

Title	PARAMETRIZATION FOR A CLASS OF RAUZY FRACTALS
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Citation	Osaka Journal of Mathematics. 2019, 56(1), p. 101-122
Version Type	VoR
URL	https://doi.org/10.18910/71138
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PARAMETRIZATION FOR A CLASS OF RAUZY FRACTALS

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(Received December 5, 2016, revised August 28, 2017)

Abstract

In this paper, we study a class of Rauzy fractals \mathcal{R}_a given by the polynomial $x^3 - ax^2 + x - 1$ where $a \geq 2$ is an integer. In particular, we give explicitly an automaton that generates the boundary of \mathcal{R}_a and using an unusual numeration system we prove that \mathcal{R}_a is homeomorphic to a topological disk.

1. Introduction

The Rauzy fractal is a compact subset of \mathbb{R}^n , $n \geq 1$. It was studied by many mathematicians and is connected to many topics such as: numeration systems [8, 11, 22, 19], geometrical representation of symbolic dynamical systems [6, 20], multidimensional continued fractions and simultaneous approximations [5, 14], auto-similar tilings [6, 22], substitutions and tilings [12] and Markov partitions of Hyperbolic automorphisms of Torus [20, 22].

Let $\beta > 1$ be a fixed real number. Any positive real number x can be expanded as

$$x = \sum_{i=N_0}^{\infty} a_{-i}\beta^{-i} = a_{-N_0}\beta^{-N_0} + a_{-N_0-1}\beta^{-N_0-1} + \dots$$

with $a_i \in \mathbb{Z} \cap [0, \beta)$ and we are assuming the greedy condition

$$\left| x - \sum_{i=N_0}^N a_{-i}\beta^{-i} \right| < \beta^{-N},$$

for all $N \geq N_0$. We call this expansion a beta expansion of x in base β . A Pisot number is an algebraic integer whose conjugates other than itself have modulus less than one. Let $Fin(\beta)$ be a set consisting of all finite beta expansions and consider the condition

$$(F) \quad Fin(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}.$$

Consider the beta expansion of the positive number

$$0 < 1 - [\beta]\beta^{-1} = c_{-2}\beta^{-2} + c_{-3}\beta^{-3} + \dots = .0c_{-2}c_{-3}\dots$$

If we put $c_{-1} = [\beta]$, we can write

$$1 = .c_{-1}c_{-2}c_{-3}\dots$$

This expansion $.c_{-1}c_{-2}c_{-3}\dots$ is called the expansion of 1 and denoted by $d(1, \beta)$. We can identify this expression with the word $c_{-1}c_{-2}c_{-3}\dots$ generated by $\mathbb{A} = \mathbb{Z} \cap [0, \beta)$. Every

finite word generated by \mathbb{A} represents a beta expansion in base β if and only if the word is lexicographically less than $d(1, \beta)$ at any starting point. This fact can be generalized to infinite words apart from certain exceptions (see [21]).

In [16] they proved that if $\beta > 1$ is an integer then (F) holds and, conversely, the condition (F) implies that β is a Pisot number. A Pisot number β is called a Pisot unit if it is also a unit of the integer ring of $\mathbb{Q}[\beta]$. In [2] we have the following results :

Theorem 1.1. *Let β be a cubic Pisot number. Then $\beta > 1$ has property (F) if and only if β is a root of the following polynomial with integer coefficients:*

$$x^3 - ax^2 - bx - 1, a \geq 0, \text{ and } -1 \leq b \leq a + 1.$$

Lemma 1.2. *Let $\beta > 1$ be a cubic Pisot number with $\text{Irr}(\beta) = x^3 - ax^2 - bx - 1$. Then the expansion of 1 in base β is given by:*

- i) $d(1, \beta) = .(a - 1)(a + b - 1)(\widetilde{a + b})$, if $-a + 1 \leq b \leq -2$;
- ii) $d(1, \beta) = .ab1$, if $0 \leq b \leq a$;
- iii) $d(1, \beta) = .(a - 1)(a - 1)01$, if $b = -1$;
- iv) $d(1, \beta) = .(a + 1)00a1$, if $b = a + 1$.

Here \widetilde{w} is the periodic expansion $www\dots$

Theorem 1.3. *A cubic Pisot unit β has property (F) if and only if $d(1, \beta)$ is finite.*

To each cubic Pisot unit satisfying (F) , we can associate a Rauzy fractal. In the case where β is a Pisot number satisfying condition (ii) of Lemma 1.2, the Rauzy fractal was studied in [17] and [18]. In [17] the authors proved that if $2b > a - 3$, then the boundary of the Rauzy fractal is not homeomorphic to a circle. If β is a cubic Pisot unit satisfying (iii) of Lemma 1.2 and $\alpha, \bar{\alpha}$ its Galois conjugates the fractal associated is given by

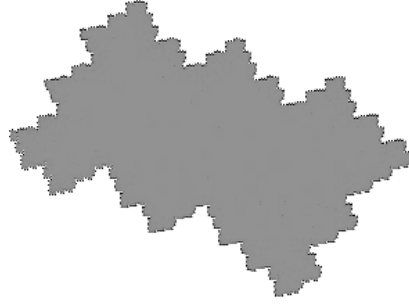
$$\mathcal{R}_a = \left\{ \sum_{i=2}^{\infty} a_i \alpha^i, a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a - 1)(a - 1)01, \forall i \geq 5 \right\},$$

where $<_{lex}$ is the lexicographic order on finite words. In [9] the authors proved the topological and arithmetical properties of \mathcal{R}_a . In particular, they proved there exists an explicit finite state automaton \mathcal{A} such that the boundary of \mathcal{R}_a is recognized by \mathcal{A} . With this automaton they proved that for $a = 2$, the boundary of \mathcal{R}_2 is homeomorphic to a circle. Their proof cannot be extended to the case $a \geq 3$. The parametrization of the boundary of \mathcal{R}_a , $a \geq 3$ is different from the case $a = 2$. It uses an unusual numeration system.

In this paper we will study the fractal associated to a number β satisfying the condition (iii) of Lemma 1.2 with $a \geq 3, b = -1$. In this case the polynomial $p(x) = x^3 - ax^2 + x - 1 = (x - \beta)(x - \alpha)(x - \bar{\alpha})$, where $\beta > 1$ and $\alpha, \bar{\alpha} \in \mathbb{C} \setminus \mathbb{R}$.

The purpose of this work is to present a complete description of the boundary of \mathcal{R}_a , $a \geq 3$. Our main result is the following.

Theorem 1.4. *$\partial \mathcal{R}_a$ is homeomorphic to S^1 .*

Fig. 1. \mathcal{R}_3

2. Background, notations and definitions

In this section we will give more informations about β -numeration, Rauzy fractal, automaton and we will present some notations that will be used in the next sections.

Assume that β is a Pisot number of degree $d \geq 3$. We denote by $\beta_2, \beta_3, \dots, \beta_r$ the real Galois conjugate of β and by $\beta_{r+1}, \dots, \beta_{r+s}, \beta_{r+s+1} = \overline{\beta_{r+1}}, \dots, \beta_{r+2s} = \overline{\beta_{r+s}}$ its complex Galois conjugates. Let

$$\psi = (\beta_2, \dots, \beta_r, \beta_{r+1}, \dots, \beta_{r+s}) \in \mathbb{R}^{r-1} \times \mathbb{C}^s$$

and put $\psi^i = (\beta_2^i, \dots, \beta_r^i, \beta_{r+1}^i, \dots, \beta_{r+s}^i), \forall i \in \mathbb{Z}$. The Rauzy fractal is by definition the set

$$\mathcal{R} = \left\{ \sum_{i=0}^{\infty} a_i \psi^i, (a_i)_{i \geq 0} \in E_\beta \right\}$$

where $E_\beta = \{(x_i)_{i \geq k}, k \in \mathbb{Z} | \forall n \geq k, (x_i)_{k \leq i \leq n} \text{ is a finite } \beta \text{ expansion}\}$.

An important class of Pisot numbers are those such that the associated Rauzy fractal has 0 as an interior point. This numbers were characterized by Akiyama ([3]), and they are exactly the Pisot numbers satisfying condition (F).

In this paper we will work with sequences $(a_n)_{n \in \mathbb{Z}}$ belonging to $\{0, 1, \dots, a-1\}^{\mathbb{Z}}$ and the following set

$$\mathcal{N} = \{(a_n)_{n \in \mathbb{Z}}, \exists k \in \mathbb{Z}, a_k > 0, a_i = 0 \text{ for all } i < k, a_i a_{i-1} a_{i-2} a_{i-3} <_{lex} (a-1)(a-1)01, \forall i \geq k\}.$$

If $(a_n) \in \mathcal{N}$ we will call it an admissible sequence.

Take $(a_n, b_n)_{n \in \mathbb{Z}}$ an infinite path on the automaton \mathcal{A} starting in the initial state. If $(a_n), (b_n) \in \mathcal{N}$ we will call it an admissible path.

In [9] the following results were proved.

