# COBORDISM CLASSIFICATION OF KNOTTED HOMOLOGY 3-SPHERES IN $S^5$

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In [8], odd dimensional knot cobordism groups $C_{2n-1}$ $(n \geq 2)$ were studied and given a purely algebraic description. $C_{2n-1}$ is the set of all cobordism classes of knotted $(2n-1)$-spheres in the $(2n+1)$-sphere, and it is an abelian group under connected sum. In this paper we extend this to knotted homology 3-spheres in the 5-sphere and consider their homology cobordism classes. The set of all such homology cobordism classes, denoted by $C_3^H$, is also an abelian group under connected sum. We shall describe the group $C_3^H$ using the usual homology cobordism group of homology 3-spheres and Levine's cobordism group of certain matrices (Theorem 1.1). Our argument heavily depends on that of J. Levine.

In §3 we introduce the bounding genus and the Murasugi number of a knotted homology 3-sphere in $S^5$. Using our techniques we can estimate these numerical invariants from above. In §4 we consider the case of algebraic 3-knots which appear around isolated singularities of complex hypersurfaces in $C^3$ (see [11]). For example, we shall show that two algebraic 3-knots which are homology 3-spheres and defined by weighted homogeneous polynomials are homology cobordant if and only if they are isotopic.

Throughout the paper, we shall work in the $C^\infty$ category. Homology groups will always be with integral coefficients.

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## 1. Statement of the main result

A 3-*knot* will denote the oriented isotopy class of an embedded homology 3-sphere in the 5-sphere $S^5$. Two 3-knots $K_0$, $K_1$ are *homology cobordant* if there exists a compact oriented 4-submanifold $W$ of $[0, 1] \times S^5$ with $\partial W = (1 \times K_1) \cup (0 \times (-K_0))$ such that the inclusion induces an isomorphism $H_*(j \times K_j) \to H_*(W)$ for $j = 0, 1$. We call $W$ a *homology cobordism* between $K_0$ and $K_1$. If $W$ is diffeomorphic to $[0, 1] \times K_0$, we say that $K_0$ is *concordant* to $K_1$. A 3-knot

is *null-cobordant* if it is homology cobordant to the standard embedding $S^3 \subset S^5$ —equivalently, if it bounds an embedded homology 4-ball in $D^6$. The homology cobordism relation is an equivalence relation. The set of all homology cobordism classes of 3-knots will be denoted by $C_3^H$. This is an abelian group under connected sum. See [5, Ch. III] for details.

Let $\mathcal{H}^3$ denote the usual homology cobordism group of homology 3-spheres (for example see [9]). Furthermore let $G_+$ denote the set of all cobordism classes of integral square matrices $L$ with $\det(L + {}^t L) = \pm 1$ (see [8]). $G_+$ is an abelian group under block sum. For a 3-knot $K$, $L_K$ will denote the Seifert matrix of $K$ (for the definition see [8], [13]). For a homology 3-sphere $\Sigma$, $\rho(\Sigma)$ ($\in \{0,1\}$) will denote the Rohlin invariant of $\Sigma$. For an integral symmetric matrix $Q$, $\text{sign}(Q)$ will denote the signature of $Q$. The symbol [ ] will always mean an appropriate equivalence class.

Then our main result is the following.

**Theorem 1.1.** $C_3^H$ *is isomorphic to the index 2 subgroup* $\Lambda$ *of* $\mathcal{H}^3 \oplus G_+$, *where* $\Lambda = \{[\Sigma] \oplus [L] \in \mathcal{H}^3 \oplus G; \text{sign}(L + {}^t L) \equiv 8\rho(\Sigma) \pmod{16}\}$. *In fact, the map which sends each element $K$ of $C_3^H$ to $[K] \oplus [L_K] \in \mathcal{H}^3 \oplus G_+$ is an isomorphism onto* $\Lambda$.

Let $C_3$ denote the usual 3-dimensional knot cobordism group. Then we have the following short exact sequence.

**Corollary 1.2.** *The sequence*

$$0 \to C_3 \xrightarrow{i} C_3^H \xrightarrow{\pi} \mathcal{H}^3 \to 0$$

*is exact, where $i$ is the canonical "inclusion map" and $\pi$ is the map which forgets embeddings.*

Proof of Corollary 1.2. Let $\pi' : \Lambda \to \mathcal{H}^3$ be the projection to the first factor. $\pi'$ is surjective and $\ker \pi' \cong G_+^0 (= \{[L] \in G_+; \text{sign}(L + {}^t L) \equiv 0 \pmod{16}\})$. By Levine [8], $C_3$ is isomorphic to $G_+^0$. This completes the proof.

