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DECAY OF SOLUTIONS OF THE WAVE EQUATION IN THE EXTERIOR OF TWO CONVEX OBSTACLES

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1. Introduction. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be compact and convex sets in $\boldsymbol{R}^3$ with smooth boundary Γ_1 and Γ_2 respectively. Suppose that

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \phi .$$

Let us set

$$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 , \quad \Gamma = \Gamma_1 \cup \Gamma_2 .$$

Consider a mixed problem

$$(P) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{3} \dfrac{\partial^2 u}{\partial x_j^2} = 0 & \text{in } \Omega \times (0, \infty) \\ Bu = 0 & \text{on } \Gamma \times (0, \infty) \\ u(x, 0) = u_0(x) \\ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \end{cases}$$

where $\Omega = \boldsymbol{R}^3 - \mathcal{O}$. Here we treat as the boundary operator B the following two operators:

$$B_1 u = u$$

and

$$B_2 u = \frac{\partial u}{\partial n} + \sigma(x) u$$

where n denotes the unit outer normal of Γ and $\sigma(x)$ is a real valued C^∞-function defined on Γ.

Concerning the initial data the compatibility condition is always assumed, that is, for $u_0, u_1 \in C^\infty(\bar{\Omega})$ defining successively $u_j \in C^\infty(\bar{\Omega})$, $j=2, 3, \cdots$ by the formula

$$u_j = \Delta u_{j-2}$$

the condition

$$B u_j = 0 \qquad \text{on } \Gamma$$

is satisfied for all j.

About the obstacles $\mathcal{O}_1$, $\mathcal{O}_2$ we set the condition

(SC) the Gaussian curvature of Γ never vanishes.

Under this condition we will consider the decay of solutions of (P). Let us set

$$\Omega_R = \Omega \cap \{x;\ |x| \leqslant R\}\ .$$

Theorem 1. *Suppose that $B=B_1$, that is, the Dirichlet boundary is posed. Then there exists a constant $\alpha_1>0$ determined by $\mathcal{O}$ only with the following properties: Let R and κ be positive numbers and let m be a positive integer. For any u_0, u_1 such that*

$$\bigcup_{j=0}^{1} \operatorname{supp} u_j \subset \bar{\Omega}_\kappa$$

we have an estimate

$$\sum_{|\gamma|\leqslant m} \sup_{x\in\Omega_R} |D^\gamma_{x,t} u(x,t)| \leqslant C_{R,\kappa,m} e^{-\alpha_1 t} \{\|u_0\|_{m+5,L^2(\Omega)} + \|u_1\|_{m+4,L^2(\Omega)}\}\ , \tag{1.1}$$

where $C_{R,\kappa,m}$ is a positive constant depending on R, κ and m but independent of u_0, u_1.

For $P\in \boldsymbol{R}^3-\Gamma$, let us set

$$\Gamma_{jP} = \{x;\ x\in\Gamma_j,\ \{P+l\overrightarrow{Px};\ l>1\}\subset\Omega\}\ .$$

DEFINITION. We say that $\sigma(x)$ satisfies the condition A when there exists a finite number of $P_j\in\mathcal{O}_1$, $j=1, 2, \cdots, n_1$ and $Q_j\in\mathcal{O}_2$, $j=1, 2, \cdots, n_2$ such that

$$\bigcup_{j=1}^{n_1}\Gamma_{1P_j}\supset\Gamma_1\ , \quad \bigcup_{j=1}^{n_2}\Gamma_{2Q_j}\supset\Gamma_2$$

and

$$\sigma(x)<(n_1+n_2)^{-1}\sum_j \frac{\overrightarrow{P_jx}\cdot n(x)}{|P_j-x|^2} \qquad \text{on } \Gamma_1$$

$$\sigma(x)<(n_1+n_2)^{-1}\sum_j \frac{\overrightarrow{Q_jx}\cdot n(x)}{|Q_j-x|^2} \qquad \text{on } \Gamma_2$$

where $\sum_j$ means the summation of j such that $x\in\Gamma_{1P_j}$ or $x\in\Gamma_{2Q_j}$.

Theorem 2. *Suppose that $B=B_2$, that is, the third kind boundary condition is posed. If $\sigma(x)$ satisfies the condition A there exists a constant $\alpha_2>0$ determined by $\mathcal{O}$ and σ with the following properties: Let R and κ be positive numbers and let m be a positive integer. For any u_0, u_1 such that*

$$\bigcup_{j=0}^{1} \operatorname{supp} u_j \subset \bar{\Omega}_\kappa$$

we have an estimate

$$(1.2)\qquad \sum_{|\gamma|\leqslant m}\sup_{x\in\Omega_R}|D_{x,t}^{\gamma}u(x,t)| \leqslant C_{R,\kappa,m}e^{-\alpha_2 t}\{||u_0||_{m+5,L^2(\Omega)}+||u_1||_{m+4,L^2(\Omega)}\}$$

where $C_{R,\kappa,m}$ *is a positive constant depending on* R, κ, m *but independent of* u_0, u_1.

It seems to us that until now studies on the uniform decay of the solutions of the wave equation in the exterior domain are made mainly concerning the existence of a function $p(t)$ such that

$$(*)\qquad \begin{cases} E(u, R, t)\leqslant p(t)E(u, \infty, 0) \\ p(t)\to 0 \quad as \quad t\to\infty \end{cases}$$

for all u_0, $u_1\in C_0(\Omega_\kappa)$, where

$$E(u, R, t)=\frac{1}{2}\int_{\Omega_R}(|\nabla u(x, t)|^2+|u_t(x, t)|^2)dx\,.$$

About the necessary condition on the obstacle for the existence of $p(t)$ we know the work of Ralston [12]. Roughly speaking [12] shows that if the obstacle admits a trapped ray there is no $p(t)$ verifying $(*)$.[1]

If $\mathcal{O}$ consists of two obstacles $\mathcal{O}_1$, $\mathcal{O}_2$ there is always a trapped ray. Indeed, let $a_j\in\mathcal{O}_j$, $j=1, 2$ be the points such that

$$|a_1-a_2|=\inf_{\substack{x\in\mathcal{O}_1\\ y\in\mathcal{O}_2}}|x-y|$$

Then $\overrightarrow{a_1a_2}$ is perpendicular to Γ_1 at a_1 and $\overrightarrow{a_2a_1}$ perpendicular to Γ_2 at a_2. Therefore the ray starting at a_1 in the direction $\overrightarrow{a_1a_2}$ hits Γ_2 at a_2, and is reflected in the direction $\overrightarrow{a_2a_1}$. Then the reflected ray hits Γ_1 at a_1, and is reflected in the direction $\overrightarrow{a_1a_2}$. Namely, the ray plies between a_1 and a_2 and never goes to the infinity. This shows that $\mathcal{O}$ is trapping. Then we see that the estimate of the type $(*)$ never holds. Therefore, when $\mathcal{O}$ consists of two obstacles, in order to estimate the uniform rate of the decay of solutions it is necessary to consider the decay in a weaker form than $(*)$. Indeed, Walker [13] shows that

$$\lim_{t\to\infty}p_{\alpha,R}(t)=0$$

holds for any $\alpha>0$, where

$$p_{\alpha,R}(t)=\sup_{\{u_0,u_1\}\in C_0^\infty(\Omega_R)}E(u, R, t)^{1/2}(||u_0||_{H_0^{1+\alpha}(\Omega_R)}+||u_1||_{H_0^{\alpha}(\Omega_R)})^{-1}\,.$$

1) Morawetz, Ralston and Strauss [11] shows that when there is no trapped ray there exists $p(t)$ verifying $(*)$ under some additional conditions.

But we cannot obtain from its proof any more informations about qualitative nature of $p_{\alpha,R}(t)$ for general obstacles.

We would like to remark about the condition SC. If only the convexity of obstacles is assumed, we can not expect in general the estimate of the form (1.1) or (1.2). Namely in the appendix we will present convex obstacles $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$\overline{\lim_{t\to\infty}}\, p_{m,R}(t)t^{m(1+\varepsilon)}>0$$

holds for all positive integer m and positive constant ε (Theorem A).

The essential part of the proof of the theorems is the construction of an asymptotic solution for an oscillatory boundary data. There, we estimate carefully the decay of the amplitude function while repeating the reflections.

2. Reduction of the problem

Let the supports of u_0, u_1 be contained in $\bar{\Omega}_\kappa$. Take $v_0, v_1 \in C_0^\infty(\boldsymbol{R}^3)$ so that

$$v_j(x) = u_j(x) \qquad \text{in } \Omega,\ i = 0,\ 1 \tag{2.1}$$

and

$$\|v_j\|_{m,L^2(\boldsymbol{R}^3)} \leqslant C_m \|u_j\|_{m,L^2(\Omega)} \tag{2.2}$$

holds. Remark that the constant C_m depends on Ω and m, but is independent of u_j. Let $w(x, t)$ be the solution of the Cauchy problem

$$\begin{cases} \Box w = 0 & \text{in } \boldsymbol{R}^3\times(0,\ \infty) \\ w(x,\ 0) = v_0(x) \\ \dfrac{\partial w}{\partial t}(x,\ 0) = v_1(x)\,. \end{cases} \tag{2.3}$$

Since

$$\operatorname{supp} v \subset \{(x);\ |x| \leqslant \kappa\}$$

the Huygens' principle assures that

$$\operatorname{supp} w \subset \{(x,\ t);\ t-\kappa \leqslant |x| \leqslant t+\kappa\}\ . \tag{2.4}$$

Let us set

$$-Bw(x,\ t)\ |_{\Gamma\times(0,\infty)} = h(x,\ t)\,.$$

If we extend h to $t<0$ by setting $h(x, t)=0$ for $t<0$, the compatibility condition of u_0, u_1 means that

$$h \in C^\infty(\Gamma\times \boldsymbol{R})\,.$$

Combining $\mathcal{O} \in \{x;\ |x| \leqslant d_1\}$ for some d_1 and (2.4) we have

(2.5) $$\operatorname{supp} h \subset \Gamma \times (0, \kappa + d_1).$$

And it follows from the energy estimate of w that

(2.6) $$\|h\|_{m+1/2, L^2(\Gamma \times \boldsymbol{R})} \leqslant C_m \{\|v_0\|_{m+1+\epsilon} + \|v_1\|_{m+\epsilon}\},$$

where

$$\epsilon = \begin{cases} 1 & \text{if } B = B_2 \\ 0 & \text{if } B = B_1. \end{cases}$$

Consider the boundary value problem with parameter $p = \mu + ik$ for boundary data $g \in C^\infty(\Gamma)$

(2.7) $$\begin{cases} (\Delta - p^2)u = 0 & \text{in } \Omega \\ Bu = g & \text{on } \Gamma. \end{cases}$$

It is well known that there exists $\mu_0 > 0$ such that for $\mu \geqslant \mu_0$ (2.7) has a solution uniquely in $H^2(\Omega)$ and an estimate

$$\|u\|_{m+2, L^2(\Omega)} \leqslant C_m \sum_{j=0}^{m} |p|^{m-j} \|g\|_{j+3/2-\epsilon, L^2(\Gamma)}$$

holds for $m = 0, 1, 2, \cdots$. Define $U(p)$ a mapping from $C^\infty(\Gamma)$ into $C^\infty(\bar{\Omega})$ by

$$U(p)g = u(x).$$

Then $U(p)$ is analytic in $\operatorname{Re} p \geqslant \mu_0$ as $\mathcal{L}(C^\infty(\Gamma), C^\infty(\bar{\Omega}))$-valued function, where $\mathcal{L}(C^\infty(\Gamma), C^\infty(\bar{\Omega}))$ denotes the set of all continuous linear mappings from $C^\infty(\Gamma)$ into $C^\infty(\bar{\Omega})$.

Theorem 2.1. *Suppose that $\mathcal{O}$ satisfies the condition SC and that σ verifies the condition A when $B = B_2$. Then $U(p)$ can be prolonged analytically into a region containing*

$$\{p \colon \operatorname{Re} p \geqslant -\alpha\}$$

for some $\alpha > 0$. And we have for Re $p \geqslant -\alpha$

(2.8) $$\sup_{x \in \Omega_R} \sum_{|\gamma| \leqslant m} |D_x^\gamma (U(p)g)(x)| \leqslant C_{m,R} \sum_{j=0}^{m+4} (|k|+1)^{m+3-j} \|g\|_{j, L^2(\Gamma)}, \quad {}^\forall g \in C^\infty(\Gamma).$$

Note that the solution of the problem

$$\begin{cases} \Box z(x, t) = 0 & \text{in } \Omega \times \boldsymbol{R} \\ Bz(x, t) = h(x, t) & \text{on } \Gamma \times \boldsymbol{R} \\ \operatorname{supp} z \subset \bar{\Omega} \times (0, \infty) \end{cases}$$

is representes as

$$z(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(ik+\mu)t}(U(ik+\mu)\hat{h}(\cdot, ik+\mu))(x)dk\,,$$

where $\mu \geqslant \mu_0$ and

$$\hat{h}(x, p) = \int_{-\infty}^{\infty} e^{-pt}h(x, t)dt\,.$$

By using Theorem 2.1 the path of the integration can be changed to $\operatorname{Re} p = -\alpha$, namely we have

$$z(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(ik-\alpha)t}(U(ik-\alpha)\hat{h}(\cdot, ik-\alpha))(x)dx\,.$$

With the aid of the estimate (2.8) we obtain immediately from this formula

$$(2.9) \qquad \sum_{|\gamma|\leqslant m} \sup_{x\in\Omega_R} |D^{\gamma}_{x,t}z(x, t)|$$
$$\leqslant C_{m,R}e^{-\alpha t}\int_{-\infty}^{\infty}\sum_{j=0}^{m+3}(|k|+1)^{m+3-j}\|\hat{h}(\cdot, ik-\alpha)\|_{j,L^2(\Gamma)}dk$$
$$\leqslant C_{m,R}e^{-\alpha t}\|h\|_{m+4,L^2(\Gamma\times R)}\,.$$

Since the solution $u(x, t)$ of (P) is represented as

$$u(x, t) = w(x, t) + z(x, t)$$

we have Theorems 1 and 2 from (2.4), (2.6) and (2.9).

Concerning the analytic continuation the following results hold without the condition SC:

(i) For the Dirichlet boundary condition $U(p)$ can be prolonged analytically into a region containing $\{p;\ \operatorname{Re} p \geqslant 0\}$.

(ii) For the third kind boundary condition $U(p)$ can be prolonged into a region containing $\{p;\ \operatorname{Re} p \geqslant 0,\ p \notin [0, \mu_0]\}$[2]. In the rest of this paper we will show the following

Theorem 2.2. *Suppose that $\mathcal{O}$ satisfies the condition SC. Let $a_j \in \Gamma_j$, $j=1, 2$ be the points such that $|a_1-a_2| = \operatorname{dis}(\mathcal{O}_1, \mathcal{O}_2)$ and let us denote by K_{ij}, $j=1, 2$ the principal curvatures of Γ_i at a_i. Set*

$$c_0 = \frac{1}{d_0}\log(1+2d_0K)$$

where $d_0 = \operatorname{dis}(\mathcal{O}_1, \mathcal{O}_2)$ and $K = \inf_{i,j=1,2} K_{ij}$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$

2) See, for example, about (i) Chapter V of Lax-Phillips [6], Chapter 8 of Mizohata [9] and about (ii) Asakura [1].

such that $U(p)$ *can be prolonged into*

$$\{\mu+ik;\ \mu\geqslant-(c_0-\varepsilon),\ |k|\geqslant k_\varepsilon\}\ .$$

Moreover an estimate

$$\sum_{|\gamma|\leqslant m}\sup_{x\in\Omega_R}|D_x^\gamma(U(p)g)(x)|$$
$$\leqslant C_{\varepsilon,R,m}\sum_{j=0}^{m=3}(|k|+1)^{m+3-j}||g||_{j,L^2(\Gamma)}\,,\quad {}^\forall g\in C^\infty(\Gamma)$$

holds.

Then in order to show Theorem 2.1, if we admit Theorem 2.2, it remains to prove that $p\in[0,\infty)$ is not the generalized eigenvalue of (2.7) for $B=B_2$, that is, there is no non-trivial solution of (2.7) for $g=0$ verifying

$$|x|\,|u(x)|+|x|^2|\nabla u(x)|\leqslant C\qquad\text{as } |x|\to\infty. \tag{2.10}$$

Therefore we show

Lemma 2.3. *Suppose that* σ *satisfies the condition* A. *Then for* $p\in[0,\infty)$ *there is no non-trivial solution of* (2.7) *for* $g=0$ *verifying* (2.10).

Proof. Suppose that $u(x)$ is a non-trivial solution of (2.7) for $g=0$ verifying (2.10). Then by the integration by parts of $(\Delta-p^2)u(x)\cdot\overline{u(x)}$ it follows that

$$\int_\Gamma\sigma(x)|u(x)|^2ds=\int_\Omega|\nabla u(x)|^2dx+p^2\int_\Omega|u|^2dx\ .$$

Therefore if we derive from the assumption on σ the estimate

$$\int_\Gamma\sigma|u|^2dS\leqslant(1-\varepsilon)\int_\Omega|\nabla u|^2dx \tag{2.11}$$

for some $\varepsilon>0$, u must be identically zero. Suppose that $P\in\mathcal{O}_1$. For $x\in\Gamma_{1P}$, setting $\omega=\frac{\overrightarrow{Px}}{|\overrightarrow{Px}|}$, $r(\omega)=|\overrightarrow{xP}|$

$$u(x)=-\int_{r(\omega)}^\infty\frac{\partial}{\partial s}u(s\omega)ds=-\int_{r(\omega)}^\infty(\omega\cdot\nabla u)(s\omega)ds\ .$$

Then

$$|u(x)|^2\leqslant\int_{r(\omega)}^\infty\frac{ds}{s^2}\int_{r(\omega)}^\infty|\nabla u(s\omega)|^2s^2ds$$
$$=r(\omega)^{-1}\int_{r(\omega)}^\infty|\nabla u(s\omega)|^2s^2ds\ .$$

Setting $\omega_{1P}=\left\{\frac{\overrightarrow{Px}}{|\overrightarrow{Px}|},\ x\in\Gamma_{1P}\right\}$

$$\int_{\omega_{1P}} |u(x)|^2 r(\omega) d\omega \leqslant \int_{\omega_{1P}} \int_{r(\omega)}^{\infty} |\nabla u(s\omega)|^2 s^2 ds d\omega$$
$$\leq \int_{\Omega} |\nabla u(x)|^2 dx .$$

Denoting by dS the surface element of Γ_1, since $dS = r(\omega)^2 \frac{r(\omega)}{\overrightarrow{Px} \cdot n(x)} d\omega$ the condition

$$\sup_{x \in \Gamma_{1P}} \sigma(x) \frac{|\overrightarrow{Px}|^2}{\overrightarrow{Px} \cdot n(x)} \leqslant C$$

implies

$$\int_{\Gamma_{1P}} \sigma(x) |u(x)|^2 dS \leqslant C \int_{\omega_{1P}} r(\omega) |u(x)|^2 d\omega$$
$$\leqslant C \int_{\Omega} |\nabla u(x)|^2 dx .$$

Suppose that P_j, $j=1, 2, \cdots, n_1$ satisfy

$$\bigcup_{j=1}^{n_1} \Gamma_{1P_j} \supset \Gamma_1 \quad \text{and} \quad \sum_j \frac{\sigma(x) |\overrightarrow{P_j x}|^2}{\overrightarrow{P_j x} \cdot n(x)} \leqslant (1-\varepsilon)(n_1+n_1)^{-1} .$$

Then it holds that

$$\int_{\Gamma_1} \sigma(x)^+ |u(x)|^2 dS \leqslant \sum_{j=1}^{n_1} \int_{\Gamma_{1P_j}} \sigma(x)^+ |u(x)|^2 dS$$
$$\leqslant \frac{n_1(1-\varepsilon)}{(n_1+n_2)} \int_{\Omega} |\nabla u(x)|^2 dx ,$$

where $\sigma(x)^+ = \max(\sigma(x), 0)$. By the same way we have

$$\int_{\Gamma_2} \sigma^+(x) |u(x)|^2 dS \leqslant \frac{n_2(1-\varepsilon)}{(n_1+n_2)} \int_{\Omega} |\nabla u(x)|^2 dx .$$

Then it follows that

$$\int_{\Gamma} \sigma(x) |u(x)|^2 dS \leqslant \int_{\Gamma} \sigma(x)^+ |u(x)|^2 dS$$
$$\leqslant (1-\varepsilon) \int_{\Omega} |\nabla u(x)|^2 dx .$$

Thus (2.11) is derived. Therefore Lemma is proved.

