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Author(s)	Song, Weiling; Zhang, Xiaojin; Huang, Zhaoyong
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THE SOCLE OF THE LAST TERM IN THE MINIMAL INJECTIVE RESOLUTION OF A GORENSTEIN MODULE

Weiling SONG, Xiaojin ZHANG and Zhaoyong HUANG

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Abstract

Let R be a left Noetherian ring, S a right Noetherian ring and ${}_RU$ a Gorenstein module with $S = \mathrm{End}({}_RU)$. If the injective dimensions of ${}_RU$ and U_S are finite, then the last term in the minimal injective resolution of ${}_RU$ has an essential socle.

1. Introduction

Recall that a left and right Noetherian ring is called *Gorenstein* if its left and right self-injective dimensions are finite. The following question still remains open.

Question 1.1. For a Gorenstein ring R, is the socle of the last term in the minimal injective resolution of ${}_RR$ non-zero?

The answer to this question is positive in any case of the following

(1) R is a left and right Artinian ring.
(2) The left and right self-injective dimensions of R are at most 2 ([7, Theorem 4.5]).
(3) R is an Auslander-Gorenstein ring ([5, Proposition 1.1]).

Furthermore, in the case (3) above, Iwanaga and Sato showed in [12, Theorem 6] that this socle is essential in the last term. As a natural generalization of Auslander's n-Gorenstein rings, Huang introduced in [8] the notion of n-Gorenstein modules such that a left and right Noetherian ring R is Auslander's n-Gorenstein if and only if it is n-Gorenstein as an R-module. Then Huang and Wang proved in [11, Theorem 3.1] that for left and right Noetherian rings R and S and a generalized tilting module ${}_RU$ with $S = \mathrm{End}({}_RU)$, if ${}_RU$ is $(n-2)$-Gorenstein with the injective dimensions of ${}_RU$ and U_S being n (where n is a non-negative integer), then the socle of the last term in the minimal injective resolution of ${}_RU$ is non-zero. In this paper we extend these results and prove the following

Theorem 1.2. *Let R be a left Noetherian ring, S a right Noetherian ring and ${}_RU$ a Gorenstein module with $S = \mathrm{End}({}_RU)$. If the injective dimensions of ${}_RU$ and U_S are finite, then the last term in the minimal injective resolution of ${}_RU$ has an essential socle.*

In Section 2, we give some terminology and some preliminary results. In Section 3, we introduce the notion of Gorenstein modules. Let R be a left Noetherian ring, S a right Noetherian ring and ${}_RU$ a Gorenstein module with $S = \mathrm{End}({}_RU)$ such that the injective dimensions of ${}_RU$ and U_S are equal to n. We first prove that $\mathrm{Ext}^n_{S^{op}}(N, U)$ is an Artinian

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left R-module for any finitely generated right S-module N. Then we get that any non-zero submodule of the last term in the minimal injective resolution of ${}_RU$ has a non-zero Artinian submodule. Theorem 1.2 follows from this result.

2. Preliminaries

Let R be an arbitrary associative ring with identity, and let $\operatorname{Mod} R$ be the category of left R-modules and $\operatorname{mod} R$ the category of finitely generated left R-modules. For a module M in $\operatorname{Mod} R$, we use $\operatorname{add}_R M$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of modules isomorphic to direct summands of finite direct sums of copies of ${}_RM$, and use $\operatorname{pd}_R M$, $\operatorname{id}_R M$ and $\operatorname{fd}_R M$ to denote the projective, injective and flat dimensions of M respectively. We use $\operatorname{gen}^*({}_RR)$ to denote the full subcategory of $\operatorname{mod} R$ consisting of modules admitting a degreewise finite R-projective resolution.

DEFINITION 2.1([17, 18]). A module ${}_RU$ is called *generalized tilting* (sometimes it is also called *Wakamatsu tilting*, see [2, 13]), if the following conditions are satisfied.

(1) ${}_RU \in \operatorname{gen}^*({}_RR)$.

(2) $\operatorname{Ext}_R^{\geq 1}(U, U) = 0$, that is, ${}_RU$ is *self-orthogonal*.

