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UNSTABILIZED WEAKLY REDUCIBLE HEEGAARD SPLITTINGS

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Abstract
In this paper, we give a sufficient condition for a (weakly reducible) Heegaard splitting to be unstabilized and uncritical. We also give a sufficient condition for a Heegaard splitting to be critical.

1. Introduction

All 3-manifolds in this paper are assumed to be compact and orientable. All surfaces in 3-manifolds are assumed to be properly embedded and orientable.

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S = \partial_+ W = \partial_+ V$, then we say M has a Heegaard splitting, denoted by $M = V \cup_S W$; and S is called a Heegaard surface of M. Moreover, if the genus $g(S)$ of S is minimal among all Heegaard surfaces of M, then $g(S)$ is called the genus of M, denoted by $g(M)$. If there are essential disks $B \subset V$ and $D \subset W$ such that $\partial B = \partial D$ (resp. $\partial B \cap \partial D = \emptyset$), then $V \cup_S W$ is said to be reducible (resp. weakly reducible). Otherwise, it is said to be irreducible (resp. strongly irreducible). If there are essential disks $B \subset V$ and $D \subset W$, such that $|B \cap D| = 1$, then $M = V \cup_S W$ is said to be stabilized; otherwise, $M = V \cup_S W$ is said to be unstabilized. If a surface F in a 3-manifold M is incompressible and not parallel to ∂M, then F is said to be essential. If a separating surface F in M is compressible on both sides of F, then F is said to be bicompressible. If every compressing disk in one side of F intersects every compressing disk in the other side, then F is said to be strongly irreducible. If F is incompressible except for $[\partial F]$, then F is said to be almost incompressible; if F is bicompressible except for $[\partial F]$, then F is said to be almost bicompressible; if F is strongly irreducible except for $[\partial F]$, then F is said to be almost strongly irreducible, where $[\partial F]$ is the isotopy class of ∂F.

Let M be a 3-manifold, and S be a closed separating compressible surface in M. S is said to be critical (see [1]), if the compressing disks for S can be partitioned into two sets C_0 and C_1, and there is at least one pair of disks $V_i, W_i \in C_i$ $(i = 0, 1)$ on opposite sides of S, such that $V_i \cap W_i = \emptyset$, and if $V \in C_i$ and $W \in C_{1-i}$ lie on opposite sides of S, then $V \cap W \neq \emptyset$. If S is not critical, then S is said to be uncritical. There are some examples, see [2]–[4], [8]–[10].

Let S be a closed surface with $g(S) \geq 2$. The curve complex of S (see [5]) is the complex whose vertices are the isotopy classes of essential simple closed curves on S, and $k + 1$

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vertices determine a k-simplex if they are represented by pairwise disjoint curves. If S is a torus, the curve complex of S (see [11], [12]) is the complex whose vertices are the isotopy classes of essential simple closed curves on S, and $k+1$ vertices determine a k-simplex if they can be represented by a collection of curves, any two of which intersect in only one point. We denote the curve complex of S by $C(S)$. For any two vertices in $C(S)$, one can define the distance $d_{C(S)}(x, y)$ to be the minimal number of 1-simplices in a simplicial path jointing x to y over all such possible paths.

If S is a surface with $\partial S \neq \emptyset$, then we can define the curve complex $C(S)$ of S and $d_{C(S)}(x, y)$ for any two vertices x and y in $C(S)$ by the same way, where the vertex of $C(S)$ is the isotopy class of non-∂-parallel essential simple closed curves on S. The distance of the Heegaard splitting $M = V \cup_S W$ with $g(S) \geq 2$ (see [6]) is $d(S) = \mathrm{Min}\{d_{C(S)}(\alpha,\beta) \mid \alpha$ bounds a disk in V and β bounds a disk in $W\}$. If S' is an almost bicompressible subsurface of S, then $d(S') = \mathrm{Min}\{d_{C(S')}(\alpha,\beta) \mid \alpha$ bounds a disk in V *and* β bounds a disk in $W\}$ is said to be local Heegaard distance of S' respect to $d(S)$ (see [7], [13]).