3. Properties of broken rays

For $x \in \Gamma$, $n(x)$ denotes the unit outer normal of Γ at x, and

$$\sum_x^+ = \{\xi;\, |\xi|=1,\, n(x)\cdot\xi \geqslant 0\}\,.$$

We denote by $\mathcal{X}(x, \xi)$ the broken ray according to the law of the geometrical optics starting at $x\in\Gamma$ in the direction $\xi\in\sum_x^+$, by $X_1(x, \xi)$, $X_0(x, \xi)$, $\cdots$ the points of reflection of the broken ray and by $\Xi_j(x, \xi)$ the direction of the ray reflected at $X_j(x, \xi)$. More precisely, if

$$\{x+l\xi;\, l>0\} \cap \Gamma = \phi$$

we set $L_0(x, \xi)=\{x+l\xi;\, l\geqslant 0\}$. If $\{x+l\xi;\, l>0\}\cap\Gamma\neq\phi$ we set

$$\begin{aligned}
l_0(x, \xi) &= \inf\{l;\, l>0,\, x+l\xi\in\Gamma\}\\
L_0(x, \xi) &= \{x+l\xi;\, 0\leqslant l\leqslant l_0(x, \xi)\}\\
X_1(x, \xi) &= x+l_0(x, \xi)\xi\\
\Xi_1(x, \xi) &= \xi-2(n(X_1(x, \xi))\cdot\xi)n(X_1(x, \xi))\,.
\end{aligned}$$

When $\{X_1+l\Xi_1;\, l>0\}\cap\Gamma=\phi$, $L_1(x, \xi)=\{X_1+l\Xi_1;\, l\geqslant 0\}$. Otherwise we set

$$\begin{aligned}
l_1(x, \xi) &= \inf\{l;\, l>0,\, X_1+l\Xi_1\in\Gamma\}\\
L_1(x, \xi) &= \{X_1+l\Xi_1;\, 0\leqslant l\leqslant l_1\}\\
X_2(x, \xi) &= X_1+l_1\Xi_1\\
\Xi_2(x, \xi) &= \Xi_1-2(n(X_2)\cdot\Xi_1)n(X_2)\,.
\end{aligned}$$

Thus we define successively $l_j(x, \xi)$, $X_j(x, \xi)$, $\Xi_j(x, \xi)$, $L_j(x, \xi)$ until $\{X_j+l\Xi_j;\, l>0\}\cap\Gamma=\phi$. If there exists j_0 such that for $j\leqslant j_0$, $l_j(x, \xi)$, $X_j(x, \xi)$, $\Xi_j(x, \xi)$ are defined and $\{X_{j_0}+l\Xi_{j_0};\, l>0\}\cap\Gamma=\phi$, then we define

$$\begin{aligned}
\mathcal{X}(x, \xi) &= \bigcup_{j=0}^{j_0} L_j(x, \xi)\\
{}^{\#}\mathcal{X}(x, \xi) &= j_0\,.
\end{aligned}$$

Otherwise

$$\begin{aligned}
\mathcal{X}(x, \xi) &= \bigcup_{j=0}^{\infty} L_j(x, \xi)\\
{}^{\#}\mathcal{X}(x, \xi) &= \infty.
\end{aligned}$$

We set for $(x, t)\in\Gamma\times\boldsymbol{R}$, $\xi\in\sum_x^+$

$$\begin{aligned}
\mathcal{L}_j(x, t, \xi) &= \{(X_j(x, \xi)+l\Xi_j(x, \xi),\, t+\bar{l}_j(x, \xi)+l);\, 0\leqslant l\leqslant l_j(x, \xi)\}\\
\bar{\mathcal{L}}_j(x, t, \xi) &= \{(X_j(x, \xi)+l\Xi_j(x, \xi),\, t+\bar{l}_j(x, \xi)+l);\, 0\leqslant l\}\\
\mathcal{L}(x, t, \xi) &= \bigcup_j \mathcal{L}_j(x, t, \xi)\,,
\end{aligned}$$

where $\bar{l}_j(x, \xi)=\sum_{p=0}^{j-1} l_p(x, \xi)$.

The following two lemmas are trivial.

Lemma 3.1. *For $\mathcal{O}_1$, $\mathcal{O}_2$ there exists $\delta_1>0$ with the following property: If $x\in\Gamma_1$, $\xi\in\sum_x^+$ and*

$$0\leqslant -n(X_1(x, \xi))\cdot\xi\leqslant\delta_1\,,$$

the reflected ray does not pass the $d_0/2$ neighborhood of $\mathcal{O}_1$, i.e., $L_1(x, \xi)$ is a half line and

$$L_1(x, \xi)\cap\{y;\ \mathrm{dis}(y, \mathcal{O}_1)\leqslant d_0/2\} = \phi\,.$$

Lemma 3.2. *For each $x\in\Gamma$, $\xi\in\sum_x^+$ verifying $^{\#}\mathcal{X}(x, \xi)\geqslant J$ and*

$$n(X_j(x, \xi))\cdot\Xi_j(x, \xi)>0\,, \qquad j=1, 2, \cdots, J\,,$$

there exists $\varepsilon>0$ such that if $y\in\Gamma$, $\eta\in\sum_y^+$ and

$$|x-y|+|\xi-\eta|<\varepsilon\,,$$

it holds that

$$^{\#}\mathcal{X}(y, \eta)\geqslant J$$

and

$$X_j(y, \eta)\to X_j(x, \xi)$$
$$\Xi_j(y, \eta)\to\Xi_j(x, \xi)$$

for $j=1, 2, \cdots, J$ when $(y, \eta)\to(x, \xi)$.

Corollary. *Let L be the line passing a_1 and a_2, where a_j, $j=1, 2$ are those in Theorem 2.2. Denote by $S_j(\delta)$ the connected component containing a_j of*

$$\Gamma_j\cap\{x;\ \mathrm{dis}(x, L)\leqslant\delta\}\,, \qquad \delta>0\,.$$

For any $\varepsilon_0>0$, if we choose $\delta_2>0$ sufficiently small, every ray starting at $x\in S_1(\delta_2)$ in the direction $\xi\in\sum_x^+$ and fitting $S_2(\delta_2)$ (starting at $x\in S_2(\delta_2)$ and hitting $S_1(\delta_2)$) satisfies

$$-\xi\cdot n(X_1(x, \xi))>1-\varepsilon_0\,.$$

Proof. Note that for $\xi_0=\dfrac{a_2-a_1}{|a_2-a_1|}\in\sum_{a_1}^+$

$$-\xi_0\cdot n(X_1(a_1, \xi_0))=1\,.$$

Then from Lemma 3.2 there exists $\varepsilon>0$ such that

$$|y-a_1|+|\eta-\xi_0|<\varepsilon, \qquad y\in\Gamma_1\,, \eta\in\sum_y^+$$

implies that

$$|\eta\cdot n(X_1(y, \eta))-\xi_0\cdot n(X_1(a_1, \xi_0))|<\varepsilon_0\,.$$

For any $y\in S_1(\delta_2)$, $\eta\in\sum_y^+$ such that $X_1(y, \eta)\in S_2(\delta_2)$ we have $|y-a_1|\leqslant 2\delta_2$ and $|X_1(y, \eta)-a_2|\leqslant 2\delta_2$. Then

$$\eta-\xi_0=\frac{X_1(y, \eta)-y}{|X_1(y, \eta)-y|}-\frac{a_2-a_1}{|a_2-a_1|}$$

must be small if δ_2 is choosen so small. Then for any $\varepsilon>0$ there exists $\delta_2>0$ such that $y\in S_1$, $\eta\in\sum_y^+$ and $X_1(y, \eta)\in S_2$ imply

$$|y-a_1|+|\eta-\xi_0|<\varepsilon\,.$$

Using the above remark we have

$$-\eta\cdot n(X_1(y, \eta))\geqslant-\xi_0\cdot n(X_1(a_1, \xi_0))-\varepsilon_0=1-\varepsilon_0\,.$$

Q.E.D.

Lemma 3.3. *For each $\delta_2>0$ of Corollary of the previous lemma there exists a positive integer K such that $x\in\Gamma-S(\delta_2)$, $\xi\in\sum_x^+$ and $\mathscr{X}(x, \xi)\cap S(\delta_2)=\phi$ imply*

$$^{\#}\mathscr{X}(x, \xi)\leqslant K\,, \tag{3.1}$$

where $S(\delta_2)=S_1(\delta_2)\cup S_2(\delta_2)$.

Proof. Let $\delta_1>0$ be that of Lemma 3.1. Suppose that $L=\{(0, 0, l); l\in\boldsymbol{R}\}$. For $x=(x_1, x_2, x_3)$ denote by x' the point $(x_1, x_2, 0)$. The strict convexity assures that

$$n(x)\cdot x'\geqslant c>0\,,\quad {}^{\forall}x\in\Gamma_l-S_l \tag{3.2}$$

if $n(x)\cdot((-1)^{l-1}(a_2-a_1))\geqslant 0$, where $c>0$ depends on δ_2. Suppose that

$$^{\#}\mathscr{X}(x, \xi)\geqslant J\,.$$

Then by Lemma 3.1 we have

$$-n(X_j)\cdot\Xi_{j-1}\geqslant\delta_1\qquad\text{for } j=1, 2, \cdots, J-1\,. \tag{3.3}$$

Note that, if $x\in\Gamma_l-S_l$,

$$n(X_j)\cdot((-1)^{j+l-1}(a_2-a_1))\geqslant 0\,,\qquad l=1, 2,\ j=1, 2, \cdots\,. \tag{3.4}$$

Let $x(s)=(x_1(s), x_2(s), x_3(s))$ be the representation of $\mathscr{X}(x, \xi)$ by the length of the broken ray from x to $x(s)$. Then for $s\notin\{\bar{l}_j; j=0, 1, 2, \cdots\}$

$$\frac{d}{ds}|x(s)'|^2=2\Big(x_1(s)\cdot\frac{d}{ds}x_1(s)+x_2(s)\cdot\frac{d}{ds}x_2(s)\Big)$$

and if $x(s)\in L_j$

$$=2(X_j+(s-\bar{l}_j)\Xi_j)'\cdot\Xi_j\geqslant 2X_j'\cdot\Xi_j\,.$$

And we have

$$X_j'\cdot\Xi_j = X_j'\cdot(\Xi_{j-1}-2(n(X_j)\cdot\Xi_{j-1})n(X_j))$$
$$= X_j'\cdot\Xi_{j-1}+2(-n(X_j)\cdot\Xi_{j-1})(X_j'\cdot n(X_j)) .$$

From (3.2) and (3.3) it follows that

$$X_j'\cdot\Xi_j \geqslant X_j'\cdot\Xi_{j-1}+2c\delta_1 . \tag{3.5}$$

Since $X_1'\cdot\xi \geqslant -|X_1| \geqslant -\max\{|x|\,;\, x\in\Gamma\}(=-A)$, we have

$$X_j'\cdot\Xi_j \geqslant -A+2jc\delta_1 .$$

Then for $j\geqslant j_0=\left[\dfrac{A}{2c\delta_1}\right]+2$, we have $X_j'\cdot\Xi_j\geqslant 2c\delta_1$, which implies

$$\frac{d}{ds}|x(s)'|^2 \geqslant 4c\delta_1 , \qquad s\geqslant \bar{l}_{j_0} .$$

Therefore for $s\geqslant \bar{l}_{j_0}$, integrating the above inequality from $\bar{l}_{j_0}$ to S, we have

$$|x(s)'|^2-|x(\bar{l}_{j_0})'|^2 \geqslant 4(s-\bar{l}_{j_0})c\delta_1 .$$

Then if $x(s)\in\Gamma$ we have $4(s-\bar{l}_{j_0})c\delta_1\leqslant A^2$, then

$$s\leqslant \bar{l}_{j_0}+\frac{A^2}{4c\delta_1} = s_0 ,$$

from which it follows that for any $X_j=x(\bar{l}_{j-1})$

$$\bar{l}_{j-1} = \sum_{h=0}^{j-1} l_h \leqslant s_0 .$$

Since $l_j\geqslant |a_2-a_1|$ for all j, ${}^{\sharp}\mathcal{X}(x, \xi)\leqslant s_0/|a_2-a_1|=K$. Thus the lemma is proved. Q.E.D.

Corollary. *If we choose δ_3 as $\delta_3>\delta_2$ and sufficiently close to δ_2, it holds that for any $x\in S(\delta_3)$, $\xi\in\sum_x^+$ such that $X_1(x, \xi)\in S(\delta_3)-S(\delta_2)$*

$${}^{\sharp}\mathcal{X}(x, \xi)\leqslant K+1 ,$$

where K is the one in Lemma 3.3.

Proof. By the process of the proof of Lemma 3.3 we see that for x, ξ such that $x'\cdot\xi\geqslant 0$ it must holds the estimate

$${}^{\sharp}\mathcal{X}(x, \xi)\leqslant K .$$

Therefore, if we can show that

$$X_1'\cdot\Xi_1\geqslant 0 , \tag{3.6}$$

we have

$$^{\#}\mathcal{X}(x,\,\xi)\leqslant K+1\,.$$

Then we will show (3.6). Suppose that $x\in S_1(\delta_3)$, $X_1(x,\,\xi)\in S_2(\delta_3)-S_2(\delta_2)$ and $x'\cdot\xi<0$. This means that

$$x_1^2+x_2^2\leqslant\delta_3^2 \tag{3.7}$$

$$\delta_2^2\leqslant(x_1+l_0\xi_1)^2+(x_2+l_0\xi_2)^2\leqslant\delta_3^2 \tag{3.8}$$

$$x_1\xi_1+x_2\xi_2<0\,. \tag{3.9}$$

Since

$$(x_1+l_0\xi_1)^2+(x_2+l_0\xi_2)^2=x_1^2+x_2^2+l_0x'\cdot\xi+l_0X_1'\cdot\xi\,,$$

(3.7) and (3.8) imply

$$\delta_2^2\leqslant\delta_3^2+l_0x'\cdot\xi+l_0X_1'\cdot\xi\,.$$

By taking account of (3.9) we have

$$(\delta_2^2-\delta_3^2)/l_0\leqslant X_1'\cdot\xi\,.$$

Since $X_1\notin S_2(\delta_2)$ we can use (3.5) and obtain

$$X_1'\cdot\Xi_1\geqslant(\delta_2^2-\delta_3^2)/l_0+2c\delta_1\,.$$

Then, if we choose δ_3 sufficiently close to δ_2, we have (3.6). Thus our assertion is proved. Q.E.D.

4. The curvatures of the wave front of reflected rays

For $\varphi(x)$ such that $|\nabla\varphi|=1$, we will call the surface $\{y;\,\varphi(y)=\varphi(x)\}$ the wave front of φ passing x and denote it by $\mathcal{C}_\varphi(x)$. Let $\mathcal{U}$ be a neighborhood in $\boldsymbol{R}^3$ of $s_0\in\Gamma$ and φ^+, φ^- be functions defined in $\mathcal{U}$ satisfying $|\nabla\varphi^\pm|=1$. Suppose that

$$\varphi^+(x)=\varphi^-(x)\qquad {}^\forall x\in\Gamma\cap\mathcal{U} \tag{4.1}$$

$$\frac{\partial\varphi^+}{\partial n}>0\,,\quad\frac{\partial\varphi^-}{\partial n}<0\qquad\text{on }\Gamma\cap\mathcal{U}\,. \tag{4.2}$$

We will consider the relation between the principal curvatures of the wave fronts $\mathcal{C}_{\varphi^+}(s_0)$ and $\mathcal{C}_{\varphi^-}(s_0)$. Hereafter the principal curvatures of $\mathcal{C}_\varphi(x)$ signify those with respect to $-\nabla\varphi(x)$. Suppose that

$$n(s_0)=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

and Γ is represented in $\mathcal{U}$ as $\{y(\eta);\ \eta\in U\}$, $U\subset \boldsymbol{R}^2$, $y(0)=s_0$

$$\frac{\partial y}{\partial \eta_1}(0)=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \frac{\partial y}{\partial \eta_2}(0)=\begin{bmatrix}0\\1\\0\end{bmatrix}. \tag{4.3}$$

Let $\{\omega(\sigma);\ \sigma\in\tilde{U}\}$ be a representation of the wave front $\mathscr{C}^-=\mathscr{C}_{\varphi^-}(s_0)$ such that $\omega(0)=s_0$ and

$$\frac{\partial \omega}{\partial \sigma_i}(0)\cdot\frac{\partial \omega}{\partial \sigma_j}(0)=\delta_{ij}, \qquad i,j=1,2 \tag{4.4}$$

$$\frac{\partial i}{\partial \sigma_j}(0)=\kappa_j\frac{\partial \omega}{\partial \sigma_j}(0), \qquad j=1,2 \tag{4.5}$$

where $i(\sigma)=(\nabla\varphi^-)(\omega(\sigma))$ and κ_j, $j=1, 2$ are the principal curvatures of $\mathscr{C}^-$ at s_0. Then $\mathscr{C}^+=\mathscr{C}_{\varphi^+}(s_0)$ is represented as

$$\tau(\sigma)=y(\eta(\sigma))-l(\sigma)r(\sigma)$$

where

$$r(\sigma)=i(\sigma)-2\{i(\sigma)\cdot n(y(\eta(\sigma)))\}n(y(\eta(\sigma))),$$

and $\eta(\sigma)=(\eta_1(\sigma),\ \eta_2(\sigma))\in U$, $l(\sigma)\in\boldsymbol{R}$ are determined by

$$y(\eta)=\omega(\sigma)+l(\sigma)i(\sigma) \qquad \text{for } \sigma\in\tilde{U}. \tag{4.6}$$

Evidently $r(\sigma)=(\nabla\varphi^+)(\tau(\sigma))$. When we denote

$$-i(0)=\begin{bmatrix}\sin\nu\\0\\\cos\nu\end{bmatrix} \tag{4.7}$$

we can write

$$\frac{\partial\omega}{\partial\sigma_1}(0)=\begin{bmatrix}-\cos\theta_0\cdot\cos\nu\\ \sin\theta_0\\ \cos\theta_0\cdot\sin\nu\end{bmatrix},\quad \frac{\partial\omega}{\partial\sigma_2}(0)=\begin{bmatrix}\sin\theta_0\cdot\cos\nu\\ \cos\theta_0\\ -\sin\theta_0\cdot\sin\nu\end{bmatrix} \tag{4.8}$$

for some θ_0. Differentiate (4.6) by σ_j and we have

$$\frac{\partial y}{\partial\eta_1}\frac{\partial\eta_1}{\partial\sigma_j}+\frac{\partial y}{\partial\eta_2}\frac{\partial\eta_2}{\partial\sigma_j}=\frac{\partial\omega}{\partial\sigma_j}+\frac{\partial l}{\partial\sigma_j}i+l\frac{\partial i}{\partial\sigma_j}.$$

Substituting (4.3), (4.5), (4.7), (4.8) and $l(0)=0$ we obtain

$$\frac{\partial l}{\partial\sigma_1}(0)=\tan\nu\cdot\cos\theta_0,\quad \frac{\partial l}{\partial\sigma_2}(0)=-\tan\nu\cdot\sin\theta_0 \tag{4.9}$$

$$(4.10)\qquad \begin{bmatrix} \dfrac{\partial\eta_1}{\partial\sigma_1} & \dfrac{\partial\eta_1}{\partial\sigma_2} \\ \dfrac{\partial\eta_2}{\partial\sigma_1} & \dfrac{\partial\eta_2}{\partial\sigma_2} \end{bmatrix}_{\sigma=0} = T\Theta,$$

where

$$T = \begin{bmatrix} (\cos\nu)^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Theta = \begin{bmatrix} -\cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix}.$$