(1) Let $z = \sum_{i=2}^{\infty} a_i \alpha^i \in \mathcal{R}_a$. Then $z \in \partial \mathcal{R}_a$ if and only if there exists $(b_i)_{i \geq l} \in \mathcal{N}, l < 1,$

$$b_l \neq 0 \text{ such that } \sum_{i=2}^{\infty} a_i \alpha^i = \sum_{i=l}^{\infty} b_i \alpha^i.$$

(2) There exists an explicit finite state automaton \mathcal{A} (see figure 2 below) such that

$$\sum_{i=l}^{\infty} \epsilon_i \alpha^i = \sum_{i=l}^{\infty} \epsilon'_i \alpha^i, (\epsilon_i), (\epsilon'_i) \in \mathcal{N} \text{ if and only if } (\epsilon_i, \epsilon'_i)_{i \geq l} \text{ is an admissible path.}$$

Let us explain the behavior of this automaton. Let $\epsilon = (\epsilon_i)_{i \geq l}$ and $\epsilon' = (\epsilon'_i)_{i \geq l}$ belonging to \mathcal{N} , $x = \sum_{i=l}^{\infty} \epsilon_i \alpha^i$ and $y = \sum_{i=l}^{\infty} \epsilon'_i \alpha^i$. Suppose $x = y$. For all $k \geq l$ we put

$$(2.1) \quad A_k(\varepsilon, \varepsilon') = \alpha^{-k+2} \sum_{i=l}^k (\varepsilon_i - \varepsilon'_i) \alpha^i.$$

In [9] the authors proved that $A_k(\varepsilon, \varepsilon') \in S = \{0, \pm\alpha, \pm\alpha^2, \pm(\alpha - \alpha^2), \pm(1 + (a-1)\alpha^2), \pm(1 + (a-2)\alpha^2), \pm(1 - \alpha + (a-1)\alpha^2), \pm(1 - 2\alpha + a\alpha^2)\}$. We can see that for all $k \geq l$,

$$(2.2) \quad A_{k+1}(\varepsilon, \varepsilon') = \frac{A_k(\varepsilon, \varepsilon')}{\alpha} + (\varepsilon_{k+1} - \varepsilon'_{k+1})\alpha^2.$$

Let s be the smallest integer such that $\varepsilon_s \neq \varepsilon'_s$. Hence $A_i(\varepsilon, \varepsilon') = 0$ for $i \in \{l, \dots, s-1\}$. Suppose $\varepsilon_s > \varepsilon'_s$. Then, $A_s = (\varepsilon_s - \varepsilon'_s)\alpha^2 = \alpha^2$. From (2.2) we deduce $A_{s+1}(\varepsilon, \varepsilon') = \alpha + (\varepsilon_{s+1} - \varepsilon'_{s+1})\alpha^2$ which should belong to S . Hence $A_{s+1}(\varepsilon, \varepsilon') = \alpha$ if $\varepsilon_{s+1} = \varepsilon'_{s+1}$ or $A_{s+1}(\varepsilon, \varepsilon') = \alpha - \alpha^2$ if $(\varepsilon_{s+1}, \varepsilon'_{s+1}) = (t_1, t_1 + 1)$, where $0 \leq t_1 \leq a-2$. Continuing by the same way and using the fact that the set of states S is finite, we obtain the following finite state automaton shown in Figure 2.

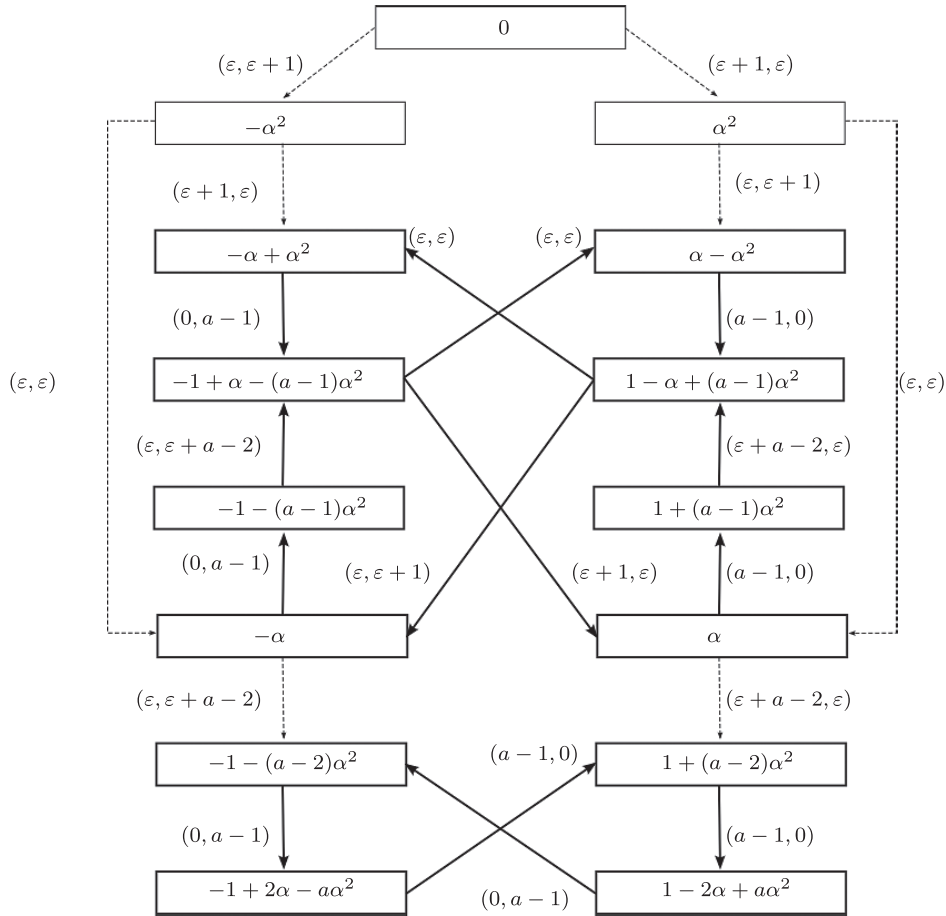


Fig.2. Automaton \mathcal{A}

Another result proved in [9] is the following.

Proposition 2.1. \mathcal{R}_a induces a periodic tiling of the plane \mathbb{C} modulo $\mathbb{Z}u + \mathbb{Z}\alpha u$ where $u = \alpha - 1$. Moreover $\partial\mathcal{R}_a = \bigcup_{v \in B} \mathcal{R}_a \cap (\mathcal{R}_a + v)$, where $B = \{\pm u, \pm\alpha u, \pm(1 + \alpha)u, \pm(\alpha - 1)u\}$ and $\mathcal{R}_a \cap (\mathcal{R}_a + (1 + \alpha)u) = \{-1\}$, $\mathcal{R}_a \cap (\mathcal{R}_a + (\alpha - 1)u) = \{-\alpha\}$.

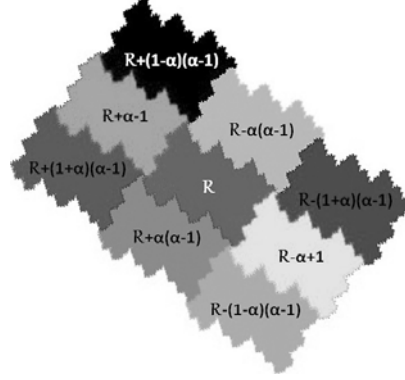


Fig.3. Tiling induced by \mathcal{R}_3

REMARK 2.2. In this paper we will use the following relations:

$$(2.3) \quad \alpha^n - a\alpha^{n-1} + \alpha^{n-2} - \alpha^{n-3} = 0, \quad \alpha^n = (a-1)\alpha^{n-1} + (a-1)\alpha^{n-2} + \alpha^{n-4}, \quad \forall n \in \mathbb{Z}.$$

Lemma 2.3. Let $z \in \mathcal{B}_{\alpha-1}$. Then $z = (\alpha - 1) + \sum_{i=2}^{\infty} a_i \alpha^i$ and $z = \sum_{i=2}^{\infty} b_i \alpha^i$ with $a_2 = 0$, $b_2 = a - 1$ and $b_4 = 0$.

Proof. Take $z \in \mathcal{B}_{\alpha-1}$. Using relation (2.3) we have $z = \alpha^{-3} + (a-1)\alpha^{-1} + (a-2) + \sum_{i=2}^{\infty} a_i \alpha^i$ and $z = \sum_{i=2}^{\infty} b_i \alpha^i$. Then the admissible path, starting from 0, in the automaton associated to z is

$$(1, 0)(0, 0)(a-1, 0)(a-2, 0)(0, 0)(a_2, b_2)(a_3, b_3) \dots$$

Using the automaton we have that $a_2 = 0$, $b_2 = a - 1$ and then we can write $z = \alpha^{-3} + (a-1)\alpha^{-1} + (a-2) + \sum_{i=3}^{\infty} a_i \alpha^i = \alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i$ and $z = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i$. We also have that $b_4 = 0$. \square

3. Parametrization of $\partial\mathcal{R}_a$, $a \geq 3$

In this section we give a complete description of $\partial\mathcal{R}_a$, $a \geq 3$. By Proposition (2.1) we have that $\partial\mathcal{R}_a = \bigcup_{v \in B} \mathcal{R}_a \cap (\mathcal{R}_a + v)$ where $B = \{\pm(\alpha^{-3} + \alpha^{-1}) = \pm(\alpha - 1), \pm(\alpha^2 - \alpha), \pm(\alpha^2 - 1), \pm(\alpha - 1)^2\}$. Since $\mathcal{R}_a \cap (\mathcal{R}_a \pm v)$ is a point if $v = \alpha^2 - 1$ or $v = (\alpha - 1)^2$, we will study the others four regions $\mathcal{B}_v = \mathcal{R}_a \cap (\mathcal{R}_a + v)$ where $v \in \{\pm(\alpha - 1), \pm(\alpha^2 - \alpha)\}$. For this we will use the set $\mathcal{B}_{\alpha-1} = \mathcal{R}_a \cap (\mathcal{R}_a + \alpha - 1)$ described by

$$\mathcal{B}_{\alpha-1} = \left\{ z \in \mathbb{C}, z = \alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i \right\}.$$

In particular, we will prove the following results.

Proposition 3.1. *Let f_i , $i = 1, 2, 3$, be the functions defined by $f_1(z) = \alpha^{-1} - 1 + \alpha^{-1}z$, $f_2(z) = -(a-1)\alpha + \alpha^{-1}z$ and $f_3(z) = 1 - \alpha + z$. Then we have the following properties:*

- (1) $\mathcal{B}_{\alpha^2-\alpha} = f_1(\mathcal{B}_{\alpha-1})$,
- (2) $\mathcal{B}_{\alpha-\alpha^2} = f_2(\mathcal{B}_{\alpha-1})$,
- (3) $\mathcal{B}_{1-\alpha} = f_3(\mathcal{B}_{\alpha-1})$.
- (4) $\mathcal{B}_{\alpha-1} \cap f_1(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{\alpha^2-\alpha} = \{-1\}$.
- (5) $f_1(\mathcal{B}_{\alpha-1}) \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha^2-\alpha} \cap \mathcal{B}_{1-\alpha} = \{-\alpha\}$.
- (6) $f_2(\mathcal{B}_{\alpha-1}) \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-\alpha^2} \cap \mathcal{B}_{1-\alpha} = \{-\alpha^2\}$.
- (7) $\mathcal{B}_{\alpha-1} \cap f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{\alpha-\alpha^2} = \{-(a-1)\alpha - \alpha^{-1}\}$.
- (8) $\mathcal{B}_{\alpha-1} \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{1-\alpha} = \emptyset$.
- (9) $f_1(\mathcal{B}_{\alpha-1}) \cap f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha^2-\alpha} \cap \mathcal{B}_{\alpha-\alpha^2} = \emptyset$.