In this paper, we give a sufficient condition for a (weakly reducible) Heegaard splitting to be unstabilized and uncritical. We also give a sufficient condition for a Heegaard splitting to be critical as follows:

Theorem 1. *Let M be a 3-manifold, $M = V \cup_S W$ be a Heegaard splitting of M, D be an essential disk in V such that ∂D cuts S into an almost incompressible surface F and an almost strongly irreducible surface S'. If $d(S') \geq 5$, then $M = V \cup_S W$ is unstabilized and uncritical.*

Corollary 2. *Let M be a 3-manifold, $M = V \cup_S W$ be a Heegaard splitting of M, ψ be an essential simple closed curve on S which cuts S into an almost incompressible surface F and an almost strongly irreducible surface S'. If $d(S') \geq 9$, then $M = V \cup_S W$ is unstabilized.*

Theorem 3. *Let M be an irreducible 3-manifold, $M = V \cup_S W$ be a Heegaard splitting of M, D be an essential disk in V such that ∂D cuts S into an almost incompressible surface F and an almost strongly irreducible surface S'.*

(1) *If S is critical, then $d(S') \leq 4$.*

(2) *If there are two essential disks $D_V \subset V$ and $D_W \subset W$, such that D_V is not isotopic to D, $D_W \cap D \neq \emptyset$ and $D_W \cap D_V = \emptyset$, then S is critical.*

2. The proof of Theorem 1

Firstly, we show that $M = V \cup_S W$ is unstabilized. Assume on the contrary that $M = V \cup_S W$ is stabilized. Then, there are two essential disks $D_V \subset V$ and $D_W \subset W$, such that $|D_V \cap D_W| = 1$. So, there is an essential simple closed curve γ on S which bounds an essential disk D^γ_V in V and an essential disk D^γ_W in W such that the 2-sphere $S^\gamma = D^\gamma_V \cup D^\gamma_W$ bounds a once-punctured standard genus one Heegaard splitting of the 3-sphere (i.e. a 3-ball).

Proposition 4. $\gamma \cap \partial D \neq \emptyset$.

Proof. Assume on the contrary that $\gamma \cap \partial D = \emptyset$. If γ is parallel to ∂D, then F and S' lie in opposite sides of S^γ. Since F is almost incompressible, S' lies in the 3-ball bounded by S^γ. Then, S' is a once-punctured torus. Hence, $d(S') \leq 1$, a contradiction. So, γ is a non-∂-parallel essential simple closed curve on F or S'. Since F is almost incompressible,

γ lies in S' and $d(S') = 0$, a contradiction. □

By Proposition 4, we may assume that $\gamma \cap \partial D \neq \emptyset$ and $|\gamma \cap \partial D|$ is minimal. So, each component of $\gamma \cap S'$ (resp. $\gamma \cap F$) is an essential arc on S' (resp. F). Recall that γ bounds an essential disk D_V^γ in V and an essential disk D_W^γ in W. If $|\gamma \cap S'| = |\gamma \cap F| = n$, then D_V^γ (resp. D_W^γ) is said to be an n-disk in V (resp. W).

Since $D_V^\gamma \cap D \neq \emptyset$, we may assume that each component of $D_V^\gamma \cap D$ is an arc on both D_V^γ and D. Let α be a component of $D_V^\gamma \cap D$. Then, α cuts a disk D_α from D_V^γ. If $intD_\alpha \cap D = \emptyset$, then D_α is said to be an outermost disk of D_V^γ, and α is said to be an outermost arc of $D_V^\gamma \cap D$ on D_V^γ. Since F is almost incompressible, all outermost disks of D_V^γ lie in the component of $cl(V - D)$ which contains S'. Let D_0 be an outermost disk of D_V^γ. Then, $|\partial D_0 \cap S'| = |\partial D_0 \cap D| = 1$, and $\partial D_0 \cap S'$ is an essential arc on S'. Let $l_1 = \partial D_0 \cap S'$ and $l_1' = \partial D_0 \cap D$. We push l_1' into ∂D and denote it by l_1''. Let $l^1 = l_1 \cup l_1''$. After isotopy, we may assume that l^1 lies in S'. Since l_1 is essential on S', l^1 is non-∂-parallel essential on S' and bounds an essential disk D_l in V. So, $d_{C(S')}(l^1, \partial D_l) = 0$.

If there is an essential disk D_h in W with $\partial D_h \subset S'$, such that ∂D_h is non-∂-parallel on S' and disjoint from a component h of $\gamma \cap S'$, then h cuts ∂D into two arcs h_1 and h_1'. Let $h^1 = h \cup h_1$. After isotopy, we may assume that h^1 lies in S' and $h^1 \cap \partial D_h = \emptyset$. Since h is essential on S', h^1 is non-∂-parallel on S'. So, $d_{C(S')}(h^1, \partial D_h) \leq 1$. Since $h \cap l_1 = \emptyset$, $d_{C(S')}(h^1, l^1) \leq 2$. So, $d(S') \leq d_{C(S')}(\partial D_l, \partial D_h) \leq d_{C(S')}(\partial D_l, l^1) + d_{C(S')}(l^1, h^1) + d_{C(S')}(h^1, \partial D_h) \leq 3$, a contradiction.