Using (4.9) we have

$$(4.11)\qquad \begin{bmatrix} \dfrac{\partial\tau}{\partial\sigma_1} & \dfrac{\partial\tau}{\partial\sigma_2} \end{bmatrix}_{\sigma=0} = [Y_1 \quad Y_2]\Theta$$

where

$$Y_1 = \begin{bmatrix} \cos\nu \\ 0 \\ \sin\nu \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Denote by K_1, K_2 the principal curvatures with respect to $-n(s_0)$ of Γ at s_0 and by ψ the angle from $\dfrac{\partial y}{\partial\eta_1}(0)$ to the principal direction corresponding to K_1. Then it holds that

$$\begin{bmatrix} \dfrac{\partial n(y(\eta))}{\partial\eta_1} & \dfrac{\partial n(y(\eta))}{\partial\eta_2} \end{bmatrix}_{\eta=0} = \begin{bmatrix} \dfrac{\partial y}{\partial\eta_1} & \dfrac{\partial y}{\partial\eta_2} \end{bmatrix}_{\eta=0} {}^t\Psi K\Psi$$

where

$$\Psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

Therefore we have

$$(4.12)\qquad \begin{bmatrix} \dfrac{\partial n(y(\eta(\sigma)))}{\partial\sigma_1} & \dfrac{\partial n(y(\eta(\sigma)))}{\partial\sigma_2} \end{bmatrix}_{\sigma=0}$$

$$= \begin{bmatrix} \dfrac{\partial n}{\partial\eta_1} & \dfrac{\partial n}{\partial\eta_2} \end{bmatrix}_{\eta=0} \begin{bmatrix} \dfrac{\partial\eta_i}{\partial\sigma_j} \end{bmatrix}_{\substack{i,j=1,2 \\ \sigma=0}} = \begin{bmatrix} \dfrac{\partial y}{\partial\eta_1} & \dfrac{\partial y}{\partial\eta_2} \end{bmatrix}_{\eta=0} {}^t\Psi K\Psi T\Theta.$$

Note that

$$\frac{\partial r}{\partial\sigma_j} = \frac{\partial i}{\partial\sigma_j} - 2\Big(\frac{\partial i}{\partial\sigma_j}\cdot n\Big)n - 2\Big(i\cdot\frac{\partial n}{\partial\sigma_j}\Big)n - 2(i\cdot n)\frac{\partial n}{\partial\sigma_j}.$$

Since at $\sigma=0$

$$\frac{\partial i}{\partial\sigma_1} - 2\Big(\frac{\partial i}{\partial\sigma_1}\cdot n\Big)n = \kappa_1 \begin{bmatrix} -\cos\theta_0\cos\nu \\ \sin\theta_0 \\ -\cos\theta_0\sin\nu \end{bmatrix} = \kappa_1[Y_1 \quad Y_2]\begin{bmatrix} -\cos\theta_0 \\ \sin\theta_0 \end{bmatrix}$$

$$-2\left(i\cdot\frac{\partial n}{\partial\sigma_1}\right)n-2(i\cdot n)\frac{\partial n}{\partial\sigma_1}=2\begin{bmatrix}\cos\nu\\0\\\sin\nu\end{bmatrix}\left(\frac{\partial n}{\partial\sigma_1}\right)_1+2\cos\nu\begin{bmatrix}0\\1\\0\end{bmatrix}\left(\frac{\partial n}{\partial\sigma_1}\right)_2,$$

where $\left(\frac{\partial n}{\partial\sigma_1}\right)_j$ denotes the j-th component of $\frac{\partial n}{\partial\sigma_1}$, we have

$$\frac{\partial r}{\partial\sigma_1}(0)=[Y_1\quad Y_2](\kappa_1 I+2\cos\nu\ T^t\Psi K\Psi T)\begin{bmatrix}-\cos\theta_0\\ \sin\theta_0\end{bmatrix}.$$

By the same way

$$\frac{\partial r}{\partial\sigma_2}(0)=[Y_1\quad Y_2](\kappa_2 I+2\cos\nu\ T^t\Psi K\Psi T)\begin{bmatrix}\sin\theta_0\\ \cos\theta_0\end{bmatrix}.$$

Namely

$$\left[\frac{\partial r}{\partial\sigma_1}\ \frac{\partial r}{\partial\sigma_2}\right]_{\sigma=0}=[Y_1\quad Y_2](\tilde{K}+2\cos\nu\ T^t\Psi K\Psi T)\Theta$$

where

$$\tilde{K}=\begin{bmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{bmatrix}.$$

Taking account of (4.11) we see that the principal curvatures of $\mathscr{E}^+$ at s_0 are the eigenvalues of the matrix

$$\tilde{K}+2\cos\nu\ T^t\Psi K\Psi T.$$

Thus we have

Lemma 4.1. *The principal curvatures κ_1^+, κ_2^+ of $\mathscr{E}^+$ at s_0 satisfy the inequalities*

$$\min_{j=1,2}\kappa_j^- +2\min_{j=1,2}K_j\leqslant\kappa_j^+\leqslant\max_{j=1,2}\kappa_j^- +\frac{2}{\cos\nu}\max_{j=1,2}K_j \tag{4.13}$$

where κ_j^-, K_j, $j=1,2$ are the principal curvatures of $\mathscr{E}^-$ and Γ at s_0 respectively and $\cos\nu=-n(s_0)\cdot\nabla\varphi^-(s_0)$.

Next we will show

Proposition 4.2. *Let $\mathcal{U}$ be a neighborhood of $x_0\in\Gamma_1$(or Γ_2) and $\varphi\in C^\infty(\mathcal{U})$ and $|\nabla\varphi|=1$ in $\mathcal{U}$. Suppose that*
(i) *for all $x\in\Gamma_1\cap\mathcal{U}$ the principal curvatures of $\mathscr{E}_\varphi(x)$ is positive.*

(ii) $$^{\sharp}\mathscr{X}(x,\nabla\varphi(x))\geqslant p+1,\qquad {}^{\forall}x\in\Gamma_1\cup\mathcal{U}.$$

Then there exist $\varphi_j(x)$, $j=0,1,2,\cdots,p$ such that (*a*) $\varphi_j \in C^\infty(\omega_j)$, $|\nabla\varphi_j| = 1$ *in* ω_j

where $\omega_j = \bigcup_{y\in\Gamma_1\cap\mathcal{U}} L_j(y, \nabla\varphi(y))$.

(*b*) $\varphi_0 = \varphi$ *in* $\omega_0 \cap \mathcal{U}$

(*c*) $\varphi_j(x) = \varphi_{j-1}(x)$, $\dfrac{\partial\varphi_j}{\partial n}(x) = -\dfrac{\partial\varphi_{j-1}}{\partial n}(x)$, $^\forall x \in \bigcup_{y\in\Gamma_1\cap\mathcal{U}} X_j(y, \nabla\varphi(y))$.

Moreover it holds that

(*d*) *the principal curvatures* $\kappa_{jl}(y)$, $l=1, 2$ *of* $\mathcal{G}_{\varphi_j}(X_j(y, \nabla\varphi(y))$ *verify*

$$2\min_{\substack{h=1,2\\ x\in\Gamma}} K_h(x) \leqslant \kappa_{jl}(y) \leqslant 1/d_0 + 2/\delta_1 \max_{\substack{h=1,2\\ x\in\Gamma}} K_h(x) \tag{4.14}$$

where d_0, δ_1 *are the constants in Theorem* 2.2 *and Lemma* 3.1 *respectively.*

Proof. In the proof we write $L_j(y, \nabla\varphi(y))$ and $X_j(y, \nabla\varphi(y))$ as $L_j(y)$ and $X_j(y)$ in brief. Note that $\varphi(x)$ can be extended as C^∞-function verifying $|\nabla\varphi|=1$ in $\{y+l\nabla\varphi(y);\ l\geqslant 0,\ y\in\Gamma_1\cap\mathcal{U}\}$ because (i) holds. Suppose that there exist $\varphi_0, \varphi_1, \cdots, \varphi_j$, $j\leqslant p-1$, verifying (a), (b), (c) and (d). Taking account of (ii) it holds that

$$-n(X_{j+1}(y))\cdot\Xi_j(y) \geqslant \delta_1 ,$$

from which it follows that

$$-\frac{\partial\varphi_j}{\partial n}(X_{j+1}(y)) \geqslant \delta_1 .$$

Then there exists $\psi(x)$ defined in a neighborhood of $\bigcup_y X_{i+1}(y)$ verifying $|\nabla\psi| =1$ and

$$\psi = \varphi_j , \quad -\frac{\partial\psi}{\partial n} = \frac{\partial\varphi_j}{\partial n} \qquad \text{on } \bigcup_y X_{j+1}(y) .$$

Since the principal curvatures of $\mathcal{G}_{\varphi_j}(X_{j+1}(y))$ at $X_{j+1}(y)$ are given by

$$\frac{\kappa_{jl}(y)}{1+l_j(y)\kappa_{jl}(y)} , \qquad l = 1, 2 ,$$

we see from Lemma 4.1 that the principal curvatures $\kappa_{j+1l}(y)$ of $\mathcal{G}_\psi(X_{j+1}(y))$ at $X_{j+1}(y)$ satisfy

$$2\min_{h,y} K_h(y) \leqslant \kappa_{j+1l}(y) \leqslant 1/l_j(y) + 2/\delta_1 \max_{h,y} K_h(y) .$$

Taking account of $l_j(y) \geqslant d_0$ $\kappa_{j+1l}(y)$ satisfies (4.14). Now since $\kappa_{j+1l}(y) > 0$ $\psi(x)$ can be extended to C^∞-function in $\{X_{j+1}(y) + l\nabla\psi(X_{j+1}(y));\ y\in\Gamma_1\cap\mathcal{U},\ l\geqslant 0\}$ verifying $|\nabla\psi|=1$. Setting this extended $\psi(x)$ as φ_{j+1}, we see that $\varphi_{j+1}(x)$ satisfies the required properties. Thus Proposition is proved.

5. Estimates of derivatives of the phase functions

Let the principal curvatures of $\mathscr{C}_0=\mathscr{C}_\varphi(x_0)$ be $\kappa_1\geqslant\kappa_2>0$ and let $\{\omega(\sigma);\ \sigma\in U\}$ be a representation of $\mathscr{C}_\varphi(x_0)$ near x_0 such that

$$(5.1)\qquad \begin{cases} \omega(0)=x_0 \\ \dfrac{\partial\omega}{\partial\sigma_i}(0)\cdot\dfrac{\partial\omega}{\partial\sigma_i}(0)=\delta_{ij} \\ \dfrac{\partial\nu}{\partial\sigma_j}(0)=\kappa_j\dfrac{\partial\omega}{\partial\sigma_j}(0)\,, \qquad j=1,2, \end{cases}$$

where $\nu(\sigma)=(\nabla\varphi)(\omega(\sigma))$. Take $l>0$ and set

$$x_1=x_0+l\nabla\varphi(x_0)\,.$$

Let $\{\tilde{\omega}(\eta);\ \eta\in\tilde{U}\}$ be a representation of $\mathscr{C}_1=\mathscr{C}_\varphi(x_1)$ near x_1 satisfying

$$(5.2)\qquad \begin{cases} \tilde{\omega}(0)=x_1 \\ \dfrac{\partial\tilde{\omega}}{\partial\eta_j}(0)=\dfrac{\partial\omega}{\partial\sigma_j}(0)\,, \qquad j=1,2\,. \end{cases}$$

Note that we can choose $\omega(\sigma)$ and $\tilde{\omega}(\eta)$ satisfying (5.1) and (5.2) in such a way

$$(5.3)\qquad \begin{aligned} &\sum_{|\gamma|\leqslant m}|(D_\sigma^\gamma\omega)(0)|\leqslant C_m\sum_{|\gamma|\leqslant m}|(D_x^\gamma\varphi)(x_0)| \\ &\sum_{|\gamma|\leqslant m}|(D_\eta^\gamma\tilde{\omega})(0)|\leqslant C_m\sum_{|\gamma|\leqslant m}|(D_x^\gamma\varphi)(x_1)| \end{aligned}$$

where C_m is a constant independent of x_0, x_1 and l.

On the other hand, since $\varphi(x+l\nabla\varphi(x))=\varphi(x)+l$ we have another representation of $\mathscr{C}_1$

$$\mathscr{C}_1=\{\omega(\sigma)+l\nu(\sigma);\ \sigma\in U\}\,.$$

Therefore for η sufficiently near 0 there corresponds $\sigma\in U$ uniquely by the relation

$$(5.4)\qquad \tilde{\omega}(\eta)=\omega(\sigma)+l\nu(\sigma)\,.$$

Denote this correspondance as $\sigma(\eta)=(\sigma_1(\eta_1,\eta_2),\ \sigma_2(\eta_1,\eta_2))$. Now from (5.1) and (5.2) we have

$$(5.5)\qquad \frac{\partial\sigma_i}{\partial\eta_j}(0)=(1+l\kappa_j)^{-1}\delta_{ij}\,, \qquad i,j=1,2\,.$$

Let us set

$$S^2=\{(\xi_1,\xi_2,\xi_3);\ \xi_1^2+\xi_2^2+\xi_3^2\leqslant 1\}\,.$$

For $a=(a_1, a_2, a_3)\in S^2$ we set

$$X_a = a_1\frac{\partial}{\partial x_1}+a_2\frac{\partial}{\partial x_2}+a_3\frac{\partial}{\partial x_3}.$$

DEFINITION 5.1. We define $|\nabla\varphi|_m(x)$ for $m=1, 2, \cdots$ by

$$|\nabla\varphi|_m(x) = \max_{j\leqslant m}\max_{a(r)\in S^2}|(X_{a(1)}X_{a(2)}\cdots X_{a(j)}\nabla\varphi)(x)|.$$

We may assume without generality that

$$\frac{\partial\omega}{\partial\sigma_1}(0) = (1, 0, 0), \quad \frac{\partial\omega}{\partial\sigma_2}(0) = (0, 1, 0), \quad \nu(0) = (0, 0, 1).$$

The correspondance from (σ, τ) near $(0, 0)$ to x near x_0 given by

$$x = \omega(\sigma)+\tau\nu(\sigma)$$

verifies a relation

$$\frac{\partial\sigma_j}{\partial x_k}(x_0) = \delta_{jk}, \qquad j, k = 1, 2, 3, \tag{5.6}$$

where we denote τ as σ_3. Then we have

$$(X_a\nabla\varphi)(x_0) = \sum_{j=1}^{2} a_j\frac{\partial\nu}{\partial\sigma_j}(0)$$

because $\dfrac{\partial\nu(\sigma)}{\partial\tau}=0$. By the same way we have

$$(X_a\nabla\varphi)(x_1) = \sum_{j=1}^{2} a_j\frac{\partial\mu}{\partial\eta_j}(0) \tag{5.7}$$

where $\mu(\eta)=(\nabla\varphi)(\tilde{\omega}(\eta))$. Since $\mu(\eta)=\nu(\sigma(\eta))$ it follows from (5.5) and (5.7) that

$$(X_a)\nabla\varphi(x_1) = \sum_{j=1}^{2} a_j(1+l\kappa_j)^{-1}\frac{\partial\nu}{\partial\sigma_j}(0).$$

Taking account of (5.1) we have

$$|(X_a\nabla\varphi)(x_0)| = (\sum_{j=1}^{2}|a_j\kappa_j|^2)^{1/2}$$

$$|(X_a\nabla\varphi)(x_1)| = (\sum_{j=1}^{2}|a_j\kappa_j(1+l\kappa_j)^{-1}|^2)^{1/2}.$$

Thus we have

Lemma 5.1. *For $l>0$ it holds that*

$$|\nabla\varphi|_1(x_0+l\nabla\varphi(x_0))\leqslant\frac{1}{1+l\kappa_2}|\nabla\varphi|_1(x_0). \tag{5.8}$$

Differentiating (5.4) by η_k and η_h we have

$$\sum_{j=1}^{2}\Big(\frac{\partial\omega}{\partial\sigma_j}+l\frac{\partial\nu}{\partial\sigma_j}\Big)\frac{\partial^2\sigma_j}{\partial\eta_k\partial\eta_h}+\sum_{i,j=1}^{2}\Big(\frac{\partial^2\omega}{\partial\sigma_i\partial\sigma_j}+l\frac{\partial^2\nu}{\partial\sigma_i\partial\sigma_j}\Big)\frac{\partial\sigma_i}{\partial\eta_k}\frac{\partial\sigma_j}{\partial\eta_h}$$
$$=\frac{\partial^2\tilde{\omega}}{\partial\eta_k\partial\eta_k}.$$

Substitute $\eta=0$ and take the scalar product with $\dfrac{\partial\omega}{\partial\sigma_j}(0)$. Then it follows that

$$(1+l\kappa_j)\frac{\partial^2\sigma_j}{\partial\eta_k\partial\eta_h}+l\frac{\partial\omega}{\partial\sigma_j}\frac{\partial^2\nu}{\partial\sigma_k\partial\sigma_h}(1+l\kappa_k)^{-1}(1+l\kappa_h)^{-1}$$
$$+\frac{\partial\omega}{\partial\sigma_j}\frac{\partial^2\omega}{\partial\sigma_h\partial\sigma_k}(1+l\kappa_k)^{-1}(1+l\kappa_h)^{-1}=\frac{\partial\omega}{\partial\sigma_j}\cdot\frac{\partial^2\tilde{\omega}}{\partial\eta_k\partial\eta_h}.$$

Taking account of (5.3) we have

$$(5.9)\quad\begin{cases}\dfrac{\partial^2\sigma_j}{\partial\eta_k\partial\eta_h}(0)=-(1+l\kappa_j)^{-1}(1+l\kappa_k)^{-1}(1+l\kappa_h)^{-1}l\dfrac{\partial\omega}{\partial\sigma_j}(0)\dfrac{\partial^2\nu}{\partial\sigma_k\partial\sigma_h}(0)+I_{j,k,h}\\ |I_{j,k,h}|\leqslant C_2\{|\nabla\varphi|_1(x_0)+|\nabla\varphi|_1(x_1)\}\end{cases}$$

where C_1 is independent of x_2, x_0 and φ. Using (5.5) we have

$$\frac{\partial^2(\nabla\varphi)(\tilde{\omega}(\eta))}{\partial\eta_k\partial\eta_k}\bigg|_{\eta=0}=\frac{\partial^2\nu}{\partial\sigma_k\partial\sigma_h}(0)(1+l\kappa_k)^{-1}(1+l\kappa_h)^{-1}+\sum_{j=1}^{2}\frac{\partial\nu}{\partial\sigma_j}\frac{\partial^2\sigma_j}{\partial\eta_k\partial\eta_h}$$

Take the scalar product with $\dfrac{\partial\omega}{\partial\sigma_j}(0)$ and substitute (5.9). Then we have

$$(5.10)\qquad\frac{\partial\omega}{\partial\sigma_j}(0)\cdot\frac{\partial^2(\nabla\varphi)(\tilde{\omega}(\eta))}{\partial\eta_k\partial\eta_h}\bigg|_{\eta=0}$$
$$=\frac{\partial\omega}{\partial\sigma_j}(0)\cdot\frac{\partial^2\nu}{\partial\sigma_k\partial\sigma_h}(0)(1+l\kappa_k)^{-1}(1+l\kappa_h)^{-1}(1+l\kappa_j)^{-1}+\kappa_jI_{j,h,k}.$$

Since $\dfrac{\partial\nu(\sigma)}{\partial\sigma_j}\cdot\nu(\sigma)=0$ we have

$$\frac{\partial^2\nu}{\partial\sigma_j\partial\sigma_k}\cdot\nu=\left|\frac{\partial\nu}{\partial\sigma_j}\frac{\partial\nu}{\partial\sigma_k}\right|\leqslant C\,|\nabla\varphi|_1^2(x_0).$$

For $a=(a_1, a_2, a_3)$, $b=(b_1, b_2, b_3)\in S^2$

$$(5.11)\quad(X_bX_a\nabla\varphi)(x)=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k,h=1}^{3}b_ja_i\Big(\frac{\partial^2\nu}{\partial\sigma_k\partial\sigma_h}\frac{\partial\sigma_k}{\partial x_j}\frac{\partial\sigma_h}{\partial x_i}+\frac{\partial\nu}{\partial\sigma_k}\frac{\partial^2\sigma_k}{\partial x_j\partial x_i}\Big).$$

Then using $\dfrac{\partial\nu}{\partial\sigma_3}=0$ we have

$$(5.12)\qquad \begin{cases}(X_aX_b\nabla\varphi)(x_0)=\sum_{i,j=1}^{2}a_ib_j\dfrac{\partial^2\nu}{\partial\sigma_i\partial\sigma_j}(0)+II_0\\ |II_0|\leqslant C\,|\nabla\varphi|_1(x_0)\,.\end{cases}$$

By the same method we have

$$(X_aX_b\nabla\varphi)(x_1)=\sum_{i,j=1}^{2}a_ib_j\frac{\partial^2(\nabla\varphi)(\tilde\omega(\eta))}{\partial\eta_i\partial\eta_j}\bigg|_{\eta=0}+II_1$$

$$|II_1|\leqslant C\nabla|\varphi|_1(x_1)\,.$$

$$(5.13)\qquad \left|\sum_{k,h=1}^{2}a_kb_h\frac{\partial^2\mu}{\partial\eta_k\partial\eta_h}(0)\right|^2$$

$$=\sum_{j=1}^{2}\left|\sum_{k,h=1}^{2}a_kb_h\left(\frac{\partial\omega}{\partial\sigma_j}(0)\cdot\frac{\partial^2\mu}{\partial\eta_k\partial\eta_h}(0)\right)\right|^2+II_1$$

using (5.10)

$$=\sum_{j=1}^{2}(1+l\kappa_j)^{-2}\left|\sum_{k,h=1}^{2}\frac{a_kb_h}{(1+l\kappa_k)(1+l\kappa_h)}\left(\frac{\partial\omega}{\partial\sigma_j}\cdot\frac{\partial^2\nu}{\partial\sigma_k\partial\sigma_h}(0)\right)\right|^2+\tilde{II}_1$$

$$\leqslant(1+l\kappa_2)^{-2}|X_{\tilde a}X_{\tilde b}\nabla\varphi(x_0)|^2+\tilde{II}_0$$

where

$$|\tilde{II}_j|\leqslant C\,|\nabla\varphi|_1(x_j)\,,\qquad j=0\,,1$$

and

$$\tilde a=\left(\frac{a_1}{1+l\kappa_1},\ \frac{a_2}{1+l\kappa_2},\ 0\right),\quad \tilde b=\left(\frac{b_1}{1+l\kappa_1},\ \frac{b_2}{1+l\kappa_2},\ 0\right).$$

Combining (5.11), (5.12) and (5.13) it holds that

$$\max_{a,b\in S^2}|(X_aX_b\nabla\varphi)(x_1)|$$

$$\leqslant\max_{a,b\in S^2}(1+l\kappa_2)^{-1}|(X_{\tilde a}X_{\tilde b}\nabla\varphi)(x_0)|+C\sum_{j=0}^{1}|\nabla\varphi|_1(x_j)$$

$$\leqslant(1+l\kappa_2)^{-3}\max_{a,b\in S^2}|(X_aX_b\nabla\varphi)(x_0)|+C\sum_{j=0}^{2}|\nabla\varphi|_1(x_j)\,.$$

Thus, with the aid of (5.8), we have

Lemma 5.2. *For $l>0$ it holds that*

$$(5.14)\qquad |\varphi|_2(x_0+l\nabla\varphi(x_0))\leqslant(1+l\kappa_2)^{-3}|\nabla\varphi|_2(x_0)+C_2|\nabla\varphi|_1(x_0)\,,$$

where C_2 is a constant independent of x_0 and φ.