(3) There exists an exact sequence

$$0 \to {}_RR \to U_0 \to U_1 \to \cdots \to U_i \to \cdots$$

in $\operatorname{mod} R$ with all U_i in $\operatorname{add}_R U$, such that after applying the functor $\operatorname{Hom}_R(-, {}_RU)$ the sequence is still exact.

Let R and S be arbitrary associative rings with identity. Recall that a bimodule ${}_RU_S$ is called *faithfully balanced* if $R = \operatorname{End}(U_S)$ and $S = \operatorname{End}({}_RU)$. By [18, Corollary 3.2], we have that ${}_RU_S$ is faithfully balanced and self-orthogonal with ${}_RU \in \operatorname{gen}^*({}_RR)$ and $U_S \in \operatorname{gen}^*(S_S)$ if and only if ${}_RU$ is generalized tilting with $S = \operatorname{End}({}_RU)$, and if and only if U_S is generalized tilting with $R = \operatorname{End}(U_S)$. Note that a faithfully balanced and self-orthogonal bimodule ${}_RU_S$ with ${}_RU \in \operatorname{gen}^*({}_RR)$ and $U_S \in \operatorname{gen}^*(S_S)$ is also called a *semidualizing bimodule* (cf. [11]).

Let ${}_RU$ be a generalized tilting module with $S = \operatorname{End}({}_RU)$. We write

$${}^*(-) := \operatorname{Hom}({}_RU_S, -) \text{ and } (-)^* := \operatorname{Hom}(-, {}_RU_S).$$

We use

$$0 \to {}_RU \to E_0 \to E_1 \to \cdots \to E_i \to \cdots$$

to denote the minimal injective resolution of ${}_RU$ and $K_i = \operatorname{Ker}(E_i \to E_{i+1})$ for any $i \geq 0$ (note: $K_0 = {}_RU$), and use

$$0 \to U_S \to E'_0 \to E'_1 \to \cdots \to E'_i \to \cdots$$

to denote the minimal injective resolution of U_S. Following [8], we use $\operatorname{add-lim}{}_RU$ (resp. $\operatorname{add-lim} U_S$) to denote the full subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{Mod} S^{op}$) consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of ${}_RU$ (resp. U_S).

Definition 2.2([8]). For a module M in $\operatorname{Mod} R$, if there exists an exact sequence

$$\cdots \to U_n \to \cdots \to U_1 \to U_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all U_i in add-lim ${}_R U$, then we define U-lim.dim$_R\, M = \inf\{n \mid$ there exists an exact sequence

$$0 \to U_n \to \cdots \to U_1 \to U_0 \to M \to 0$$

in $\operatorname{Mod} R$ with all U_i in add-lim ${}_R U\}$. We set U-lim.dim$_R\, M$ infinity if no such an integer exists. For S^{op}-modules, we may define such a dimension similarly.

Let M be in $\operatorname{mod} R$ and $i \geq 0$. We say that the *grade* of M with respect to U, written $\operatorname{grade}_U M$, is at least i if $\operatorname{Ext}_R^{0 \leq j < i}(M, U) = 0$. We say that the *strong grade* of M with respect to U, written s.grade$_U\, M$, is at least i if $\operatorname{grade}_U X \geq i$ for any finitely generated R-submodule X of M (cf. [8]). The following result was proved in [8, Theorem 17.1.11] when R and S are two-sided Noetherian rings. Because the argument there remains valid in the setting here, we omit it.

Theorem 2.3. *Let R be a left Noetherian ring, S a right Noetherian ring and ${}_R U$ a generalized tilting module with $S = \operatorname{End}({}_R U)$. Then for any $n \geq 0$, the following statements are equivalent.*

(1) U-lim.dim$_R\, E_i \leq i$ *for any* $0 \leq i \leq n-1$.
(2) $\operatorname{fd}_S {}^*E_i \leq i$ *for any* $0 \leq i \leq n-1$.
(3) s.grade$_U \operatorname{Ext}_R^i(M, U) \geq i$ *for any* $M \in \operatorname{mod} R$.
(4) $\operatorname{fd}_{R^{op}} {}^*E'_i \leq i$ *for any* $0 \leq i \leq n-1$.
(5) U-lim.dim$_{S^{op}}\, E'_i \leq i$ *for any* $0 \leq i \leq n-1$.
(6) s.grade$_U \operatorname{Ext}_{S^{op}}^i(N, U) \geq i$ *for any* $N \in \operatorname{mod} S^{op}$.