By the argument as above, we may assume that for any essential disk D^W in W with $\partial D^W \subset S'$ and any component η of $\gamma \cap S'$, if ∂D^W is non-∂-parallel on S', then $\partial D^W \cap \eta \neq \emptyset$. If D_W^γ (which is bounded by γ) is a 1-disk in W, then $|\gamma \cap S'| = 1$. Then, $|D_V^\gamma \cap D| = 1$. Hence, there are two outermost disks of D_V^γ which lie in different components of $cl(V - D)$, a contradiction. So, we may assume that D_W^γ is an n-disk with $n \geq 2$.

Proposition 5 ([2])**.** *There are an essential disk D_k in W with $\partial D_k \subset S'$ and a component l_2 of $\gamma \cap S'$, such that ∂D_k is non-∂-parallel on S' and $d_{C(S')}(l^2, \partial D_k) \leq 3$, where l^2 is obtained from l_2 by attaching a component of $cl(\partial D - \partial l_2)$, after isotopy, l^2 is non-∂-parallel essential on S'.*

Proof. Recall that for any essential disk D^W in W with $\partial D^W \subset S'$ and any component α of $\partial D_W^\gamma \cap S'$, if ∂D^W is non-∂-parallel on S', then $\partial D^W \cap \alpha \neq \emptyset$. We may assume that $|D^W \cap D_W^\gamma|$ is minimal among all essential disks in W, whose boundaries lie in S' and are non-∂-parallel. So, each component of $D^W \cap D_W^\gamma$ is an arc on both D^W and D_W^γ. Since $|D^W \cap D_W^\gamma|$ is minimal, and for each component α of $\partial D_W^\gamma \cap S'$, $\alpha \cap \partial D^W \neq \emptyset$, both endpoints of each arc of $D_W^\gamma \cap D^W$ on D_W^γ lie in different components of $\partial D_W^\gamma \cap S'$. For each subdisk D_W' of D_W^γ which is cut by D^W, if $\partial D_W'$ contains m components or subcomponents of $\partial D_W^\gamma \cap S'$, then D_W' is said to be a pseudo m-disk. For each component α of $\partial D_W^\gamma \cap S'$, there are two components α_1 and α_2 of $\partial D_W^\gamma \cap F$, which are adjacent to α. Let $L_\alpha = \{l \mid l$ is an arc of $D_W^\gamma \cap D^W$ on D_W^γ, such that $l \cap \alpha \neq \emptyset\}$.

Suppose $\alpha \in \partial D_W^\gamma \cap S'$ and l_α is a component of L_α. Then, l_α cuts D_W^γ into two disks D' and D''. We may assume that D' is a pseudo m_1-disk, and D'' is a pseudo m_2-disk. Then, $m_2 = n - m_1 + 2$, see Figure 1. If D' (resp. D'') is a pseudo 2-disk, then l_α is said to be ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ. If all components of L_α are ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ, then

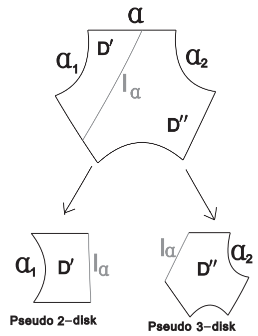

Fig. 1. D' and D'' cut by l_α

L_α is said to be ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W.

Lemma 6. *There are at least two components α and β of $\partial D^\gamma_W \cap S'$, such that both L_α and L_β are ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W.*

Proof. If D^γ_W is an n-disk with $n = 2, 3$, then the Lemma holds, see Figure 2. So, we may assume that D^γ_W is an n-disk with $n \geq 4$. If all components of $D^\gamma_W \cap D^W$ on D^γ_W are ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W, then the Lemma holds. So, we may assume that there is a component k_1 of $D^\gamma_W \cap D^W$ on D^γ_W, such that k_1 is not ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W. Then, k_1 cuts D^γ_W into two disks D^1_k and $D^{1'}_k$. Suppose D^1_k is a pseudo n_1-disk and $D^{1'}_k$ is a pseudo $n_1^{'}$-disk. Since k_1 is not ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W, $3 \leq n_1, n_1^{'} < n$.

