \) and \( L_2 \) be two diagrams of \( L \). Then Kyle [8] showed that \( G(L_1) \) and \( G(L_2) \) are equivalent, i.e., they can be transformed into each other by a finite number of transformations of the following types and their inverses:

1. \( G \to UGU^t \), where \( U \) is a unimodular matrix of integers,
2. \( G \to \begin{pmatrix} G & 0 \\ 0 & \pm 1 \end{pmatrix} \),
3. \( G \to \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \).

In that paper he also showed that a non-singular matrix \( B \) equivalent to the Goeritz matrix \( G(L) \) associated to any diagram \( L \) of a link \( l \) is a relation matrix for the torsion group of \( H_1(\mathcal{M}_2(l); \mathbb{Z}) \) of the 2-fold branched cyclic cover \( \mathcal{M}_2(l) \) of \( S^3 \) branched along the link \( l \), and that \( B^{-1} \pmod{1} \) is the linking matrix of \( \mathcal{M}_2(l) \).
On the other hand, let $S(L)$ denote the compact surface with boundary $L$, which is built up out of disks and bands. Each disk lies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and is a closed black region less a small neighborhood of each crossing. Each crossing gives a small half-twisted band. Let $\beta_0(L)$ denote the number of connected components of the surface $S(L)$.

Let $C_{\Pi}(L) = \{c_1, c_2, \ldots, c_s\}$ denote the set of all crossings of type II in $L$ and let $A(L) = \text{diag}(-\eta(c_1), -\eta(c_2), \ldots, -\eta(c_s))$, an $s \times s$ diagonal matrix. Then Traldi [18] defined the modified Goeritz matrix $H(L)$ of $L$ associated to $L$ by $H(L) = G(L) \oplus A(L) \oplus B(L)$, where $B(L)$ denotes the $(\beta_0(L) - 1) \times (\beta_0(L) - 1)$ zero matrix, and showed that the signature $\sigma(l)$ and the Murasugi nullity $\mathcal{N}(l)$ [12] of an oriented link $l$ in $S^3$ are given by the formula: $\sigma(l) = \sigma(H(L))$ and $\mathcal{N}(l) = \mathcal{N}(H(L)) + 1$, where $\sigma(H(L))$ and $\mathcal{N}(H(L))$ denote the signature and the nullity of the symmetric matrix $H(L)$, respectively.

3. $\mathbb{Z}_n$-equivariant Goeritz matrix

Let $L_1 = L \cup f$ be an oriented link in $S^3$ with an unknotted component $f$ such that $\lambda = Lk(l, f)$ is an odd integer. Applying an isotopy deformation if necessary, we can choose an oriented link diagram $L_1 = L \cup F$ in $\mathbb{R}^2 \subset \mathbb{R}^3 \cup \{\infty\}$ which has the form shown in Fig. 2, in which $L$ and $F$ represent the diagrams of $l$ and $f$, respectively, in the link $l_1$ and $\alpha_l$ is identified with $b_i$ for each $i = 1, 2, \ldots, m$. Note that $\lambda = Lk(l, f) = 2r - m$ and $m$ is an odd integer.

Color the regions of $\mathbb{R}^2 - L_1$ alternately black and white. Without loss of generality we may assume that the surfaces $S(L_1)$ and $S(L) = S(L_1 - F)$ are connected and the orientations of $l$ and $f$ are as indicated in Fig. 2. (If not, by applying Reidemeister moves to $L_1$, deform $L_1$ to $L'_1 = L' \cup F$ so that $L'_1$ is equivalent to the
diagram $L_1$, which has the required orientation and $S(L_1')$ and $S(L') = S(L_1' - F)$ are connected.) We denote the unbounded white region by $X_0$ and denote the other white regions as follows. Let $X_1, X_2, \ldots, X_w$ denote the white regions of $L_1$ each of which does not meet the component $F$. The white regions of $L_1$ each of which meets the component $F$ are denoted by $X_{{w+1}}, X_{{w+2}}, \ldots, X_{{w+(m-1)/2}}, X_{{w+(m-1)/2+1}}, X_{{w+(m-1)/2+2}}, \ldots, X_{{w+m-1}}, X_{{w+m}}, X_{{w+m+1}}, \ldots, X_{{w+(3m-1)/2}}$ as indicated in Fig. 2.

Let $G(L_1) = (g_{ij})_{1 \leq i,j \leq w+m}$ be the Goeritz matrix of $l_1$ associated to $L_1$ and we denote submatrices of $G(L_1)$ as follows: $M = (g_{ij})_{1 \leq i,j \leq w}$, $N_1 = (g_{ij})_{1 \leq i,j \leq w+(m-1)/2}$, $N_2 = (g_{ij})_{w+(m-1)/2+1 \leq i,j \leq w+m-1}$, $P = (g_{ij})_{1 \leq i \leq w, w+1 \leq j \leq w+(m-1)/2}$, $Q = (g_{ij})_{1 \leq i \leq w, w+(m-1)/2+1 \leq j \leq w+m-1}$, and $R_1 = (g_{ij})_{w+1 \leq i \leq w+(m-1)/2, w+1 \leq j \leq w+m-1}$. In this situation, we obtain the following two lemmas:

**Lemma 3.1.** ([9]).

\[
VH(L_1)V' = \begin{pmatrix} M & P - Q \\ P' - Q' & N - R \end{pmatrix} \oplus A(L) \oplus Y \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix} \begin{pmatrix} O & 0 \\ O & 0 \end{pmatrix} Y^{-1},
\]

where $V$ is a unimodular integral matrix, $N = N_1 + N_2$, $R = R_1 + R'_1$, $a - b = -Lk(l, f) - 1$, and $Y$ is an invertible rational matrix.

**Lemma 3.2.** The Goeritz matrix of $l$ associated to the link diagram $L(= L_1 - F)$ is equivalent to the matrix:

\[
G(L) = \begin{pmatrix} M & P + Q \\ P' + Q' & N + R \end{pmatrix}.
\]

Proof. Let $L = L_1 - F$, the diagram in $\mathbb{R}^2$ obtained from $L_1$ in Fig. 2 by deleting the unknotted component $F$. Then $L$ is a diagram of the link $l$. The coloring of $L_1$ then induces a coloring of $L$ such that the white regions $X_{{w+i}}$ and $X_{{w+(m-1)/2+i}}$ in $L_1$ become the same region in $L$, denoted by $X_{{w+i}}$, for each $i = 1, 2, \ldots, (m-1)/2$. Then it is not difficult to see that the matrix $G(L) = (g_{ij})_{1 \leq i,j \leq w+(m-1)/2}$ is of the required form.