REMARK. We do not know whether the above sequence splits or not.

## 2. Proof of Theorem 1.1

A 3-knot $K$ is *simple* if $\pi_1(S^5 - K) \cong \mathbf{Z}$. It is proved that $K$ is simple if and only if it bounds a 1-connected 4-submanifold of $S^5$ (see [7]).

Theorem 1.1 will follow from the following five lemmas.

**Lemma 2.1.** *If $K$ is a null-cobordant 3-knot, then its Seifert matrix $L_K$ is null-cobordant.*

**Lemma 2.2.** *Let $K$ be a 3-knot. Then* $\text{sign}(L_K + {}^t L_K) \equiv 8\rho(K) \pmod{16}$

*and* $\det(L_K + {}^t L_K) = \pm 1$.

**Lemma 2.3.** *Let $\Sigma$ be a homology 3-sphere and let $L$ be an integral square matrix with $\text{sign}(L + {}^t L) \equiv 8\rho(\Sigma) \pmod{16}$ and $\det(L + {}^t L) = \pm 1$. Then $L$ can be changed in its cobordism class so that it can be realized as a Seifert matrix of some simple 3-knot $K$, where $K$ is diffeomorphic to $\Sigma$.*

**Lemma 2.4.** *Every 3-knot is concordant to a simple 3-knot.*

**Lemma 2.5.** *Let $K$ be a simple 3-knot such that $[L_K] = 0$ in $G_+$ and $[K] = 0$ in $\mathcal{H}^3$. Then $K$ is null-cobordant, i.e., $[K] = 0$ in $C_3^H$.*

Lemmas 2.1, 2.2 and 2.4 can be proved by the same argument as in §8, §10 and §11 of [8] respectively. Lemma 2.3 follows from [13, Theorem 2.2]. For a compact oriented 4-manifold $M$, let $\sigma(M)$ denote its signature. To prove Lemma 2.5, we need the following lemma.

**Lemma 2.6.** *Let $M$ and $F$ be compact oriented spin 4-manifolds with diffeomorphic boundaries such that $H_1(M) = 0$ and $\pi_1(F) = 1$. Suppose that $\partial M \cong \partial F$ is a homology 3-sphere and $\sigma(M) = \sigma(F)$. Then a surgery on some finite collection of disjoint circles in $M$ produces $M'$ diffeomorphic to $F \sharp k(S^2 \times S^2)$ for some non-negative integer $k$.*

Proof. Surgery on some finite collection of disjoint circles in $M$ produces $M_1$ with $\pi_1(M_1)$ trivial; if the circles are framed correctly then $w_2(M_1) = 0$. Note that $\sigma(M_1) = \sigma(M)$. Then by [12, §4], $M_1 \sharp k'(S^2 \times S^2)$ is diffeomorphic to $F \sharp k$ $(S^2 \times S^2)$ for some $k'$ and $k$. Obviously $M' = M_1 \sharp k'(S^2 \times S^2)$ is obtained by surgery on the $k'$-component trivial link in $M_1$. This completes the proof.

Proof of Lemma 2.5. Since $[K] = 0$ in $\mathcal{H}^3$, $K$ bounds a compact homology 4-ball $M$. Let $F$ be a 1-connected compact 4-submanifold of $S^5$ bounded by $K$. Since we can perform connected sums with copies of $S^2 \times S^2$ in $S^5$, we may assume, by Lemma 2.6, that $F$ is diffeomorphic to the manifold obtained by surgery on some finite collection of disjoint circles in $M$. Using the surgery framings of the circles, we attach 2-handles to $M \times 1$ of $M \times [0, 1]$. Let $X$ denote the resulting 5-manifold, then $\partial X = (M \times 0) \cup (\partial M \times [0, 1]) \cup F$.