If one of the equivalent conditions in Theorem 2.3 is satisfied, then ${}_R U$ (equivalent U_S) is called *n-Gorenstein* ([8, 10]). So a left and right Noetherian ring R is (Auslander) n-Gorenstein ([12]) if and only if ${}_R R$ is n-Gorenstein, and if and only if R_R is n-Gorenstein.

3. Main Results

In this section, we give the proof of Theorem 1.2. We begin with the following

Lemma 3.1. *Let R be a ring and*

$$0 \to K \xrightarrow{f} M \to N \to 0$$

an exact sequence in $\operatorname{Mod} R$ *with $N \neq 0$ and f an essential monomorphism. Then* $\operatorname{Ext}_R^1(X, K) \neq 0$ *for any non-zero R-submodule X of N.*

Proof. Let X be a non-zero R-submodule of N and $\alpha : X \hookrightarrow N$ the inclusion. Then we have the following pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K & \dashrightarrow & W & \dashrightarrow & X & \dashrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & K & \xrightarrow{f} & M & \longrightarrow & N & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Coker}\,\alpha & = & \mathrm{Coker}\,\alpha & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

We claim that the upper row does not split. Otherwise, if it splits, then K is isomorphic to a non-trivial direct summand of W. So K is not isomorphic to an essential submodule of W, and hence K is not an essential submodule of M. It contradicts that f is an essential monomorphism. The claim is proved. Thus we have $\mathrm{Ext}_R^1(X, K) \neq 0$. □

From now on, R is a left Noetherian ring, S is a right Noetherian ring and ${}_RU$ is a generalized tilting module with $S = \mathrm{End}({}_RU)$. By [9, Theorem 2.7], we have that $\mathrm{id}_R\, U = \mathrm{id}_{S^{op}}\, U$ provided both of them are finite.

Lemma 3.2. *If* $\mathrm{id}_R\, U = n$, *then for any non-zero R-submodule X of E_n, we have* $\mathrm{Ext}_R^n(X, U) \neq 0$.

Proof. Because $\mathrm{id}_R\, U = n$, we have the following exact sequence

$$0 \to K_{n-1} \to E_{n-1} \to E_n \to 0$$

in $\mathrm{Mod}\, R$ with $K_{n-1} \to E_{n-1}$ an essential monomorphism. Let X be a non-zero R-submodule of E_n. Then $\mathrm{Ext}_R^1(X, K_{n-1}) \neq 0$ by Lemma 3.1. Thus we have $\mathrm{Ext}_R^n(X, U) \cong \mathrm{Ext}_R^1(X, K_{n-1}) \neq 0$. □

By Lemma 3.2, we have the following

Proposition 3.3.

(1) *If* $\mathrm{id}_R\, U = n(\geq 1)$, *then E_0 and E_n have no isomorphic non-zero direct summands.*

(2) *If* $\mathrm{id}_R\, U = \mathrm{id}_{S^{op}}\, U = n$, *then* $\mathrm{pd}_S\, {}^*E = n$ *for any non-zero direct summand E of E_n.*

Proof. (1) Assume that $E \neq 0$ is an indecomposable direct summand of E_0 and E is isomorphic to a direct summand of E_n. Then E is the injective envelope of some finitely generated non-zero R-submodule Y of E by [14, Theorem 2.4]. Since U is an essential submodule of E_0, we have $X := U \cap Y \neq 0$. From the exact sequence

$$0 \to X \to U \to U/X \to 0,$$

we get the following exact sequence

$$0 = \mathrm{Ext}_R^n(U, U) \to \mathrm{Ext}_R^n(X, U) \to \mathrm{Ext}_R^{n+1}(U/X, U).$$

Because $\mathrm{id}_R\, U = n$, we have $\mathrm{Ext}_R^{n+1}(U/X, U) = 0$. It induces that $\mathrm{Ext}_R^n(X, U) = 0$. On the other hand, because $0 \neq X < E \hookrightarrow E_n$, we have $\mathrm{Ext}_R^n(X, U) \neq 0$ by Lemma 3.2. It is a contradiction.