REMARK. By the same reasonning we can show for $m=3,\,4,\,\cdots$

$$(5.15)\qquad |\nabla\varphi|_m(x_0+l\nabla\varphi(x_0))\leqslant(1+l\kappa_2)^{-(2m-1)}|\nabla\varphi|_m(x_0)+C_m|\nabla\varphi|_{m-2}(x_0)\,.$$

Next consider the estimate of reflected rays. Let $\varphi^{\pm}$ satisfy (4.1) and (4.2), and $\omega(\sigma)$ and $\tau(\sigma)$ be the same ones in §4. We also suppose that $\omega(\sigma)$ verifies (5.3). Since $|(D_\sigma^\gamma i)(0)|\leqslant C_\gamma|\varphi^-|_{|\gamma|}(s_0)$ we have from (4.6)

$$|(D_\sigma^\gamma l)(0)|\leqslant C_\gamma|\varphi^-|_{|\gamma|-1}(s_0)\,.$$

Using this estimate we have

$$|(D_\sigma^\gamma r)(0)-(D_\sigma^\gamma i)(0)+2((D_\sigma^\gamma i)(0)\cdot n(s_0))n(s_0)|\leqslant C_\gamma|\nabla\varphi^-|_{|\gamma|-1}(s_0)\,.$$

Evidently

$$\left|(X_a\nabla\varphi^-)(s_0)-\sum_{j=1}^{2}a_j\frac{\partial i}{\partial\sigma_j}(0)\right|\leqslant C\,|\nabla\varphi^-|_0(s_0)$$

$$\left|(X_aX_b\nabla\varphi^-)(s_0)-\sum_{k,j=1}^{2}a_jb_k\frac{\partial^2 i}{\partial\sigma_j\partial\sigma_k}(0)\right|\leqslant C\,|\nabla\varphi^-|_1(s_0).$$

Concerning derivatives of φ^+, denoting

$$X_a=\xi_1\frac{\partial}{\partial y_1}+\xi_2\frac{\partial}{\partial y_2}+\xi_3\frac{\partial}{\partial y_3}$$

where $\xi_1\dfrac{\partial\tau}{\partial\sigma_1}(0)+\xi_2\dfrac{\partial\tau}{\partial\sigma_2}(0)+\xi_3 r(0)=a$, we have

$$\left|(X_a\nabla\varphi^+)(s_0)-\sum_{j=1}^{2}\xi_j\frac{\partial r}{\partial\sigma_j}(0)\right|\leqslant C\,|\nabla\varphi^+|_0(s_0)$$

$$\left|(X_aX_b\nabla\varphi^+)(s_0)-\sum_{k,j=1}^{2}\xi_j\gamma_k\frac{\partial^2 r}{\partial\sigma_j\partial\sigma_k}(0)\right|\leqslant C\,|\nabla\varphi^+|_1(s_0)\,.$$

Taking account of $|a|=|\xi|$ and that fact

$$|D_\sigma^\gamma i(0)|=|D_\sigma^\gamma i(0)-2(D_\sigma^\gamma i(0)\cdot n(s_0))n(s_0)|$$

we have immediately

Lemma 5.3.

$$|\nabla\varphi^+|_1(s_0)\leqslant|\nabla\varphi^-|_1(s_0)+C_1|\nabla\varphi^+|_0(s_0) \tag{5.16}$$
$$|\nabla\varphi^+|_2(s_0)\leqslant|\nabla\varphi^-|_2(s_0)+C_2|\nabla\varphi^-|_1(s_0)$$

REMARK. By the same reasonning we can show for $m=2,\ 3$,

$$|\nabla\varphi^+|_m(s_0)\leqslant|\nabla\varphi^-|_m(s_0)+C_m|\nabla\varphi^-|_{m-1}(s_0)\,. \tag{5.17}$$

Using the above lemmas let us show

Proposition 5.4. *Suppose that $\varphi(x)$ satisfies the assumptions of Proposition 4.2. Let φ_j, $j=0,\ 1,\ 2,\ \cdots,p$ be functions verifying (a)$\sim$(d) of Proposition 4.2. Then an estimate*

(5.18) $$|\nabla\varphi_j|_m(x)\leqslant C_m|\nabla\varphi|_m(y)$$

holds for $x\in L_j(y, \nabla\varphi(y))$, $j=1, 2, \cdots, p$, *where* C_m *is a constant independent of* φ *and* p.

Proof. Note that (5.18) holds for $m=0$ because $|\nabla\varphi_j|=1$, ${}^{\forall}j$. Suppose that (5.18) holds for $m=1, 2, \cdots, h$. By Proposition 4.2 the principal curvatures $\kappa_{jl}(y)$ of $\mathscr{G}_{\varphi_j}(X_j(y))$ at $X_j(y)$ satisfy

$$\kappa_{jl}(y)\geqslant K_0=2\min_{\substack{j=1,2\\ x\in\Gamma}} K_j(x)\,.$$

Then by Lemma 5.2 and its remark we see that for $j=1, 2, \cdots, p$

$$|\nabla\varphi_j|_{h+1}(X_{j+1}(y))\leqslant(1+d_0K_0)^{-2h-1}|\nabla\varphi_j|_{h+1}(X_j(y))+C_h|\nabla\varphi_j|_h(X_j(y))\,,$$

and

$$|\nabla\varphi_0|_{h+1}(X_1(y))\leqslant|\varphi|_{h+1}(y)\,.$$

On the other hand Lemma 5.3 and its remark assure that

$$|\nabla\varphi_{j+1}|_{h+1}(X_{j+1}(y))\leqslant|\nabla\varphi_j|_{h+1}(X_{j+1}(y))+C_h|\nabla\varphi_j|_h(X_{j+1}(y))$$

holds for $j=1, 2, \cdots, p$. Using the assumption on the estimate of $|\varphi_j|_h$ we have an estimate

$$\begin{aligned}|\nabla\varphi_{j+1}|_{h+1}(X_{j+1}(y))&\leqslant(1+lK_0)^{-j(2h+1)}|\nabla\varphi|_{h+1}(y)\\&\quad+2C_h(1+(1+lK_0)^{-(2m+1)}+\cdots+(1+lK)_0^{-j(2m+1)})|\nabla\varphi|_h(y)\,.\end{aligned}$$

Thus our assuration is proved.

6. On the transport equation

First we will arrange some properties of asymptotic solution for $\Box w=0$ in the free space.

Let $\varphi\in C^\infty(D)$, $D\subset\boldsymbol{R}^3$ such that $|\nabla\varphi|=1$. Let us set

(6.1) $$w(x, t; k)=e^{ik(\varphi(x)-t)}v(x, t; k)$$

(6.2) $$v(x, t; k)=\sum_{j=0}^{\infty}v_j(x, t; k)k^{-j}\,.$$

Apply $\Box$ to w of (6.1) and we have

(6.3) $$\begin{aligned}\Box w=-e^{ik(\varphi-t)}\Big\{&(ik)^2((\nabla\varphi)^2-1)v\\&+ik\Big(2\frac{\partial v}{\partial t}+2\nabla\varphi\cdot\nabla v+\Delta\varphi\cdot v\Big)-\Box v\Big\}\,.\end{aligned}$$

Substituting (6.2) into the right hand side of (6.3) and setting all the coefficients of k^{-j} equal to zero we obtain so called transport equations, that is,

$$(6.4)_0 \qquad 2\frac{\partial v_0}{\partial t}+2\nabla\varphi\cdot\nabla v_0+\Delta\varphi\cdot v_0=0\,,$$

$$(6.4)_j \qquad 2\frac{\partial v_j}{\partial t}+2\nabla\varphi\cdot\nabla v_j+\Delta\varphi\cdot v_j+i\Box v_{j-1}=0\,, \quad \text{for } j\geqslant 1\,.$$

In an attempt to obtain successively the estimate of v_j, we consider the solution of the equation

$$(6.5) \qquad 2\frac{\partial u}{\partial t}+2\nabla\varphi\cdot\nabla u+\Delta\varphi\cdot u=h(x,t)\,.$$

When $\{x+l\nabla\varphi(x);\ l\in[0,\ l_0]\}\subset D$ the solution of (6.5) is represented as

$$(6.6) \qquad u(x+l\xi(x),\ t+l)=\left[\frac{G(x,l)}{G(x,0)}\right]^{1/2}u(x,t)+\int_0^l\left[\frac{G(x,l)}{G(x,s)}\right]^{1/2}h(x+s\xi(x),\ t+s)ds$$

where $\xi(x)=\nabla\varphi(x)$ and $G(x,l)$ denotes the Gaussian curvature of $\mathscr{C}_\varphi(x+l\xi(x))$ at $x+l\xi(x)$. This formula is due to Luneberg [7], Keller, Lewis and Seckler [4]. Let us denote the principal curvature of $\mathscr{C}_\varphi(x)$ by $\kappa_1(x)$ and $\kappa_2(x)$. Then those of $\mathscr{C}_\varphi(x+l\xi(x))$ are $\kappa_j(1+l\kappa_j)^{-1}$, $j=1,2$. Thus we have

$$(6.7) \qquad \frac{G(x,l)}{G(x,s)}=\frac{(1+s\kappa_1)(1+s\kappa_2)}{(1+l\kappa_1)(1+l\kappa_2)}=\frac{1+2sH(x)+s^2G(x)}{1+2lH(x)+l^2G(x)}\,,$$

where $H(x)$ and $G(x)$ denote the mean and Gaussian curvature of $\mathscr{C}_\varphi(x)$ at x.

In the same way as §5, we introduce the following norms. Set $S^3=\{(\xi_1,\xi_2,\xi_3,\xi_4);\ \sum_{j=1}^4\xi_j^2\leqslant 1\}$ and for $a=(a_1,a_2,a_3,a_4)\in S^3$

$$X_a=\sum_{j=1}^3 a_j\frac{\partial}{\partial x_j}+a_4\frac{\partial}{\partial t}\,.$$

DEFINITION. For $w\in C^\infty(\mathcal{U}\times\boldsymbol{R}^1)$, $\mathcal{U}\subset\boldsymbol{R}^3$ we set

$$|w|_m(x,t)=\max_{p\leqslant m}\max_{a^{(j)}\in S^3}|(X_{a^{(1)}}X_{a^{(2)}}\cdots X_{a^{(p)}}w)(x,t)|\,.$$

When $x\in\mathcal{U}\cap\Gamma$ we also define another norm by

$$|w|'_m(x,t)=\max_{p\leqslant m}\max_{a^{(j)}\in S^{3\prime}}|(X_{a^{(2)}}X_{a^{(2)}}\cdots X_{a^{(p)}}w)(x,t)|\,,$$

where $S^{3'}=S^3\cap T_{(x,t)}(\Gamma\times\boldsymbol{R})$, where $T_{(x,t)}(\Gamma\times\boldsymbol{R})$ denotes the tangent space of $\Gamma\times\boldsymbol{R}$ at (x, t). And we set also for a suset $\omega\subset\mathcal{U}\times\boldsymbol{R}$

$$|w|_m(\omega)=\max_{(x,t)\in\omega}|w|_m(x, t)$$

and for $\omega'\subset\Gamma\times\boldsymbol{R}$

$$|w|'_m(\omega')=\max_{(x,t)\in\omega'}|w|'_m(x, t)\,.$$

Lemma 6.1. *Let m be a non negative integer. For the solution* (6.6) *it holds that an estimate*

$$\begin{aligned}(6.8)\qquad &|w|_m(x+l\xi(x), t+l)\\ &\leqslant(1+l\kappa_2(x))^{-1}|w|_m(x, t)+\int_0^l|h|_m(x+s\xi(x), t+s)ds\\ &+C_m|\nabla\varphi|_m(x)\Big\{|w|_{m-1}(x, t)+|w|_{m-1}(x+l\xi(x), t+l)\\ &+|h|_{m-1}(x+l\xi(x), t+l)+\int_0^l|h|_{m-1}(x+s\xi(x), t+s)ds\Big\},\end{aligned}$$

where C_m is a positive constant independent of w and φ. Especially, for $h=0$ and $m=0$ we have

$$(6.9)\qquad |w(x+l\xi(x), t+l)|\leqslant(1+l\kappa_2(x))^{-1}|w(x)|\,.$$

Proof. Note that (6.9) follows immediately from (6.6) and (6.7). Let $\{\omega(\sigma);\ \sigma\in U\}$ be a representation of $\mathscr{C}_\varphi(x_0)$ which verifies $\omega(0)=x_0$ and the condition (4.4), (4.5) and (5.3), where $i(\sigma)=(\nabla\varphi)(\omega(\sigma))$. Suppose that $\dfrac{\partial\omega}{\partial\sigma_j}(0)$, $j=1, 2$ and $\xi(x_0)$ coincide with x_1, x_2 and x_3 axis respectively. Consider a mapping $\boldsymbol{R}^4\ni(\sigma_1, \sigma_2, l, \tau)\to(x_1, x_2, x_3, t)\in\boldsymbol{R}^4$ defined by

$$\begin{cases}x=x(\sigma, l)=\omega(\sigma)+li(\sigma)\\ t=\tau+l\,.\end{cases}$$

Then the Jacobian matrix of the mapping at $\sigma=0$ is

$$(6.10)\qquad \begin{pmatrix}1+l\kappa_1 & 0 & 0 & 0\\ 0 & 1+l\kappa_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}$$

Moreover, the inverse mapping satisfies

$$(6.11)\qquad |D^\gamma_{x,t}\sigma_j|\leqslant C_\gamma|\nabla\varphi|_{|\gamma|}\,,\qquad j=1, 2, 3, 4$$

where we denote $l=\sigma_3$, $\tau=\sigma_4$. Let us set for some $l>0$

$$x_1 = x_0 + l\xi(x_0), \qquad t_1 = t + l.$$

First we show (6.8) for $x = x_0$ in the case $h \equiv 0$. Suppose that for $a^{(j)} = (a_i^{(j)})_{i=1,2,3,4} \in S^3$, $j = 1, 2, \cdots, p \leqslant m$ it holds that

$$|w|_m(x_1, t_1) = |(X_{a^{(1)}} X_{a^{(2)}} \cdots X_{a^{(p)}} w)(x_1, t_1)|. \tag{6.12}$$

On the other hand, taking account of (6.10) and (6.11) we have

$$\begin{aligned} &\Big|((\prod_{j=1}^{p} X_{a^{(j)}}) w)(x_1, t_1) - \Big[\Big(\prod_{j=1}^{p}\Big(\sum_{i=1}^{p} a_i^{(j)} \frac{1}{1 + l\kappa_i(x_0)} \frac{\partial}{\partial \sigma_i} \\ &\quad + a_3^{(j)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial \tau}\Big) + a_4^{(j)} \frac{\partial}{\partial \tau}\Big)\Big) w(\omega(\sigma) + li(\sigma), \tau + l)\Big]_{\substack{\tau = t \\ \sigma = 0}}\Big| \\ &\leqslant C_p |\nabla\varphi|_p(x_1) |w|_{p-1}(x_1, t_1). \end{aligned} \tag{6.13}$$

Note that it follows from (6.7) that for $0 \leqslant s \leqslant l$.

$$\Big| D_{\sigma,s}^{\gamma} \frac{G(\omega(\sigma), l)}{G(\omega(\sigma), s)} \Big| \leqslant C_\gamma |\nabla\varphi|_{|\gamma|}(\omega(\sigma)).$$

Then using (6.6)

$$\begin{aligned} &\Big|[Y_1 Y_2 \cdots Y_p w(\omega(\sigma) + li(\sigma), \tau + l)]_{\substack{\tau = t \\ \sigma = 0}} \\ &\quad - \Big(\frac{G(x_0, l)}{G(x_0, 0)}\Big)^{1/2} \Big[\prod_{j=1}^{p}\Big(\sum_{k=1}^{2} a_k^{(j)} \frac{1}{1 + l\kappa_k(x_0)} \frac{\partial}{\partial \sigma_k} + (a_4^{(j)} - a_3^{(j)}) \frac{\partial}{\partial \tau}\Big)\Big) w(\omega(\sigma), \tau)\Big]_{\substack{\sigma = 0 \\ \tau = t}}\Big| \\ &\leqslant C_p |\nabla\varphi|_p |w|_{p-1}(x_j, t), \end{aligned} \tag{6.14}$$

where

$$Y_j = \sum_{k=1}^{2} a_k^{(j)} \frac{1}{1 + l\kappa_k(x_0)} \frac{\partial}{\partial \sigma_k} + a_3^{(j)}\Big(\frac{\partial}{\partial l} - \frac{\partial}{\partial \tau}\Big) + a_4^{(j)} \frac{\partial}{\partial \tau}.$$

By the same consideration we have

$$\begin{aligned} &\Big|((\prod_{j=1}^{p} X_{b(j)}) w(x_0, t) - \Big[\Big(\prod_{j=1}^{p}\Big(\sum_{k=1}^{2} b_k^{(j)} \frac{\partial}{\partial \sigma_k} + (b_4^{(j)} - b_3^{(j)}) \frac{\partial}{\partial \tau}\Big)\Big) w(\omega(\sigma), \tau)\Big]_{\substack{\tau = 0 \\ \sigma = t}}\Big| \\ &\leqslant C_p |\nabla\varphi|_{p-1}(x_0) |w|_{p-1}(x_0, t). \end{aligned}$$

Therefore, setting

$$b^{(j)} = \Big(a_1^{(j)} \frac{1}{1 + l\kappa_1(x_0)}, \; a_2^{(j)} \frac{1}{1 + l\kappa_2(x_0)}, \; a_3^{(j)}, \; a_4^{(j)}\Big)$$

we have with the aid of (6.13) and (6.14)

$$\begin{aligned} &\Big|((\sum_{j=1}^{p} X_{a^{(j)}}) w)(x_1, t_1) - \Big[\frac{G(x_0, l)}{G(x_0, 0)}\Big]^{1/2} (\prod_{j=1}^{p} X_{b^{(j)}}) w)(x_0, t)\Big| \\ &\leqslant C_p |\nabla\varphi|_p(x_0) \{|w|_{p-1}(x_0, t) + |w|_{p-1}(x_1, t_1)\}. \end{aligned}$$

From (6.12) we have

$$|w|_m(x_1, t_1) \leqslant \left[\frac{G(x_0, l)}{G(x_0, 0)}\right]^{1/2} |((\prod_{j=1}^{p} X_{b(j)})w)(x_0, t)$$
$$+C_p|\nabla\varphi|_p(x_0)\{|w|_{p-1}(x_0, t)+|w|_{p-1}(x_1, t_1)\} .$$

Since $\left(\frac{G(x_0, l)}{G(x_0, 0)}\right)_{1/2} \leqslant \frac{1}{1+l\kappa_2(x_0)}$ and $|w|_m(x_0, t) \geqslant |((\prod_{j=1}^{p} X_{b(j)})w)(x_0, t)|$ we haye (6.8).

