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PERFECT FLUID SPACETIMES WITH HARMONIC GENERALIZED CURVATURE TENSOR

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Abstract

We show that n-dimensional perfect fluid spacetimes with divergence-free conformal curvature tensor and constant scalar curvature are generalized Robertson Walker (GRW) spacetimes; as a consequence a perfect fluid Yang pure space is a GRW spacetime. We also prove that perfect fluid spacetimes with harmonic generalized curvature tensor are, under certain conditions, GRW spacetimes. As particular cases, perfect fluids with divergence-free projective, concircular, conharmonic or quasi-conformal curvature tensor are GRW spacetimes. Finally, we explore some physical consequences of such results.

1. Introduction

Generalized Robertson-Walker spacetimes were introduced in 1995 by Alías, Romero and Sánchez (see [1, 2]). A Lorentzian manifold M of dimension $n \geq 3$ is named generalized Robertson-Walker (GRW) spacetime if it is the warped product

$$M = I \times_{q^2} M^*$$

with base $(I, -dt^2)$, warping function q and fiber (M^*, g^*), where M^* is an $(n-1)$-dimensional Riemannian manifold [1, 2, 34, 35].
If M^* is a 3-dimensional Riemannian manifold of constant curvature, the spacetime is called Robertson-Walker spacetime. Therefore, GRW spacetimes are a wide generalization of Robertson-Walker spacetimes on which standard cosmology is modelled. They include the Einstein-de Sitter spacetime, the Friedman cosmological models, the static Einstein spacetime, the de Sitter spacetime, and have applications as inhomogeneous spacetimes admitting an isotropic radiation (see [34]). We refer to the works by Romero et al. [32, 33], Sánchez [34], Gutiérrez and Olea [17], and the review [27] for an exhaustive presentation of geometric and physical properties.

The following deep result was recently proved by B-Y Chen (for similar results see the works by Yano [40, 41] and the recent paper [8]):

Theorem 1.1 (Chen, [3]). *Let (M, g) be an n-dimensional Lorentzian manifold, $n \geq 3$. The spacetime is a GRW spacetime if and only if it admits a timelike vector such that $\nabla_k X_j = \rho g_{kj}$.*

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According to Yano, a vector field is named torse-forming if $\nabla_k v_j = \omega_k v_j + f g_{kj}$, being f a scalar function and ω_k a non vanishing one-form, [40, 41]. Its properties in pseudo-Riemannian manifolds were studied by Mikeš and Rachůnek, [28, 31]. The vector is named concircular if ω_k is a gradient.

Lorentzian manifolds with Ricci tensor of the form

$$R_{kl} = \alpha g_{kl} + \beta u_k u_l$$

where α, β are scalar fields and u_k is a unit timelike vector (that is $u_j u^j = -1$), are called perfect fluid spacetimes and are of interest in general relativity; in the language of pure differential geometry they are named quasi-Einstein.

Riemannian quasi-Einstein spaces were investigated by Defever and Deszcz [9] and by Chaki et al. [5] (see also [11]); pseudo-Riemannian quasi-Einstein spaces arose in the study of exact solutions of Einstein's equations and in the investigation of quasi-umbilical hypersurfaces of pseudo-Euclidean spaces [12, 13]. Robertson-Walker spacetimes are quasi Einstein (see [4, 38] and references therein for details on these spacetimes; for further recent results see [6]).

In paper [10] (Lemma 4.1, Theorem 4.1 and Corollary 4.1) R. Deszcz proved that a quasi-Einstein Riemannian manifold with harmonic Weyl tensor is, under certain conditions, the warped product $I \times_{q^2} M^*$, where M^* is a $(n-1)$-dimensional Riemannian manifold of constant curvature.