(2) Let E be a non-zero direct summand of E_n. Then $\mathrm{Ext}_R^n(E, U) \neq 0$ by Lemma 3.2. Because $\mathrm{id}_{S^{op}} U = n$, we have $\mathrm{fd}_S {}^*E \leq n$ by [9, Lemma 2.6(1)]. Because $\mathrm{id}_R U = n$, we have $\mathrm{pd}_S {}^*E \leq n$ by [9, Theorem 2.11)]. If $\mathrm{pd}_S {}^*E = m < n$, then by [9, Theorem 2.9], there exists an exact sequence

$$0 \to U_m \to \cdots \to U_1 \to U_0 \to E \to 0$$

in $\mathrm{Mod}\, R$ with all U_i in $\mathrm{Add}_R U$, where $\mathrm{Add}_R U$ is the full subcategory of $\mathrm{Mod}\, R$ consisting of modules isomorphic to direct summands of direct sums of copies of ${}_RU$. So $\mathrm{Ext}_R^n(E, U) \cong \mathrm{Ext}_R^{n-m}(U_m, U) = 0$ by [9, Proposition 2.2(2)]. It is a contradiction. Thus we have $\mathrm{pd}_S {}^*E = n$. □

Lemma 3.4. *The following statements are equivalent.*

(1) *There exists* $0 \neq M \in \mathrm{mod}\, R$ *such that* $\mathrm{Ext}_R^{\geq 0}(M, U) = 0$.

(2) *There exists an exact sequence*

$$0 \to Q_0 \xrightarrow{k_1} Q_1 \xrightarrow{k_2} Q_3 \to \cdots$$

in $\mathrm{mod}\, S^{op}$ *with all* Q_i *in* $\mathrm{add}\, U_S$, *such that* $\mathrm{Ext}_{S^{op}}^1(L_1, U) \neq 0$ *and* $\mathrm{Ext}_{S^{op}}^1(L_i, U) = 0$ *with* $i \geq 2$, *where* $L_i = \mathrm{Coker}\, k_i$ *for any* $i \geq 1$.

Proof. (2) ⇒ (1) By (2), we get the following exact sequence

$$\cdots \to {Q_2}^* \xrightarrow{{k_2}^*} {Q_1}^* \xrightarrow{{k_1}^*} {Q_0}^* \to M \to 0$$

in $\mathrm{mod}\, R$, where $M = \mathrm{Coker}\, {k_1}^*$. Then $M \cong \mathrm{Ext}_{S^{op}}^1(L_1, U) \neq 0$ by assumption. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & Q_0 & \xrightarrow{k_1} & Q_1 & \xrightarrow{k_2} & Q_2 & \longrightarrow & \cdots \\
 & & & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & M^* & \longrightarrow & Q_0^{**} & \xrightarrow{{k_1}^{**}} & Q_1^{**} & \xrightarrow{{k_2}^{**}} & Q_2^{**} & \longrightarrow & \cdots .
\end{array}$$

Because the upper row is exact by assumption, so is the lower row. It implies $\mathrm{Ext}_R^{\geq 0}(M, U) = 0$.