When $h \not\equiv 0$, setting for each $0 \leqslant s \leqslant l$

$$h_s(x+l\xi(x), t+l) = \left[\frac{G(x, l)}{G(x, s)}\right]^{1/2} h(x+s\xi(x), t+s)$$

we have from the above result for all $0 \leqslant s \leqslant l$

$$|h_s|_m(x_1, t_1) \leqslant \left(\frac{G(x_0, l)}{G(x_0, s)}\right)^{1/2} |h_s|_m(x_0+s\xi(x_0), t+s)$$
$$+C_m|\nabla\varphi|_{m-1}(x_0)\{|h_s|_{m-1}(x_1, t_1)+|h_s|_{m-1}(x_0+s\xi(x_0), t+s)\} .$$

Using this estimate we have immediately (6.8) from (6.6).

Lemma 6.2. *Let φ^+, φ^- be functions verifying $|\varphi^\pm|=1$, (4.1) and (4.2). Suppose that h^+, $h^- \in C^\infty(\mathcal{U}\times \boldsymbol{R})$ and w^+, I^- satisfy*

$$2\frac{\partial w^\pm}{\partial t}+2\nabla\varphi^\pm\cdot\nabla w^\pm+\Delta\varphi^\pm\cdot w^\pm = h^\pm \qquad in\ \mathcal{U}\times\boldsymbol{R} \tag{6.15}$$

$$w^+ = aw^- + f(x, t) \qquad on\ (\Gamma\cap\mathcal{U})\times\boldsymbol{R}\,, \tag{6.16}$$

where $|a|=1$. Then it holds that for all $(x, t)\in(\Gamma\cap\mathcal{U})\times\boldsymbol{R}$

$$|w^+|_m(x, t) \leqslant |w^-|_m(x, t)+|f|'_m(x, t)+C_m|\nabla\varphi^-|_m(x)\{|w^+|_{m-1}(x, t)$$
$$+|w^-|_{m-1}(x, t)+|h^+|(x, t)_{m-1}+|h^-|_{m-1}(x, t)\}$$

where C_m is a constant independent of $w^\pm$, $h^\pm$ and $\varphi_\pm$.

Since we can prove this lemma as almost the same way as Lemma 5.3, we omit the proof.

7. Construction of asymptotic solutions

Let $s_0\in\Gamma_1$ and $\mathcal{U}$ be a small neighborhood of s_0 in $\boldsymbol{R}^3$. Suppose that $\theta(x, \eta, \beta)$, $\rho(x, \eta, \beta)$ are defined for $x\in\mathcal{U}$, $\eta\in\sum=\{(\eta_1, \eta_2);\ \eta_1^2+\eta_2^2=1,\ \beta\in[-\beta_0, \beta_0]\}$ $(\beta_0>0)$ and they satisfy

$$\begin{cases}(\nabla\theta)^2+\rho(\nabla\rho)^2\equiv 1 & (\mathrm{mod}\ \beta^\infty)\ \text{in}\ \Omega^{(1)}\cap\mathcal{U}\\ \nabla\theta\cdot\nabla\rho\equiv 0 & (\mathrm{mod}\ \beta^\infty)\ \text{in}\ \Omega^{(1)}\cap\mathcal{U}\\ \rho=-\beta & \text{on}\ \Gamma_1\cap\mathcal{U}\end{cases} \tag{7.1}$$

$$(7.2)\qquad (\nabla\theta)(s_0, \eta, 0) = \eta_1\frac{\partial}{\partial\sigma_1}+\eta_2\frac{\partial}{\partial\sigma_2},$$

where $\Omega^{(j)}=\boldsymbol{R}^3-\mathcal{O}_j$, $j=1, 2$ and $\{y(\sigma);\ \sigma\in U\}$ be a representation of Γ_1 near s_0.

Lemma 7.1. *Let u be an oscillatory data given by*

$$(7.3)\qquad \begin{aligned} &u(x, t;\ \eta, \beta, k) = e^{ik(\theta(x,\eta,\beta)-t)}f(x, t;\ k)\\ &f(x, t;\ k)\in C_0^\infty((\Gamma_1\cap\mathcal{U})\times\boldsymbol{R}).\end{aligned}$$

For any positive integer N, there exists an asymptotic solution $w^{(N)}(x, t;\ \eta, \beta, k)$ with the following properties:

$$(7.4)\qquad \operatorname{supp} w^{(N)}\subset \bigcup_{(x,t)\in\operatorname{supp} f}\{x+l\nabla(\theta+2/3\,\rho^{3/2})(x),\ t+l);\ l\geqslant 0\}$$

$$(7.5)\qquad |w^{(N)}|_m(\Omega_R^{(1)}\times\boldsymbol{R})\leqslant C_{N,m,R}k^m\Big(\sum_{j=1}^{N}|f|'_{20j+m}(\Gamma_1\times\boldsymbol{R})k^{-j}\Big)$$

(7.6) *In* $\{x;\ \operatorname{dis}(x, \mathcal{O}_1)>d_0/2\}$ $w^{(N)}$ *is of the form*

$$w^{(N)} = e^{ik(\varphi-t)}v\,,\quad v=\sum_j v_jk^{-j}$$

$$|v_j|_m(\Omega_R^{(1)}\times\boldsymbol{R})\leqslant C_{j,m,R}|f|'_{20j}(\Gamma_1\times\boldsymbol{R})$$

and the principal curvatures of $\mathcal{G}_\varphi(x)$ are positive.

$$(7.7)\qquad |\Box w^{(N)}|_m(\Omega_R^{(1)}\times\boldsymbol{R})\leqslant C_{N,m,R}k^{-N+m}|f|_{N'+m}(\Gamma_1\times\boldsymbol{R})$$

$$(7.8)\qquad |Bw^{(N)}-u|'_m(\Gamma_1\times\boldsymbol{R})\leqslant C_{N,m}k^{-N+m}|f|'_{N'+m}(\Gamma_1\times\boldsymbol{R})$$

where $\Omega_R^{(1)}=\Omega^{(1)}\cap\{x;\ |x|\leqslant R\}$ and $N'=20N$.

Since this lemma may be proved by the same process treated in §4 and 6 of [2] we omit the proof.

Lemma 7.2. *Let $\mathcal{U}$ be a neighborhood of $\mathcal{O}_1$ and let $y(x, t;\ k)$ be a function defined in $\mathcal{U}\times\boldsymbol{R}$ of the form*

$$\begin{aligned} &y(x, t;\ k) = e^{ik(\varphi-t)}v(x, t;\ k)\\ &\operatorname{supp} y\subset\mathcal{U}\times[0, \infty)\,,\quad |\nabla\varphi(x)| = 1\,.\end{aligned}$$

Suppose that for some $\delta_4>0$ it holds that

$$(7.9)\qquad |\nabla\varphi(x)\cdot n(x)|\leqslant 2\delta_4\,,\quad {}^\forall x\in\operatorname{Proj}(\operatorname{supp} y\cap\Gamma_1\times\boldsymbol{R})\,,$$

where Proj *denotes the projection to the x-space. Then we can construct for any N positive integer a function $z^{(N)}(x.\ t;\ k)$ defined in $\Omega^{(1)}\times\boldsymbol{R}$ with the following properties:*

(7.10) $$|z^{(N)}|_m(\Omega_R^{(1)}\times \boldsymbol{R})\leqslant C_{N,m}k^{m+1}|v|'_m(\Gamma_1\times \boldsymbol{R})$$

(7.11) $$\operatorname{supp} z^{(N)}\subset \bigcup_{(x,t)\in \operatorname{supp} v}\ \bigcup_{\xi\in\Sigma_x^+}\mathcal{L}_0(x, t; \xi)$$

(7.12) $$|\Box z^{(N)}|_m(\Omega_R^{(1)}\times \boldsymbol{R})\leqslant C_{N,m,R}k^{-N+m}|v|'_{N'+m}(\Gamma_1\times \boldsymbol{R})$$

(7.13) $$|B(y+z^{(N)})|'_m(\Gamma_1\times \boldsymbol{R})\leqslant C_{N,m}k^{-N+m}|v|'_{N'+m}(\Gamma_1\times \boldsymbol{R}).$$

Proof. We may suppose that $\operatorname{supp} y\cap(\Gamma_1\times \boldsymbol{R})$ is sufficiently small. Then, by using $\mathcal{V}_j$ of § 4 of [3] we have

$$y(x, t; k)=\sum_{j=1}^4 \mathcal{V}_j y \qquad \text{on } \Gamma_1\times \boldsymbol{R}.$$

The definition $\mathcal{V}_j$ gives

$$(\mathcal{V}_j y)(s(\sigma), t)=\omega(s(\sigma), t)\int_{R^1}d\tau\int_{-1}^{\infty}d\alpha\int_{\Sigma}d\xi'\int_{I_\sigma}d\sigma'\int dt'$$
$$\exp\{i\tau(-t+t'+(1+\alpha)\langle\sigma-\sigma', \xi'\rangle\}\chi_j(1+\alpha^2)k^2(1+\alpha)$$
$$\tilde{\omega}(s(\sigma'), t')e^{ik}v(s(\sigma'), t'; k).$$

Taking out the integraiion with respect to t', if $|\tau-k|\geqslant \varepsilon k$ we have an estimate for any M

$$\left|\int e^{i(\tau-k)t'}v(s(\sigma'), t'; k)dt'\right|\leqslant C_{\varepsilon,M}k^{-M}|v|'_M(\Gamma_1\times \boldsymbol{R}).$$

For verifying $|\tau-k|\leqslant \varepsilon k$, since φ verifies (7.9) it holds that for $j\neq 2$

$$|\operatorname{grad}_{\sigma'}(\varphi(s(\sigma'))-(1+\alpha)\langle\sigma', \xi'\rangle)|\geqslant \alpha_0/2$$

when $\chi_j(1+\alpha)\neq 0$ if we choose ε and δ_4 sufficiently small. Therefore we have for $j\neq 2$

$$\left|\int i^{i\tau\{\varphi(s(\sigma'))-(1+\alpha)\langle\sigma',\xi\rangle\}}\chi_j(1+\alpha^2)v(s(\sigma'))\ t'; k)d\sigma'\right|$$
$$\leqslant C_M|\tau|^{-M}|v|'_M(\Gamma_1\times \boldsymbol{R}).$$

These estimtae shows that when (7.9) holds for small $\delta_4>0$

$$|\tilde{\mathcal{V}}_2 y-y|'_M(\Gamma_1\times \boldsymbol{R})\leqslant C_{N,m}k^{-N+m}|v|'_N(\Gamma_1\times \boldsymbol{R}),$$

where $\tilde{\mathcal{V}}_2$ denotes an operator replacing $\int_{-\infty}^{\infty}d\tau$ in the definition of $\mathcal{V}$ by $\int_{|\tau-k|\leqslant\varepsilon k}d\tau$. By using the change of variables of page 87 of [3] it follows that

$$(\tilde{\mathcal{V}}_2 y)(s(\sigma), t)=\omega(s(\sigma), t)\int_{|\tau-k|\leqslant\varepsilon k}d\tau\int_{\Sigma}d\eta\int_{|\beta|\leqslant\beta_0}d\beta\int_{I_\sigma}d\sigma'\int dt'$$
$$\exp\{i\tau(\theta(s(\sigma), \eta, \beta)-\theta(s(\sigma'), \eta, \beta))+t-t')\}$$

$$\cdot \chi_2(1+\alpha)^2 k^2(1+\alpha)\frac{D(\xi',\alpha)}{D(\eta,\beta)}\tilde{\omega}(s(\sigma'),t')e^{ik(\varphi(s(\sigma'))-t')}$$
$$\cdot v(s(\sigma'),t';k).$$

By applying Lemma 7.1 to an oscillatory boundary data

$$u(x,t;\beta,\eta,k)=e^{ik(\theta(x,\eta,\beta)-t)}\omega(x,t)\chi_2(1+\alpha)k^2(1+\alpha)\frac{D(\xi',\alpha)}{D(\eta,\beta)}$$

we obtain $w^{(N)}(x,t;\eta,\beta,k,\sigma')$ verifying (7.4)~(7.8). Let us set

$$z^{(N)}(x,t;k)=\int_{|\tau-k|\leq\varepsilon k}d\tau\int_{\Sigma}d\eta\int_{|\beta|\leq\beta_0}d\beta\int_{I_\sigma}d\sigma'\int dt'$$
$$w^{(N)}(x,t;\eta,\beta,k,\sigma')e^{i\tau(\theta(s(\sigma'),\eta,\beta)-t')}e^{ik(\varphi(s(\sigma'))-t')}v(s(\sigma'),t';k).$$

Then we see immediately that $z^{(N)}$ is the rquired solution.

DEFINITION. Let $\mathcal{U}$ be a neighborhood of $s_0\in\Gamma_1$ and let $\psi(x)$ be a real valued function on $\Gamma_1\cap\mathcal{U}$. We say that $\psi(x)$ satisfies Condition C in $\mathcal{U}$ if there exists a real valued function $\varphi(x)$ defined in $\mathcal{U}$ verifying

$$\begin{cases}\varphi(x)=\psi(x) & \text{in } \Gamma_1\cap\mathcal{U}\\ |\nabla\varphi|=1 & \text{in } \mathcal{U}\\ \dfrac{\partial\varphi}{\partial n}(x)\geq\delta_4 & \text{on } \Gamma_1\cap\mathcal{U}\\ \text{the principal curvatures of } \mathcal{C}_\varphi(x) \text{ are positive for all } x\in\Gamma_1\cap\mathcal{U}.\end{cases}$$

Lemma 7.3. *Let u be an oscillatory data on $\Gamma_1\times\boldsymbol{R}$ in the form*

$$u(x,t;k)=e^{ik(\psi(x)-t)}f(x,t;k) \tag{7.14}$$

where ψ satisfies the condition C in $\mathcal{U}$ and $f(x,t;k)\in C_0^\infty((\Gamma_1\cap\mathcal{U})\times\boldsymbol{R})$. Then we have $v_j(x,t;k)\in C^\infty(\Omega^{(1)}\times\boldsymbol{R})$, $j=0,1,2,\cdots$ with the following properties:

$$\operatorname{supp} v_j\subset\bigcup_{(x,t)\in\operatorname{supp} f}\mathcal{L}_0(x,t;\nabla\varphi(x)) \tag{7.15}$$

$$|v_j|_m(\Omega_R^{(1)}\times\boldsymbol{R})\leq C_{j,m,R}|\nabla\varphi|_{2m+j}(\Omega^{(1)})|f|'_{m+2j}(\Gamma_1\times\boldsymbol{R}) \tag{7.16}$$

For an N positive integer, if we set

$$w^{(N)}(x,t;k)=e^{ik(\varphi(x)-t)}\sum_{j=0}^{N}v_j(x,t;k)k^{-j} \tag{7.17}$$

it holds that

$$\square w^{(N)}=e^{ik(\varphi-t)}k^{-N}\square v_N \quad \textit{in } \Omega^{(1)}\times\boldsymbol{R} \tag{7.18}$$

$$Bw^{(N)}-u=e^{ik(\psi-t)}k^{-N}Bv_N \quad \textit{on } \Gamma_1\times\boldsymbol{R}. \tag{7.19}$$

REMARK. Combining (7.16) and (7.18) we have

$$|\Box w^{(N)}|_m(\Omega_R^{(1)}\times \boldsymbol{R})\leqslant C_{N,m,R}k^{-N+m}|f|'_{m+2N+2}(\Gamma_1\times \boldsymbol{R}) \tag{7.20}$$

and from (7.16) and (7.19) we have

$$|Bw^{(N)}-u|_m(\Gamma_1\times \boldsymbol{R})\leqslant C_{N,m,R}k^{-N+m}|f|'_{m+2N+2}(\Gamma_1\times \boldsymbol{R})\,. \tag{7.21}$$

When $B=B_1$ from method of the construction we have $Bv_N=0$.

Proof. If we look for any asmptotic solution in a form (6.1) from the consideration in the beginning of § 6, we require that v_j is the solution of $(6.4)_j$ in $\Omega^{(1)}$. In the case of $B=B_1$ we set a boundary condition

$$v_0=f\,,\quad v_j=0\qquad \text{for } j\geqslant 1\quad \text{on } \Gamma_1\times \boldsymbol{R}\,.$$

In the case of $B=B_2$, since for $w^{(N)}$ of (7.17)

$$Bw^{(N)}=e^{ik(\varphi-t)}\sum_{j=1}^{N}\left(ik\frac{\partial\varphi}{\partial n}v_j+\frac{\partial v_j}{\partial n}+\sigma v_j\right)k^{-j}$$

we set the following boundary condition for v_j on $\Gamma_1\times \boldsymbol{R}$

$$v_0=(ik)^{-1}f$$

and for $j\geqslant 1$

$$v_j=i\left(\frac{\partial v_{j-1}}{\partial n}+\sigma v_{j-1}\right)\left(\frac{\partial\varphi}{\partial n}\right)^{-1}$$

With the aid of Lemma 6.1 we obtain successively the estimate (7.16), and using (6.6) we have also successively (7.15). On the other hand, since v_j satisfies $(6.4)_j$ we have (7.18) from (6.3). And (7.19) follows from the boundary condition which v_j satisfy. Q.E.D.

Lemma 7.4. *Let $w^{(N)}(x, t; k)$ be a function constructed in the previous lemma. Suppose that*

$$-n(X_1(x,\nabla\varphi(x)))\cdot\nabla\varphi(x)\geqslant\delta_4\qquad {}^{\forall}x\in \mathrm{Proj}(\mathrm{supp}\, f)\,. \tag{7.22}$$

Then there exists a function $\tilde{w}^{(N)}(x, t; k)$ defined in $\Omega^{(2)}\times \boldsymbol{R}$ in the form

$$\tilde{w}^{(N)}=e^{ik(\tilde{\varphi}-t)}\sum_{j=0}^{N}\tilde{v}_j(x,t;k)k^{-j} \tag{7.23}$$

satisfying

$$\mathrm{supp}\,\tilde{v}_j\subset\bigcup_{(x,t)\in\mathrm{supp}\,f}\mathcal{L}_1(x,t;\nabla\varphi(x)) \tag{7.24}$$

$$|\tilde{v}_j|_m(\Omega_R^{(2)}\times \boldsymbol{R})\leqslant C_{j,m,R}|f|'_{m+2j}(\Gamma_1\times \boldsymbol{R}) \tag{7.25}$$

$$\Box\tilde{w}^{(N)}=e^{ik(\tilde{\varphi}-t)}k^{-N}\Box\tilde{v}_N \tag{7.26}$$

$$B(w^{(N)}+\tilde{w}^{(N)})=e^{ik(\varphi-t)}k^{-N}(Bv_N+B\tilde{v}_N)\,. \tag{7.27}$$

Proof. Taking account of Proposition 4.2 there exists $\tilde{\varphi}$ defined in $\omega_1 = \bigcup_{\mathrm{Proj}\, x \in \mathrm{supp}\, f} L_1(x, \nabla\varphi(x))$, which verifies

$$\begin{cases} |\nabla\tilde{\varphi}| = 1 & \text{in } \omega_1 \\ \tilde{\varphi} = \varphi & \text{on } \Gamma_2 \\ \dfrac{\partial\tilde{\varphi}}{\partial n} = -\dfrac{\partial\varphi}{\partial n} & \text{on } \Gamma_2 . \end{cases}$$

Let $\tilde{v}_j$ be a solution of the equation $(6.4)_j$ replaced φ by $\tilde{\varphi}$. We set a boundary condition for $\tilde{v}_j$ on Γ_2.

$$\tilde{v}_j = \begin{cases} -v_j & \text{when } B = B_1 \\ v_j + i\left(\dfrac{\partial\tilde{\varphi}}{\partial n}\right)^{-1}(Bv_{j-1} + B\tilde{v}_{j-1}) & \text{when } B = B_2 . \end{cases} \tag{7.28}$$

By using Lemma 6.2 we have

$$\begin{aligned} |\tilde{v}_j|_m(\Gamma_2\times \boldsymbol{R}) \leqslant |v_j|_m(\Gamma_2\times \boldsymbol{R}) + |v_{j-1}|'_m(\Gamma_2\times \boldsymbol{R}) \\ + C_m |\nabla\varphi|_m \{|v_{j-1}|_m(\Gamma_2\times \boldsymbol{R}) + |v_{j-1}|_{m-1}(\Gamma_2\times \boldsymbol{R}) \\ + |\Box\tilde{v}_{j-1}|_{m-1}(\Gamma_2\times \boldsymbol{R})\} . \end{aligned}$$

And from Lemma 6.1 we have

$$|\tilde{v}_j|_m(\Omega_R^{(2)}\times \boldsymbol{R}) \leqslant \tilde{C}_{m,R} |\nabla\tilde{\varphi}|_m \{|\tilde{v}|_m(\Gamma_2\times \boldsymbol{R}) + |\Box\tilde{v}_{j-1}|_m(\Omega_R^{(2)}\times \boldsymbol{R})\} .$$

Then the estimate (7.25) can be derived inductively from the above two estimates. Taking account of

$$\bar{\mathcal{L}}_0(X_1(x), \Xi_1(x)) = \bar{\mathcal{L}}_1(x, \nabla\varphi(x)) ,$$

we have immediately (7.24). (7.26) and (7.27) follows from the equations and boundary conditions which v_j, $j=0, 1, 2$, satisfy.