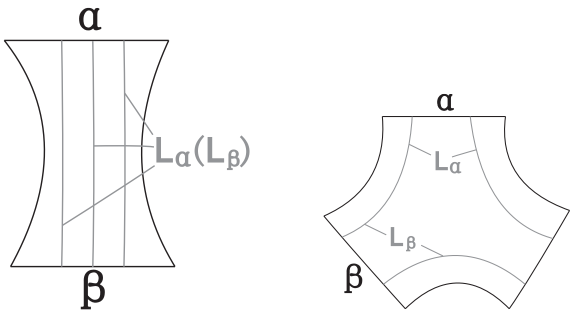

Fig. 2. n-disk with $n = 2, 3$

First, we consider D^1_k. Note that $D^1_k \cap D^W \subsetneq D^\gamma_W \cap D^W$. If D^1_k is a pseudo 3-disk, then there is only one component α of $\partial D^\gamma_W \cap S'$ on ∂D^1_k, such that $\alpha \cap k_1 = \emptyset$. Hence, L_α is ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W. So, we may assume that D^1_k is a pseudo n_1-disk with $4 \leq n_1 < n$. If all components of $D^1_k \cap D^W$ on D^1_k are ∂-parallel to $(\partial D^\gamma_W \cap F) \cup k_1$ in D^1_k, then there is a component α of $\partial D^\gamma_W \cap S'$, such that $\alpha \cap k_1 = \emptyset$ and L_α is ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W. So, we may assume that there is a component k_2 of $D^1_k \cap D^W$ on D^1_k, such that k_2 is not ∂-parallel to $(\partial D^\gamma_W \cap F) \cup k_1$ in D^1_k. Then, k_2 cuts a disk D^2_k from D^1_k, such that ∂D^2_k does not contain k_1. Hence, $D^2_k \cap D^W \subsetneq D^1_k \cap D^W \subsetneq D^\gamma_W \cap D^W$.

Since k_2 is not ∂-parallel to $(\partial D^\gamma_W \cap F) \cup k_1$ in D^1_k, we may assume that D^2_k is a pseudo n_2-disk with $3 \leq n_2 < n_1 < n$. By the same argument as D^1_k, either there is a component α of $\partial D^\gamma_W \cap S'$, which is disjoint from k_2, such that L_α is ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W, or there is a component k_3 of $D^2_k \cap D^W$ on D^2_k, such that k_3 is not ∂-parallel to $(\partial D^\gamma_W \cap F) \cup k_2$ in

D_k^2. Then, k_3 cuts a disk D_k^3 from D_k^2, such that ∂D_k^3 does not contain k_2. Then, $D_k^3 \cap D^W \subsetneq D_k^2 \cap D^W \subsetneq D_k^1 \cap D^W \subsetneq D_W^\gamma \cap D^W$. Since k_3 is not ∂-parallel to $(\partial D_W^\gamma \cap F) \cup k_2$ in D_k^2, we may assume that D_k^3 is a pseudo n_3-disk with $3 \leq n_3 < n_2 < n_1 < n$.

We continue this procedure as above, either there is a component α of $\partial D_W^\gamma \cap S'$, such that L_α is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ, or there is a component k_m of $D_k^{m-1} \cap D^W$ on D_k^{m-1}, such that k_m is not ∂-parallel to $(\partial D_W^\gamma \cap F) \cup k_{m-1}$ in D_k^{m-1} $(m \geq 2)$. Then, k_m cuts a disk D_k^m from D_k^{m-1}, such that ∂D_k^m does not contain k_{m-1}. Hence, $D_k^m \cap D^W \subsetneq D_k^{m-1} \cap D^W \subsetneq \ldots \subsetneq D_k^1 \cap D^W \subsetneq D_W^\gamma \cap D^W$. Since k_m is not ∂-parallel to $(\partial D_W^\gamma \cap F) \cup k_{m-1}$ in D_k^{m-1}, we may assume that D_k^m is a pseudo n_m-disk with $3 \leq n_m < n_{m-1} < \cdots < n_2 < n_1 < n$. Since n is finite, either there is a component α of $\partial D_W^\gamma \cap S'$, such that L_α is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ, or $n_m = 3$. If D_k^m is a pseudo n_m-disk with $n_m = 3$, then there is only one component α of $\partial D_W^\gamma \cap S'$, which is disjoint from k_m, such that L_α is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ. Finally, we obtain a component α of $\partial D_W^\gamma \cap S'$, such that L_α is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ.

Second, we consider $D_k^{1'}$. By the same argument as D_k^1, there is a component β $(\neq \alpha)$ of $\partial D_W^\gamma \cap S'$, such that L_β is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ. So, the Lemma holds. □

By Lemma 6, there is a component l_2 of $\partial D_W^\gamma \cap S'$, such that L_{l_2} is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ. Let l_2' and l_2'' be two components of $\partial D_W^\gamma \cap F$, such that l_2' and l_2'' are adjacent to l_2. Since $|\gamma \cap \partial D|$ is minimal, both l_2' and l_2'' are essential on F.