Proposition 3.2. *Let g_i , $i = 0, 1, \dots, 2(a-1)$, be the functions defined by $g_{2k+1}(z) = -1 - k\alpha^3 + \alpha^3 z$ for $k = 0, \dots, a-2$, and $g_{2k}(z) = \alpha - 1 + (a-1-k)\alpha^3 + \alpha^2 z$ for $k = 0, \dots, a-1$. Then*

$$\mathcal{B}_{\alpha-1} = \bigcup_{i=0}^{2(a-1)} g_i(X_i),$$

where $X_i = \mathcal{B}_{\alpha-1}$ if i is an odd number or $i = 2(a-1)$ and $X_i = \mathcal{B}'_{\alpha-1} = \{z \in \mathcal{B}_{\alpha-1}; a_3 \neq a-1\}$ if i is an even number.

REMARK 3.3. Using Proposition 3.2 we will construct an explicit continuous and bijective application from $[0, 1]$ to $\mathcal{R}_{\alpha-1}$. Using this fact and Proposition 3.1 we obtain an explicit homeomorphism between the circle and the boundary of \mathcal{R}_a .

Proof of Proposition 3.1. According to (2.3) we have $\alpha^2 - \alpha = \alpha^{-2} + (a-1) + (a-2)\alpha$.

- (1) Take $z \in \mathcal{B}_{\alpha-1}$. According to Lemma 2.3 $z = \alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i$.

Then

$$f_1(z) = \alpha^{-1} - 1 + \alpha^{-1}((a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i) = \alpha^{-2} + (a-1) + (a-2)\alpha + \sum_{i=3}^{\infty} b_i \alpha^{i-1}$$

$$\in \mathcal{R}_a + \alpha^2 - \alpha.$$

We also have

$$f_1(z) = \alpha^{-1} - 1 + \alpha^{-1}(\alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i) = \sum_{i=3}^{\infty} a_i \alpha^{i-1} \in \mathcal{R}_a.$$

Therefore $f_1(\mathcal{B}_{\alpha-1}) \subseteq \mathcal{B}_{\alpha^2-\alpha}$.

Take $z \in \mathcal{B}_{\alpha^2-\alpha}$, $z = \alpha^{-2} + (a-1) + (a-2)\alpha + \sum_{i=2}^{\infty} a_i \alpha^i = \sum_{i=2}^{\infty} b_i \alpha^i$. Then

$$f_1^{-1}(z) = \alpha - 1 + \alpha(\alpha^{-2} + (a-1) + (a-2)\alpha + \sum_{i=2}^{\infty} a_i \alpha^i) = (a-1)\alpha^2 + \sum_{i=2}^{\infty} a_i \alpha^{i+1} \in \mathcal{R}_a.$$

We also have

$$f_1^{-1}(z) = \alpha - 1 + \alpha \left(\sum_{i=2}^{\infty} b_i \alpha^i \right) = \alpha - 1 + \sum_{i=2}^{\infty} b_i \alpha^{i+1} \in \mathcal{R}_a + \alpha - 1.$$

Therefore $f_1^{-1}(\mathcal{B}_{\alpha^2-\alpha}) \subseteq \mathcal{B}_{\alpha-1}$ and then

$$f_1(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha^2-\alpha}.$$

- (2) Take $z \in \mathcal{B}_{\alpha-\alpha^2}$. Then $z + \alpha^2 - \alpha$ belongs to $\mathcal{B}_{\alpha^2-\alpha}$ and according to what was done before there exists $w \in \mathcal{B}_{\alpha-1}$ such that $z + \alpha^2 - \alpha = g_1(w)$. Then $z + \alpha^2 - \alpha = \alpha^{-1} - 1 + \alpha^{-1}(w) \Rightarrow z = \alpha^{-1} - 1 + \alpha - \alpha^2 + \alpha^{-1}(w) = -(a-1)\alpha + \alpha^{-1}(w)$. Therefore

$$f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-\alpha^2}.$$

We also know that $f_1^{-1}(z + \alpha^2 - \alpha) = \alpha - 1 + \alpha(z + \alpha^2 - \alpha) = (a-1)\alpha^2 + \alpha z = f_2^{-1}(z) \in \mathcal{B}_{\alpha-1}$ and then $f_2^{-1}(\mathcal{B}_{\alpha-\alpha^2}) \subseteq \mathcal{B}_{\alpha-1}$. Therefore $f_2(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-\alpha^2}$.

- (3) This item can be done by the same manner of item (2).
 (4) Take $z \in \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{\alpha^2-\alpha} = \mathcal{R} \cap (\mathcal{R} + \alpha - 1) \cap (\mathcal{R} + \alpha^2 - \alpha)$. Then $z - \alpha + 1 \in \mathcal{R} \cap (\mathcal{R} - \alpha + 1) \cap (\mathcal{R} + (\alpha - 1)^2) \subseteq \mathcal{R} \cap (\mathcal{R} + (\alpha - 1)^2) = \{-1\}$. Therefore, $z - \alpha + 1 = -\alpha$, and $z = -1$.
 (8) Take $z \in \mathcal{B}_{\alpha-1} \cap f_3(\mathcal{B}_{\alpha-1})$. Then there is $z_1 \in \mathcal{B}_{\alpha-1}$ such that $z = 1 - \alpha + z_1$. Then

$$\alpha + z = 1 + z_1.$$

Since $z, z_1 \in \mathcal{B}_{\alpha-1}$ we know that $z = (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i \alpha^i$ and $z_1 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i$.

Then the equality above becomes

$$1 + (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i \alpha^i = \alpha + (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i.$$

So, we conclude that $(1,0)(0,1)(a-1,a-1)\dots$ is an admissible path on the automaton \mathcal{A} starting from 0. But there is no such path on the automaton and then

$$\mathcal{B}_{\alpha-1} \cap f_3(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1} \cap \mathcal{B}_{1-\alpha} = \emptyset.$$

Following the ideas of items (4) and (8) we can prove (5), (6), (7) and (9). For more details see [10] □

$\mathcal{R} + \alpha^2 - 1$	$\mathcal{R} + \alpha - 1$	$\mathcal{R} - (\alpha - 1)^2$
$\mathcal{R} + \alpha^2 - \alpha$	$\begin{matrix} -1 & & -(a-1)\alpha - \alpha^{-1} \\ & \mathcal{R} & \end{matrix}$	$\mathcal{R} + \alpha - \alpha^2$
$\mathcal{R} + (\alpha - 1)^2$	$\begin{matrix} -\alpha & & -\alpha^2 \\ & \mathcal{R} + 1 - \alpha & \end{matrix}$	$\mathcal{R} + 1 - \alpha^2$

Fig.4. Boundary of \mathcal{R}_a

Proof of Proposition 3.2. Let z be an element of $\mathcal{B}_{\alpha-1}$. Using the automaton \mathcal{A} we can write $z = \alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i$ where $(a_3, b_3) = (t, t)$, $t = 0, 1, \dots, a-1$ or

$(a_3, b_3) = (t, t - 1)$, $t = 1, \dots, a - 1$. Let $\mathcal{B}_{\alpha-1}^{1,t}$, $\mathcal{B}_{\alpha-1}^{2,t}$ be the following sets:

$$\mathcal{B}_{\alpha-1}^{1,t} = \{z \in \mathcal{B}_{\alpha-1}; (a_3, b_3) = (t, t), t = 0, 1, \dots, a - 1\},$$

$$\mathcal{B}_{\alpha-1}^{2,t} = \{z \in \mathcal{B}_{\alpha-1}; (a_3, b_3) = (t, t - 1), t = 1, 2, \dots, a - 1\}. \text{ Since}$$

$$\mathcal{B}_{\alpha-1} = \left[\bigcup_{t=0}^{a-1} \mathcal{B}_{\alpha-1}^{1,t} \right] \cup \left[\bigcup_{t=1}^{a-1} \mathcal{B}_{\alpha-1}^{2,t} \right],$$

in order to prove this theorem we need to show that $g_{2k+1}(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1}^{2,a-1-k}$, $k = 0, \dots, a - 2$, $g_{2(a-1)}(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1}^{1,0}$ and $g_{2k}(\mathcal{B}'_{\alpha-1}) = \mathcal{B}_{\alpha-1}^{1,a-1-k}$, $k = 0, \dots, a - 2$.