Let $\{S_i'\}$ be the family of disjoint embedded 2-spheres in $F$ which consists of the belt spheres of the 2-handles of $X$. Let $a_i' \in H_2(F)$ be the homology class of $S_i'$. Since $\partial F$ is a homology 3-sphere, the intersection pairing of $F$ is unimodular. Thus by the same argument as in [8, §13], we can find a self-diffeomorphism $h$ of $F$ such that $\Gamma_K(h_* \alpha_i', h_* \alpha_j') = 0$ for all $i$ and $j$, where $\Gamma_K$ is the Seifert form of the 3-knot $K$. Let $S_i = h(S_i')$. Then we can perform surgery on $F$ in $D^6$ using $\{S_i\}$ as in [8, §12]. The resulting 4-submanifold of $D^6$ is easily seen to be diffeomorphic to the homology 4-ball $M$. Since $\partial M = K$,

$[K]{=}0$ in $C_3^H$.   This completes the proof of Lemma 2.5, and Theorem 1.1.

As the above proof shows, a null-cobordant simple 3-knot $K$ bounds in $D^6$ any homology 4-ball $M$ with $\partial M$ diffeomorphic to $K$.   Combining this with Lemma 2.4, we obtain the following.

**Corollary 2.7.**   *Let $K_0$ and $K_1$ be homology cobordant 3-knots.   If $K_0$ is diffeomorphic to $K_1$, then $K_0$ is concordant to $K_1$.*

## 3.   The bounding genus and the Murasugi number of a 3-knot

Let $K$ be a 3-knot.   Let $\mathscr{B}_K$ denote the set of all compact oriented 4-submanifolds $\Delta$ of $D^6$ bounded by $K$ with the following properties.
(1)   $\Delta$ is connected and $H_1(\Delta){=}0$.
(2)   The intersection form of $\Delta$ is isomorphic to an orthogonal sum of $n$ copies of hyperbolic planes, $nU{=}n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for some non-negative integer $n$.

Define $b_K{=}\dfrac{1}{2}$ min $\{\mathrm{rank}\ H_2(\Delta);\ \Delta{\in}\mathscr{B}_K\}$ if $\mathscr{B}_K{\neq}\phi$, and define $b_K{=}\infty$ if $\mathscr{B}_K{=}$ $\phi$.   $b_K$ is called the *bounding genus* of $K$.   This is an analogue of Matsumoto's bounding genus of a homology 3-sphere ([9]).   To avoid confusion we call the latter the *abstract bounding genus*, which will be denoted by $|\Sigma|$ for every homology 3-sphere $\Sigma$.

Note that $b_K$ is a homology cobordism invariant of $K$.   Thus the bounding genus gives a function $C_3^H{\to}\{0, 1, 2, \cdots, \infty\}$.

DEFINITION.   Let $L$ be an integral square matrix of size $n$ with $\det(L{+}{}^tL){=}$ $\pm 1$.   $L$ is *b-cobordant* ($b{\geq}0$) if $L$ is congruent to a matrix of the form $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$ where $N_1, N_2, N_3$ are $k{\times}l, l{\times}k, l{\times}l$ matrices respectively and $k{\geq}\dfrac{n}{2}{-}b, l{=}n{-}k$. Note that $n$ is always even.

Note that a matrix is 0-cobordant if and only if it is null-cobordant.

**Lemma 3.1.**   *Let $L_1$ and $L_2$ be cobordant integral square matrices with $\det(L_i{+}{}^tL_i){=}\pm 1$ ($i{=}1, 2$).   Assume that $L_1$ is b-cobordant.   Then $L_2$ is also b-cobordant.*

This lemma can be proved by using an argument similar to that of [8, §2].

We now estimate the bounding genus of a 3-knot from above.

**Theorem 3.2.**   *For a 3-knot $K$, the following two are equivalent.*
(i)   $b_K{\leq}b{<}\infty$.

(ii)  *The abstract bounding genus $|K|$ of the homology 3-sphere $K$ satisfies $|K| \le$*
      *b. Furthermore, $L_K$ is b-cobordant and $\mathrm{sign}(L_K + {}^t L_K) = 0$.*

Proof.  (i)$\Rightarrow$(ii)  $|K| \le b$ is obvious.  Using the same argument as in [8,
§8] and [10, §4. Lemme 1], we see easily that $L_K$ is $b$-cobordant.  Let $F$ be a
compact oriented 4-submanifold of $S^5$ bounded by $K$.  Let $\Delta$ be a 4-submanifold
of $D^6$ belonging to $\mathscr{B}_K$.  Then $F \cup (-\Delta)$ bounds an oriented 5-submanifold of
$D^6$.  Hence $\mathrm{sign}(L_K + {}^t L_K) = \sigma(F) = \sigma(\Delta) = 0$.