**Theorem 3.3.** Let $n$ be any integer greater than or equal to 3. Then the Goeritz matrix of the $n$-periodic covering link $l^n$ over $l_1 = I \cup f$ is equivalent to the symmetric
block-wise circulant matrix of the form:

\[
G(L^{(n)}) = \begin{pmatrix}
M & P & O & Q & O & O & \cdots & O & O & O & O \\
N & O & R_1 & O & O & \cdots & O & O & Q^t & R_1^t \\
& & & & & \ddots & & & & & \\
O & O & M & P & Q & \cdots & O & O & O & O \\
O & O & Q^t & R_1^t & P^t & N & \cdots & O & O & O \\
& & & & & \ddots & & & & & \\
O & Q & O & O & O & O & \cdots & O & O & M & P \\
O & R_1 & O & O & O & O & \cdots & Q^t & R_1^t & P^t & N \\
\end{pmatrix}
\]

Consequently, \(G(L^{(n)})\) is a relation matrix for \(H_1(\mathcal{M}_2(l^{(n)}); \mathbb{Z})\).

Proof. Let \(L_1 = L \cup F\) be a diagram of \(I_1\) given by Fig. 2. We may assume that \(F\) represents the \(\mathbb{Z}\)-axis \(\cup \{\infty\}\) and \(L\) lies in an annulus \(A \subset \mathbb{R}^2\). Let \(\varphi_n : \mathbb{R}^2 \to \mathbb{R}^2\) be the \(n\)-fold branched cyclic cover branched at the origin and let \(L^{(n)} = \varphi_n^{-1}(L)\). Then \(L^{(n)}\) is an \(n\)-periodic diagram in an annulus \(A \subset \mathbb{R}^2\) of the \(n\)-periodic covering link \(l^{(n)}\). Also the coloring of \(\mathbb{R}^2 - L\) induces a coloring of \(\mathbb{R}^2 - L^{(n)}\). Let \(\varphi_n : \mathbb{R}^2 \to \mathbb{R}^2\) be the \((2\pi/n)\)-rotation of \(\mathbb{R}^2\) about the origin. We denote the white regions of \(\mathbb{R}^2 - L^{(n)}\) as follows.

For each \(i = 1, 2, \ldots, (m - 1)/2\), let \(X_{\delta+i}\) denote the white region in \(\mathbb{R}^2 - L^{(n)}\) which meets the line \(\theta = 0\) in the polar coordinate system of \(\mathbb{R}^2\) such that \(\varphi_n(X_{\delta+i}) = X_{\delta+i}\). Now let \(D\) be the closed domain in \(A\) bounded by two half lines \(\theta = 0\) and \(\theta = 2\pi/n\). Then for the white region \(X_i(i = 1, 2, \ldots, w)\) of \(\mathbb{R}^2 - L\), we denote the white region \(\varphi_n^{-1}(X_i) \cap D \ni X_i\). Finally, for each \(p = 2, \ldots, n\) and \(i = 1, 2, \ldots, w + (m - 1)/2\), \(X_i^p = \varphi_n^{-1}(X_i)\) and \(X_0 = \varphi_n^{-1}(X_0)\).

For \(p, q = 0, 1, \ldots, n\), let \(G_{pq} = (g_{pq}^{ij})_{1 \leq i, j \leq w + (m - 1)/2}\) be the matrix defined as follows. If \(p \neq q\) or \(i \neq j\), then \(g_{pq}^{ij} = -\sum_{c \in C_{pq}(X_i^p, X_j^q)} \eta(c)\). If \(p = q\) and \(i = j\), then \(g_{pp}^{ij} = -\sum_{q \neq p, j \neq i} g_{ij}^{pq}\). Then it is not difficult to see that the Goeritz matrix of \(l^{(n)}\) associated to \(L^{(n)}\) is equivalent to the symmetric block matrix given by \(G(L^{(n)}) = (G_{pq})_{1 \leq p, q \leq n}\), where

\[
G_{11} = \begin{pmatrix} M & P \\ P^t & N \end{pmatrix}, \quad G_{12} = \begin{pmatrix} O & Q \\ O & R_1 \end{pmatrix}, \quad G_{1n} = \begin{pmatrix} O & Q^t \\ R_1^t & P^t \end{pmatrix},
\]

\[
G_{1q} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} (3 \leq q \leq n - 1), \quad G_{pq} = G_{q,p}, \quad G_{pq} = G_{p+q+1}(1 \leq p, q \leq n),
\]

and \(G_{1n-i} = G_{1i+2}(i = 1, \ldots, n - 2)\). This completes the proof. \(\square\)
4. Signature and nullity

Lemma 4.1. Let \( L_1 = \mathcal{L} \cup f \) be an oriented link in \( S^3 \) with an unknotted component \( f \) such that \( \lambda = Lk(I, f) \) is an odd integer. For any integer \( n \geq 2 \), let \( \mathcal{L}^{(n)} \) be the \( n \)-periodic covering link over \( L_1 \). Let \( \xi = e^{2\pi i/n} \), where \( i = \sqrt{-1} \), and define the Hermitian matrix

\[
\Lambda_L(n; \xi^j) = \begin{pmatrix} M & P + \xi^j Q \\ P^t + \xi^{-j} Q^t & N + \xi^j R_1 + \xi^{-j} R_1^t \end{pmatrix},
\]

where \( M, N, P, Q, \) and \( R_1 \) are matrices as in Section 3.