(1) ⇒ (2) Let $0 \neq M \in \mathrm{mod}\, R$ with $\mathrm{Ext}_R^{\geq 0}(M, U) = 0$ and let

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

be an exact sequence in $\mathrm{mod}\, R$ with all P_i projective. Then we get the following exact sequence

$$0 \to P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} P_2^* \to \cdots \tag{3.1}$$

in $\mathrm{mod}\, S^{op}$ with all P_i^* in $\mathrm{add}\, U_S$. Notice that $P_i \cong {P_i}^{**}$ naturally for any $i \geq 0$, so the sequence

$$\cdots \to P_2^{**} \xrightarrow{d_2^{**}} P_1^{**} \xrightarrow{d_1^{**}} P_0^{**}$$

in $\operatorname{mod} R$ is exact. Set $L_i := \operatorname{Coker} d_i^*$ for any $i \geq 1$. Then $\operatorname{Ext}^1_{S^{op}}(L_1, U) \cong M \neq 0$ and $\operatorname{Ext}^1_{S^{op}}(L_i, U) = 0$ for any $i \geq 2$, and so (3.1) is the desired exact sequence. □

Following [4], an injective resolution

$$0 \to N \xrightarrow{\varepsilon_0} I_0 \xrightarrow{\varepsilon_1} I_1 \xrightarrow{\varepsilon_2} I_2 \to \cdots$$

of a module N in $\operatorname{Mod} S^{op}$ is said to *have a redundant image* if some $\operatorname{Im} \varepsilon_n = \oplus_{j=1}^m W_j$ such that each W_j is isomorphic to a direct summand of some $\operatorname{Im} \varepsilon_{i_j}$ with $i_j \neq n$. It is clear that the minimal injective resolution of N_S has a redundant image if $\operatorname{id}_{S^{op}} N < \infty$.

Lemma 3.5. *If U_S has an injective resolution with a redundant image, then $M = 0$ for any $M \in \operatorname{mod} R$ with $\operatorname{Ext}^{\geq 0}_R(M, U) = 0$.*

Proof. Let $M \in \operatorname{mod} R$ with $\operatorname{Ext}^{\geq 0}_R(M, U) = 0$. If $M \neq 0$, then by Lemma 3.4 and its proof, we get an exact sequence

$$0 \to Q_0 \xrightarrow{k_1} Q_1 \xrightarrow{k_2} Q_3 \to \cdots$$

in $\operatorname{mod} S^{op}$ with all Q_i in $\operatorname{add} U_S$ such that $\operatorname{Ext}^1_{S^{op}}(L_1, U) \cong M \neq 0$ and $\operatorname{Ext}^1_{S^{op}}(L_i, U) = 0$ for any $i \geq 2$, where $L_i = \operatorname{Coker} k_i$ for any $i \geq 1$. It yields that $\operatorname{Ext}^{\geq 2}_{S^{op}}(L_1, U) = 0$ and $\operatorname{Ext}^{\geq 1}_{S^{op}}(L_i, U) = 0$ for any $i \geq 2$.

Because U_S has an injective resolution with a redundant image, there exists an injective resolution

$$0 \to U_S \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} I_1 \xrightarrow{\alpha_2} I_2 \to \cdots$$

of U_S in $\operatorname{Mod} S^{op}$ with some $\operatorname{Im} \alpha_n = \oplus_{j=1}^m W_j$ such that each W_j is isomorphic to a direct summand of some $\operatorname{Im} \alpha_{i_j}$ with $i_j \neq n$. Then we have

$$\operatorname{Ext}^1_{S^{op}}(L_1, U) \cong \operatorname{Ext}^{n+1}_{S^{op}}(L_{n+1}, U) \cong \operatorname{Ext}^1_{S^{op}}(L_{n+1}, \operatorname{Im} \alpha_n) \cong \oplus_{j=1}^m \operatorname{Ext}^1_{S^{op}}(L_{n+1}, W_j).$$

Since

$$\operatorname{Ext}^1_{S^{op}}(L_{n+1}, \operatorname{Im} \alpha_{i_j}) \cong \operatorname{Ext}^{i_j+1}_{S^{op}}(L_{n+1}, U) = 0$$

by the above argument, we have $\operatorname{Ext}^1_{S^{op}}(L_{n+1}, W_j) = 0$ for any $1 \leq j \leq m$. It follows that $\operatorname{Ext}^1_{S^{op}}(L_1, U) = 0$, a contradiction. Consequently we conclude that $M = 0$. □

We introduce the notion of Gorenstein modules as follows.