(ii)$\Rightarrow$(i)  By Lemmas 2.4, 2.1 and 3.1, we may assume that $K$ is simple.
Let $F$ be a 1-connected 4-submanifold of $S^5$ bounded by $K$.  Furthermore let
$\Delta$ be a 4-manifold attaining the abstract bounding genus $|K|$.  By Lemma 2.6,
we may assume that $F$ is diffeomorphic to the 4-manifold obtained by surgery
on some finite collection of disjoint circles in $\Delta$.  Then by the argument given
in the rpoof of Lemma 2.5, we can prove that $K$ bounds $\Delta$ embedded in $D^6$.
Thus $b_K \le b$.  This completes the proof.

REMARK.  For a 3-knot $K$, let $\sigma_K = \mathrm{sign}(L_K + {}^t L_K)$, which is called the *sig-*
*nature* of $K$.  By Theorem 3.2, $b_K < \infty$ if and only if $\sigma_K = 0$.

Next we define the Murasugi number of a 3-knot.  For a 3-knot $K$, let $\mathscr{G}_K$
denote the set of all compact oriented 1-connected 4-submanifolds of $D^6$ bounded
by $K$.  By Lemma 2.4, $\mathscr{G}_K \neq \phi$.  We now define $g_K = \min\{\mathrm{rank}\, H_2(\Delta); \Delta \in \mathscr{G}_K\}$.
$g_K$ is called the *Murasugi number* of $K$ (see [10]).  Note that $g_K$ is always even.
This is a concordance invariant of $K$.  However, we do not know whether this is
a homology cobordism invariant of $K$.  Note that $g_K \ge |\sigma_K|$.

**Theorem 3.3.**  *For a 3-knot $K$, the following two are equivalent.*
(i)  $g_K \le g$, *where $g > |\sigma_K|$.*
(ii)  *There exists a compact 1-connected oriented spin 4-manifold $\Delta$ with $\partial\Delta$*
      *diffeomorphic to $K$ such that $\mathrm{rank}\, H_2(\Delta) \le g$, $\sigma(\Delta) = \sigma_K$ and the intersection*
      *pairing of $\Delta$ is indefinite.  Furthermore $L_K$ is $\dfrac{g}{2}$-cobordant.*

The conditions that $\Delta$ is indefinite and $g > |\sigma_K|$ are needed only in the
proof of (ii)$\Rightarrow$(i).

Proof.  (i)$\Rightarrow$(ii)  This is proved by the same argument as in the proof of
Theorem 3.2.

(ii)$\Rightarrow$(i)  We may assume that $K$ is simple.  Furthermore we may assume
that there exists a 1-connected 4-submanifold $F$ of $S^5$ bounded by $K$ which
is diffeomorphic to the manifold obtained by surgery on some disjoint circles
in $\Delta$.

Now as in [8, §13], for the given two families of linearly independent ele-
ments $\{\alpha_i; \beta_i\}_{i=1}^k$ and $\{\alpha_i'; \beta_i'\}_{i=1}^k$ of $H_2(F)$ ($2k \le \mathrm{rank}\, H_2(F)$) with $\alpha_i \cdot \alpha_j =$

$\alpha_i' \cdot \alpha_j' = \beta_i \cdot \beta_j = \beta_i' \cdot \beta_j' = 0$ and $\alpha_i \cdot \beta_j = \alpha_i' \cdot \beta_j' = \delta_{ij}$, we need to extend these to two bases of $H_2(F)$ for which the intersection matrices agree. Let $A$ and $A'$ be the orthogonal complements of $\{\alpha_i; \beta_i\}$ and $\{\alpha_i'; \beta_i'\}$ respectively. Then by the indefiniteness of $\Delta$, the intersection pairings restricted to $A$ and $A'$ are both indefinite. Since their ranks, signatures and types agree, they are isomorphic by Serre's theorem [15]. Thus $\{\alpha_i; \beta_i\}$ and $\{\alpha_i'; \beta_i'\}$ can be extended in the required manner.

The remaining argument is the same as in the proof of Theorem 3.2. This completes the proof.