1. If \( n \) is an odd integer, then

\[
\sigma(L^{(n)}) = n\sigma(I) + 2 \sum_{j=1}^{(n-1)/2} \sigma(\Lambda_L(n; \xi^j)) \oplus -G(L),
\]

\[
\mathcal{N}(L^{(n)}) = \mathcal{N}(I) + 2 \sum_{j=1}^{(n-1)/2} \mathcal{N}(\Lambda_L(n; \xi^j)) - 1.
\]

2. If \( n \) is an even integer, then

\[
\sigma(L^{(n)}) = (n-1)\sigma(I) + \sigma(L \cup f) + Lk(I, f) + 2 \sum_{j=1}^{(n-2)/2} \sigma(\Lambda_L(n; \xi^j)) \oplus -G(L)
\]

\[
\mathcal{N}(L^{(n)}) = \mathcal{N}(I) + \mathcal{N}(L \cup f) + 2 \sum_{j=1}^{(n-2)/2} \mathcal{N}(\Lambda_L(n; \xi^j)) - 1.
\]

Proof. Let \( I_n \) denote the \( n \times n \) identity matrix and let \( T \) and \( U \) be the \( n \times n \) matrices:

\[
T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{2(n-1)} & \cdots & \xi^{(n-1)^2} \end{pmatrix}.
\]

Let \( G(L^{(n)}) \) be the Goeritz matrix of \( L^{(n)} \) given by Theorem 3.3. Then \( G(L^{(n)}) = G_{11} \otimes I_n + G_{12} \otimes T + \cdots + G_{1n} \otimes T^{n-1} \) and \( U^{-1} T^j U = \text{diag}(1, \xi^j, \xi^{2j}, \ldots, \xi^{(n-1)j}) \) \( (j = 1, \ldots, n) \).
2, . . . , n − 1). Hence, combining Lemma 3.2, we obtain that

\[(I_n \otimes U)^{-1} G(L^{(n)})(I_n \otimes U)\]
\[= G_{11} \otimes U^{-1} I_n U + G_{12} \otimes U^{-1} T U + \cdots + G_{1n} \otimes U^{-1} T^{n-1} U\]
\[= \text{diag}(G_{11} + G_{12} + \cdots + G_{1n}, G_{11} + \xi G_{12} + \cdots + \xi^{n-1} G_{1n}, \ldots, G_{11} + \xi^{n-1} G_{12} + \cdots + \xi^{(n-1)^2} G_{1n})\]
\[= G(L) \oplus \left( \bigoplus_{j=1}^{n-1} \Lambda_L(n; \xi^j) \right).\]

(4.5)

It is clear that \(A(L^{(n)}) = \bigoplus_{j=1}^{n} A(L)\) and \(A(L^{(n)})\) is nonsingular. Since the surface \(S(L^{(n)})\) is connected, \(B(L^{(n)})\) is the empty matrix. It thus follows from (4.5) that

\[N(L^{(n)}) = N(G(L^{(n)})) + 1 = N(G(L) \oplus \bigoplus_{j=1}^{n-1} \Lambda_L(n; \xi^j)) + 1 = N(I) + \sum_{j=1}^{n-1} N(\Lambda_L(n; \xi^j))\]

and

\[\sigma(L^{(n)}) = \sigma(H(L^{(n)})) = \sigma(G(L^{(n)} \oplus A(L^{(n)}))\]
\[= \sigma \left( \bigoplus_{j=1}^{n} (G(L) \oplus A(L)) \oplus \left( \bigoplus_{j=1}^{n-1} (\Lambda_L(n; \xi^j) \oplus -G(L)) \right) \right).\]

Therefore we obtain

\[\sigma(L^{(n)}) = n \sigma(I) + \sum_{j=1}^{n-1} \sigma(\Lambda_L(n; \xi^j) \oplus -G(L)).\]

(4.6)

Since \(\xi^{n-j} = \xi^j\), \(\Lambda_L(n; \xi^j) = \Lambda_L(n; \xi^j)^t\), i.e., \(\Lambda_L(n; \xi^j)\) is a Hermitian matrix, and hence \(N(\Lambda_L(n; \xi^j)) = N(\Lambda_L(n; \xi^{n-j}))\) and \(\sigma(\Lambda_L(n; \xi^j)) = \sigma(\Lambda_L(n; \xi^{n-j}))\). Thus assertion (1) follows.

If \(n = 2\), then it follows from [9] that \(\sigma(L^{(2)}) = \sigma(I) + \sigma(I \cup f) + Lk(I, f)\) and \(N(L^{(2)}) = N(I) + N(I \cup f) - 1\). If \(n\) is an even integer with \(n > 2\), then, by Lemma 3.1, \(N(\Lambda_L(n; \xi^{n/2})) = N(H(L \cup F)) = N(I \cup f) - 1\) and \(\sigma(\Lambda_L(n; \xi^{n/2}) \oplus A(L)) = \sigma(I \cup f) + Lk(I, f)\). This implies assertion (2).

Theorem 4.2. Let \(I_1 = I \cup f\) be an oriented link in \(S^3\) of \(\mu + 1\) components such that \(f\) is unknotted. For any integer \(n \geq 2\), let \(L^{(n)}\) be the \(n\)-periodic covering link over \(I_1\). We assume that \(N(I^{(n)}) = N(I)\).

(1) If either \(Lk(I, f)\) and \(N(I)\) are odd or \(Lk(I, f)\) and \(N(I)\) are even, then

\[\sigma(L^{(n)}) \equiv \begin{cases} n \sigma(I) & \text{if } n \text{ is odd,} \\ (n-1) \sigma(I) + \sigma(I \cup f) + Lk(I, f) & \text{if } n \text{ is even.} \end{cases} \]

(2) If either \(Lk(I, f)\) is odd and \(N(I)\) is even or \(Lk(I, f)\) is even and \(N(I)\) is odd,
then

\[
\sigma(f^n) \equiv \begin{cases} 
  n\sigma(l) + n - 1 & \text{mod } 4 \text{ if } n \text{ is odd,} \\
  (n - 1)\sigma(l) + \sigma(l \cup f) + \text{Lk}(l, f) + n - 2 & \text{mod } 4 \text{ if } n \text{ is even.}
\end{cases}
\]

Proof.