Definition 3.6. We called ${}_RU$ (resp. U_S) *Gorenstein* if U-lim.dim$_R E_i \leq i$ (resp. U-lim.dim$_{S^{op}} E'_i \leq i$) for any $i \geq 0$.

By Theorem 2.3, we have that ${}_RU$ is Gorenstein if and only if U_S is Gorenstein, and if and only if ${}_RU$ (equivalent U_S) is n-Gorenstein for all n. Following Theorem 2.3 and [16, Theorem 3.6], we know that the notion defined above is a non-commutative version of that of Gorenstein modules in [16]. The following proposition is useful in proving the main result.

Proposition 3.7. *Let ${}_RU$ be a Gorenstein module with $\operatorname{id}_R U = \operatorname{id}_{S^{op}} U = n$. Then $\operatorname{Ext}^n_{S^{op}}(N, U) \in \operatorname{mod} R$ is Artinian for any $N \in \operatorname{mod} S^{op}$.*

Proof. Let $X \subseteq Y$ be left R-submodules of $\mathrm{Ext}^n_{S^{op}}(N, U)$. Then by Theorem 2.3, we have

$$\mathrm{Ext}^{0\le i<n}_R(X, U) = 0 = \mathrm{Ext}^{0\le i<n}_R(Y, U).$$

So from the exact sequence

$$0 \to X \to Y \to Y/X \to 0,$$

we get $\mathrm{Ext}^{0\le i<n}_R(Y/X, U) = 0$ and the following exact sequence

$$0 \to \mathrm{Ext}^n_R(Y/X, U) \to \mathrm{Ext}^n_R(Y, U) \to \mathrm{Ext}^n_R(X, U) \to 0.$$

Put $A_0 := \mathrm{Ext}^n_{S^{op}}(N, U)$ and let

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

be a descending chain of left R-submodules of A_0. Then from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{i+1} & \longrightarrow & A_0 & \longrightarrow & A/A_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & A_i & \longrightarrow & A_0 & \longrightarrow & A/A_i & \longrightarrow & 0,
\end{array}$$

we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^n_R(A_0/A_i, U) & \longrightarrow & \mathrm{Ext}^n_R(A_0, U) & \longrightarrow & \mathrm{Ext}^n_R(A_i, U) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}^n_R(A_0/A_{i+1}, U) & \longrightarrow & \mathrm{Ext}^n_R(A_0, U) & \longrightarrow & \mathrm{Ext}^n_R(A_{i+1}, U) & \longrightarrow & 0.
\end{array}$$

We can regard $\mathrm{Ext}^n_R(A_0/A_i, U)$ as a right S-submodule of $\mathrm{Ext}^n_R(A_0, U)$. Then the following sequence is the ascending chain of right S-submodules of $\mathrm{Ext}^n_R(A_0, U)$:

$$\mathrm{Ext}^n_R(A_0/A_1, U) \subseteq \mathrm{Ext}^n_R(A_0/A_2, U) \subseteq \cdots \subseteq \mathrm{Ext}^n_R(A_0, U). \tag{3.2}$$

Since $A_0 \in \mathrm{mod}\, R$, $\mathrm{Ext}^n_R(A_0, U) \in \mathrm{mod}\, S^{op}$ is Noetherian. So (3.2) terminates at some step m. Then from the exact sequence

$$0 \to A_m/A_{m+1} \to A_0/A_{m+1} \to A_0/A_m \to 0,$$

we get the following exact sequence

$$\mathrm{Ext}^n_R(A_0/A_m, U) \xrightarrow{\cong} \mathrm{Ext}^n_R(A_0/A_{m+1}, U) \to \mathrm{Ext}^n_R(A_m/A_{m+1}, U) \to 0$$

and $\mathrm{Ext}^n_R(A_m/A_{m+1}, U) = 0$. Then by the above argument, we have that $\mathrm{Ext}^{0\le i\le n}_R(A_m/A_{m+1}, U) = 0$. Because $\mathrm{id}_R\, U = n$, we have $\mathrm{Ext}^{\ge 0}_R(A_m/A_{m+1}, U) = 0$. Because $\mathrm{id}_{S^{op}}\, U = n$, by Lemma 3.5 we have that $A_m/A_{m+1} = 0$ and $A_m = A_{m+1}$. Thus $A_0 (= \mathrm{Ext}^n_{S^{op}}(N, U))$ is an Artinian left R-module. □