1)- Indeed since $z \in \mathcal{B}_{\alpha-1}$ then:

$$g_{2k+1}(z) = -1 - k\alpha^3 + \alpha^3(\alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i) = (a - 1)\alpha^2 + (a - 2 - k)\alpha^3 + \sum_{i=3}^{\infty} a_i \alpha^{i+3} \text{ and}$$

$$g_{2k+1}(z) = -1 - k\alpha^3 + \alpha^3((a - 1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i) = \alpha - 1 + (a - 1 - k)\alpha^3 + (a - 1)\alpha^4 + (a -$$

$$2)\alpha^5 + \sum_{i=3}^{\infty} b_i \alpha^{i+3}, \text{ that is } g_{2k+1}(\mathcal{B}_{\alpha-1}) \subseteq \mathcal{B}_{\alpha-1}^{2,a-1-k}. \text{ On the other hand if we take } w \in \mathcal{B}_{\alpha-1}^{2,a-1-k},$$

$$w = \alpha - 1 + (a - 1 - k)\alpha^3 + (a - 1)\alpha^4 + (a - 2)\alpha^5 + \sum_{i=6}^{\infty} u_i \alpha^i = (a - 1)\alpha^2 + (a - 2 - k)\alpha^3 + \sum_{i=6}^{\infty} v_i \alpha^i \text{ then}$$

$$z = \alpha - 1 + \sum_{i=6}^{\infty} v_i \alpha^{i-3} = (a - 1)\alpha^2 + \sum_{i=6}^{\infty} u_i \alpha^{i-3} \text{ is an element of } \mathcal{B}_{\alpha-1} \text{ such that } g_{2k+1}(z) = w.$$

Therefore we conclude that

$$g_{2k+1}(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1}^{2,a-1-k}, k = 0, \dots, a - 2.$$

$$g_{2(a-1)}(z) = \alpha - 1 + \alpha^2(\alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i) = (a - 1)\alpha^2 + \sum_{i=3}^{\infty} a_i \alpha^{i+2} \text{ and}$$

$$g_{2(a-1)}(z) = \alpha - 1 + \alpha^2((a - 1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i) = \alpha - 1 + (a - 1)\alpha^4 + \sum_{i=3}^{\infty} b_i \alpha^{i+2}, \text{ that is}$$

$$g_{2(a-1)}(\mathcal{B}_{\alpha-1}) \subseteq \mathcal{B}_{\alpha-1}^{1,0}. \text{ On the other hand if we take } w \in \mathcal{B}_{\alpha-1}^{1,0}, w = \alpha - 1 + (a - 1)\alpha^4 + \sum_{i=5}^{\infty} u_i \alpha^i =$$

$$(a - 1)\alpha^2 + \sum_{i=5}^{\infty} v_i \alpha^i \text{ then } z = \alpha - 1 + \sum_{i=5}^{\infty} v_i \alpha^{i-2} = (a - 1)\alpha^2 + \sum_{i=5}^{\infty} u_i \alpha^{i-2} \text{ is an element of } \mathcal{B}_{\alpha-1}$$

such that $g_{2(a-1)}(z) = w$. Therefore

$$g_{2(a-1)}(\mathcal{B}_{\alpha-1}) = \mathcal{B}_{\alpha-1}^{1,0}.$$

2)- Let $z \in \mathcal{B}'_{\alpha-1}$ given by $z = \alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i = (a - 1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i$. Since $a_3 \neq a - 1$ then using the automaton we have $b_3 \neq a - 1$. Then

$$g_{2k}(z) = \alpha - 1 + (a - 1 - k)\alpha^3 + \alpha^2(\alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i) = (a - 1)\alpha^2 + (a - 1 - k)\alpha^3 + \sum_{i=3}^{\infty} a_i \alpha^{i+2} \text{ and}$$

$$g_{2k}(z) = \alpha - 1 + (a - 1 - k)\alpha^3 + \alpha^2((a - 1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i) = \alpha - 1 + (a - 1 - k)\alpha^3 + (a - 1)\alpha^4 +$$

$\sum_{i=3}^{\infty} b_i \alpha^{i+2}$. So we have $g_{2k}(\mathcal{B}'_{\alpha-1}) \subseteq \mathcal{B}_{\alpha-1}^{1, a-1-k}$. On the other hand if we take $w \in \mathcal{B}_{\alpha-1}^{1, a-1-k}$, $w = \alpha - 1 + (a-1-k)\alpha^3 + (a-1)\alpha^4 + \sum_{i=5}^{\infty} u_i \alpha^i = (a-1)\alpha^2 + (a-1-k)\alpha^3 + \sum_{i=5}^{\infty} v_i \alpha^i$ then we have $u_5, v_5 \neq a-1$ (again use the automaton) and $z = \alpha - 1 + \sum_{i=5}^{\infty} v_i \alpha^{i-2} = (a-1)\alpha^2 + \sum_{i=5}^{\infty} u_i \alpha^{i-2}$ is an element of $\mathcal{B}'_{\alpha-1}$ such that $g_{2k}(z) = w$. Therefore

$$g_{2k}(\mathcal{B}'_{\alpha-1}) = \mathcal{B}_{\alpha-1}^{1, a-1-k}, k = 0, \dots, a-2.$$

□

Using the previous notation and taking $u = -1, v = -(a-1)\alpha - \alpha^{-1}, w = -1 - \alpha^3$, we have the following lemmas.

Lemma 3.4.

(1) Take $k_1 \leq k_2$. Then

$$g_{2k_1}(\mathcal{B}'_{\alpha-1}) \cap g_{2k_2+1}(\mathcal{B}_{\alpha-1}) = \begin{cases} \emptyset, & \text{if } k_2 > k_1 \\ -1 - \alpha^2 - k\alpha^3 - (a-1)\alpha^4, & \text{if } k_2 = k_1. \end{cases}$$

Therefore $-1 - \alpha^2 - k\alpha^3 - (a-1)\alpha^4 = g_{2k}(w) = g_{2k+1}(v)$.

(2) Take $k_1 < k_2$. Then

$$g_{2k_1+1}(\mathcal{B}_{\alpha-1}) \cap g_{2k_2}(\mathcal{B}'_{\alpha-1}) = \begin{cases} \emptyset, & \text{if } k_2 > k_1 + 1 \\ -1 - (k_1 + 1)\alpha^3, & \text{if } k_2 = k_1 + 1. \end{cases}$$

Therefore $-1 - (k_1 + 1)\alpha^3 = g_{2k_1+1}(u) = g_{2(k_1+1)}(v)$.

(3) $g_{2k_1}(\mathcal{B}'_{\alpha-1}) \cap g_{2k_2}(\mathcal{B}'_{\alpha-1}) = \emptyset$, if $k_1 \neq k_2$.

(4) $g_{2k_1+1}(\mathcal{B}_{\alpha-1}) \cap g_{2k_2+1}(\mathcal{B}_{\alpha-1}) = \emptyset$, if $k_1 \neq k_2$.

(5) $\lim_{n \rightarrow \infty} (g_0 \circ g_{2(a-1)})^n(z) = u = -1, \forall z \in \mathcal{B}_{\alpha-1}$.

(6) $\lim_{n \rightarrow \infty} (g_{2(a-1)} \circ g_0)^n(z) = v = -(a-1)\alpha - \alpha^{-1}, \forall z \in \mathcal{B}'_{\alpha-1}$.

Proof. 1)– Take $z \in g_{2k_1}(\mathcal{B}'_{\alpha-1}) \cap g_{2k_2+1}(\mathcal{B}_{\alpha-1}), k_1 \leq k_2$.

Then $z = g_{2k_1}(z_1) = g_{2k_2+1}(z_2), z_1 \in \mathcal{B}'_{\alpha-1}, z_2 \in \mathcal{B}_{\alpha-1}$ and

$$\alpha - 1 + (a-1-k_1)\alpha^3 + \alpha^2 z_1 = -1 - k_2 \alpha^3 + \alpha^3 z_2.$$

If we suppose $k_2 = k_1 + k$ we have

$$\alpha + (a-1)\alpha^3 + \alpha^2 z_1 = -k\alpha^3 + \alpha^3 z_2,$$

and multiplying by α^{-3}

$$\alpha^{-2} + (a-1) + \alpha^{-1} z_1 = -k + z_2.$$

Since $z_1 \in \mathcal{B}'_{\alpha-1}, z_2 \in \mathcal{B}_{\alpha-1}$ we know that $z_1 = \alpha - 1 + \sum_{i=3}^{\infty} a_i \alpha^i, a_3 \neq a-1$ and $z_2 = \alpha - 1 + \sum_{i=3}^{\infty} b_i \alpha^i$. Then the equality above becomes

$$\alpha^{-2} + a - \alpha^{-1} + \sum_{i=3}^{\infty} a_i \alpha^{i-1} = -k + \alpha - 1 + \sum_{i=3}^{\infty} b_i \alpha^i,$$

and since $\alpha^{-2} + a - 1 = \alpha$ then

$$(k+1) + \sum_{i=3}^{\infty} a_i \alpha^{i-1} = \sum_{i=3}^{\infty} b_i \alpha^i.$$

So we conclude that $(k+1, 0)(0, 0)(a_3, 0)(a_4, b_3)(a_5, b_4)\dots$ is an admissible path on the automaton \mathcal{A} starting from 0. Using the automaton, since $a_3 \neq a - 1$, we see that the only possibility is

$$(1, 0)(0, 0)(a-2, 0)(a-1, 0)(0, a-1)(0, a-1)(a-1, 0)(a-1, 0)\dots$$

Then $k = 0$,

$$z_1 = \alpha - 1 + (a-2)\alpha^3 + (a-1)\alpha^4 + \sum_{i=2}^{\infty} [(a-1)\alpha^{4i-1} + (a-1)\alpha^{4i}] = -1 - \alpha^3,$$

and

$$z_2 = \alpha - 1 + \sum_{i=1}^{\infty} [(a-1)\alpha^{4i} + (a-1)\alpha^{4i+1}] = -(a-1)\alpha - \alpha^{-1}.$$

2)– Take $z \in g_{2k_1+1}(\mathcal{B}_{\alpha-1}) \cap g_{2k_2}(\mathcal{B}'_{\alpha-1})$, $k_1 < k_2$.

Then $z = g_{2k_1+1}(z_1) = g_{2k_2}(z_2)$, $z_1 \in \mathcal{B}_{\alpha-1}$, $z_2 \in \mathcal{B}'_{\alpha-1}$ and

$$-1 - k_1\alpha^3 + \alpha^3 z_1 = \alpha - 1 + (a-1-k_2)\alpha^3 + \alpha^2 z_2.$$

If we suppose $k_2 = k_1 + k$ we have

$$\alpha^3 z_1 = \alpha + (a-1-k)\alpha^3 + \alpha^2 z_2,$$

and multiplying by α^{-3}

$$z_1 = \alpha^{-2} + (a-1-k) + \alpha^{-1} z_2.$$

Since $z_1 \in \mathcal{B}_{\alpha-1}$, $z_2 \in \mathcal{B}'_{\alpha-1}$ we know that $z_1 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i \alpha^i$ and $z_2 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i$. Then the equality above becomes

$$(a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i \alpha^i = \alpha^{-2} + (a-1-k) + (a-1)\alpha + \sum_{i=3}^{\infty} b_i \alpha^{i-1}.$$

So we conclude that $(1, 0)(0, 0)(a-1-k, 0)(a-1, 0)(b_3, a-1)(b_4, a_3)(b_5, a_4)\dots$ is an admissible path on the automaton \mathcal{A} starting from 0. Using the automaton, we see that the only possibility is

$$(1, 0)(0, 0)(a-2, 0)(a-1, 0)(0, a-1)(0, a-1)(a-1, 0)(a-1, 0)\dots$$

Then $k = 1$,

$$z_2 = (a-1)\alpha^2 + \sum_{i=1}^{\infty} [(a-1)\alpha^{4i+1} + (a-1)\alpha^{4i+2}] = -(a-1)\alpha - \alpha^{-1},$$

and

$$z_1 = \sum_{i=1}^{\infty} [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] = -1.$$

3)–Take $z \in g_{2k_1}(\mathcal{B}'_{\alpha-1}) \cap g_{2k_2}(\mathcal{B}'_{\alpha-1})$, $k_1 < k_2$.