Case I. \(\text{Lk}(l, f) \equiv 1 \pmod{2}\):
(i) If \(N(f^n) = N(l)\) is odd, then, from (4.2) and (4.4), \(N(\Lambda_L; n; \xi^j)) = 0\) for each \(j\). Notice that \(N(G(L)) = N(l) - 1\) is even and \(\Lambda_L; (n; \xi^j) \oplus -G(L)\) is a \((2w+m-1) \times (2w+m-1)\) square matrix for each \(j\). Since \(2w+m-1\) is also even, \(\sigma(\Lambda_L; (n; \xi^j) \oplus -G(L))\) must be even. By (4.1) and (4.3), the desired result follows.
(ii) If \(N(f^n) = N(l)\) is even, then, from (4.2) and (4.4), \(N(\Lambda_L; n; \xi^j)) = 0\) for each \(j\). In this case, \(N(G(L)) = N(l) - 1\) is odd and so \(\sigma(\Lambda_L; (n; \xi^j) \oplus -G(L))\) must be odd, say \(2k_j + 1\) for some \(k_j \in \mathbb{Z}\). From (4.1) and (4.3), we obtain that

\[
\sigma(f^n) = \begin{cases} 
  n\sigma(l) + 2 \sum_{j=1}^{(n-1)/2}(2k_j + 1) & \text{if } n \text{ is odd,} \\
  (n - 1)\sigma(l) + \sigma(l \cup f) + \text{Lk}(l, f) + 2 \sum_{j=1}^{(n-2)/2}(2k_j + 1) & \text{if } n \text{ is even.}
\end{cases}
\]

This implies the desired result.

Case II. \(\text{Lk}(l, f) \equiv 0 \pmod{2}\): Let \(l_2 = l_1 \sharp h^-\) denote the connected sum of \(l_1 = l \cup f\) and the left handed Hopf link \(h^-\) as shown in Fig. 3. It is easy to see that \(l_2 = l_1 \sharp h^- = (l \cup f) \sharp h^-\) is ambient isotopic to the link \((l \circ u) \cup f\). The link \((l \circ u)^{n_1}\) is also ambient isotopic to the link \(l_2^{n_1} \circ u\). Note that \(\text{Lk}(l \circ u, f) = \text{Lk}(l, f) - 1\) is odd, where \(l \circ u\) denotes the split link consisting of \(l\) and the unknot \(u\) which is one of the components of \(h^-\). Observe that \(N(l \circ u) = N(l) + 1, N((l \circ u)^{n_1}) = N(l^{n_1} \circ u) = N(l^{n_1}) + 1\) [12, Lemma 6.4].

If \(N(f^n) = N(l)\) is even, then, from (4.2) and (4.4), \(N(\Lambda_L; n; \xi^j)) = 0\) for each \(j\) and \(N((l \circ u)^{n_1}) = N(l \circ u)\) is odd. By the argument in (i) above, we have that
σ((l \circ u)^{(n)}) \equiv \\
\begin{cases}
(n-1)σ(l \circ u) + σ((l \circ u) \cup f) + Lk(l \circ u, f) & \text{mod } 4 \text{ if } n \text{ is even},
\end{cases}

Finally, if \(N(f^{(n)}) = N(l)\) is odd, then \(N((l \circ u)^{(n)}) = N(l \circ u)\) is even. By the argument in (ii) above, we obtain that \(σ((l \circ u)^{(n)}) \equiv \\
\begin{cases}
(n-1)σ(l \circ u) + σ((l \circ u) \cup f) + Lk(l \circ u, f) + n-2 & \text{mod } 4 \text{ if } n \text{ is even},
\end{cases}

Note that \(σ(l \circ u) = σ(I, σ((l \circ u) \cup f) = σ((l \cup f) \sharp h^-) = σ(l \cup f) + σ(h^-) = σ(l \cup f) + 1\) [12, Lemma 7.2, 7.4] and \(Lk(l \circ u, f) = Lk(l, f) - 1\). This completes the proof.  

**Theorem 4.3.** Let \(l = k_1 \cup \cdots \cup k_\mu\) be an oriented link in \(S^3\) of \(\mu\) components and let \(l_1 = l \cup f\) be an oriented link in \(S^3\) of \(\mu + 1\) components such that \(f\) is unknotted. For any integer \(n \geq 2\), let \(f^{(n)}\) be the \(n\)-periodic covering link over \(l_1\). Then

\[N(f^{(n)}) = \begin{cases}
N(l) + 2\rho_1 & \text{if } n \text{ is odd},
N(l) + N(l \cup f) + 2\rho_2 - 1 & \text{if } n \text{ is even},
\end{cases}\]

where \(\rho_1\) and \(\rho_2\) are some integers with the following properties:

\[
\frac{1 - N(l)}{2} \leq \rho_1 \leq \frac{1}{2} \left( \sum_{i=1}^{\mu} \nu_i - N(l) \right),
\]

\[
1 - \frac{N(l) + N(l \cup f)}{2} \leq \rho_2 \leq \frac{1}{2} \left( \sum_{i=1}^{\mu} \nu_i - N(l) - N(l \cup f) + 1 \right),
\]

where \(\nu_i\) denotes the greatest common divisor of \(n\) and \(Lk(k_i, f)\).

Proof.

**Case I.** \(Lk(l, f) \equiv 1 \pmod{2}\). By (4.2) and (4.4), if \(n\) is odd, then \(N(f^{(n)}) = N(l) + 2\rho_1\) for some integer \(\rho_1\), and if \(n\) is even, then \(N(f^{(n)}) = N(l) + N(l \cup f) + 2\rho_2 - 1\) for some integer \(\rho_2\).

**Case II.** \(Lk(l, f) \equiv 0 \pmod{2}\). Let \(l_2 = l \sharp h^-\) denote the connected sum of \(l_1 = l \cup f\) and the left handed Hopf link \(h^-\) as shown in Fig. 3. As Case I, if \(n\) is odd, then \(N((l \circ u)^{(n)}) = N(l \circ u) + 2\rho_1\), and if \(n\) is even, then \(N((l \circ u)^{(n)}) = N(l \circ u) + N((l \circ u) \cup f) + 2\rho_2 - 1\). Note that \(N(l \circ u) = N(l) + 1, N((l \circ u)^{(n)}) = N(f^{(n)} \circ u) = N(f^{(n)}) + 1,\) and \(N((l \circ u) \cup f) = N((l \cup f) \sharp h^-) = N(l \cup f) + N(h^-) - 1 = N(l \cup f)\) [12, Lemmas 6.3, 6.4]. Since the number of components of \(f^{(n)}\) is equal to \(\sum_{i=1}^{\mu} \nu_i\), \(1 \leq N(f^{(n)}) \leq \sum_{i=1}^{\mu} \nu_i\). So \(1 \leq N(l) + 2\rho_1 \leq \sum_{i=1}^{\mu} \nu_i (n: \text{ odd})\) and \(1 \leq N(l) + N(l \cup f) + 2\rho_2 - 1 \leq \sum_{i=1}^{\mu} \nu_i (n: \text{ even})\). This implies the results. 