REMARK. Applying (7.25) to (7.26) we have

$$|\Box\tilde{w}^{(N)}|_m(\Omega_R^{(2)}\times \boldsymbol{R}) \leqslant C_{N,m,R} k^{-N+m} |f|'_{m+2N+2}(\Gamma_1\times \boldsymbol{R}) . \tag{7.29}$$

Similary we have

$$|B(w^{(N)} + \tilde{w}^{(N)})|_m(\Gamma_2\times \boldsymbol{R}) \leqslant C_{N,m} k^{-N+m} |f|'_{m+2N+2}(\Gamma_1\times \boldsymbol{R}) . \tag{7.30}$$

Proposition 7.5. *Let J be a positive integer and let $u(x, t; k)$ an oscillatory data on $\Gamma_1\times \boldsymbol{R}$ of the form (7.3) or of the form (7.24) with ψ satisfying the condition C. Suppose that*

$$^{\sharp}\mathcal{X}(x, \nabla\varphi(x)) \leqslant J \qquad \textit{for all } x \in \mathrm{Proj\ supp} f , \tag{7.31}$$

where for u of the form (7.3) $\varphi(x)$ means $\theta + 2/3\, \rho^{3/2}$. Then for any N positive *integer there exists a function* $w^{(N)}(x, t; k) \in C^\infty(\bar{\Omega} \times R)$ *such that*

(7.32) $$\operatorname{supp} w^{(N)} \subset \bigcup_{(x,t)\in \operatorname{supp} f} \mathcal{L}(x, t\colon \nabla\varphi(x))$$

(7.33) $$|w^{(N)}|_m(\Omega_R \times \boldsymbol{R}) \leqslant C_{J,N,m\ R} |\nabla\varphi|_{m+N'} k^{m+1} \sum_{j=0}^{N'} k^{-j} |f|_{m+2j+40}(\Gamma_1 \times \boldsymbol{R})$$

(7.34) $$|\square^{(N)}|_m(\Omega_R \times \boldsymbol{R}) \leqslant C_{J,N,m,R} |\nabla\varphi|_{m+N'} k^{-N+n} |f|_{N'+m}(\Gamma_1 \times \boldsymbol{R})$$

(7.35) $$|B^{(N)} - u|_m(\Gamma \times \boldsymbol{R}) \leqslant C_{J,N,m,R} |\nabla\varphi|_{m+N'} k^{-N+m} |f|_{N'+m}(\Gamma_1 \times \boldsymbol{R})$$

where $N' = 20N + 40$ *and for* u *of the form* (7.3) *we set* θ *in the place of* φ *of the above estimates.*

Proof. First we consider the case of u of the form (7.24). Let $(s_0, t_0) \in \operatorname{supp} f$ and let us set

$$^{\sharp}\mathcal{X}(s_0, \nabla\varphi(s_0)) = K_0$$

and $X_j^0 = X_j(s_0, \nabla\varphi(s_0))$, $\Xi_j^0 = \Xi_j(s_0, \nabla\varphi(s_0))$. Taking account of Lemma 3.1, for fixed $\varepsilon > 0$ there are three cases:

Case 1. $n(X_j^0) \cdot \Xi_j^0 \geqslant \delta_1$, $j = 1, 2, \cdots, K_0$ and if $X_{K_0}^0 \in \Gamma_1(\Gamma_2)$ $L_{K_0}(s_0, \nabla\varphi(s_0))$ does not pass the ε-neighborhood of $\mathcal{O}_2(\mathcal{O}_1)$.

Case 2. $n(X_j^0) \cdot \Xi_j^0 \geqslant \delta_1$, $j = 1, 2, \cdots, K_0$ and if $X_{K_0}^0 \in \Gamma_1(\Gamma_2)$ $L_{K_0}(s_0, \nabla\varphi(s_0))$ passes the ε-neighborhood of $\mathcal{O}_2(\mathcal{O}_1)$.

Case 3. $n(X_j^0) \cdot \Xi_j^0 \geqslant \delta_1$, $j = 1, 2, \cdots, K_0 - 1$ and

$$0 \leqslant n(X_{K_0}^0) \cdot \Xi_{K_0}^0 \leqslant \delta_4\,.$$

Case 1. Taking account of Lemmas 3.1 and 3.2 we can take $\mathcal{U}_0$ a neighborhood of s_0 such that

$$^{\sharp}\mathcal{X}(x, \nabla\varphi(x)) = K_0 \quad \text{and} \quad n(X_j) \cdot \Xi_j \geqslant \frac{1}{2}\delta_1\,, \qquad j = 1, 2, \cdots, K$$

hold for all $x \in \mathcal{U}_0 \cap \Gamma_1$ and $L_{K_0}(x, \nabla\varphi(x))$ does not pass the $\varepsilon/2$ neighborhood of $\mathcal{O}_2$. Suppose that $\operatorname{supp} f \subset \mathcal{U}_0 \cap \Gamma_1$. Applying Lemma 7.3 we have

$$w_0^{(N)} = e^{ik(\varphi - t)} \sum_{j=0}^{N} v_{0j}(x, t; k) k^{-j}$$

satisfying (7.15)$\sim$(7.19). Applying Lemma 7.4 to $w_0^{(N)}$ we have $w_1^{(N)}$ with the properties (7.23)$\sim$(7.27). Next apply to $w_1^{(N)}$ we have $w_2^{(N)}$. Repeating this process, we obtain $w_j^{(N)}$, $j = 0, 1, 2, \cdots, K_0$ such that for $j \geqslant 1$

$$|w_j^{(N)}|_m(\Omega_R\times\boldsymbol{R})\leqslant C_{j,m,N,R}\sum_{l=0}^{2N}|f|'_{m+2l}k^{-l+m}$$

$$\operatorname{supp} w_j^{(N)}\subset\bigcup_{(x,t)\in\operatorname{supp} f}\mathcal{L}_j(x,t;\nabla\varphi(x))$$

$$|\Box w_j^{(N)}|_m(\Omega_R\times\boldsymbol{R})\leqslant C_{j,m,N,R}k^{-N+m}|f|'_{2N+2m}(\Gamma_1\times\boldsymbol{R})$$

$$|B(w_j^{(N)}+w_{j+1}^{(N)})|_m(\Gamma_{(j)}\times\boldsymbol{R})\leqslant C_{j,m,N,R}k^{-N+m}|f|'_{2N+2+m}(\Gamma_1\times\boldsymbol{R}),$$

where we set

$$\Gamma_{(j)}=\{X_j(x,\nabla\varphi(x));\,x\in\mathcal{U}_0\}\,.$$

Taking account of

$$\operatorname{supp} w_{K_0}^{(N)}\cap(\Gamma_\epsilon\times\boldsymbol{R})=\phi\,,\quad \epsilon=(1+(-1)^{K_0+1})/2+1$$

we see immediately

$$w^{(N)}=\sum_{j=0}^{K_0}w_j^{(N)}|_{\omega_j\times\boldsymbol{R}}$$

is the desired one.

For the case 2, we construct $w_j^{(N)}$, $j=0, 1, \cdots, K_0$ by the same way as case 1. And for $w_{K_0}^{(N)}|_{\Gamma_\epsilon\times\boldsymbol{R}}$ using Lemma 7.2 we have $w_{K_0+1}^{(N)}$ such that

$$|B(w_{K_0}^{(N)}+w_{K_0+1}^{(N)})|_m(\Gamma_\epsilon\times\boldsymbol{R})\leqslant C_{K_0,m,N}k^{-N+m}\sum|f|_{N'+m}(\Gamma_1\times\boldsymbol{R})\,.$$

Then

$$w^{(N)}=\sum_{j=0}^{K_0}w_j^{(N)}|_{\omega_j\times\boldsymbol{R}}$$

satisfies (7.32) (7.35).

For the case 3, we may show by the same process.

For u of the form (7.3), first construct $w_0^{(N)}$ according to the process of Lemma 7.1. Then concerning

$$w_0^{(N)}|_{\Gamma_{(1)}\times\boldsymbol{R}}=e^{ik(\psi-t)}\sum_{j=0}^{N}v_{0j}k^{-j}\,,$$

$\psi=\varphi|_{\Gamma_{(1)}}$ satisfies the condition C. Then we may apply the consideration for u of the form (7.24). On the other hand from (7.6)

$$|v_{0j}|_m(\Gamma_2\times\boldsymbol{R})\leqslant C_{j,m}|f|'_{20j}(\Gamma_1\times\boldsymbol{R})\,.$$

Combining this estimate and the results for u of the form (7.24) (7.32)~(7.35) follow immediately.

8. Decay of asymptotic solutions

In this section we will prove the following

Proposition 8.1. *Let u be an oscillatory data of the form* (7.3) *or* (7.14). *Suppose that*

$$\text{supp}\, u \subset \Gamma_1 \times (T, T+1)\,. \tag{8.1}$$

Then for any N positive integer there exists a function $z^{(N)} \in C^\infty(\bar{\Omega} \times \boldsymbol{R})$ with the following properties:

$$\text{supp}\, z^{(N)} \subset \bar{\Omega} \times (T, \infty) \tag{8.2}$$

$$|z^{(N)}|_m(\Omega_R, t) \leqslant C_{N,m,R,\varepsilon} e^{-(c_0-\varepsilon)(t-T)} \times k \sum_{j=0}^{N} k^{-j} |f|'_{m+2j+40}(\Gamma_1 \times \boldsymbol{R}) \tag{8.3}$$

$$|\square z^{(N)}|_m(\Omega_R, t) \leqslant C_{N,m,R,\varepsilon} e^{-(c_0-\varepsilon)(t-T)} k^{-N+m} |f|'_{m+2N+40}(\Gamma_1 \times \boldsymbol{R}) \tag{8.4}$$

$$|Bz^{(N)} - u|_m(\Gamma, t) \leqslant C_{N,m,\varepsilon} e^{-(c_0-\varepsilon)(t-T)} k^{-N+m} |f|'_{m+2N+40}(\Gamma_1 \times \boldsymbol{R})\,, \tag{8.5}$$

where c_0 is the constant in Theorem 2.2, ε is an arbitrary positive constant and $C_{N,m,R,\varepsilon}$ depends on N, m, R, ε and $\{|\psi|_m;\ m=1, 2, \cdots\}$.

First we note a simple lemma without proof.

Lemma 8.1. *Let α be a positive constant <1. Suppose that $a_j, j=0, 1, 2, \cdots$ be a sequence of positive numbers verifying for some $A>0$*

$$a_j \leqslant \alpha a_{j-1} + A\alpha^{j-1}\,, \qquad j = 1, 2, \cdots\,.$$

Then for any $\varepsilon>0$ we have an estimate

$$a_j \leqslant (a_0 + AC_\varepsilon)(\alpha e^\varepsilon)^j\,, \qquad j = 0, 1, \cdots$$

where C_ε depends only on ε.

Let us fix $\varepsilon_0>0$ and we will show (8.2)~(8.5) for this ε_0. Let us choose $\delta>0$ so that we have

$$\left| \frac{K(y)}{l(1+lK(y))} - \frac{K(a_j)}{l_0(1+l_0K(a_j))} \right| < \delta\,, \qquad {}^\forall y \in S_j(\delta),\ j = 1, 2\,,$$

where $l = \max\limits_{\substack{x\in S_1(\delta)\\ y\in S_2(\delta)}} |x-y|$, $K(y)=\min(\kappa_j(y))$, $\kappa_j(y), j=1, 2$ are the principal curvatures of Γ at y. Fix δ_2 and δ_3 so that $\delta>\delta_3>\delta_2$ and Corollary of Lemma 3.3 holds. Hereafter K denotes the one of Lemma 3.3 determined by δ_2. Let us set $S=S(\delta_2)$ and $\tilde{S}=S(\delta_3)$. Let $v_{ij}(x)$, $i, j=1, 2$ be functions defined on Γ_i such that

$$v_{i1} = \begin{cases} 1 & x \in S_i(\delta_2) \\ 0 & x \notin S_i(\delta_3) \end{cases}$$

and $v_{i1}+v_{i2}=1$ on Γ_i, $i=1, 2$.

First we construct $z^{(N)}$ for u of the form (7.14) verifying

$$\text{supp} f \subset S_1(\delta_2) \times (T, T+1)\,.$$

Let us set for $j=1, 2, 3, \cdots$

$$\Gamma_{(j)} = \tilde{S} \cap \bigcup_{x \in S_1(\delta_2)} X_j(x, \nabla\varphi(x))$$

$$\tilde{\omega}_j = \{X_j(x, \nabla\varphi(x)) + l\Xi_j(x, \nabla\varphi(x));\ l \geqslant 0$$
$$x \in S_1(\delta_2) \text{ such that } X_j(x, \nabla\varphi(x)) \in \tilde{S}\}\ .$$

Then using Proposition 4.2 we have $\varphi_j(x)$ defined in $\tilde{\omega}_j$ such that

$$|\nabla\varphi_j| = 1 \qquad \text{in } \tilde{\omega}_j \tag{8.6}$$

and

$$\begin{cases} \varphi_j = \varphi_{j-1} & \text{on } \Gamma_{(j)} \\ \dfrac{\partial\varphi_j}{\partial n} = -\dfrac{\partial\varphi_{j-1}}{\partial n} > 0 & \text{on } \Gamma_{(j)}\ . \end{cases} \tag{8.7}$$

Applying Proposition 5.4 we have

$$|\nabla\varphi_j|_m(\tilde{\omega}_j) \leqslant C_m |\nabla\varphi|_m(\tilde{\omega}_0) \leqslant \tilde{C}_m |\psi|'_{m+1}(\Gamma_1)\ . \tag{8.8}$$

Let

$$w_0^{(N)} = e^{ik(\varphi-t)} \sum_{j=0}^{N} v_{0j}(x, t; k) k^{-j}$$

be the function constructed for u according to the process of Lemma 7.3. Set

$$u_{1p} = v_{2p} B w_0^{(N)}|_{\Gamma_2 \times \boldsymbol{R}}\ , \qquad p = 1, 2\ .$$

In the case of the Dirichlet boundary condition

$$u_{1p} = e^{ik(\varphi-t)} \sum_{j=0}^{N} v_{2p} v_{0j} k^{-j}\ .$$

In the case of the third kind boundary condition

$$u_{1p} = e^{ik(\varphi-t)} \sum_{j=0}^{N} v_{2p} \left(i \frac{\partial\varphi}{\partial n} v_{0j} + B v_{0j-1} \right) k^{-j+1}\ .$$

For u_{11} we can apply Lemma 7.3 and we have

$$w^{(N)}(x, t; k) = e^{ik(\varphi_1 - t)} \sum_{j=0}^{N} v_{1j}(x, t; k) k^{-j}\ .$$

Set

$$u_{2p}(x, t; k) = v_{1p}(x) B w_1^{(N)}|_{\Gamma_1 \times \boldsymbol{R}}\ , \qquad p = 1, 2\ ,$$

and construct for u_{21}

$$w_2^{(N)}(x, t; k) = e^{ik(\varphi_2 - t)} \sum_{j=0}^{N} v_{2j}(x, t; k) k^{-j}$$

following the process of Lemma 7.3. Repeating this process construct for $q=0, 1, 2, \cdots$.

$$w_q^{(N)}(x, t; k) = e^{ik(\varphi_q(x)-t)} \sum_{j=0}^{N} v_{qj}(x, t; k)k^{-j} .$$

Lemma 8.2. *Let us set*

$$a = \max_{y \in S} \frac{1}{1+2lK(y)}, \qquad l = \min_{\substack{x \in S_1(\delta_2) \\ y \in S_2(\delta_2)}} |x-y| .$$

Then it holds that for $q=0, 1, 2, \cdots$

$$|v_{qj}|_m(\Omega_R \times \boldsymbol{R}) \leqslant C_{j,R,m,\varepsilon}(ae^{\varepsilon})^q |f|'_{m+2j}(\Gamma_1 \times \boldsymbol{R}) . \tag{8.9}$$

Proof. First we show (8.9) for $j=0$ and $m=0$. Note that v_{q0} satisfy

$$2\frac{\partial v_{q0}}{\partial t} + 2\nabla\varphi_q \cdot \nabla v_{q0} + \nabla\varphi_q \cdot v_{q0} = 0 \qquad \text{in } \tilde{\omega}_q$$

and for $q \geqslant 1$ they verify the coundary condition on $\Gamma_{(q)}$

$$v_{q0} = \begin{cases} v_{\epsilon 1} v_{q-1,0} & \text{if } B = B_1 \\ -v_{\epsilon 1} v_{q-1,0} & \text{if } B = B_2 , \end{cases}$$

where $\epsilon = ((-1)^q + 1)/2$. Consider the case of $B = B_1$. Then

$$v_{00} = f \qquad \text{on } \Gamma_1 .$$

Then applying (6.9) we have

$$|v_{00}|_0(\Gamma_2 \times \boldsymbol{R}) \leqslant |f|'_0(\Gamma_1 \times \boldsymbol{R}) .$$

Note that form Lemma 4.1 we see the principal curvatures of $\mathscr{C}_{\varphi_j}(X_j) \geqslant 2K(X_j)$. From the definition we have for $x \in \Gamma_2$

$$v_{10}(x, t) = v_{21}(x) v_{00}(x, t) .$$

Then

$$|v_{10}|_0(\Gamma_2 \times \boldsymbol{R}) \leqslant |f|'_0(\Gamma_1 \times \boldsymbol{R}) .$$

Applying (6.9) for v_{20} we have

$$|v_{20}|_0(\Gamma_1 \times \boldsymbol{R}) \leqslant a |v_{10}|_0(\Gamma_2 \times \boldsymbol{R}) \leqslant a |f|'_0(\Gamma_1 \times \boldsymbol{R}) .$$

Repeating this process we have for all q

$$|v_{q0}|_0(\Gamma_\epsilon \times \boldsymbol{R}) \leqslant a^{q-1} |f|'_0(\Gamma_1 \times \boldsymbol{R}) .$$

Next, suppose that (8.9) holds for m when $j=0$. Note that

$$|v_{00}|_{m+1}(\Gamma_1 \times \boldsymbol{R}) \leqslant C\, |f|'_{m+1}(\Gamma_1 \times \boldsymbol{R}) .$$

Then applying Lemma 6.1

$$|v_{21}v_{00}|_{m+1}(\Gamma_2\times\boldsymbol{R})\leqslant C|f|'_{m+1}(\Gamma_1\times\boldsymbol{R}),$$

from which it follows that

$$|v_{21}v_{00}|_{m+1}(\Gamma_2\times\boldsymbol{R})\leqslant C\{|f|'_{m+1}(\Gamma_1\times\boldsymbol{R})+|v_{00}|_m(\Gamma_1\times\boldsymbol{R})\}.$$

Since from the definition

$$v_{q0}=v_{\epsilon(q)1}v_{q-1,0}\qquad\text{on }\Gamma_{(q)}$$

we have by applying Lemma 6.2

$$|v_{q0}|_{m+1}(\Gamma_{(q)}\times\boldsymbol{R})\leqslant|v_{q-1,0}|_{m+1}(\Gamma_{(q)}\times\boldsymbol{R})\\ +C\{|v_{q0}|_m(\Gamma_{(q)}\times\boldsymbol{R})+|v_{q-1,0}|_m(\Gamma_{(q)}\times\boldsymbol{R})\}.$$

And from Lcmma 6.1

$$|v_{q0}|_{m+1}(\Gamma_{(q+1)}\times\boldsymbol{R})\leqslant a|v_{q0}|_{m+1}(\Gamma_{(q)}\times\boldsymbol{R})\\ +C\{|v_{q0}|_m(\Gamma_{(q+1)}\times\boldsymbol{R})+|v_{q0}|_m(\Gamma_{(q)}\times\boldsymbol{R})\}.$$

Combining these estimates and using the assumption

$$|v_{q0}|_{m+1}(\Gamma_{(q+1)}\times\boldsymbol{R})\leqslant a|v_{q-1,0}|_{m+1}(\Gamma_{(q)}\times\boldsymbol{R})+4C\cdot C_\varepsilon(ae^\varepsilon)^{q-1}.$$

Applying Lemma 8.1 we have

$$|v_{q0}|_{m+1}(\Gamma_{(q+1)}\times\boldsymbol{R})\leqslant C_{m,\varepsilon}(ae^\varepsilon)^q\qquad\text{for all }q.$$

Tıking account of (6.8) the estimate (8.9) for $j=0$ follows from the above estimate.