Then $z = g_{2k_1}(z_1) = g_{2k_2}(z_2)$, $z_1 \in \mathcal{B}'_{\alpha-1}$, $z_2 \in \mathcal{B}'_{\alpha-1}$ and

$$\alpha - 1 + (a - 1 - k_1)\alpha^3 + \alpha^2 z_1 = \alpha - 1 + (a - 1 - k_2)\alpha^3 + \alpha^2 z_2.$$

If we suppose $k_2 = k_1 + k$ we have

$$k\alpha^3 + \alpha^2 z_1 = \alpha^2 z_2,$$

and multiplying by α^{-2}

$$k\alpha + z_1 = z_2.$$

Since $z_1 \in \mathcal{B}'_{\alpha-1}$, $z_2 \in \mathcal{B}'_{\alpha-1}$ we know that $z_1 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i \alpha^i$, $z_2 = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i$, and, by Example 2.3, $a_4 = b_4 = 0$. Then the equality above becomes

$$k\alpha + (a-1)\alpha^2 + \sum_{i=3}^{\infty} a_i \alpha^i = (a-1)\alpha^2 + \sum_{i=3}^{\infty} b_i \alpha^i.$$

So we conclude that if the intersection is not empty, $(k, 0)(a-1, a-1)(a_3, b_3)(0, 0)(a_5, b_5) \dots$ is an admissible path on the automaton \mathcal{A} starting from 0. But there is no such path on the automaton and then

$$g_{2k_1}(\mathcal{B}'_{\alpha-1}) \cap g_{2k_2}(\mathcal{B}'_{\alpha-1}) = \emptyset.$$

Using the same ideas we can prove that $g_{2k_1+1}(\mathcal{B}_{\alpha-1}) \cap g_{2k_2+1}(\mathcal{B}_{\alpha-1}) = \emptyset$.

5)- Using induction we can prove that

$$(g_0 \circ g_{r-1})^n(z) = \sum_{i=1}^n [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4n}z.$$

Then $\lim_{n \rightarrow \infty} (g_0 \circ g_{r-1})^n(z) = \sum_{i=1}^{\infty} [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] = -1$.

Indeed for $n = 1$ and $z \in \mathcal{B}_{\alpha-1}$ by proposition 3.2 we have $g_{r-1}(z) = \alpha - 1 + \alpha^2 z \in \mathcal{B}_{\alpha-1}^{1,0}$. By definition we have

$$\begin{aligned} g_0(g_{r-1}(z)) &= \alpha - 1 + (a-1)\alpha^3 + \alpha^2(\alpha - 1 + \alpha^2 z) = \alpha - 1 + (a-1)\alpha^3 + \alpha^3 - \alpha^2 + \alpha^4 z \\ &= (a-1)\alpha^2 + (a-1)\alpha^3 + \alpha^4 z. \end{aligned}$$

Suppose the formula is true for $k \geq 1$, that is

$$(g_0 \circ g_{r-1})^k(z) = \sum_{i=1}^k [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4k}z. \quad (*)$$

We have to prove the formula for $n = k+1$. Since $(g_0 \circ g_{r-1})^{k+1}(z) = (g_0 \circ g_{r-1}) \circ (g_0 \circ g_{r-1})^k(z)$ using (*) we have to prove that

$$g_0(g_{r-1}\left(\sum_{i=1}^k [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4k}z\right)) = \sum_{i=1}^{k+1} [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4(k+1)}z.$$

Indeed

$$\begin{aligned} & g_{r-1}\left(\sum_{i=1}^k [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4k}z\right) \\ &= \alpha - 1 + \alpha^2\left(\sum_{i=1}^k [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4k}z\right) \\ &= \alpha - 1 + \sum_{i=1}^k [(a-1)\alpha^{4i} + (a-1)\alpha^{4i+1}] + \alpha^{4k+2}z, \end{aligned}$$

and

$$\begin{aligned} & g_0(g_{r-1}\left(\sum_{i=1}^k [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4k}z\right)) \\ &= g_0\left(\alpha - 1 + \sum_{i=1}^k [(a-1)\alpha^{4i} + (a-1)\alpha^{4i+1}] + \alpha^{4k+2}z\right) \\ &= \alpha - 1 + (a-1)\alpha^3 + \alpha^2\left(\alpha - 1 + \sum_{i=1}^k [(a-1)\alpha^{4i} + (a-1)\alpha^{4i+1}] + \alpha^{4k+2}z\right) \\ &= (a-1)\alpha^2 + (a-1)\alpha^3 + \sum_{i=1}^k [(a-1)\alpha^{4i+2} + (a-1)\alpha^{4i+3}] + \alpha^{4(k+1)}z \\ &= \sum_{i=1}^{k+1} [(a-1)\alpha^{4i-2} + (a-1)\alpha^{4i-1}] + \alpha^{4(k+1)}z. \end{aligned}$$

6)– Using induction we can prove that

$$(g_{r-1} \circ g_0)^n(z) = (a-1)\alpha^2 + \sum_{i=1}^{n-1} [(a-1)\alpha^{4i+1} + (a-1)\alpha^{4i+2}] + (a-1)\alpha^{4n+1} + \alpha^{4n}z,$$

$$\text{and then } \lim_{n \rightarrow \infty} (g_{r-1} \circ g_0)^n(z) = (a-1)\alpha^2 + \sum_{i=1}^{\infty} [(a-1)\alpha^{4i+1} + (a-1)\alpha^{4i+2}] = -(a-1)\alpha - \alpha^{-1}.$$

□

Proposition 3.5. *Let $t \in [0, 1]$, $a \geq 3$, $r = 2a - 1$. Then there exists an unusual expansion $(a_i)_{i \geq 1} \in \{0, 1, \dots, r-1\}^{\mathbb{N}}$, $n_i, m_i \in \mathbb{N}$ such that we can write*

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=k}}^{\infty} \frac{a_i}{r^{n_i}(r-2)^{m_i}},$$

where the digits a_i and the numbers n_i, m_i satisfy the following properties:

- (1) if $a_1 \in \{1, 3, 5, \dots, r-2, r-1\}$ then $a_2 \in \{0, 1, \dots, r-1\}$, $n_2 = 2$, $m_2 = 0$;
 - (2) if $a_1 \in \{0, 2, 4, \dots, r-3\}$ then $a_2 \in \{0, 1, \dots, r-3\}$, $n_2 = 1$, $m_2 = 1$;
- and for $i \geq 3$ we have:

- (3) $a_i \in \{0, 1, 2, \dots, r-1\}$, $m_i = m_{i-1}$, $n_i = n_{i-1} + 1$ if one of the following conditions are satisfied
- (a) $a_{i-1} = 0$ and $i-1$ even;
 - (b) $a_{i-1} = r-1$ and $i-1$ odd;
 - (c) $a_{i-1} = 2n-1$, $n = 1, \dots, a-1$;
 - (d) $a_{i-1} = r-3$, $i-1$ odd, $n_{i-1} = n_{i-2}$ and $m_{i-1} = m_{i-2} + 1$
- (4) $a_i \in \{0, 1, 2, \dots, r-3\}$, $m_i = m_{i-1} + 1$, $n_i = n_{i-1}$ if one of the following conditions are satisfied
- (a) $a_{i-1} = 0$ and $i-1$ odd;
 - (b) $a_{i-1} = r-1$ and $i-1$ even;
 - (c) $a_{i-1} = 2n$, $n = 1, \dots, a-3$ or $a_{i-1} = r-3$, $i-1$ even or odd, $n_{i-1} = n_{i-2} + 1$ and $m_{i-1} = m_{i-2}$;
 - (d) $a_{i-1} = r-3$, $i-1$ even, $n_{i-1} = n_{i-2}$ and $m_{i-1} = m_{i-2} + 1$

Proof. We can write

$$1 = \frac{r-1}{r} + \sum_{i=0}^{\infty} \frac{r-1}{r^{2+i}(r-2)^i} + \frac{r-3}{r^{2+i}(r-2)^{1+i}}.$$

Given $t \in [0, 1)$, we can prove by induction that for each $k \geq 1$ we can write

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^k \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k, \quad 0 \leq c_k < \frac{1}{r^{n_k}(r-2)^{m_k}}.$$

Indeed if $t \in [0, 1)$, then there exist $a_1 \in \{0, 1, \dots, r-1\}$ such that $rt = a_1 + t_1$, with $0 \leq t_1 < 1$.

Then $t = \frac{a_1}{r} + \frac{t_1}{r} = \frac{a_1}{r} + c_1$ with $c_1 < \frac{1}{r}$.

If $a_1 \in \{1, 3, \dots, r-2, r-1\}$ then there exist $a_2 \in \{0, 1, \dots, r-1\}$ such that $rc_1 = \frac{a_2}{r} + t_2$, with $0 \leq t_2 < \frac{1}{r}$ and then

$$t = \frac{a_1}{r} + c_1 = \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{t_2}{r} = \frac{a_1}{r} + \frac{a_2}{r^2} + c_2, \quad 0 < c_2 < \frac{1}{r^2}.$$

If $a_1 \in \{0, 2, \dots, r-3\}$ then there exist $a_2 \in \{0, 1, \dots, r-3\}$ such that $rc_1 = \frac{a_2}{r-2} + t_2$, with $0 \leq t_2 < \frac{1}{r-2}$ and then

$$t = \frac{a_1}{r} + c_1 = \frac{a_1}{r} + \frac{a_2}{r(r-2)} + \frac{t_2}{r} = \frac{a_1}{r} + \frac{a_2}{r(r-2)} + c_2, \quad 0 < c_2 < \frac{1}{r(r-2)}.$$

Then the result is true for $k = 1, 2$. Suppose that it is true for $2 < k$, that is

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^k \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k,$$

where $0 \leq c_k < \frac{1}{r^{n_k}(r-2)^{m_k}}$.