□
5. The reduced Alexander polynomial

Let $I = k_1 \cup \cdots \cup k_\mu$ be an oriented link in $S^3$ of $\mu$ components, let $E$ be the exterior of $I$, and let $\pi_I(E)$ be the link group of $I$. Let $t_i$ be the homology class in $H_i(E)$ represented by a meridian of $k_i (1 \leq i \leq \mu)$. Then $H_i(E)$ is a free abelian group of rank $\mu$ generated by $t_1, \ldots, t_\mu$. Let $\gamma : \pi_I(E) \to H_1(E)$ be the Hurewicz epimorphism and let $E_\gamma$ be the universal abelian covering space of $E$ corresponding to the kernel of $\gamma$. Then $H_1(E)$ acts on $E_\gamma$ as the covering transformation group and so $H_1(E_\gamma)$ can be regarded as a module over the integral group ring $\mathbb{Z}H_1(E)$ of $H_1(E)$. By regarding $H_1(E)$ as the multiplicative free abelian group $F_\mu$ with basis $t_1, \ldots, t_\mu$, we can identify $\mathbb{Z}H_1(E)$ with the Laurent polynomial ring $\Lambda$ in the variables $t_1, \ldots, t_\mu$, so that we can regard $H_1(E_\gamma)$ as a $\Lambda$-module. The 0-th characteristic polynomial of $H_1(E_\gamma)$, i.e., the greatest common divisor of the elements of the 0-th elementary ideal of $H_1(E_\gamma)$, is called the Alexander polynomial of $I$ on $\mu$ variables, and denoted by $\Delta_\gamma(t_1, \ldots, t_\mu)$.

Now let $\nu : H_1(E) \to F_r$ be an epimorphism from $H_1(E)$ to the free abelian group $F_r$ of rank $r$ with basis $t_1, \ldots, t_r$ and let $E_\nu$ be the covering space over $E$ corresponding to the kernel of the composite homomorphism $\nu \circ \gamma : \pi_I(E) \to F_r$. Then $H_1(E_\nu)$ can be regarded as a $\mathbb{Z}F_r$-module. The reduced Alexander polynomial of $I$ on $r$ variables associated to $\nu$ is defined to be the 0-th characteristic polynomial of the $\mathbb{Z}F_r$-module $H_1(E_\nu)$ and denoted by $\tilde{\Delta}_\nu(t_1, \ldots, t_r)$. If $I$ is a knot, we have $\tilde{\Delta}_\nu(t) \equiv \Delta_\nu(t)$. For $\mu \geq 2$, the relationship between the Alexander polynomial $\Delta_\gamma(t_1, \ldots, t_\mu)$ and the reduced one $\tilde{\Delta}_\nu(t_1, \ldots, t_r)$ is as follow [7, Proposition 7.3.10]:

\[
\begin{align*}
\tilde{\Delta}_\nu(t_1) & \equiv (t_1 - 1)\Delta_\nu(\nu(t_1), \ldots, \nu(t_\mu)) & \text{if } r = 1, \\
\tilde{\Delta}_\nu(t_1, \ldots, t_r) & \equiv \Delta_\nu(\nu(t_1), \ldots, \nu(t_\mu)) & \text{if } r \geq 2.
\end{align*}
\]

Now let $I_1 = k \cup f$ be a two component link in $S^3$, where $f$ is unknotted and $Lk(k, f) = \lambda$. In [14, Theorem 2], Murasugi showed that the Alexander polynomial $\Delta_{k_\nu}(t)$ of the $n$-periodic covering knot $k_\nu$ over $I_1 = k \cup f$, where $n = p^r (r \geq 1)$ and $p$ is an odd prime with $(\lambda, p) = 1$, satisfies the congruence:

\[
\Delta_{k_\nu}(t) \equiv (1 + t + \cdots + t^{\lambda-1})^{p-1}\Delta_\nu(t)^p \pmod{p},
\]

The following theorem is a natural generalization of Murasugi’s congruence (5.2) on periodic knots.

**Theorem 5.1.** Let $I$ be an oriented link in $S^3$ of $\mu$ components, let $I_1 = I \cup f$, where $f$ is unknotted, and let $\lambda = Lk(I, f)$. Let $I_\nu$ be the oriented $n$-periodic covering link in $S^3$ over $I_1$ of period $n = p^r (r \geq 1)$, where $p$ is an odd prime. Then the reduced Alexander polynomials $\tilde{\Delta}_{k_\nu}(t)$ and $\tilde{\Delta}_\nu(t)$, where a meridian of each component of $I_\nu$ and $I$ corresponds to $t$, satisfy the congruence:

\[
\tilde{\Delta}_{k_\nu}(t) \equiv (1 + t + \cdots + t^{\lambda-1})^{p-1}\tilde{\Delta}_\nu(t)^p \pmod{p}.
\]
Proof. Let \( I = k_1 \cup \cdots \cup k_\mu \) be an oriented link in \( S^3 \) of \( \mu \) components and let \( \tilde{I}^{(n)} = k_1^{(n)} \cup \cdots \cup k_\mu^{(n)} = k_1^{(n)} \cup \cdots \cup k_\mu^{(n)} \cup \cdots \cup k_\mu^{(n)} \) be the oriented \( n \)-periodic covering link in \( S^3 \) over \( I \). If \( \mu \nu \mu = 1 \), then the congruence (5.3) is just the Murasugi’s congruence (5.2). Assume that \( \mu \nu \mu \geq 2 \). Let \( \tilde{\Delta}_{\tilde{I}^{(n)}}(t_1, \ldots, t_\mu) \) be the reduced Alexander polynomial of \( n \)-periodic covering link \( \tilde{I}^{(n)} \) such that for \( 1 \leq i \leq \mu \), a meridian of each component of \( k_i^{(n)} = \pi^{-1}(k_i) \) corresponds to \( t_i \). By [16], the following formula holds:

\[
\tilde{\Delta}_{\tilde{I}^{(n)}}(t_1, \ldots, t_\mu) \equiv \Delta_I(t_1, \ldots, t_\mu) \prod_{j=1}^{n-1} \Delta_{I \cup f}(t_1, \ldots, t_\mu, \xi^j),
\]

where \( \xi \) is a primitive \( n \)-th root of 1. From (5.1),

\[
\begin{cases}
\tilde{\Delta}_{\tilde{I}^{(n)}}(t_1) \equiv (t_1 - 1)\Delta_{I^{(n)}}(t_1) & \text{if } \mu = 1 \text{ and } \mu \nu \mu \geq 2, \\
\tilde{\Delta}_{\tilde{I}^{(n)}}(t_1, \ldots, t_\mu) \equiv \Delta_{I^{(n)}}(t_1, \ldots, t_\mu, \xi^j) & \text{if } \mu \geq 2.
\end{cases}
\]