Next we will show (8.9) for $j\geqslant 1$. Note that v_{qj}, $q=0, 1, 2, \cdots$ satisfy

$$2\frac{\partial v_{qj}}{\partial t}+2\nabla\varphi_q\cdot\nabla v_{qj}+\Delta\varphi_q\cdot v_{qj}=i\Box v_{qj-1}\qquad\text{in }\tilde{\omega}_q$$

and on $\Gamma_{(q)}$

$$v_{qj}=\begin{cases}v_{\epsilon 1}v_{gj-1} & \text{if } B=B_1\\ -v_{\epsilon 1}v_{qj-1}+(Bv_{qj-1}-v_{\epsilon 1}Bv_{q-1,j-1})\left(i\dfrac{\partial\varphi_q}{\partial n}\right) & \text{if } B=B_2.\end{cases}$$

Suppose that (8.9) holds for j and $m=0, 1, \cdots$. From the assumption

$$|\Box v_{qj}|_m(\Omega_R^{(\epsilon)}\times\boldsymbol{R})\leqslant C_{j,m,R}(ae^\varepsilon)^q|f|'_{m+2j+2}(\Gamma_1\times\boldsymbol{R})$$

and

$$|Bv_{qj}-Bv_{q-1,j}|_m(\Gamma_{(q)}\times\boldsymbol{R})\leqslant C_{j,m}(ae^\varepsilon)^q|f|'_{m+2j+1}(\Gamma_1\times\boldsymbol{R}).$$

Then we have from Lemma 6.1

$$|v_{q,j+1}|_m(\Gamma_{(q+1)}\times\boldsymbol{R})\leqslant a|v_{q,j+1}|_m(\Gamma_{(q)}\times\boldsymbol{R})$$
$$+C(ae^{\varepsilon})^q|f|'_{m+2j+2}(\Gamma_1\times\boldsymbol{R})\,.$$

And applying Lemma 6.2 to $v_{q+1,j+1}$ and $v_{q,j+1}$ on $\Gamma_{(q+1)}$ we have

$$|v_{q+1,j+1}|_m(\Gamma_{(q+1)}\times\boldsymbol{R})\leqslant|v_{qj+1}|_m(\Gamma_{(q+1)}\times\boldsymbol{R})+C(ae^{\varepsilon})^q|f|_{m+2j+2}(\Gamma_1\times\boldsymbol{R})\,.$$

Then we have (8.9) for $j+1$ by combining the above two estimates and by using Lemma 8.1. By the induction we have (8.9) for all j and m. Q.E.D.

Since
$$w_q^{(N)}=e^{ik(\varphi_q-t)}\sum_{j=0}^{N}v_{qj}k^{-j}\quad\text{and}$$
$$\Box w_q^{(N)}=e^{ik(\varphi_q-t)}k^{-N}\Box v_{qN},$$

we have

$$|w_q^{(N)}|_m(\Omega_R^{(\epsilon)}\times\boldsymbol{R})\leqslant C_{N,m,R,\varepsilon}(ae^{\varepsilon})^q\sum_{j=0}^{N}|f|'_{m+2j+2}(\Gamma_1\times\boldsymbol{R})k^{-j+m}\tag{8.10}$$

$$|\Box w_q^{(N)}|_m(\Omega_R^{(\epsilon)}\times\boldsymbol{R})\leqslant C_{N,m,R,\varepsilon}(ae^{\varepsilon})^q|f|'_{m+2N+2}(\Gamma_1\times\boldsymbol{R})k^{-N+m}\,.\tag{8.11}$$

Taking account of

$$\operatorname{supp} v_{qj}\subset\bigcup_{(x,t)\in\operatorname{supp} f}\mathcal{L}_q(x,t;\nabla\varphi(x))$$

we have

$$\operatorname{supp} w_q^{(N)}\cap(\Omega_R\times\boldsymbol{R})\subset\Omega_R\times[ql_0,(q+1)l_1+R]\,,$$

therefore we can rewrite (8.10) and (8.11) as

$$|w_q^{(N)}|_m(\Omega_R^{(\epsilon)}\times\boldsymbol{R})\leqslant C_{N,m,R,\varepsilon}e^{-(c_0-\varepsilon)t}\sum_{j=0}^{N}|f|'_{m+2j+2}(\Gamma_1\times\boldsymbol{R})k^{-j+m}\tag{8.12}$$

$$|\Box w_q^{(N)}|_m(\Omega_R^{(\epsilon)}\times\boldsymbol{R})|\leqslant C_{N,m,R,\varepsilon}e^{-(c_0-\varepsilon)t}k^{-N+m}|f|'_{m+2N+2}(\Gamma_1\times\boldsymbol{R})\,.\tag{8.13}$$

Concerning the estimate on the boundary, we see that from the process of the construction of $w_q^{(N)}$

$$Bw_0^{(N)}|_{\Gamma_1\times\boldsymbol{R}}=f+e^{ik(\varphi-t)}k^{-N}Bv_{0N}\tag{8.14}$$

and for $q\geqslant1$

$$[Bw_{q+1}^{(N)}+v_{\epsilon1}Bw_q^{(N)}]_{\Gamma\times\boldsymbol{R}}=e^{ik(\varphi_q-t)}k^{-N}(Bv_{q+1,N}+v_{\epsilon1}Bv_{qN})\,.\tag{8.15}$$

Remark that for $q\geqslant1$ it holds that

$$^{\#}\mathcal{X}(x,\nabla\varphi_q(x))\leqslant K\,,\qquad {}^{\forall}x\in\text{Proj supp }u_{q2}\,.$$

Indeed, $x\in$Proj supp u_{q2} is represented as

$$x=X_1(\tilde{x},\nabla\varphi_{q-1}(\tilde{x}))\,,\qquad\tilde{x}\in\text{Proj supp }u_{q-1,1}$$

and $x\notin S$. Then Lemma 3.3 and its corollary assure our assertion.

Now we can apply Proposition 7.5 to u_{q2}, $q\geqslant 1$ taking $J=K$. Then we have $y_q^{(N)}(x, t; k)$ such that

$$
\begin{aligned}
&|y_q^{(N)}+u_{q2}|'_m(\Gamma\times\boldsymbol{R})\leqslant Ck^{-N+m}|f_{q2}|'_{m+2N+40}(\Gamma_\epsilon\times\boldsymbol{R})\\
&|y_q^{(N)}|_m(\Omega_R\times\boldsymbol{R})\leqslant Ck^{m+1}\sum_{j=0}^{N}|f_{q2}|'_{m+2j+4}(\Gamma_\epsilon\times\boldsymbol{R})k^{-j}\\
&|\Box y_q^{(N)}|_m(\Omega_R\times\boldsymbol{R})\leqslant Ck^{-N+m}|f_{q2}|'_{m+2N+40}(\Gamma_\epsilon\times\boldsymbol{R})
\end{aligned}
$$

$$
\text{(8.16)}\qquad \operatorname{supp} y_q^{(N)}\subset\bigcup_{(x,t)\in\operatorname{supp} f_{q2}}\mathcal{L}(x, t; \nabla\varphi_{q+1}(x)),
$$

where we set $u_{q2}=\iota^{ik(\varphi_q-t)}f_{q2}$. Then we have from the definition

$$
f_{q2}=\begin{cases} v_{\epsilon2}\sum_{j=1}^{N} v_{qj}k^{-j} & \text{if } B=B_1\\ v_{\epsilon2}\sum_{j=0}^{N}\left(ik\dfrac{\partial\varphi_q}{\partial n}v_{qj}+Bv_{q,j-1}\right)k^{-j-1} & \text{if } B=B_2\,. \end{cases}
$$

Using (8.9) it follows that

$$
\text{(8.17)}\qquad |f_{q2}|'_m(\Gamma_\epsilon\times\boldsymbol{R})\leqslant C(ae^{\varepsilon})^q\sum_{j=0}^{N}|f|'_{m+2j}(\Gamma_1\times\boldsymbol{R})k^{-j}\,.
$$

Taking account of the support of u_{q2} and (8.16) we have

$$
\operatorname{supp} y_q^{(N)}\cap(\Omega_R\times\boldsymbol{R})\subset\Omega_R\times[l_0q, l_1(q+K)+R]\,.
$$

Then it follows that

$$
\text{(8.18)}\qquad |y_q^{(N)}+u_{q2}|'_m(\Gamma\times\boldsymbol{R})\leqslant e^{-(c_0-\varepsilon)t}k^{-N+m}\sum_{j=0}^{N}|f|'_{m+2N+40+2j}(\Gamma_1\times\boldsymbol{R})k^{-j}
$$

$$
\text{(8.19)}\qquad |y_q^{(N)}|_m(\Omega_R\times\boldsymbol{R})\leqslant Ck^{m+1}e^{-(c_0-\varepsilon)t}\sum_{j=0}^{2N}|f|'_{m+40+2j}(\Gamma_1\times\boldsymbol{R})k^{-j}
$$

$$
\text{(8.20)}\qquad |\Box y_q^{(N)}|_m(\Omega_R\times\boldsymbol{R})\leqslant Ce^{-(c_0-\varepsilon)t}k^{-N+m}\sum_{j=0}^{N}|f|'_{m+40+2(N+j)}k^{-j}
$$

Then we see immediately that

$$
z^{(N)}(x, t; k)=\sum_{q=0}^{\infty}w_q^{(N)}(x, t; k)+y_q^{(N)}(x, t; k)
$$

is well defined and it verifies the properties of Proposition because for each t only $w_q^{(N)}$ such that $tl_0\geqslant q\geqslant t-R/l_1-K$ do not vanish.

Now we will remove the assumption

$$
\operatorname{supp} f\subset S_1(\delta_2)\times(T, T+1)\,.
$$

Let $s_0\in$ Proj supp f. If

$$
{}^{\sharp}\mathcal{X}(s_0, \nabla\varphi(s_0))\leqslant K\,,
$$

Proposition 7.5 proves Proposition 8.1 if supp f contained in a small neighborhood of s_0. If ${}^{\sharp}\mathscr{X}(s_0, \nabla\varphi(s_0))\geqslant K$, Lemma 3.3 assures for some $j\leqslant K$

$$X_j(s_0, \nabla\varphi(s_0))\in S .$$

Then we may suppose that

$$X_j(s, \nabla\varphi(s))\in S , \qquad {}^{\forall}s\in \text{Proj supp} f .$$

Using the process of Proposition 7.5 we construct $w_0^{(N)}, w_1^{(N)}, \cdots, w_j^{(N)}$, then we have

$$Bw_j^{(N)}|_{\Gamma_{\epsilon(j)}\times \boldsymbol{R}}= e^{ik(\varphi_j-t)}f_j$$
$$\text{supp} f_j\subset S_{\epsilon(j)}(\delta_2)\times \boldsymbol{R}$$

and the estimate

$$|f_j|'_m\leqslant C\sum_{l=1}^{N}|f|'_{2l+40+m}(\Gamma_1\times \boldsymbol{R})k^{-l}$$

holds. Thus we may apply the process of the construction of $z^{(N)}$ to $e^{ik(\varphi_j-t)}f_j$ and we have the desired function.

Corollary. *For an oscillatory boundary data of Proposition there exists* $\mathfrak{z}^{(N)}(x, t; k)$ *satisfying*

$$\square \mathfrak{z}^{(N)} = 0 \qquad in\ \Omega\times \boldsymbol{R}$$

and (8.2), (8.3) a*and* (8.5).

Proof. Let $z^{(N)}$ be a function verifying (8.2)~(8.5). Take a function $h(x, t; k)$ such that

$$h(x, t; k) = -\square z^{(N)} \qquad \text{in } \Omega\times \boldsymbol{R}$$
$$|h|_m(\mathcal{O}, t)\leqslant C_{N,m,R,\varepsilon}e^{-(c_0-\varepsilon)(t-T)}k^{-N+m}|f|'_{2N+m+40}(\Gamma_1\times \boldsymbol{R}) .$$

Let $g(x, t; k)$ be te the solution of

$$\begin{cases}\square g = h & \text{in } \boldsymbol{R}^3\times \boldsymbol{R}\\ g(x, t; k) = 0 & \text{for } t<T .\end{cases}$$

And we have the estimate

$$|g|_m(x, t)\leqslant \int_{t-2R}^{t}|h|_m(|x|\leqslant R, s)ds .$$

Then we see immediately that

$$\mathfrak{z}^{(N)} = z^{(N)}+g$$

satisfies the required properties.

9. Proof of Theorem 2.2

Let $s_0 \in \Gamma_0$ and let $\mathcal{U}$ be a small neighborhood in $\boldsymbol{R}^3$ of s_0 such that for all

$$h(x, t) \in C_0^\infty((\Gamma_1 \cap \mathcal{U}) \times \boldsymbol{R}) \tag{9.1}$$

the representation

$$h(x, t) = \sum_{j=1}^{4} (\mathcal{V}_j h)(x, t)$$

of § 4 of [3] is applicable. We will define $\mathcal{W}_j$ an operator from $C_0^\infty((\Gamma_1 \cap \mathcal{U}) \times \boldsymbol{R})$ into $C^\infty(\bar{\Omega} \times \boldsymbol{R})$. We set

$$(\mathcal{W}_1^{(N)} h)(x, t) = \int dk \int_{-1}^{\infty} d\alpha \int_{\Sigma} d\xi'$$
$$z^{(N)}(x, t; \alpha, \xi', k) \chi_j (1+\alpha)^2 k^2 (1+\alpha) \hat{h}(\xi, k),$$

where

$$\hat{h}(\xi, k) = \int d\sigma' \int dt' e^{i(\xi \cdot \sigma' - t'k)} h(s(\sigma'), t'),$$

and $z^{(N)}(z, t; \alpha, \xi', k)$ is the function of Corollary of Proposition 8.1 constructed for an oscillatory data

$$u(s(\sigma), t; k) = e^{ik((1+\alpha)\langle \sigma, \xi' \rangle - t)} \omega(s(\sigma), t).$$

Indeed, $\psi(s(\sigma)) = (1+\alpha)\langle \sigma, \xi' \rangle$ satisfies the condition C for $\alpha \leqslant -\alpha_0$. And we set

$$(\mathcal{W}_2^{(N)} h)(x, t) = \int dk \int_{\Sigma} d\eta \int_{|\beta| \leqslant \beta_0} d\beta \int d\sigma' \int dt' z^{(N)}(x, t; \beta, \eta, k)$$
$$\cdot k^2 e^{ik(\theta(s(\sigma'), \eta, \beta) - t')} \tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t')$$

where $z^{(N)}$ is the function of Corollary of Proposition 8.1 constructed for an oscillatory boundary data

$$u(s(\sigma), t') = e^{ik(\theta(s(\sigma), \eta, \beta) - t)} \omega(s(\sigma), t) \chi_2 (1+\alpha(\sigma, \sigma', \eta, \beta))^2$$
$$\cdot (1+\alpha(\sigma, \sigma', \eta, \beta)) \frac{D(\xi', \alpha)}{D(\eta, \beta)} (\sigma, \sigma', \eta, \beta).$$

And we set $\mathcal{W}_j + \mathcal{Z}_j$, $j=3, 4$ of §4 of [3] as $\mathcal{W}_j^{(N)}$, $j=3, 4$ respectively. Remark that $\mathcal{W}_j + \mathcal{Z}_j$ of [3] satisfies

$$\square(\mathcal{W}_j + \mathcal{Z}_j) = 0 \qquad \text{in } \Omega^{(1)}$$

and for any N'

$$|(\mathcal{W}_j + \mathcal{Z}_j) h|_m(\Gamma_2 \times \boldsymbol{R}) \leqslant C_{m,N'} |k|^{-N'} |h|_0(\Gamma_1 \times \boldsymbol{R})$$
$$(\mathcal{W}_j + \mathcal{Z}_j) h = 0 \qquad \text{for } |x| \geqslant t - 2l_1.$$

Then we have for $j=1, 3$

$$|\mathcal{W}_j^{(N)}h|_m(\Omega_R, t)\leqslant C_{m,R,\varepsilon}e^{-(c_0-\varepsilon)(t-1)}$$
$$\cdot\int dk\int d\alpha\int d\xi'(\sum_{j=0}^{N'}k^{-j}A_{m+2j+40})|k|^{m+1}\chi_j(1+\alpha)^2k^2(1+\alpha)|\hat{h}(\xi, k)|,$$

where A_j depends on the derivatives of $\omega(x, t)$, and

$$|\mathcal{W}_2^{(N)}h|_m(\Omega_R, t)\leqslant C_{m,R,\varepsilon}e^{-(c_0-\varepsilon)(t-1)}$$
$$\cdot\int dk\int d\eta\int d\beta(\sum_{j=0}^{N'}|k|^{-j}A_{m+2j+40})k^{m+1}\chi_2(1+\alpha)^2$$
$$(\int|\tilde{h}(s(\sigma'), k)|d\sigma'),$$

where $\tilde{h}(s(\sigma), k)=\int e^{-ikt'}h(s(\sigma), t')dt'$. Simillary it holds that

$$|\mathcal{W}_4h|_m(\Omega_R, t)\leqslant C_{m,R,\varepsilon}e^{-(c_0-\varepsilon)(t-1)}$$
$$\cdot\int dk\int d\alpha\int d\xi'|\xi|^m\chi_4(1+\alpha)^2k^2(1+\alpha)\hat{h}(\xi, k).$$

By the same way we have, for $j=1, 3$

$$|\mathcal{W}_j^{(N)}h-\mathcal{V}_jh|'_m(\Gamma, t)\leqslant C_{N,m}e^{-(c_0-\varepsilon)(t-1)}\int dk\int d\alpha\int d\xi'$$
$$\cdot A_{m+2N+40}|k|^{-N+m+1}\chi_j(1+\alpha)^2k^2(1+\alpha)|\hat{h}(\xi, k)|,$$
$$|\mathcal{W}_2^{(N)}h-\mathcal{V}_2h|_m(\Gamma, t)\leqslant C_{N,m}e^{-(c_0-\varepsilon)(t-1)}\int dk\int d\beta\int d\eta$$
$$\cdot A_{m+2N+40}k^{m+1}\chi_2(1+\alpha)^2(\int|\tilde{h}(s(\sigma'), k)|d\sigma'),$$

and

$$|\mathcal{W}_4^{(N)}h-\mathcal{V}_4h|'_m(\Gamma, t)\leqslant C_{N,m,\varepsilon}e^{-(c_0-\varepsilon)(t-1)}\int dk\int d\alpha\int d\xi'$$
$$\cdot|k|^{-N}A_{m+2N+40}|\xi|^m\chi_4(1+\alpha)^2(1+\alpha)k^2|\hat{h}(\xi, k)|.$$

Set

$$\mathcal{W}^{(N)}h=\sum_{j=1}^{4}\mathcal{W}_j^{(N)}h.$$

Then we have

(9.2) $\mathcal{W}^{(N)}h=0 \quad$ for $t<0$

(9.3) $|\mathcal{W}^{(N)}h|_m(\Omega_R, t)\leqslant C_{N,m,R,\varepsilon}e^{-(c_0-\varepsilon)(t-1)}$
$$\cdot\int\{(1+|k|)^{m+3}\int|\tilde{h}(s(\sigma), k)|d\sigma+\int|\xi|^m|\hat{h}(\xi, k)|d\xi\}dk$$

(9.4) $\square\mathcal{W}^{(N)}h=0 \quad$ in $\Omega\times\boldsymbol{R}$

(9.5) $$|D_t(B\mathcal{W}^{(N)}h-h)|'_m(\Gamma, t)\leqslant C_{N,m,R,\varepsilon}e^{-(c_0-\varepsilon)(t-1)}||h||_{-N+m+2,L^2(\Gamma\times\boldsymbol{R})}\,.$$

For $p=ik+\mu$, $\mu>-c_0$ define $\mathcal{Z}^{(N)}(p)h$ by

$$(\mathcal{Z}^{(N)}(p)h)(x)=\int_{-\infty}^{\infty}e^{-pt}(\mathcal{W}^{(N)}h)(x, t)dt\,.$$

From (9.4) it follows that

(9.6) $$(p^2-\Delta)\mathcal{Z}^{(N)}(p)h=0 \qquad \text{in } \Omega\,.$$

Taking account of (9.2) and (9.3) we have for $\operatorname{Re} p>0$

$$\mathcal{Z}^{(N)}(p)h\in\bigcap_{m\geqslant 0}H^m(\Omega)$$

and

(0.7) $$||\mathcal{Z}^{(N)}(p)h||_{m,L^2(\Omega)}\leqslant C_m(\operatorname{Re} p)^{-1}||h||_{m,L^2(\Gamma\times\boldsymbol{R})}$$

From (9.3) we have for $\operatorname{Re} p\geqslant-(c_0-\varepsilon)$

(9.8) $$|\mathcal{Z}^{(N)}(p)h|_m(\Omega_R)$$
$$\leqslant C_{N,m,R}\int_{-\infty}^{\infty}\{|k|^{m+3}||\tilde{h}(\cdot, k)||_{L^2(\Gamma)}+||\tilde{h}(\cdot, k)||_{m+3,L^2(\Gamma)}\}\,dk\,,$$

and also from (9.5) we have for $\operatorname{Re} p\geqslant-(c_0-\varepsilon)$

(9.9) $$|p\{B(\mathcal{Z}^{(N)}(p)h-\tilde{h}(\cdot, p)\}|'_m(\Gamma)$$
$$\leqslant C_{N,m,\varepsilon}||h||_{-N+m+1,L^2(\Gamma\times\boldsymbol{R})}$$

We see immediately that by using the partition of the unity we can define $\mathcal{W}^{(N)}$ for all $h\in C_0^\infty(\Gamma\times(0, 1))$ and (9.8) and (9.9) hold.