If a_k and k satisfy condition (3a) or (3b) or (3c) or (3d) then there exist $a_{k+1} \in \{0, 1, \dots, r-1\}$ such that

$r^{n_k}(r-2)^{m_k}c_k = \frac{a_{k+1}}{r} + t_{k+1}$, $0 \leq t_{k+1} < \frac{1}{r}$ and then

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^k \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^k \frac{a_k}{r^{n_i}(r-2)^{m_i}} + \frac{a_{k+1}}{r^{n_{k+1}}(r-2)^{m_k}} + c_{k+1},$$

where $0 \leq c_{k+1} = \frac{t_{k+1}}{r^{n_k}(r-2)^{m_k}} < \frac{1}{r^{n_{k+1}}(r-2)^{m_k}}$. Then the result is true for $k+1$.

If a_k and k satisfy condition (4a) or (4b) or (4c) or (4d) then there exist $a_{k+1} \in \{0, 1, \dots, r-3\}$ such that

$r^{n_k}(r-2)^{m_k}c_k = \frac{a_{k+1}}{r-2} + t_{k+1}$, $0 \leq t_{k+1} < \frac{1}{r-2}$ and then

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^k \frac{a_k}{r^{n_i}(r-2)^{m_i}} + c_k = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^k \frac{a_k}{r^{n_i}(r-2)^{m_i}} + \frac{a_{k+1}}{r^{n_k}(r-2)^{m_{k+1}}} + c_{k+1},$$

where $0 \leq c_{k+1} = \frac{t_{k+1}}{r^{n_k}(r-2)^{m_k}} < \frac{1}{r^{n_k}(r-2)^{m_{k+1}}}$. Then the result is true for $k+1$. Therefore the result is true for every $k \geq 1$. \square

REMARK 3.6. Let $t \in [0, 1]$ written as $t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^k \frac{a_k}{r^{n_i}(r-2)^{m_i}}$. In order to simplify the

demonstration of some of the results in this paper, a simpler notation will be used, i.e., t will be represented as

$$t = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{\infty} \frac{a_i}{(n_i, m_i)}.$$

Proposition 3.7. Let t, t' be elements in $[0, 1]$, $t = \frac{a_1}{r} + \sum_{i=2}^{\infty} \frac{a_i}{(n_i, m_i)}$, $t' = \frac{a'_1}{r} + \sum_{i=2}^{\infty} \frac{a'_i}{(n'_i, m'_i)}$, a_i and a'_i as in Proposition (3.4). Suppose that $a_i = a'_i$, $i = 1, 2, \dots, k-1$ and $a_k < a'_k$.

If $|t' - t| < r^{-N}$ with $k < N$ then $t = T_1 + T_2 + T_3$, $t' = T_1 + T'_2 + T'_3$ where

$$T_1 = \frac{a_1}{r} + \sum_{\substack{i=2 \\ m_i+n_i=i}}^{k-1} \frac{a_i}{(n_i, m_i)}, \quad T_3 = \sum_{\substack{i \geq N+1 \\ m_i+n_i=i}}^{\infty} \frac{a_i}{(n_i, m_i)}, \quad T'_3 = \sum_{\substack{i \geq N+1 \\ m_i+n_i=i}}^{\infty} \frac{a'_i}{(n'_i, m'_i)} \text{ and}$$

1) if k is even and a_k satisfies items (3a) or (3b) or (3c) of Proposition 3.5 then

$$T_2 = \frac{a_k}{(n_k, m_k)} + \frac{r-1}{(n_{k+1}, m_k)} + \frac{r-1}{(n_{k+2}, m_k)} + \frac{r-3}{(n_{k+2}, m_{k+1})} + \frac{r-1}{(n_{k+3}, m_{k+1})} + \frac{r-3}{(n_{k+3}, m_{k+2})} + \dots + \frac{a_N}{(n_N, m_N)},$$

$$T'_2 = \frac{a_k+1}{(n_k, m_k)} + \frac{0}{(n'_{k+1}, m'_{k+1})} + \frac{0}{(n'_{k+2}, m'_{k+2})} + \frac{0}{(n'_{k+3}, m'_{k+3})} + \frac{0}{(n'_{k+4}, m'_{k+4})} + \frac{0}{(n'_{k+5}, m'_{k+5})} + \dots + \frac{0}{(n'_N, m'_N)},$$

Moreover we have $t = t'$ if and only if

$$t = T_1 + \frac{a_k}{(n_k, m_k)} + \frac{r-1}{(n_{k+1}, m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(n_{k+2+i}, m_{k+i})} + \frac{(r-3)}{(n_{k+2+i}, m_{k+1+i})} \right) \text{ and}$$

$$t' = T_1 + \frac{a_k+1}{(n_k, m_k)} + \sum_{i>k+1}^{\infty} \frac{0}{(n'_i, m'_i)}.$$

2) if k is odd and a_k satisfies item (3c) of Proposition 3.5 then

$$T_2 = \frac{a_k}{(n_k, m_k)} + \frac{r-1}{(n_{k+1}, m_k)} + \frac{r-3}{(n_{k+1}, m_{k+1})} + \frac{r-1}{(n_{k+2}, m_{k+1})} + \frac{r-3}{(n_{k+2}, m_{k+2})} + \dots + \frac{a_N}{(n_N, m_N)},$$

$$T'_2 = \frac{a_k+1}{(n_k, m_k)} + \frac{0}{(n'_{k+1}, m'_{k+1})} + \frac{0}{(n'_{k+2}, m'_{k+2})} + \frac{0}{(n'_{k+3}, m'_{k+3})} + \frac{0}{(n'_{k+4}, m'_{k+4})} + \dots + \frac{0}{(n'_N, m'_N)},$$

Moreover we have $t = t'$ if and only if

$$t = T_1 + \frac{a_k}{(n_k, m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(n_k+1+i, m_k+i)} + \frac{(r-3)}{(n_k+1+i, m_k+1+i)} \right) \text{ and}$$

$$t' = T_1 + \frac{a_k+1}{(n_k, m_k)} + \sum_{i>k+1}^{\infty} \frac{0}{(n'_i, m'_i)}.$$

3) if k is odd and a_k satisfies items (4a) or (4b) of Proposition 3.5 then

$$T_2 = \frac{a_k}{(n_k, m_k)} + \frac{r-3}{(n_k, m_k+1)} + \frac{r-3}{(n_k, m_k+2)} + \frac{r-1}{(n_k+1, m_k+2)} + \frac{r-3}{(n_k+1, m_k+3)} + \frac{r-1}{(n_k+2, m_k+3)} + \dots + \frac{a_N}{(n_N, m_N)},$$

$$T'_2 = \frac{a_k+1}{(n_k, m_k)} + \frac{0}{(n'_{k+1}, m'_{k+1})} + \frac{0}{(n'_{k+2}, m'_{k+2})} + \frac{0}{(n'_{k+3}, m'_{k+3})} + \frac{0}{(n'_{k+4}, m'_{k+4})} + \frac{0}{(n'_{k+5}, m'_{k+5})} + \dots + \frac{0}{(n'_N, m'_N)},$$

Moreover we have $t = t'$ if and only if

$$t = T_1 + \frac{a_k}{(n_k, m_k)} + \frac{r-3}{(n_k, m_k+1)} + \sum_{i=0}^{\infty} \left(\frac{r-3}{(n_k+i, m_k+2+i)} + \frac{(r-1)}{(n_k+1+i, m_k+2+i)} \right) \text{ and}$$

$$t' = T_1 + \frac{a_k+1}{(n_k, m_k)} + \sum_{i>k+1}^{\infty} \frac{0}{(n'_i, m'_i)}.$$

4) if k is even and a_k satisfies item (4c) of Proposition 3.5 then

$$T_2 = \frac{a_k}{(n_k, m_k)} + \frac{r-3}{(n_k, m_k+1)} + \frac{r-1}{(n_k+1, m_k+1)} + \frac{r-3}{(n_k+1, m_k+2)} + \frac{r-1}{(n_k+2, m_k+2)} + \dots + \frac{a_N}{(n_N, m_N)},$$

$$T'_2 = \frac{a_k+1}{(n_k, m_k)} + \frac{0}{(n'_{k+1}, m'_{k+1})} + \frac{0}{(n'_{k+2}, m'_{k+2})} + \frac{0}{(n'_{k+3}, m'_{k+3})} + \frac{0}{(n'_{k+4}, m'_{k+4})} + \dots + \frac{0}{(n'_N, m'_N)},$$

Moreover we have $t = t'$ if and only if

$$t = T_1 + \frac{a_k}{(n_k, m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-3}{(n_k+i, m_k+1+i)} + \frac{(r-1)}{(n_k+1+i, m_k+1+i)} \right) \text{ and}$$

$$t' = T_1 + \frac{a_k+1}{(n_k, m_k)} + \sum_{i>k+1}^{\infty} \frac{0}{(n'_i, m'_i)}.$$

Proof. Take $t, t' \in [0, 1]$ such that $|t' - t| < r^{-N}$, $t = \frac{a_1}{r} + \sum_{\substack{k=2 \\ m_k+n_k=k}}^{\infty} \frac{a_k}{(n_k, m_k)}$, $t' = \frac{a'_1}{r} +$

$\sum_{\substack{k=2 \\ m'_k+n'_k=k}}^{\infty} \frac{a'_k}{(n'_k, m'_k)}$, $a_i = a'_i$, $\forall i = 1, 2, \dots, k-1$ and $a_k < a'_k$. Then

$$\begin{aligned} t' - t &= \frac{(a'_k - a_k)}{(n_k, m_k)} + \sum_{i>k}^{\infty} \left[\frac{a'_i}{(n'_i, m'_i)} - \frac{a_i}{(n_i, m_i)} \right] \\ &= \frac{(a'_k - a_k - 1)}{(n_k, m_k)} + \frac{1}{(n_k, m_k)} + \sum_{i>k}^{\infty} \left[\frac{a'_i}{(n'_i, m'_i)} - \frac{a_i}{(n_i, m_i)} \right], \end{aligned}$$

and since $m_k + n_k = k$, $|t' - t| < r^{-N}$ then $a'_k - a_k - 1 = 0$, that is, $a'_k = a_k + 1$.