(1) Let \( \mu = 1 \) and \( \mu \nu \mu \geq 2 \), i.e., \( I = k_1 \cup f \) and \( \tilde{I}^{(n)} = k_1^{(n)} \cup \cdots \cup k_1^{(n)} \). By Torres condition [17], \( \Delta_{k_1 \cup f}(t_1, 1) = (t_1^\lambda - 1)(t_1 - 1)^{-1}\Delta_{k_1}(t_1) \). From (5.4) and (5.5), we obtain that

\[
(t_1^\lambda - 1)\Delta_{k_1}(t_1) \Delta_{\tilde{I}^{(n)}}(t_1, \ldots, t_\mu) \equiv \Delta_{k_1}(t_1) \prod_{j=1}^{n-1} \Delta_{k_1 \cup f}(t_1, \xi^j).
\]

From [14, Proposition 4.2] and the fact that \( \Delta_{k_1}(t) \not\equiv 0 \) (mod \( p \)), we have that

\[
\Delta_{\tilde{I}^{(n)}}(t_1, \ldots, t_\mu) \equiv (t_1^\lambda - 1)^{n-1}(t_1 - 1)^{-1}\Delta_{k_1}(t_1)^n \pmod{p}.
\]

Therefore, by (5.1), we obtain the congruence:

\[
\tilde{\Delta}_{\tilde{I}^{(n)}}(t) \equiv (1 + t + \cdots + t^{n-1})^{n-1}\Delta_{k_1}(t)^n \pmod{p},
\]

(2) Let \( \mu \geq 2 \) and denote \( \lambda_i = Lk(k_i, f) \) and so \( \lambda = Lk(I, f) = \sum_{i=1}^{\mu} \lambda_i \). By Torres condition [17], \( \Delta_{I \cup f}(t_1, \ldots, t_\mu, 1) \equiv (t_1^\lambda \cdots t_\mu^\lambda - 1)\Delta_f(t_1, \ldots, t_\mu) \). From (5.4) and (5.5), we obtain that

\[
(t_1^\lambda - 1)\Delta_f(t_1, \ldots, t_\mu) \Delta_{\tilde{I}^{(n)}}(t_1, \ldots, t_\mu, 1, \xi^j) \equiv \Delta_f(t_1, \ldots, t_\mu) \prod_{j=0}^{n-1} \Delta_{I \cup f}(t_1, \ldots, t_\mu, \xi^j).
\]

From [14, Proposition 4.2], we obtain the congruence:

\[
\Delta_f(t_1, \ldots, t_\mu) \Delta_{\tilde{I}^{(n)}}(t_1, \ldots, t_\mu, 1, \xi^j) \equiv \Delta_f(t_1, \ldots, t_\mu)(t_1^\lambda - 1)^{n-1}\Delta_f(t_1, \ldots, t_\mu)^n \pmod{p},
\]
By (5.1),
\[ \tilde{\Delta}_{\nu(n)}(t) \equiv \tilde{\Delta}_f(t)(1 + t + \cdots + t^{\lambda^{-1})^{n-1}} \tilde{\Delta}_f(t)^n (\mod p). \]

Therefore we have that either \( \tilde{\Delta}_f(t) \equiv 0 (\mod p) \) or
\[ \tilde{\Delta}_{\nu(n)}(t) \equiv (1 + t + \cdots + t^{\lambda^{-1})^{n-1}} \tilde{\Delta}_f(t)^n (\mod p). \]

If \( \tilde{\Delta}_f(t) \equiv 0 (\mod p) \), then by (5.4) it is obvious that \( \tilde{\Delta}_{\nu(n)}(t) \equiv 0 (\mod p) \). This completes the proof. \( \square \)

**Theorem 5.2.** Let \( l \) be an oriented link in \( S^3 \) of \( \mu \) components, let \( I_1 = l \cup f \), where \( f \) is unknotted, and let \( \lambda = \text{Lk}(I, f) \). Let \( l^{(n)} \) be the oriented \( n \)-periodic covering link in \( S^3 \) over \( I_1 \) of period \( n = p^r (r \geq 1) \), where \( p \) is an odd prime. Suppose that the reduced Alexander polynomial \( \tilde{\Delta}_{\nu(n)}(t) \) of \( l^{(n)} \) satisfies the

(i) \( \tilde{\Delta}_{\nu(n)}(t) \) is not a product of non-trivial link polynomials,

(ii) \( \tilde{\Delta}_{\nu(n)}(t) \not\equiv 0, \pm 1 (\mod p) \).

Then

(1) \( \tilde{\Delta}_{\nu(n)}(t) \equiv (1 + t + \cdots + t^{\lambda^{-1})^{n-1}} (\mod p). \)

(2) If \( \tilde{\Delta}_{\nu(n)}(-1) \not\equiv 0, \) then

\[ \sigma(l^{(n)}) \equiv \begin{cases} 
0 & (\text{mod } 4) \text{ if } \lambda \text{ is odd,} \\
n - 1 & (\text{mod } 4) \text{ if } \lambda \text{ is even.} 
\end{cases} \]

**Proof.** (1) From (5.1), (5.4) and (5.5), we obtain that

\[ (5.6) \quad \tilde{\Delta}_{\nu(n)}(t) \equiv \tilde{\Delta}_f(t) \prod_{j=1}^{n-1} \Delta_{\nu(j)}(t, \ldots, t, \xi^j), \]

By condition (i), either \( \tilde{\Delta}_{\nu(n)}(t) \equiv \tilde{\Delta}_f(t) \) or \( \tilde{\Delta}_f(t) \equiv 1. \)

If \( \tilde{\Delta}_{\nu(n)}(t) \equiv \tilde{\Delta}_f(t) \), then, by Theorem 5.1,

\[ \tilde{\Delta}_f(t) \equiv (1 + t + \cdots + t^{\lambda^{-1})^{n-1}} \tilde{\Delta}_f(t)^n (\mod p). \]

From condition (ii), we obtain that \( \tilde{\Delta}_f(t) \not\equiv 0 (\mod p) \). So

\[ 1 \equiv (1 + t + \cdots + t^{\lambda^{-1})^{n-1}} \tilde{\Delta}_f(t)^n (\mod p), \]

Hence all the polynomials \( 1 + t + \cdots + t^{\lambda^{-1}} \) and \( \tilde{\Delta}_f(t) \) are congruent to \( \pm 1 \) modulo \( p. \)

Hence \( \tilde{\Delta}_{\nu(n)}(t) \equiv \pm 1 (\mod p) \). This contradicts condition (ii). Therefore \( \tilde{\Delta}_f(t) \equiv 1. \)

By Theorem 5.1, the result follows.