By Proposition 3.7, we get the following

Corollary 3.8. *Let ${}_RU$ be a Gorenstein module with* $\mathrm{id}_R\, U = \mathrm{id}_{S^{op}}\, U = n$ *and* $N \in \mathrm{mod}\, S^{op}$. *Then we have*

(1) $\mathrm{Ext}^n_{S^{op}}(N, U)$ *embeds in* $E^{(t)}_n$ *for some* $t \ge 1$.

(2) *If M is a non-zero R-submodule of* $\mathrm{Ext}^n_{S^{op}}(N,U)$*, then* $\mathrm{Ext}^n_R(M,U)\neq 0$.

Proof. (1) Let $N\in \mathrm{mod}\, S^{op}$. Then $\mathrm{Ext}^n_{S^{op}}(N,U)\in \mathrm{mod}\, R$ embeds in a finite direct sum of copies of $\oplus_{i=0}^n E_i$ by [8, Lemma 17.2.5]. On the other hand, by [3, Proposition VI.5.3] and Theorem 2.4, we have

$$\mathrm{Hom}_R(\mathrm{Ext}^n_{S^{op}}(N,U),E_i)\cong \mathrm{Tor}^S_n(N,{}^*E_i)=0$$

for any $0\le i\le n-1$. So there exists some $t\ge 1$ such that $\mathrm{Ext}^n_{S^{op}}(N,U)$ embeds in $E_n^{(t)}$.

(2) Let M be a non-zero R-submodule of $\mathrm{Ext}^n_{S^{op}}(N,U)$. Because we have the following exact sequence

$$0\to K^{(t)}_{n-1}\to E^{(t)}_{n-1}\to E^{(t)}_n\to 0$$

in $\mathrm{Mod}\, R$ with $K^{(t)}_{n-1}\to E^{(t)}_{n-1}$ an essential monomorphism, by (1) and Lemma 3.1 we have that $\mathrm{Ext}^n_R(M,U)^{(t)}\cong \mathrm{Ext}^1_R(M,K_{n-1})^{(t)}\cong \mathrm{Ext}^1_R(M,K^{(t)}_{n-1})\neq 0$ and $\mathrm{Ext}^n_R(M,U)\neq 0$. □

We now are in a position to prove the following

Theorem 3.9. *Let ${}_RU$ be a Gorenstein module with* $\mathrm{id}_R\, U=\mathrm{id}_{S^{op}}\, U=n$*. Then any non-zero submodule of E_n has a non-zero Artinian submodule.*

Proof. Let V be a non-zero submodule of E_n and E a non-zero indecomposable direct summand of the injective envelope Q of V. Then E is the injective envelope of some finitely generated non-zero R-submodule X of E by [14, Theorem 2.4]. By Lemma 3.2, we have $\mathrm{Ext}^n_R(X,U)\neq 0$. Let I be an injective cogenerator for $\mathrm{Mod}\, S^{op}$. Then by [3, Proposition VI.5.3], we have

$$\mathrm{Tor}^R_n({}^*I,X)\cong \mathrm{Hom}_{S^{op}}(\mathrm{Ext}^n_R(X,U),I)\neq 0.$$