(1) Let k be an even number and $a_k = 0$ or $2n - 1$, $n = 1, \dots, a - 1$. Then $a_{k+1} \in \{0, 1, \dots, r - 1\}$ and we can write

$$\frac{1}{(n_k, m_k)} = \frac{r-1}{(n_k+1, m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(n_k+2+i, m_k+i)} + \frac{(r-3)}{(n_k+2+i, m_k+1+i)} \right).$$

Therefore

$$\begin{aligned} t' - t &= \frac{a'_{k+1}}{(n'_{k+1}, m'_{k+1})} - \frac{a_{k+1}}{(n_k+1, m_k)} + \frac{(r-1)}{(n_k+1, m_k)} + \dots \\ &= \frac{a'_{k+1}}{(n'_{k+1}, m'_{k+1})} + \frac{r-1-a_{k+1}}{(n_k+1, m_k)} + \dots, \end{aligned}$$

where $m'_{k+1} + n'_{k+1} = m_k + n_k + 1 = k + 1$. As $|t' - t| < r^{-N} < (r - 2)^{-m} r^{-n}$, $m + n = N \geq k + 1$ then $\frac{a'_{k+1}}{(n'_{k+1}, m'_{k+1})} + \frac{r-1-a_{k+1}}{(n_{k+1}, m_k)} = 0$ and it is possible only with $a'_{k+1} = 0$ and $a_{k+1} = r - 1$.

As $a_{k+1} = r - 1$ and $k + 1$ is an odd number, then $a_{k+2} \in \{0, 1, \dots, r - 1\}$ and we have

$$\begin{aligned} t' - t &= \frac{a'_{k+2}}{(n'_{k+2}, m'_{k+2})} - \frac{a_{k+2}}{(n_k + 2, m_k)} + \frac{r - 1}{(n_k + 2, m_k)} + \dots \\ &= \frac{a'_{k+2}}{(n'_{k+2}, m'_{k+2})} + \frac{r - 1 - a_{k+2}}{(n_k + 2, m_k)} + \dots \end{aligned}$$

with $m'_{k+2} + n'_{k+2} = m_k + n_k + 2 = k + 2$. Again we have $a'_{k+2} = 0$ and $a_{k+2} = r - 1$. Now $a_{k+2} = r - 1$ and $k + 2$ is an even number. Then $a_{k+3} \in \{0, 1, \dots, (r - 3)\}$ and

$$\begin{aligned} t' - t &= \frac{a'_{k+3}}{(n'_{k+3}, m'_{k+3})} - \frac{a_{k+3}}{(n_k + 2, m_k + 1)} + \frac{(r - 3)}{(n_k + 2, m_k + 1)} + \dots \\ &= \frac{a'_{k+3}}{(n'_{k+3}, m'_{k+3})} + \frac{(r - 3) - a_{k+3}}{(n_k + 2, m_k + 1)} + \dots \end{aligned}$$

with $m'_{k+3} + n'_{k+3} = m_k + n_k + 3 = k + 3$. Therefore $a'_{k+3} = 0$ and $a_{k+3} = (r - 3)$. Following this idea we have the result.

(2) To prove this part we use the same ideas of (1) and the equality

$$\frac{1}{(n_k, m_k)} = \sum_{i=0}^{\infty} \left(\frac{(r - 1)}{(n_k + 1 + i, m_k + i)} + \frac{(r - 3)}{(n_k + 1 + i, m_k + 1 + i)} \right).$$

(3) To prove this part we use the same ideas of (1) and the equality

$$\frac{1}{(n_k, m_k)} = \frac{(r - 3)}{(n_k, m_k + 1)} + \sum_{i=0}^{\infty} \left(\frac{(r - 3)}{(n_k + i, m_k + 2 + i)} + \frac{r - 1}{(n_k + 1 + i, m_k + 2 + i)} \right).$$

(4) To prove this part we use the same ideas of (1) and the equality

$$\frac{1}{(n_k, m_k)} = \sum_{i=0}^{\infty} \left(\frac{(r - 3)}{(n_k + i, m_k + 1 + i)} + \frac{r - 1}{(n_k + 1 + i, m_k + 1 + i)} \right).$$

□

Now we will give an explicit parametrization of $\mathcal{B}_{\alpha-1}$ and hence for the boundary $\partial\mathcal{R}_\alpha$. Let z be an element of $\mathcal{B}_{\alpha-1}$. Using Proposition 3.2, there exists a sequence $(z_n)_{n \geq 1}$ in $\mathcal{B}_{\alpha-1}$, such that

$$z = g_{a_1} \circ g_{a_2} \circ \dots \circ g_{a_n}(z_n), \forall n \geq 1.$$

If x is an element of $\mathcal{B}'_{\alpha-1}$, the sequence $y_n = g_{a_1} \circ g_{a_2} \circ \dots \circ g_{a_n}(x)$ converges to z because the functions $g_i, i = 0, 1, \dots, r - 1$ are contractions.

Let $A = \{0, 1, \dots, r - 1\}$ be a subset of \mathbb{N} and consider the function

$$\begin{aligned} \psi : A^{\mathbb{N}} &\longrightarrow A^{\mathbb{N}} \\ (a_i) &\longmapsto \psi((a_i)) = (b_i) \end{aligned}$$

given by:

$$b_1 = a_1;$$

$$b_{2k} = r - 1 - a_{2k};$$

$$b_{2k+1} = a_{2k+1} \text{ if } a_{2k} \in \{0\} \cup \{2n - 1 : n = 1, \dots, a - 1\};$$

$$b_{2k+1} = a_{2k+1} + 2 \text{ if } a_{2k} \in \{2n : n = 1, \dots, a - 1\}.$$

Take $x_0 \in \mathcal{B}'_{\alpha-1}$ and consider $f : [0, 1] \rightarrow \mathcal{B}_{\alpha-1}$ defined as follows:

$$\text{if } t = \frac{a_1}{r} + \sum_{\substack{k=2 \\ m_k+n_k=k}}^{\infty} \frac{a_k}{r^{n_k}(r-2)^{m_k}}, (a_i) \in A^{\mathbb{N}}, \text{ then } f(t) = \lim_{n \rightarrow \infty} g_{b_1} \circ g_{b_2} \circ \dots \circ g_{b_n}(x_0) \text{ where } \psi((a_i)) = (b_i).$$

Theorem 3.8. *f is a continuous, bijective function satisfying $f(0) = u = -1$ and $f(1) = v$.*

Proof. (1)– *f is a well defined function.*

We are going to use the following notation:

$$g_{b_1} \circ g_{b_2} \circ \dots \circ g_{b_{k-1}} \circ g_{b_k}(x_0) = g_{b_1 \dots b_k}(x_0).$$

According Lemma 3.4 we have

$$\begin{aligned} u &= -1 = g_{0(r-1)0(r-1)\dots}(x_0) = g_{\overline{0(r-1)}}(x_0). \\ v &= -(a-1)\alpha - \alpha^{-1} = g_{(r-1)0(r-1)0\dots}(x_0) = g_{\overline{(r-1)0}}(x_0). \\ w &= -1 - \alpha^3 = g_{2(r-1)0(r-1)0\dots}(x_0) = g_{\overline{2(r-1)0}}(x_0). \end{aligned}$$

Taking $t, t' \in [0, 1]$ such that $t = t'$. We have to prove that $f(t) = f(t')$ and for this we use Proposition 3.7 and the definition of ψ . We have to consider some cases.

- k be an even number and $a_k = 0$ or $2n - 1$, $n = 1, \dots, a - 1$.

Then $a_k + 1 = 1$ or $2n$ and by Proposition 3.7 we have

$$t = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{a_k}{(n_k, m_k)} + \frac{r-1}{(n_k+1, m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(n_k+2+i, m_k+i)} + \frac{r-3}{(n_k+2+i, m_k+1+i)} \right)$$

and

$$t' = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{a_k+1}{(n_k, m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n'_i, m'_i)}. \text{ Using the definition of } \psi \text{ we have}$$

$$f(t) = g_{b_1 \dots b_{k-1}(r-1)\overline{(r-1)0}}(x_0) = g_{b_1 \dots b_{k-1}(r-1)}(v),$$

$$f(t') = g_{b_1 \dots b_{k-1}(r-2)\overline{0(r-1)}}(x_0) = g_{b_1 \dots b_{k-1}(r-2)}(u),$$

if $a_k = 0$ or

$$f(t) = g_{b_1 \dots b_{k-1}(r-2n)\overline{(r-1)0}}(x_0) = g_{b_1 \dots b_{k-1}(r-2n)}(v)$$

$$f(t') = g_{b_1 \dots b_{k-1}(r-2n-1)\overline{2(r-1)0}}(x_0) = g_{b_1 \dots b_{k-1}(r-2n)}(w) \text{ if } a_k = 2n - 1, n = 1, \dots, a - 1.$$

By Lemma 3.4 $f(t) = f(t')$.

Using the same ideas we can prove the following cases (see [10]).

- k be an odd number and $a_k = 2n - 1$, $n = 1, \dots, a - 1$,

- k be an odd number and $a_k = 0$ or $2n$, $n = 1, \dots, a - 2$,

- k be an even number and $a_k = 2n$, $n = 1, \dots, a - 2$.