(2) Since \( \tilde{\Delta}_{\nu(n)}(-1) \not\equiv 0, \) it follows from (5.6) that \( \Delta_{\nu(j)}(-1, \ldots, -1, \xi) \not\equiv 0 \) for each \( j = 0, 1, \ldots, n - 1. \) So \( \mathcal{N}(l) = 1 = \mathcal{N}(l^{(n)}). \) Since \( \tilde{\Delta}_f(t) \equiv 1, \sigma(l) = 0. \) By Theorem 4.2, the result follows. \( \square \)
6. Applications

Let \( l \) be a null-homologous oriented link in a closed oriented 3-manifold \( \mathcal{M} \) and let \( \pi: \mathcal{M}_n \to \mathcal{M} \) be an \( n \)-fold branched cyclic cover of \( \mathcal{M} \) branched along \( l \). We shall always assume that each oriented meridian of \( l \) corresponds to a fixed generator of the group of covering transformations. Let \( F \) be a surface properly embedded in a 4-manifold \( N \) with \( \partial(N, F) = (\mathcal{M}, l) \) and suppose \( \pi \) extends to a covering \( N_n \to N \) branched over \( F \). Then the integer

\[
\tau_n(l, \pi) = \sigma(N_n) - n\sigma(N) + \frac{(n^2 - 1)}{3n}[F, \partial F] \cdot [F, \partial F]
\]

is an invariant of \( l \) and \( \pi \), where \( [F, \partial F] \cdot [F, \partial F] \) denotes algebraic intersection number of homology class \( [F, \partial F] \) in \( H_2(N, \partial N) \).

If \( \mathcal{M} \) is a homology 3-sphere, then Viro [19] shows that \( \tau_n(l, \pi) \) can be calculated from a Seifert matrix for \( l \). In general, let \( l \) be a null-homologous oriented link in a closed oriented 3-manifold \( \mathcal{M} \) and let \( \pi_\infty \) be an infinite cyclic cover of \( \mathcal{M} - l \) such that each oriented meridian of \( l \) corresponds to a fixed generator of the group of covering transformations. Then, in [4], the authors observed that this invariant \( \tau_n(l, \pi) \) can be calculated from a surface \( F \subset \mathcal{M} \), called a spanning surface for \((l, \pi_\infty)\), such that \( \partial F = l \) and the epimorphism \( H_1(\mathcal{M} - l) \to \mathbb{Z} \) which determines \( \pi_\infty \) is given by intersection number with \( F \).

Now let \( l_1 = l \cup f \) be an oriented link in \( S^3 \) such that \( f \) is unknotted and \( \lambda = \text{LK}(l, f) \) is an odd integer. Let \( L_1 = L \cup F \) be a diagram of \( l_1 = l \cup f \) which has the form as shown in Fig. 2 and let \( M, N, P, Q \) and \( R_1 \) be the matrices defined in the Section 3. For any given integer \( n \geq 2 \) and \( \xi = e^{2\pi i/n} (i = \sqrt{-1}) \), define \( S_{L_1}(n; \xi^j) \) to be the Hermitian matrix given by

\[
S_{L_1}(n; \xi^j) = \begin{pmatrix} M & P + \xi^j Q \\ P^\dagger + \xi^{-j} Q^\dagger N + \xi j R_1 + \xi^{-j} R_1^\dagger \end{pmatrix} \oplus \begin{pmatrix} -M & -P - Q \\ -P^\dagger - Q^\dagger N - R_1 - R_1^\dagger \end{pmatrix}.
\]

**Theorem 6.1.** Let \( l_1 = l \cup f \) be an oriented link in \( S^3 \) such that \( f \) is unknotted and \( \lambda = \text{LK}(l, f) \) is an odd integer. Let \( \pi_2: \mathcal{M}_2(l) \to S^3 \) be the 2-fold branched cyclic cover branched along \( l \) and let \( f^{(2)} = \pi_2^{-1}(f) \subset \mathcal{M}_2(l) \). Then for any integer \( n \geq 2 \),

\[
\tau_n(f^{(2)}, \pi) = \sum_{j=1}^{n-1} \sigma(S_{L_1}(n; \xi^j)).
\]

**Proof.** By Theorem 3.1 in [4], \( \tau_n(f^{(2)}, \pi) = \sigma(f^{(n)}) - n\sigma(l) + 2\sigma(f) \). Note that \( \sigma(f) = 0 \) since \( f \) is unknotted. By (4.6), we have that

\[
\tau_n(f^{(2)}, \pi) = \sigma(f^{(n)}) - n\sigma(l) = \sum_{j=1}^{n-1} \sigma(S_{L_1}(n; \xi^j)).
\]
This completes the proof. □

Example 6.2. Let \( \beta = \sigma_1 \sigma_2 \sigma_2 \sigma_1^{-1} \sigma_2 \in B_3 \) be a braid of 3-strings and let \( I = I \cup f \), where \( I = \beta^p \) denotes the closed braid with braid axis \( f \) (cf. \( I = \beta^p \) is the prime knot 5_2). Then \( \lambda = Lk(\beta^p, f) = 3 \) and \( M = \begin{pmatrix} -2 & 1 \\ 1 & -3 \\ 1 & \xi^{-j} \end{pmatrix}, P = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, Q = \begin{pmatrix} 1 \end{pmatrix}, R_1 = (0), \) and \( N = N_1 + N_2 = (1) + (-1) = (0) \). For any integer \( n \geq 2 \),

\[
S_L(n; \xi^j) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & \xi^j \\ 1 & \xi^{-j} & 0 \end{pmatrix} \oplus \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 0 \end{pmatrix}.
\]

So \( \tau_n(f^{(2)}, \pi) = \sum_{j=1}^{n-1} \sigma(S_L(n; \xi^j)) = 0 \) for any integer \( n \geq 2 \). Since \( \sigma(\beta^p) = 2 \), it follows from (6.1) that \( \sigma((\beta^p)^\wedge) = n \sigma(\beta^p) = 2n \) for any integer \( n \geq 2 \). On the other hand, \( N((\beta^p)^\wedge) = 1 \) for any integer \( n \geq 2 \).