REMARK. In the definition of the Murasugi number of a 3-knot $K$, replacing $\mathcal{G}_K$ by $\mathcal{G}_K' = \{$compact connected oriented 4-submanifolds $\Delta$ of $D^6$ bounded by $K$ with $H_1(\Delta) = 0\}$, we obtain the *homological Murasugi number* $g_K'$. Then an analogue of Theorem 3.3 also holds for $g_K'$. Note that $g_K'$ is a homology cobordism invariant of $K$.

In the next section we shall give some examples of the Murasugi number of algebraic 3-knots.

## 4. Application to algebraic 3-knots

Let $f$ be an analytic function defined on some neighborhood of the origin $\mathbf{0}$ in $\mathbf{C}^{n+1}$ with $f(\mathbf{0}) = 0$. We assume that $f$ has at most an isolated critical point at the origin. The *algebraic* $(2n-1)$-*knot* associated with $f$ is defined to be the knot given by $K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1} \subset S_\varepsilon^{2n+1}$, where $\varepsilon > 0$ is sufficiently small and $S_\varepsilon^{2n+1}$ is the sphere of radius $\varepsilon$ centered at the origin (for more details see [11]). Note that $K_f$ is a smooth closed $(2n-1)$-manifold.

By [11], the algebraic knot $K_f$ is fibered for any function $f$ as above. Furthermore the fiber has the homotopy type of a bouquet of $n$-dimensional spheres. The number of these spheres are called the *Milnor number* of $f$, denoted by $\mu(f)$.

In this section, we confine ourselves to the case of algebraic 3-knots ($n=2$).

**Proposition 4.1.** *Let $K_f$ be an algebraic 3-knot which is a homology 3-sphere. If $K_f$ is zero or finite order in $C_3^H$, then $f^{-1}(0)$ is smooth at the origin. In particular $K_f$ is the trivial knot.*

Proof. Let $L$ be a Seifert matrix of the algebraic knot $K_f$. Then by Michel [10], if $L$ is zero or finite order in $G_+$, $f^{-1}(0)$ is smooth at the origin. The proposition now follows from this and Lemma 2.1.

Let $\{f_t\}_{t \in J}$ be a family of polynomials such that each $f_t$ has an isolated critical point at the origin and $f_t(\mathbf{0}) = 0$, where $J$ is an open interval and $\{f_t\}$ depends smoothly on $t$. We say that $\{f_t\}$ is a *μ-constant deformation* if the Milnor number

$\mu(f_t)$ of $f_t$ does not depend on $t$ (see [6]).

**Proposition 4.2.** *Let $\{f_t\}$ be a $\mu$-constant deformation. Suppose that the algebraic knots $K_{f_t}$ associated with $f_t$ are homology 3-spheres. Then $K_{f_t}$ and $K_{f_{t'}}$ are homology cobordant for any $t$, $t' \in J$.*

Proof. We may assume that $|t - t'|$ is sufficiently small. If $R > 0$, $\varepsilon > 0$ are sufficiently small, the Milnor fibration of $f_t$

$$f_t / |f_t| : \{z; \|z\| \leq R, |f_t(z)| = \varepsilon\} \to S^1 \tag{A}$$

is isomorphic to the fibration

$$f_{t'} / |f_{t'}| : \{z; \|z\| \leq R, |f_{t'}(z)| = \varepsilon\} \to S^1 \tag{B}$$

(see [6, p. 73]). Note that (B) is not always the Milnor fibration of $f_{t'}$. Choose $\delta, \eta$ with $0 < \delta < R, 0 < \eta < \varepsilon$ such that

$$f_{t'} / |f_{t'}| : \{z; \|z\| \leq \delta, |f_{t'}(z)| = \eta\} \to S^1 \tag{C}$$

is the Milnor fibration of $f_{t'}$. This fibration is contained in the fibration

$$f_{t'} / |f_{t'}| : \{z; \|z\| \leq R, |f_{t'}(z)| = \eta\} \to S^1 \tag{D}$$

and (B) and (D) are isomorphic fibrations. Let $F$ and $F'$ be the fibers of (D) and (C) respectively. Note that $\partial F \subset S_R^5$ is isotopic to $K_{f_t}$ and $\partial F' \subset S_\delta^5$ is isotopic to $K_{f_{t'}}$, where $S_R^5$ and $S_\delta^5$ are the spheres of radii $R$, $\delta$ respectively with centers at the origin. Then $F'$ is contained in $F$ and, by Lê and Ramanujam [6], the inclusion map is a homotopy equivalence. Thus, it is easy to see that $F - \mathrm{Int}\, F'$ is a homology cobordism between $\partial F$ and $\partial F'$. Thus $K_{f_t}$ and $K_{f_{t'}}$ are homology cobordant.