Define $U_q^{(N)}(p)g$ an operator from $C^\infty(\Gamma)$ into $C^\infty(\bar{\Omega})$ for p and $q(t)$ such that $\operatorname{Re} p>-c_0$, $\hat{q}(p)\neq 0$ by

(9.10) $$\tilde{U}_q^{(N)}(p)g=\frac{1}{\hat{q}(p)}(\mathcal{Z}^{(N)}(p)h)\,,$$

where

$$h(x, t)=g(x)q(t)\,,\qquad g\in C^\infty(\Gamma)\,,\ q\in C_0^\infty(0, 1)\,.$$

Let $m(t)$ be a function of $C_0^\infty(0, 1)$ verifying

$$\int m(t)dt=1\,.$$

Set

$$q_k(t)=e^{-ikt}m(t)\,.$$

Since $\hat{q}_k(ik'+\mu)=\hat{m}(\mu+i(k'-k))$ there exist $a_0>0$ and $C_0>0$ such that for all $|k'-k|\leqslant a_0$, $1\geqslant\mu\geqslant-c_0$

$$(9.11)\qquad |\hat{q}_k(ik'+\mu)|\geqslant C_0 .$$

From the definition we have

$$(9.12)\qquad \int |k'^m q_k(ik'+\mu)|\,dk'\leqslant C_m|k|^m, \qquad {}^\forall k\in \boldsymbol{R}$$

where C_m depends only on $m(t)$. Then by using (9.11) and (9.12) we have from (9.8) for $1\geqslant \mathrm{Re}\, p\geqslant -(c_0-\varepsilon)$, $|k'-k|\leqslant a_0$

$$(9.13)\qquad |\tilde{U}^{(N)}_{q_k}(\mu+ik')g|_m(\Omega_R)\leqslant C_{N,m,R,\varepsilon}\{|k|^{m+3}||g||_{L^2(\Gamma)}+||g||_{m+3,L^2(\Gamma)}\}$$

and from (9.9)

$$(9.14)\qquad |\mu+ik'||B\tilde{U}^{(N)}_{q_k}(\mu+ik')g-g|_m(\Gamma)\leqslant C_{N,m,\varepsilon}||g||_{L^2(\Gamma)} .$$

Then, when $C_{N,m,\varepsilon}/|k|\leqslant 1/2$, for $p=ik'+\mu$ such that $1\geqslant\mu\geqslant(c_0-\varepsilon)$ and $|k'-k|\leqslant a_0$

$$A^{(N)}_k(p) = I+\sum_{j=1}^{\infty}(I-B\tilde{U}^{(N)}_{q_k}(p))^j$$

is well defined as a mapping from $C^m(\Gamma)$ into $C^m(\Gamma)$ and for all $g\in C^m(\Gamma)$ we have

$$(9.15)\qquad B\tilde{U}^{(N)}_{q_k}A^{(N)}_k(p)g = g .$$

Define $U^{(N)}_k(p)g=\tilde{U}^{(N)}_{q_k}(p)A^{(N)}_{k_k}(p)g$. Then it holds that

$$(9.16)\qquad \begin{cases}(p^2-\Delta)U^{(N)}_k(p)g = 0 & \text{in } \Omega\\ BU^{(N)}_k(p)g = g & \text{on } \Gamma ,\end{cases}$$

$$(9.17)\qquad |U^{(N)}_k(p)g|_m(\Omega_R)\leqslant 2C_{N,m,R,\varepsilon}\{|k|^{m+3}||g||_{L^2(\Gamma)}+||g||_{m+3,L^2(\Gamma)}\} .$$

For $1\geqslant \mathrm{Re}\, p>0$ we have from (9.7)

$$(9.18)\qquad ||U^{(N)}_k(p)g||_{m,L^2(\Omega)}\leqslant C_m(\mathrm{Re}\, p)^{-1}\{|p|^m||g||_{L^2(\Gamma)}+||g||_{m,L^2(\Gamma)}\} .$$

For each k verifying $C_{N,m,\varepsilon}/|k|<1/2$, $U^{(N)}_k(p)$ is analytic in $1\geqslant\mu\geqslant-(c_0-\varepsilon)$, $|k'-k|\leqslant a_0$. Since in $\mathrm{Re}\, p>0$ the solution of

$$\begin{cases}(p^2-\Delta)u(x) = 0 & \text{in } \Omega\\ Bu = g & \text{on } \Gamma\end{cases}$$

is unique in $L^2(\Omega)$, we see that $U^{(N)}_k(p)$ is independent of k and N in $\mathrm{Re}\, p>0$. Thus $U^{(N)}_k(p)g$ is the analytic continuation of $U(p)g$. Then we have

Proposition 9.1. *For any $\varepsilon>0$, there exists $k_\varepsilon>0$ such that for any $g\in C^m(\Gamma)$, $m\geqslant 3$, $U(p)$ can be prolonged analytically into a region*

$$\{\mu+ik;\ \mu\geqslant-(c_0-\varepsilon),\ |k|\geqslant k_\varepsilon\}$$

and the estimate

$$|U(p)g|_m(\Omega_R)\leqslant C_{m,R,\varepsilon}\{|k|^{m+3}\|g\|_{L^2(\Gamma)}+\|g\|_{m+3,L^2(\Gamma)}\}$$

holds.

Proof. First set $m=0$. Then we can choose k_ε as the proposition holds. By using the regularity theorem for Δ we get the estimate for all $m\geqslant 1$ if p belongs to this region.

Appendix

Let $\mathcal{O}_1$ be a convex and bounded obstacle with smooth boundary Γ_1 such that

$$\Gamma_1\supset\{(x, y, z);\ x=0,\ y^2+z^2\leqslant 1\}$$
$$\mathcal{O}_1\subset\{(x, y, z);\ x\leqslant 0\}$$

and $\mathcal{O}_2$ be a convex and bounded obstacle with smooth boundary Γ_2 such that

$$\Gamma_2\supset\{(x, y, z);\ x=1,\ y^2+z^2\leqslant 1\}$$
$$\mathcal{O}_2\subset\{(x, y, z);\ x\geqslant 1\}\ .$$

For these $\mathcal{O}_j$ we consider the problem with the Dirichlet boundary condition. Set

$$p_{m,R}(t)=\sup_{\{u_0,u_1\}\in C_0^\infty(\Omega_R)} E(u, R, t)^{1/2}(\|u_0\|_{m+1,L^2(\Omega)}+\|u_1\|_{m,L^2(\Omega)})^{-1}\ .$$

We like to show

Theorem A. *Let $R\geqslant 2$. For every positive integer m it holds that*

$$\overline{\lim_{t\to\infty}}\, p_{m,R}(t)t^{m(1+\varepsilon)}>0$$

for any positive number ε.

To prove this theorem we consider the behavior of an asymptotic solution $U^{(N)}(x, y, z, t; k)$ of the problem

$$\text{(A.1)}\qquad \begin{cases}\Box U=0 & \text{in } \Omega\times(0, \infty)\\ BU=0 & \text{on } \Gamma\times(0, \infty)\ .\end{cases}$$

First we construct an asymptotic solution of the problem

$$\text{(A.2)}\qquad \Box w=0 \qquad \text{in } \boldsymbol{R}^3\times(0, \infty)$$

in the form

$$\text{(A.3)}\qquad w=e^{ik(x-t)}\sum_{j=0}^{\infty}v_j(x, y, z, t)k^{-j}\ .$$

Then v_j must satisfy

$$(A.4)_0 \qquad 2\frac{\partial v_0}{\partial t}+2\frac{\partial v_0}{\partial x}=0 \qquad \text{in } \boldsymbol{R}^3\times(0, \infty)$$

and for $j\geqslant 1$

$$(A.4)_j \qquad 2\frac{\partial v_j}{\partial t}+2\frac{\partial v_j}{\partial x}=-i\Box v_{j-1} \qquad \text{in } \boldsymbol{R}^3\times(0, \infty)\,.$$

Lemma A.1. *Let $f(x, y, z)\in C_0^\infty(\boldsymbol{R}^3)$. If we require*

$$(A.5)_0 \qquad v_0(x, y, z, 0)=f(x, y, z)$$

and for $j\geqslant 1$

$$(A.5)_j \qquad v_j(x, y, z, 0)=0$$

v_j, $j=0, 1, 2, \cdots$ the solutions of $(A.4)_j$ are determined successivley and they are represented as

$$(A.6)_j \qquad v_j(x, y, z, t)=\sum_{l=0}^{j} t^l(P_{jl}f)(x-t, y, z)$$

where P_{jl} are differential operators with constant coefficients of order $\leqslant 2j$.

Proof. Let v_0 satisfy $(A.4)_0$ and $(A.5)_0$. Then for any $(\xi, y, z)\in\boldsymbol{R}^3$ and $s\geqslant 0$

$$\frac{\partial}{\partial s}v_0(\xi+s, y, z, s)=\Big(\frac{\partial v_0}{\partial t}+\frac{\partial v_0}{\partial x}\Big)(\xi+s, y, z, s)=0\,.$$

Then we have for all $s\geqslant 0$

$$v_0(\xi+s, y, z, s)=v_0(\xi, y, z, 0)=f(\xi, y, z)\,.$$

Then taking $\xi=x-t$, $s=t$ it follows that

$$v_0(x, y, z, t)=f(x-t, y, z)\,.$$

This is nothing but (A.6) for $j=0$. Suppose that (A.6) holds for $j=h$. Then we have

$$\begin{aligned}\Box v_h(x, y, z, t)&=\sum_{l=0}^{h} t^l\Big(\Big(-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\Big)P_{hl}f\Big)(x-t, y, z)\\&\quad+\sum_{l=0}^{h}\Big\{l(l-1)t^{l-2}(P_{pl}f)(x-t, y, z)-2lt^{l-1}\Big(\frac{\partial}{\partial x}P_{hl}f\Big)(x-t, y, z)\Big\}\\&=\sum_{l=0}^{h} t^l(\tilde{P}_{h+1,l}f)(x-t, y, z)\,,\end{aligned}$$

where $\tilde{P}_{h+1,l}$ is a differential operator of order $\leqslant 2(h+1)$ with constant coefficients. Now from $(A.4)_{h+1}$ and $(A.5)_{h+1}$ it follows that

$$\begin{cases} 2\dfrac{\partial}{\partial s} v_{h+1}(\xi+s, y, z, s) = \sum_{l=0}^{h} s^l (\tilde{P}_{h+1,l} f)(\xi, y, z) \\ v_{h+1}(\xi, y, z, 0) = 0 , \end{cases}$$

from which we have

$$v_{h+1}(\xi+s, y, z, s) = \sum_{l=0}^{h} s^{l+1}(2(l+1))^{-1}(\tilde{P}_{h+1,l} f)(\xi, y, z) .$$

Setting $s=t$, $\xi=x-t$ and $P_{h+1,l}=(2(l+1))^{-1}\tilde{P}_{h+1,l}$ we see that (A.6) holds for $j=h+1$. Q.E.D.

Suppose that

(A.7) $$\operatorname{supp} f \subset \{(x, y, z); y^2+z^2 \leqslant 1/16, 1/4 \leqslant x \leqslant 3/4\}$$

(A.8) $$\int |f(x, y, z)|^2 dxdydz = 1 .$$

Let us define $w^{(N)}$ by

$$w^{(N)}(x, y, z, t; k) = e^{ik(x-t)} \sum_{j=0}^{N} v_j(x, y, z, t) k^{-j} .$$

Then by using (A.6) we have from (A.7)

(A.9) $$\operatorname{supp} w^{(N)} \subset \{(x, y, z, t); y^2+z^2 \leqslant 1/16, 1/4 \leqslant x-t \leqslant 3/4\} .$$

We have immediately

(A.10) $$\begin{aligned} \square w^{(N)} &= e^{ik(x-t)} \square v_N \\ &= e^{ik(x-t)} k^{-N} \sum_{l=0}^{N} t^l (\tilde{P}_{N+1,l} f)(x-t, y, z) . \end{aligned}$$

Set

$$\omega = \{(x, y, z); 0 \leqslant x \leqslant 1, y^2+z^2 \leqslant 1\} .$$

Then for any $(x, y, z, t) \in \omega \times \boldsymbol{R}$ (A.9) implies that the number of positive integer p such that

$$w^{(N)}(2p-x, y, z, t) - w^{(N)}(2p+x, y, z, t) \neq 0$$

is at most one. Now we can define a function $\tilde{U}^{(N)}$ in $\omega \times \boldsymbol{R}$ by

$$\begin{aligned} &\tilde{U}^{(N)}(x, y, z, t; k) \\ &= w^{(N)}(x, y, z, t; k) - \sum_{p=1}^{\infty} \{w^{(N)}(2p-x, y, z, t) - w^{(N)}(2p+x, y, z, t; k)\} . \end{aligned}$$

We see immediately

$$U^{(N)} = 0 \qquad \text{on } x = 0 \text{ and } x = 1$$

$$\Box U^{(N)} = g^{(N)}(x, y, z, t; k) \\ -\sum_{p=1}^{\infty} \{g^{(N)}(2p-x, y, z, t) - g^{(N)}(2p+x, y, z, t)\}$$

where $g^{(N)} = \Box w^{(N)}$. Since we have

$$\int_{\omega} |D^{\gamma} g^{(N)}(2p-x, y, z, t; k)|^2 dxdydz \leqslant C_{N,\gamma} t^{N+|\gamma|} k^{-N+|\gamma|}$$

it holds that

(A.11) $$\int_{\omega} |D^{\gamma} \Box \tilde{U}^{(N)}(x, y, z, t; k)|^2 dxdydz \leqslant C_{N,\gamma} t^{N+|\gamma|} k^{-N+|\gamma|} .$$

Let us set

$$U^{(N)}(x, y, z, t; k) = \begin{cases} \tilde{U}^{(N)}(x, y, z, t; k) & \text{if } (x, y, z) \in \omega \\ 0 & \text{if } (x, y, z) \in \Omega - \omega . \end{cases}$$

Evidently we see that $U^{(N)} \in C^{\infty}(\Omega \times (0, \infty))$ and

$$\Box U^{(N)} = \begin{cases} \Box \tilde{U}^{(N)} & \text{in } \omega \times (0, \infty) \\ 0 & \text{in } (\Omega - \omega) \times (0, \infty) . \end{cases}$$

Let $V^{(N)}$ be the solution of

$$\begin{cases} \Box V^{(N)} = -\Box U^{(N)} & \text{in } \Omega \times (0, \infty) \\ V^{(N)} = 0 & \text{on } \Gamma \times (0, \infty) \\ V^{(N)}|_{t=0} = V_t^{(N)}|_{t=0} = 0 . \end{cases}$$

Then we have from (A.11)

(A.12) $$\|V^{(N)}(\cdot, t)\|_{l, L^2(\Omega)} \leqslant C_{N,l} t^{N+l+2} k^{-N-l} \qquad t \geqslant 0 .$$

Setting $W^{(N)} = U^{(N)} + V^{(N)}$ we have

$$\begin{cases} \Box W^{(N)} = 0 & \text{in } \Omega \times (0, \infty) \\ W^{(N)} = 0 & \text{on } \Gamma \times (0, \infty) . \end{cases}$$

Taking account of the properties of $U^{(N)}$ and $V^{(N)}$ we have

$$W^{(N)}(x, y, z, 0; k) = e^{ikx} f(x, y, z)$$
$$W_t^{(N)}(x, y, z, 0; k) = -(ikf(x, y, z) + \frac{\partial f}{\partial x}(x, y, z)) e^{ikx} .$$

Then it follows that

(A.13) $$\|W^{(N)}(\cdot, 0)\|_{m+1, L^2(\Omega)} + \|W_t^{(N)}(\cdot, 0)\|_{m, L^2(\Omega)} \leqslant C_m k^{m+1} .$$

From the form of $U^{(N)}$ we have

$$U^{(N)}(x, y, z, 2n; k)$$
$$= e^{ik(x-2n)}(f(x, y, z)-\sum_{j=1}^{N}\sum_{l=0}^{j}(2n^l)(F_{jl}f)(x, y, z)k^{-j}).$$

Then we have

$$E(U^{(N)}, R, 2n)^{1/2} \geqslant k(1-\sum_{j=1}^{N} C_{N,j}n^j k^{-j}). \tag{A.14}$$

Therefore it follows from (A.12) and (A.14) that

$$E(W^{(N)}, R, 2n)^{1/2} \geqslant E(U^{(N)}, R, 2n)^{1/2} - E(V^{(N)}, \infty, 2n)^{1/2}$$
$$\geqslant k(1-\sum_{j=1}^{N} C_{N,j}n^j k^{-j}) - C_N(2n)^{N+1}k^{-N+1}.$$

Then we have

$$p_{m,R}(2n) \geqslant \max_{k,N} \{k(1-\sum_{j+1}^{N} C_{N,j}n^j k^{-j}) - C_N(2n)^{N+1}k^{-N+1}\}(C_m k^{m+1})^{-1}.$$

Suppose that $\varepsilon > 0$ is given. Then take $N > (1+\varepsilon)/\varepsilon$ and $k = n^{1+\varepsilon}$. Then we have for large n

$$p_{m,R}(2n) \geqslant (2C_m)^{-1}(2n)^{-m(1+\varepsilon)}$$

from which we have

$$\overline{\lim_{t\to\infty}}\, p_{m,R}(t)t^{m(1+\varepsilon)} \geqslant \overline{\lim_{n\to\infty}}\, p_{m,R}(2n)(2n)^{m(1+\varepsilon)} \geqslant (2C_m)^{-1}.$$

Thus Theorem A is proved.

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