Example 6.3. Let \( \beta = \sigma_1 \sigma_2 \sigma_2 \sigma_2 \in B_3 \) be a braid of 3-strings and let \( I = I \cup f \), where \( I = \beta^p \) (cf. \( I = \beta^p \) is the right handed trefoil knot). Then \( \lambda = Lk(\beta^p, f) = 3 \) and \( M = \begin{pmatrix} -2 & 1 \\ 1 & -2 \\ 1 & \xi^{-j} \end{pmatrix}, P = \begin{pmatrix} 1 \end{pmatrix}, Q = \begin{pmatrix} 0 \end{pmatrix}, R_1 = (0), \) and \( N = N_1 + N_2 = (0) + (-1) = (-1) \). For any integer \( n \geq 2 \),

\[
S_L(n; \xi^j) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & \xi^j \\ 1 & \xi^{-j} & -1 \end{pmatrix} \oplus \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}.
\]

So \( \tau_n(f^{(2)}, \pi) = -a_n(j) - 2b_n(j) \), where \( a_n(j) \) and \( b_n(j) \) denote the numbers of the integers \( j(1 \leq j \leq n - 1) \) such that \(-1 - 2 \cos 2\pi j/n = 0 \) and \(-1 - 2 \cos 2\pi j/n > 0 \), respectively. Since \( \sigma(\beta^p) = -2 \), it follows from (6.1) that \( \sigma((\beta^p)^\wedge) = -2n - a_n(j) - 2b_n(j) \). On the other hand, we obtain from (4.2) and (4.4) of Lemma 4.1 that \( N((\beta^p)^\wedge) = 3 \) or 1 according as \( n \) is a multiple of 3 or not.

Example 6.4. Let \( \beta = \sigma_1 \sigma_1 \sigma_2 \in B_3 \) be a braid of 3-strings and let \( I = I \cup f \), where \( I = \beta^p \) (cf. \( I = \beta^p \) is the right handed Hopf link). Then \( \lambda = Lk(\beta^p, f) = 3 \) and \( M = P = Q = (0), R_1 = (1), \) and \( N = N_1 + N_2 = (1) + (-1) = (0) \). For any integer \( n \geq 2 \), \( S_L(n; \xi^j) = (2 \cos (2\pi j/n)) \oplus (-2) \). So \( \tau_n(f^{(2)}, \pi) = \sum_{j=1}^{n-1} \sigma(S_L(n; \xi^j)) = \sum_{j=1}^{n-1} \epsilon_j - n + 1 \) and consequently, \( \sigma((\beta^p)^\wedge) = \sum_{j=1}^{n-1} \epsilon_j - 2n + 1 \) for any integer \( n \geq 2 \), where \( \epsilon_j \) is the sign of the real number \( \cos (2\pi j/n) \). On the other hand, we obtain from (4.2) and (4.4) of Lemma 4.1 that \( N((\beta^p)^\wedge) = 3 \) or 1 according as \( n \) is a multiple of 4 or not.

Remarks 6.5. (1) Example 6.2, 6.3, and 6.4 show that Theorem 6.1 gives a method to calculate the signature and the nullity of a closed \( n \)-periodic braid \( \beta^p \) \((n \geq 2) \) from the braid \( \beta \in B_{2n+1}(m \geq 0) \).
(2) Let \( l = k_1 \cup \cdots \cup k_\mu \) be an oriented link of \( \mu \) components and let \( l_1 = l \cup f \) be an oriented link in \( S^3 \) such that \( f \) is unknotted and \( \lambda = Lk(l, f) \) is an odd integer. For any integer \( n \geq 2 \), let \( \tilde{l}^{(n)} \) be the \( n \)-periodic covering link over \( l_1 \). Suppose that \( \Delta_{l \cup f}(-1, \ldots, -1, \xi^j) \neq 0 \) for each \( j = 0, 1, \ldots, n - 1 \). Let \( \Lambda_{L,i}(n; \xi^j) \) be the matrix in Lemma 4.1. Then, from (4.5), we obtain the followings:

(i) Let \( \mathcal{O}[H_1(M)] \) denote the order of \( H_1(M) \) with integral coefficients. Then

\[
\mathcal{O}[H_1(M_2(\tilde{l}^{(n)}))] = \mathcal{O}[H_1(M_2(l))] \prod_{j=1}^{(n-1)/2} |\det(\Lambda_{L,i}(n; \xi^j))|,
\]

More precisely, if \( n \) is odd, then

\[
\mathcal{O}[H_1(M_2(\tilde{l}^{(n)}))] = \mathcal{O}[H_1(M_2(l))] \prod_{j=1}^{(n-1)/2} |\det(\Lambda_{L,i}(n; \xi^j))|^2.
\]

If \( n \) is even, then

\[
\mathcal{O}[H_1(M_2(\tilde{l}^{(n)}))] = \frac{1}{2} \mathcal{O}[H_1(M_2(l))] \mathcal{O}[H_1(M_2(l \cup f))] \prod_{j=1}^{(n-2)/2} |\det(\Lambda_{L,i}(n; \xi^j))|^2,
\]

where \( |\det(\Lambda_{L,i}(n; \xi^j))| \) is the absolute value of the determinant of the Hermitian matrix \( \Lambda_{L,i}(n; \xi^j) \).

(ii) It follows from (6.2), [6, Theorem 1] and [16, Theorem 2] that

\[
\prod_{j=1}^{(n-1)} |\det(\Lambda_{L,i}(n; \xi^j))| = \prod_{j=1}^{n-1} |\Delta_{l \cup f}(-1, \ldots, -1, \xi^j)|.
\]

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References


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