Lemma 7. *There is a* 1*-disk* D^1 *in* W*, such that* $(\partial D^1 \cap S') \cap l_2 = \emptyset$*, and* $\partial D^1 \cap F$ *is parallel to* l_2' *or* l_2''*.*

Proof. Let k be a component of L_{l_2}. Since L_{l_2} is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ, k cuts a pseudo 2-disk D^k from D_W^γ. If $intD^k \cap L_{l_2} = \emptyset$, then D^k is said to be an outermost disk of D_W^γ, and k is said to be an outermost arc of $D^W \cap D_W^\gamma$ on D_W^γ. Let k_1 be a component of L_{l_2}, such that k_1 is an outermost arc of $D^W \cap D_W^\gamma$ on D_W^γ. Then, k_1 cuts an outermost disk D_1^k from D_W^γ, such that $intD_1^k \cap L_{l_2} = \emptyset$. So, D_1^k is a pseudo 2-disk. Since L_{l_2} is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ, we may assume that k_1 is parallel to l_2', where l_2' is adjacent to l_2 on ∂D_W^γ. Note that k_1 also cuts D^W into two disks $D_k^{1'}$ and $D_k^{1''}$. Let $D_{k_1} = D_k^{1'} \cup D_1^k$ and $D_{k_1}' = D_k^{1''} \cup D_1^k$. Since k_1 is parallel to l_2' in D_W^γ, after isotopy, both $\partial D_{k_1} \cap F$ and $\partial D_{k_1}' \cap F$ are parallel to l_2'. Since l_2' is essential on F and F is almost incompressible, both $\partial D_{k_1} \cap S'$ and $\partial D_{k_1}' \cap S'$ are essential on S'. Hence, D_{k_1} and D_{k_1}' are 1-disks in W. After isotopy, $|D_{k_1} \cap D_W^\gamma| < |D^W \cap D_W^\gamma|$, $|D_{k_1}' \cap D_W^\gamma| < |D^W \cap D_W^\gamma|$, $D_{k_1} \cap D_W^\gamma \subsetneq D^W \cap D_W^\gamma$, and $D_{k_1}' \cap D_W^\gamma \subsetneq D^W \cap D_W^\gamma$.

Suppose $|D_{k_1} \cap D_W^\gamma| \leq |D_{k_1}' \cap D_W^\gamma|$, we only consider D_{k_1}. Let $L_{l_2}^1 = \{k \mid k$ is a component of $D_W^\gamma \cap D_{k_1}$ on D_W^γ, such that $k \cap l_2 \neq \emptyset\}$. Then, $L_{l_2}^1 \subsetneq L_{l_2}$. Hence, $L_{l_2}^1$ is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ. If $L_{l_2}^1 = \emptyset$, let $D^1 = D_{k_1}$, then $l_2 \cap (\partial D^1 \cap S') = \emptyset$ and $\partial D^1 \cap F$ is parallel to l_2'. Hence, the Lemma holds. If $L_{l_2}^1 \neq \emptyset$, let k_2 be a component of $L_{l_2}^1$, such that k_2 is an outermost arc of $D_{k_1} \cap D_W^\gamma$ on D_W^γ. Then, k_2 cuts an outermost disk D_2^k from D_W^γ, such that $intD_2^k \cap L_{l_2}^1 = \emptyset$. So, D_2^k is a pseudo 2-disk. Since $L_{l_2}^1$ is ∂-parallel to $\partial D_W^\gamma \cap F$ in D_W^γ, we may assume that k_2 is parallel to l_2', where l_2' is adjacent to l_2 in ∂D_W^γ. Let $D_k^{2'}$ be a subdisk of D_{k_1}, which is cut by k_2, such that $\partial D_k^{2'}$ does not contain $\partial D_{k_1} \cap F$, and $D_{k_2} = D_2^k \cup D_k^{2'}$.

By the same argument as D_{k_1}, D_{k_2} is a 1-disks in W and $\partial D_{k_2} \cap F$ is parallel to l_2'. After isotopy, $|D_{k_2} \cap D_W^\gamma| < |D_{k_1} \cap D_W^\gamma| < |D^W \cap D_W^\gamma|$ and $D_{k_2} \cap D_W^\gamma \subsetneq D_{k_1} \cap D_W^\gamma \subsetneq D^W \cap D_W^\gamma$. Let $L_{l_2}^2 = \{k \mid k$ is a component of $D_W^\gamma \cap D_{k_2}$ on D_W^γ, such that $k \cap l_2 \neq \emptyset\}$. Then, $L_{l_2}^2 \subsetneq L_{l_2}^1 \subsetneq L_{l_2}$.