(2)– *f is injective.*

Suppose that $f(t) = f(t')$. According to Lemma 3.4 we have two possibilities:

$$- f(t) = g_{b_1 \dots b_{k-1}b_k}(u), b_k \in \{1, 3, 5, \dots, r - 2\} \text{ and } f(t') = g_{b_1 \dots b_{k-1}(b_k+1)}(v).$$

Using the above notations we have

$$f(t) = g_{b_1 \dots b_{k-1}b_k\overline{0(r-1)}}(x_0) \text{ and } f(t') = g_{b_1 \dots b_{k-1}(b_k+1)\overline{(r-1)0}}(x_0). \text{ We need to consider the following cases:}$$

- k is an even number, $b_k \neq r-2$. In this case $b_k = r-1-a_k$ and then $a_k = r-1-b_k$ is an odd number. By the definition of ψ we have:

- $a_{k+1} = 0$ because $b_{k+1} = 0$, a_k odd number,
- $a_{k+2} = 0$ because $b_{k+2} = r-1$, $k+2$ even number.

Following this idea is easy to see that $a_i = 0$, $\forall i \geq k+1$.

Therefore $t = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{r-1-b_k}{(n_k, m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n_i, m_i)}$.

We also have $b'_k = b_k + 1 = r-1-a'_k$ and then $a'_k = r-2-b_k \neq 0$ is an even number. By the definition of ψ and Proposition 3.5 we have:

- $a'_{k+1} = r-3$, $n'_{k+1} = n'_k$, $m'_{k+1} = m'_k + 1$ because $b'_{k+1} = r-1$, a'_k even,
- $a'_{k+2} = r-1$ because $b'_{k+2} = 0$, $k+2$ even,
- $a_{k+3} = r-3$, $n'_{k+3} = n'_{k+2}$, $m'_{k+3} = m'_{k+2} + 1$.

Following this idea we have

$$t' = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{r-2-b_k}{(n_k, m_k)} + \sum_{i=0}^{\infty} \left(\frac{r-3}{(n'_k+i, m'_k+1+i)} + \frac{r-1}{(n'_k+1+i, m'_k+1+i)} \right) \text{ and then } t = t'.$$

Using the same ideas we can prove the following cases (see [10]).

- k is an even number, $b_k = r-2$,
 - k is an odd number.
- $f(t) = g_{b_1 \dots b_{k-1} b_k}(-1 - \alpha^3)$, $b_k \in \{0, 2, \dots, r-3\}$ and $f(t') = g_{b_1 \dots b_{k-1} (b_k+1)}(v)$. Using the above notations we have $f(t) = g_{b_1 \dots b_{k-1} b_k \overline{(r-1)0}}(x_0)$ and $f(t') = g_{b_1 \dots b_{k-1} (b_k+1) \overline{(r-1)0}}(x_0)$. We have to consider the following cases:

- k is an even number. In this case $b_k = r-1-a_k$ and $a_k = r-1-b_k$ is an even number too and we can prove that

$$t = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{r-1-b_k}{(n_k, m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n_i, m_i)}$$

and

$$t' = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{r-2-b_k}{(n_k, m_k)} + \frac{r-1}{(n'_k+1, m'_k)} + \sum_{i=0}^{\infty} \left(\frac{r-1}{(n'_k+2+i, m'_k+i)} + \frac{r-3}{(n'_k+2+i, m'_k+1+i)} \right).$$

Then $t = t'$.

- k is an odd number. In this case $b_k = a_k$ or $b_k = a_k + 2$ and then $a_k = b_k$ or $a_k = b_k - 2$. We can prove that

$$t = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{a_k}{(n_k, m_k)} + \frac{r-3}{(n_k, m_k+1)} + \sum_{i=0}^{\infty} \left(\frac{r-3}{(n_k+i, m_k+2+i)} + \frac{r-1}{(n_k+1+i, m_k+2+i)} \right)$$

and

$$t' = \frac{a_1}{r} + \dots + \frac{a_{k-1}}{(n_{k-1}, m_{k-1})} + \frac{a_k+1}{(n_k, m_k)} + \sum_{i=k+1}^{\infty} \frac{0}{(n'_i, m'_i)}.$$

Then $t = t'$.

(3)- f is a continuous function.

Let us consider $t, t' \in [0, 1]$, $|t' - t| < r^{-N}$ as in Proposition 3.7. We have to consider the following cases:

- t and t' satisfying (1) of Proposition 3.7.

Here we have:

- (1) $f(t) = g_{b_1 \dots b_{k-1} (r-1) (r-1) 0 (r-1) 0 \dots b_{N+1} \dots}(x_0)$ and $f(t') = g_{b_1 \dots b_{k-1} (r-2) 0 (r-1) 0 (r-1) \dots b'_{N+1} \dots}(x_0)$ if $a_k = 0$. Then

$$\begin{aligned} |f(t) - f(t')| &= |g_{b_1 b_2 \dots b_{k-1} r-1}(z_1) - g_{b_1 b_2 \dots b_{k-1} r-2}(z_2)| \\ &\leq |\alpha|^{2(k-1)} |g_{r-1}(z_1) - g_{r-2}(z_2)|. \end{aligned}$$

As $g_{r-2}(u) = g_{r-1}(v)$ then

$$\begin{aligned} |f(t) - f(t')| &\leq |\alpha|^{2(k-1)} (|g_{r-1}(z_1) - g_{r-1}(v)| + |g_{r-2}(z_2) - g_{r-2}(u)|) \\ &\leq |\alpha|^{2(k-1)} (|\alpha|^2 + |\alpha|^3) \text{diam}(\mathcal{B}_{\alpha-1}) = |\alpha|^{2k} (1 + |\alpha|) \text{diam}(\mathcal{B}_{\alpha-1}), \end{aligned}$$

where $\text{diam}(\mathcal{B}_{\alpha-1})$ is the diameter of $\mathcal{B}_{\alpha-1}$.

- (2) $f(t) = g_{b_1 \dots b_{k-1}(r-2n)(r-1)0(r-1)0 \dots b_{N+1} \dots}(x_0)$ and
 $f(t') = g_{b_1 \dots b_{k-1}(r-2n-1)2(r-1)0(r-1)0 \dots b'_{N+1} \dots}(x_0)$ if $a_k = 2n - 1$. Then

$$\begin{aligned} |f(t) - f(t')| &= |g_{b_1 b_2 \dots b_{k-1}(r-2n)}(z_1) - g_{b_1 b_2 \dots b_{k-1}(r-2n-1)}(z_2)| \\ &\leq |\alpha|^{2(k-1)} |g_{r-2n}(z_1) - g_{r-2n-1}(z_2)|. \end{aligned}$$

As $g_{r-2n-1}(w) = g_{r-2n}(v)$ then

$$\begin{aligned} |f(t) - f(t')| &\leq |\alpha|^{2(k-1)} (|g_{r-2n}(z_1) - g_{r-2n}(v)| + |g_{r-2n-1}(z_2) - g_{r-2n-1}(w)|) \\ &\leq |\alpha|^{2(k-1)} (|\alpha|^2 + |\alpha|^3) \text{diam}(\mathcal{B}_{\alpha-1}) = |\alpha|^{2k} (1 + |\alpha|) \text{diam}(\mathcal{B}_{\alpha-1}). \end{aligned}$$

Using the same ideas we can prove the next cases (see [10]).

- t and t' satisfying item (2) of Proposition 3.7.
- t and t' satisfying item (3) of Proposition 3.7.
- t and t' satisfying item (4) of Proposition 3.7.

In all that cases we conclude that f is a continuous function. □

Now we can finally prove Theorem 1.4.

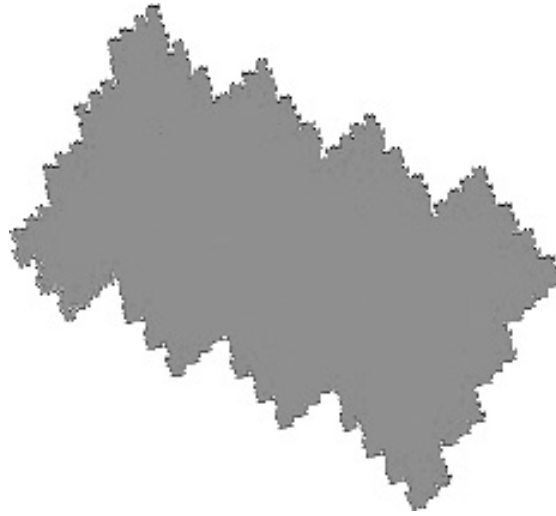
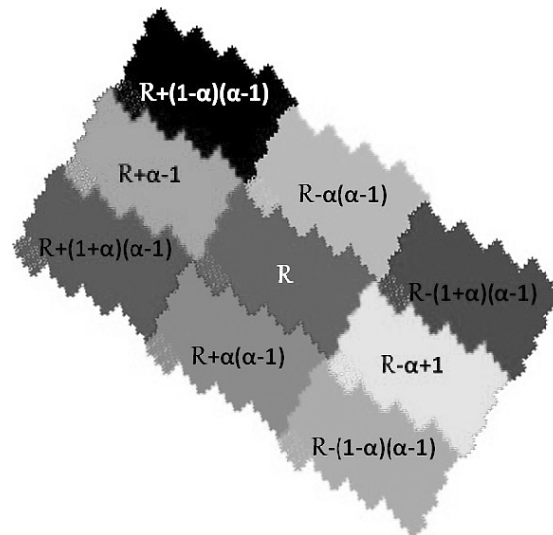
Proof. Let $\mathcal{Q} \subseteq \mathbb{R}^2$ be the set

$$\mathcal{Q} = \{(0, y), 0 \leq y \leq 1\} \cup \{(x, 1), 0 \leq x \leq 1\} \cup \{(1, y), 0 \leq y \leq 1\} \cup \{(x, 0), 0 \leq x \leq 1\}.$$

Using Proposition 3.1, Theorem 3.8 and Figure 4 we can prove that $F : \mathcal{Q} \rightarrow \partial \mathcal{R}_a$ given by

$$F(x, y) = \begin{cases} f(y), & \text{if } (x, y) = (0, y), 0 \leq y \leq 1 \\ (f_2 \circ f)(x), & \text{if } (x, y) = (x, 1), 0 \leq x \leq 1 \\ (f_3 \circ f)(y), & \text{if } (x, y) = (1, y), 0 \leq y \leq 1 \\ (f_1 \circ f)(x), & \text{if } (x, y) = (x, 0), 0 \leq x \leq 1 \end{cases}$$

is an homeomorphism. □

Fig.5. \mathcal{R}_4 Fig.6. Tiling induced by \mathcal{R}_4

References

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