REMARK. For $n \neq 2$, a $\mu$-constant deformation is topologically constant ([6]). However, we do not know whether this is also true for $n = 2$. Note that, even when $n = 2$, Seifert matrices of a $\mu$-constant deformation are constant modulo congruence.

Next we consider the following problem (see [4, Problem 5]).

PROBLEM. If two algebraic 3-knots, which are homology 3-spheres, are homology cobordant, are they isotopic?

A pratial answer is given as follows.

DEFINITION. A polynomial $f(z_1, z_2, \cdots, z_{n+1})$ in $C^{n+1}$ is said to be *weighted homogeneous* if there exist positive rational numbers $(w_1, w_2, \cdots, w_{n+1})$ with $\sum_{i=1}^{n+1} \dfrac{a_i}{w_i}$

$=1$ for every monomial $cz_1^{q_1} z_2^{q_2} \cdots z_{n+1}^{q_{n+1}}(c \neq 0)$ of $f(z_1, z_2, \cdots, z_{n+1})$ (see [11, §9]).

**Proposition 4.3.** *Let $f_i(x, y, z)$ $(i=1, 2)$ be weighted homogeneous polynomials having isolated critical points at the origin. Suppoes that the algebraic 3-knots $K_{f_i}$ associated with $f_i$ are homology 3-spheres. Then $K_{f_1}$ and $K_{f_2}$ are homology cobordant if and only if they are isotopic.*

REMARK. If we forget the embeddings, the above proposition does not hold. For example, the Brieskorn manifolds $\Sigma(2, 3, 13)$, $\Sigma(2, 5, 7)$ and $\Sigma(3, 4, 5)$ are all homology cobordant to $S^3$, i.e., they are zero in $\mathcal{H}^3$ ([1]). However, they are not diffeomorphic to one another.

Proof of Proposition 4.3. By [14], we may assume that $f_i$ is a Brieskorn type polynomial. Let $f_i(x, y, z) = x^{a_i} + y^{b_i} + z^{c_i}$ $(i=1, 2)$. Since $K_{f_i}$ is a homology 3-sphere, $a_i$, $b_i$ and $c_i$ are pairwise relatively prime ([2]). Let $\Delta_i(t)$ be the characteristic polynomial of the monodromy for $f_i$, which agrees with the Alexander polynomial of $K_{f_i}$. Then by [11, §9],

$$\Delta_i(t) = \frac{(t^{a_i b_i c_i}-1)(t^{a_i}-1)(t^{b_i}-1)(t^{c_i}-1)}{(t^{a_i b_i}-1)(t^{b_i c_i}-1)(t^{c_i a_i}-1)(t-1)}.$$

Let $\phi_d(t)$ be the $d$-th cyclotomic polynomial. Then $t^n-1 = \prod_{d|n} \phi_d(t)$. Using this formula, we obtain the equation

$$(*) \quad \Delta_i(t) = \prod_{d \in D_i} \phi_d(t),$$

where $D_i = \{d \in N; d \mid a_i b_i c_i, d \nmid a_i b_i, d \nmid b_i c_i, d \nmid c_i a_i\}$. Note that $(*)$ is the irreducible factorization of $\Delta_i(t)$.

Now assume that $K_{f_1}$ and $K_{f_2}$ are homology cobordant. Then $K=K_{f_1} \# (-K_{f_2})$ is a null-cobordant 3-knot. For every $\gamma(t) \in Z[t]$, define $\gamma^*(t) = t^m \gamma(t^{-1})$, where $m = \deg \gamma(t)$. Let $\Delta(t)$ be the Alexander polynomial of $K$, then $\Delta(t) = \pm \Delta_1(t) \Delta_2^*(t)$. Since $\phi_d^*(t) = \pm \phi_d(t)$ and $\Delta_2(t)$ is a product of cyclotomic polynomials, $\Delta_2^*(t) = \pm \Delta_2(t)$. Thus $\Delta(t) = \Delta_1(t) \Delta_2(t)$.