Hence, $L^2_{l_2}$ is ∂-parallel to $\partial D^\gamma_W \cap F$ in D^γ_W. By the same proof as D_{k_1}, either $D^1 = D_{k_2}$ such that $l_2 \cap (D^1 \cap S') = \emptyset$ and $D^1 \cap F$ is parallel to l'_2, or we obtain a 1-disk D_{k_3} in W, such that $\partial D_{k_3} \cap F$ is parallel to l'_2, where l'_2 is adjacent to l_2 in ∂D^γ_W, $D_{k_3} \cap D^\gamma_W \subsetneq D_{k_2} \cap D^\gamma_W \subsetneq D_{k_1} \cap D^\gamma_W \subsetneq D^W \cap D^\gamma_W$, and $\{k \mid k$ is a component of $D^\gamma_W \cap D_{k_3}$ on D^γ_W, such that $k \cap l_2 \neq \emptyset\} = L^3_{l_2} \subsetneq L^2_{l_2} \subsetneq L^1_{l_2} \subsetneq L_{l_2}$. Continue this procedure as above, since $|D^W \cap D^\gamma_W|$ is finite, finally, we obtain a 1-disk D_{k_m} ($m \geq 1$) in W, such that $\partial D_{k_m} \cap F$ is parallel to l'_2, where l'_2 is adjacent to l_2 in ∂D^γ_W, $D_{k_m} \cap D^\gamma_W \subsetneq D_{k_{m-1}} \cap D^\gamma_W \subsetneq ... \subsetneq D_{k_1} \cap D^\gamma_W \subsetneq D^W \cap D^\gamma_W$, and $\emptyset = \{k \mid k$ is a component of $D^\gamma_W \cap D_{k_m}$ on D^γ_W, such that $k \cap l_2 \neq \emptyset\} = L^m_{l_2} \subsetneq L^{m-1}_{l_2} \subsetneq ... \subsetneq L^1_{l_2} \subsetneq L_{l_2}$. Let $D^1 = D_{k_m}$. Then, $l_2 \cap (D^1 \cap S') = \emptyset$ and $D^1 \cap F$ is parallel to l'_2. Hence, the Lemma holds. □

Lemma 8. *If D^1 is a 1-disk in W, then there is an essential disk D_k in W with $\partial D_k \subset S'$, such that $D_k \cap D^1 = \emptyset$.*

Proof. Assume on the contrary that for each essential disk D_k in W with $\partial D_k \subset S'$, $D_k \cap D^1 \neq \emptyset$. We may assume that $|D_k \cap D^1|$ is minimal among all essential disks in W with $\partial D_k \subset S'$. If ∂D_k is parallel to $\partial S'$, then $|D_k \cap D^1| = 1$. Let $\delta = D_k \cap D^1$. Then, there is a subdisk D_δ of D^1 which is cut by δ, such that D_δ contains $\partial D^1 \cap F$. We can push δ into F. After isotopy, we denote D_δ by D'_δ. So, D'_δ is an essential disk in W with $\partial D'_\delta \subset F$ and $\partial D'_\delta$ is not parallel to ∂F. It is a contradiction to the fact that F is almost incompressible.

So, we may assume that ∂D_k is not parallel to $\partial S'$. Since $|D_k \cap D^1|$ is minimal, each component of $D_k \cap D^1$ is an arc on both D_k and D^1. Let λ be an outermost arc of $D^1 \cap D_k$ on D^1, such that λ cuts a subdisk D_λ from D^1 with $intD_\lambda \cap D_k = \emptyset$, and ∂D_λ does not contain $\partial D^1 \cap F$. Also, λ cuts D_k into D^1_k and D^2_k. Let $D^1_\lambda = D_\lambda \cup D^1_k$ and $D^2_\lambda = D_\lambda \cup D^2_k$. Since D_k is essential in W with $\partial D_k \subset S'$ and ∂D_k is not parallel to $\partial S'$, at least one of D^1_λ and D^2_λ is essential in W whose boundary lies in S' and is not parallel to $\partial S'$. We may assume that D^1_λ is essential in W with $\partial D^1_\lambda \subset S'$ and ∂D^1_λ is not parallel to $\partial S'$. So, $|D^1_\lambda \cap D^1| < |D_k \cap D^1|$, a contradiction. □