Because $\mathrm{fd}_{R^{op}}\, {}^*I=\mathrm{id}_R\, U=n$ by [9, Lemma 2.6(2)] and assumption, the inclusion $X\hookrightarrow E$ induces a monomorphism $\mathrm{Tor}^R_n({}^*I,X)\rightarrowtail \mathrm{Tor}^R_n({}^*I,E)$. It yields $\mathrm{Tor}^R_n({}^*I,E)\neq 0$. By [11, Lemmas 5.1(c) and 4.1], we have that $\mathrm{Tor}^R_{\ge 1}({}^*I,U)=0$ and both I and E are in the Bass class with respect to ${}_RU_S$. So we get the following isomorphisms

$$\begin{aligned}
&\mathrm{Tor}^S_n(I,{}^*E)\\
&\cong \mathrm{Tor}^S_n({}^*I\otimes_R U,{}^*E)\ \text{(by [11, Lemma 4.1])}\\
&\cong \mathrm{Tor}^R_n({}^*I,E)\ \text{(by [11, Theorem 6.4(c)])},
\end{aligned}$$

and hence $\mathrm{Tor}^S_n(I,{}^*E)\neq 0$. Let $\{N_i\}$ be the set of all finitely generated right S-submodules of I. Because $I=\varinjlim N_i$ and the functor Tor commutes with $\varinjlim$ by [15, Example 5.32(iii) and Proposition 7.8], there exists some N_i such that $\mathrm{Tor}^S_n(N_i,{}^*E)\neq 0$. Then by [3, Proposition VI.5.3] again, we have

$$\mathrm{Hom}_R(\mathrm{Ext}^n_{S^{op}}(N_i,U),E)\cong \mathrm{Tor}^S_n(N_i,{}^*E)\neq 0.$$

So there exists a non-zero homomorphism $f:\mathrm{Ext}^n_{S^{op}}(N_i,U)\to E$ in $\mathrm{Mod}\, R$. By Proposition 3.7, $\mathrm{Ext}^n_{S^{op}}(N_i,U)\in \mathrm{mod}\, R$ is Artinian. Thus as an R-quotient module of $\mathrm{Ext}^n_{S^{op}}(N_i,U)$, $\mathrm{Im}\, f$ is a non-zero Artinian R-submodule of $E(<Q)$. Because V is essential in Q, we have that $V\cap \mathrm{Im}\, f$ is a non-zero Artinian R-submodule of V. □

Theorem 1.2 is a special case of the following result.

Corollary 3.10. *Let $_RU$ be a Gorenstein module with* $\mathrm{id}_R U = \mathrm{id}_{S^{op}} U = n$. *Then any non-zero R-submodule of E_n has an essential socle.*

Proof. By Theorem 3.9, we have that any non-zero R-submodule of E_n has a non-zero Artinian R-submodule. Now the assertion follows from [1, Corollary 9.10]. □

The second assertion in the following result is a supplement to Proposition 3.3(1).

Corollary 3.11. *Let $_RU$ be a Gorenstein module with* $\mathrm{id}_R U = \mathrm{id}_{S^{op}} U = n$. *Then we have*

(1) *Any non-zero direct summand of E_n is a direct sum of the injective envelopes of some simple left R-modules.*

(2) *If $n \geq 1$ and S is further a right Artinian ring, then $\oplus_{i=0}^{n-1} E_i$ and E_n have no isomorphic non-zero direct summands.*

Proof. (1) By Corollary 3.10 and [14, Proposition 2.1].

(2) If S is a right Artinian ring, then $\mathrm{pd}_S {}^*(\oplus_{i=0}^{n-1} E_i) = \mathrm{fd}_S {}^*(\oplus_{i=0}^{n-1} E_i) \leq n-1$ by Theorem 2.3. So $\mathrm{pd}_S {}^*E' \leq n-1$ for any direct summand E' of $\oplus_{i=0}^{n-1} E_i$. On the other hand, we have $\mathrm{pd}_S {}^*E = n$ for any non-zero direct summand E of E_n by Proposition 3.3(2). Thus the assertion follows. □

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Weiling Song
Department of Applied Mathematics
College of Science, Nanjing Forestry University
Nanjing 210037, Jiangsu Province
P.R. China
e-mail: songwl@njfu.edu.cn

Xiaojin Zhang
School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, Jiangsu Province
P.R. China
e-mail: xjzhang@nuist.edu.cn

Zhaoyong Huang
Department of Mathematics
Nanjing University, Nanjing 210093, Jiangsu Province
P.R. China
e-mail: huangzy@nju.edu.cn