Since $K$ is null-cobordant, $\Delta(t) = \pm \lambda(t) \lambda^*(t)$ for some $\lambda(t) \in Z[t]$ (see [8]). Furthermore $\lambda^*(t) = \pm \lambda(t)$, since $\Delta(t) = \Delta_1(t) \Delta_2(t)$ is a product of cyclotomic polynomials. Thus $\Delta(t) = \Delta_1(t) \Delta_2(t) = \lambda(t)^2$. Using the equation $(*)$, we see easily $\Delta_1(t) = \Delta_2(t)$, i.e., $D_1 = D_2$. This implies $\{a_1, b_1, c_1\} = \{a_2, b_2, c_2\}$. Thus $K_{f_1}$ is isotopic to $K_{f_2}$. This completes the proof.

Finally we give some examples of the Murasugi number of algebraic 3-knots. Note that $g_{K_f} \leq \mu(f)$ for every algebraic 3-knot $K_f$.

EXAMPLE 4.4. Let $f(x, y, z) = x^2 + y^3 + z^{13}$, and let $K_f$ be the algebraic 3-knot associated with $f$. $K_f$ is the Brieskorn manifold $\Sigma(2, 3, 13)$, which is a homology

3-sphere ([2]). Let $L$ be its unimodular Seifert matrix. Using a formula of Brieskorn [2], one obtains $\text{sign}(L+{}^tL)=-16$. Since rank $L=\mu(f)=24$, $L+{}^tL$ is indefinite. In particular $L$ is 11-cobordant. Furthermore, $\Sigma(2,3,13)$ bounds a compact contractible 4-manifold $M$ ([1]). Let $\Delta=M\,\#\,V_4$, where $V_4$ is a manifold diffeomorphic to a $K3$-surface. Then $\Delta$ is a 1-connected spin 4-manifold with $\partial\Delta\cong K_f$, rank $H_2(\Delta)=22$, and $\sigma(\Delta)=-16$. Thus $K_f$ satisfies the conditions of Theorem 3.3 (ii). Hence $g_{K_f}\leq 22$. Note that this result has already been obtained by S. Akbulut ([10, p. 176]).

On the other hand, by Donaldson [3], $\Sigma(2,3,13)$ cannot bound any compact 1-connected spin 4-manifold with signature $-16$ and second betti number strictly less than 22. Thus $g_{K_f}=22$.

The same argument as above can also be used in the cases of $\Sigma(2,5,7)$ and $\Sigma(3,4,5)$.

In fact, there exist infinitely many algebraic 3-knots $K_f$ with $g_{K_f}<\mu(f)$. For instance, $f(x,y,z)=x^2+y^3+z^{12k-1}$ ($k\geq 4$) is such an example. See [2, p. 13] and [9, p. 307].

Next we consider a theorem of F. Michel [10, p. 176], which states that an algebraic $(2n-1)$-knot $K_f$, which is a homology $(2n-1)$-sphere, satisfies $g_{K_f}=\mu(f)$ if and only if $f$ has a simple critical point at the origin, provided that $n>2$. We now give a counter-example of this theorem in the case of $n=2$. We use the deep result of S. Donaldson [3]. See also Problem E [16, p. 250] by Michel.

EXAMPLE 4.5. Let $f(x,y,z)=x^2+y^3+z^{11}$, and let $K_f$ be the algebraic 3-knot associated with $f$. By [2], $K_f$ is a homology 3-sphere and $\sigma_{K_f}=-16$. Furthermore $K_f$ bounds a compact 1-connected 4-manifold $N$ with intersection form isomorphic to $U$ ([9]).

Now suppose $g_{K_f}<\mu(f)$. Let $\Delta\in\mathcal{G}_{K_f}$ be a submanifold of $D^6$ attaining $g_{K_f}$. Then we have $\sigma(\Delta)=\sigma_{K_f}=-16$ and the second betti number of $\Delta$ is strictly less than $\mu(f)=20$. Let $V=\Delta\cup N$ be the 4-manifold given by identifying the boundaries of $\Delta$ and $N$. Then $V$ is a compact 1-connected spin 4-manifold with $\sigma(V)=-16$ and second betti number strictly less than 22. This contradicts the result of S. Donaldson [3]. Thus $g_{K_f}=\mu(f)$.

Note that the singularity of $f$ at the origin is not simple, because the intersection form of its Milnor fiber is indefinite.

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