By Lemma 7, we may assume that D^1 is a 1-disk in W, such that $l_2 \cap (\partial D^1 \cap S') = \emptyset$, and $\partial D^1 \cap F$ is parallel to l'_2, where l'_2 is adjacent to l_2 in ∂D^γ_W and l'_2 is essential on F. For convenience, let $\gamma_1 = \partial D^1 \cap S'$ and $\gamma_2 = \partial D^1 \cap F$. So, $l_2 \cap \gamma_1 = \emptyset$, and γ_2 is parallel to l'_2. By Lemma 8, there is an essential disk D_k in W with $\partial D_k \subset S'$, such that $\partial D_k \cap \gamma_1 = \emptyset$. Let l^2 be a non-∂-parallel essential simple closed curve on S', which is obtained from l_2 by attaching a component of $cl(\partial D - \partial l_2)$, γ^1 be a non-∂-parallel essential simple closed curve on S', which is obtained from γ_1 by attaching a component of $cl(\partial D - \partial\gamma_1)$. Since $l_2 \cap \gamma_1 = \emptyset$, $|l^2 \cap \gamma^1| \leq 1$. So, $d_{C(S')}(l^2, \gamma^1) \leq 2$. Since $\partial D_k \cap \gamma_1 = \emptyset$, $\partial D_k \cap \gamma^1 = \emptyset$. Then, $d_{C(S')}(\gamma^1, \partial D_k) \leq 1$. Hence, $d_{C(S')}(l^2, \partial D_k) \leq d_{C(S')}(l^2, \gamma^1) + d_{C(S')}(\gamma^1, \partial D_k) \leq 3$. So, the Proposition holds. □

By Proposition 5, there are an essential disk D_k in W with $\partial D_k \subset S'$ and a component l_2 of $\gamma \cap S'$, such that ∂D_k is non-∂-parallel on S' and $d_{C(S')}(l^2, \partial D_k) \leq 3$, where l^2 is obtained from l_2 by attaching a component of $cl(\partial D - \partial l_2)$, after isotopy, l^2 is non-∂-parallel essential on S'. Since both l_1 and l_2 are components of $\gamma \cap S'$, $l_1 \cap l_2 = \emptyset$. Then, $|l^1 \cap l^2| \leq 1$. Since l^1 bounds an essential disk D_l in V with $\partial D_l \subset S'$ and ∂D_l is not ∂-parallel, there is an essential disk D^l in V with $\partial D^l \subset S'$, such that ∂D^l is non-∂-parallel on S' and $d_{C(S')}(\partial D^l, l^2) \leq 1$. So,

$d(S') \leq d_{C(S')}(\partial D^l, \partial D_k) \leq d_{C(S')}(\partial D^l, l^2) + d_{C(S')}(l^2, \partial D_k) \leq 4$, a contradiction.

Secondly, we show that the Heegaard surface S is uncritical. Assume on the contrary that S is critical. Then, all compressing disks for S can be partitioned into two sets C_0 and C_1, and there is at least one pair of disks V_i, $W_i \in C_i$ ($i = 0, 1$) on opposite sides of S, such that $V_i \cap W_i = \emptyset$, and if $V \in C_i$ and $W \in C_{1-i}$ lie on opposite sides of S, then $V \cap W \neq \emptyset$.

We may assume that D lies in C_0, D_V and D_W lie in C_1 and $D_V \cap D_W = \emptyset$. By definition, $D \cap D_W \neq \emptyset$. Since ∂D cuts S into an almost incompressible surface F and an almost strongly irreducible surface S', by the argument as above, there are essential disks $D^V \subset V$, $D^W \subset W$ and a component $l_2 \subset (\partial D_W \cap S')$, such that ∂D^V is non-∂-parallel on S', ∂D^W is non-∂-parallel on S', $d_{C(S')}(\partial D^V, l^2) \leq 1$ and $d_{C(S')}(\partial D^W, l^2) \leq 3$, where l^2 is obtained from l_2 by attaching a component of $cl(\partial D - \partial l_2)$, after isotopy, l^2 is non-∂-parallel essential on S'. So, $d(S') \leq d_{C(S')}(\partial D^V, \partial D^W) \leq d_{C(S')}(\partial D^V, l^2) + d_{C(S')}(l^2, \partial D^W) \leq 4$, a contradiction. □

3. The proof of Corollary 2

Assume on the contrary that $M = V \cup_S W$ is stabilized. Then, there are two essential disks $D_V \subset V$ and $D_W \subset W$, such that $|D_V \cap D_W| = 1$. So, there is an essential simple closed curve γ on S which bounds an essential disk D_V^γ in V and an essential disk D_W^γ in W such that the 2-sphere $S^\gamma = D_V^\gamma \cup D_W^\gamma$ bounds a once-punctured standard genus one Heegaard splitting of the 3-sphere (i.e. a 3-ball). By arguments similar to those for Proposition 4, we may assume that $\gamma \cap \psi \neq \emptyset$ and $|\gamma \cap \psi|$ is minimal. So, each component of $\gamma \cap S'$ (resp. $\gamma \cap F$) is an essential arc on S' (resp. F).

If D_V^γ (resp. D_W^γ) is a 1-disk in V (resp. W), then $|\gamma \cap S'| = 1$. Let $l = \gamma \cap S'$. By Lemma 10 in [2], there are essential disks $D^V \subset V$ and $D^W \subset W$, such that ∂D^V is non-∂-parallel on S', ∂D^W is non-∂-parallel on S', $d_{C(S')}(\partial D^V, l^1) \leq 1$ and $d_{C(S')}(\partial D^W, l^1) \leq 1$, where l^1 is obtained from l by attaching a component of $cl(\psi - \partial l)$, after isotopy, l^1 is non-∂-parallel essential on S'. So, $d(S') \leq d_{C(S')}(\partial D^V, \partial D^W) \leq d_{C(S')}(\partial D^V, l^1) + d_{C(S')}(l^1, \partial D^W) \leq 2$, a contradiction.

So, we may assume that D_V^γ (resp. D_W^γ) is an n-disk in V (resp. W) with $n \geq 2$. By arguments in the proof of Theorem 1, there are essential disks $D^V \subset V$, $D^W \subset W$, and components l_1 and l_2 of $\gamma \cap S'$, such that ∂D^V is non-∂-parallel on S', ∂D^W is non-∂-parallel on S', $d_{C(S')}(\partial D^V, l^1) \leq 3$ and $d_{C(S')}(\partial D^W, l^2) \leq 3$, where l^i ($i = 1, 2$) is obtained from l_i by attaching a component of $cl(\psi - \partial l_i)$, after isotopy, l^i is non-∂-parallel essential on S'. Since both l_1 and l_2 are components of $\gamma \cap S'$, $l_1 \cap l_2 = \emptyset$. Then, $|l^1 \cap l^2| \leq 1$. Hence, $d_{C(S')}(l^1, l^2) \leq 2$. So, $d(S') \leq d_{C(S')}(\partial D^V, \partial D^W) \leq d_{C(S')}(\partial D^V, l^1) + d_{C(S')}(l^1, l^2) + d_{C(S')}(l^2, \partial D^W) \leq 8$, a contradiction. □

4. The proof of Theorem 3

(1) By arguments in the proof of Theorem 1, if S is critical, then $d(S') \leq 4$.

(2) For all compressing disks for S, we partition them into two sets C_0 and C_1. Let $V \cap C_0 = \{D\}$, $W \cap C_0 = \{D_W |\ D_W$ is an essential disk in W and $D_W \cap D = \emptyset\}$, $V \cap C_1 = \{D_V |\ D_V$ is an essential disk in V and D_V is not isotopic to $D\}$ and $W \cap C_1 = \{D_W |\ D_W$ is an essential disk in W and $D_W \cap D \neq \emptyset\}$. Since S' is almost strongly irreducible, $V \cap C_1 \neq \emptyset$ and

$W \cap C_0 \neq \emptyset$. Since there is an essential disk $D_W \subset W$ with $D_W \cap D \neq \emptyset$, $W \cap C_1 \neq \emptyset$.

In C_0, for any disk D_W^0 in $W \cap C_0$, $D_W^0 \cap D = \emptyset$. In C_1, there are two essential disks $D_V^1 \subset (V \cap C_1)$ and $D_W^1 \subset (W \cap C_1)$, such that $D_W^1 \cap D_V^1 = \emptyset$. For any disk D_W^1 in $W \cap C_1$, $D_W^1 \cap D \neq \emptyset$. For any disks $D_W^0 \subset (W \cap C_0)$ and $D_V^1 \subset (V \cap C_1)$, since M is irreducible, F is almost incompressible and S' is almost strongly irreducible, ∂D_W^0 lies in S' and ∂D_W^0 is non-∂-parallel on S'. If $D_V^1 \cap D = \emptyset$, since S' is almost strongly irreducible, $D_W^0 \cap D_V^1 \neq \emptyset$. If $D_V^1 \cap D \neq \emptyset$, we may assume that $|D_V^1 \cap D|$ is minimal and each component of $D_V^1 \cap D$ is an arc on both D_V^1 and D. Assume on the contrary that $D_W^0 \cap D_V^1 = \emptyset$. By arguments in the proof of Theorem 1, all outermost disks of D_V^1 lies in the component of $cl(V - D)$ which contains S'. Let D_0 be an outermost disk of D_V^1. We can push ∂D_0 into S'. After isotopy, we still denote it by D_0. Since ∂D_0 is non-∂-parallel on S' and $D_W^0 \cap D_0 = \emptyset$, it is a contradiction to the fact that S' is almost strongly irreducible. □

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