The Weyl tensor is the traceless part of the Riemann tensor [30]:

$$C_{jklm} = R_{jklm} + \frac{g_{jm}R_{kl} - g_{km}R_{jl} + R_{jm}g_{kl} - R_{km}g_{jl}}{n-2} - R\frac{g_{jm}g_{kl} - g_{km}g_{jl}}{(n-1)(n-2)},$$

where $R_{jl} = R_{jml}{}^m$ and $R = R^j{}_j$. The number of algebraically independent components of the Ricci and the Weyl tensors equals that of the Riemann tensor. Since in general relativity only the Ricci tensor is coupled to matter by the Einstein equations, the Weyl tensor describes the pure gravity degrees of freedom.

Perfect fluid spacetimes in four dimensions with divergence-free Weyl tensor (i.e. $\nabla_m C_{jkl}{}^m = 0$) were firstly investigated by Shepley and Taub [37], and successively by Sharma [36] and Coley [7]. Recently in [23] Mantica, Molinari and De extended some results to n-dimensional perfect fluids and proved the following

Theorem 1.2 ([23], Theorem 2.1)**.** *Let (M, g) be a perfect fluid spacetime. If $\nabla_k u_j = \nabla_j u_k$ and $\nabla_m C_{jkl}{}^m = 0$ then:*
i) *u_j is a concircular vector field and it is rescalable to a timelike vector X_j such that*

$$\nabla_k X_j = \rho g_{kj} \quad \text{and} \quad \nabla_k \rho = \frac{\alpha - \beta}{1-n} X_k; \tag{1.1}$$

ii) *(M, g) is a GRW spacetime whose fiber is Einstein, that is, $R^*_{\alpha\beta} = \frac{R^*}{n-1} g^*_{\alpha\beta}$;*
iii) *the vector u_i annihilates the Weyl tensor, that is,*

$$C_{jklm} u^m = 0. \tag{1.2}$$

In [26] Mantica, De and Suh proved that a Ricci simple spacetime (that is, a Lorentzian manifold for which the Ricci tensor takes the form $R_{ij} = -R u_i u_j$) with vanishing divergence of the Weyl tensor admits a proper concircular vector field and it is necessarily a GRW spacetime. Further it was shown that stiff matter perfect fluid spacetimes or massless scalar fields with timelike gradient, with divergence-free Weyl tensor, are GRW spacetimes.

In the framework of Yang's gravitational theory, Guilfoyle and Nolan [16] introduced the notion of Yang pure space, i.e. an n-dimensional Lorentzian manifold with the properties $\nabla_m C_{jkl}{}^m = 0$ and $\nabla_k R = 0$. They proved that a 4-dimensional perfect fluid spacetime with $p + \mu \neq 0$ is a Yang pure space if and only if it is a Robertson-Walker spacetime with state equation $p = \frac{\mu}{3} + c$, being c a constant.

The present paper is organized as follows. In section 2 we prove that for an n-dimensional perfect fluid spacetime satisfying $\nabla_m C_{jkl}{}^m = 0$ with constant scalar curvature, the condition $\nabla_k u_j = \nabla_j u_k$ is implied and thus the space is a GRW spacetime. Therefore, the condition $\nabla_k R = 0$ is more stringent than the closedness of the covector u_j. Next, we prove that an n-dimensional perfect fluid spacetime with divergence-free generalized curvature tensor, under certain conditions satisfies $\nabla_m C_{jkl}{}^m = 0$ and $\nabla_k R = 0$, and thus is a GRW spacetime. Generalized curvature tensors were introduced by Kobayashi and Nomizu [19]; they share the algebraic properties of the Riemann and the Weyl tensors: $K_{ijkl} = -K_{jikl} = -K_{ijlk}$ and $K_{ijkl} + K_{jkil} + K_{kijl} = 0$.
As particular cases, perfect fluids with divergence-free projective, concircular, conharmonic or quasi-conformal curvature tensor are GRW spacetimes. In section 3, we give a look to some physical consequences of the above mentioned results.

2. Perfect fluid with harmonic generalized curvature tensor

We outline the main results of this section. We prove that an n-dimensional perfect fluid spacetime with $\nabla_m C_{jkl}{}^m = 0$ and $\nabla_k R = 0$ is a GRW spacetime. As a consequence, a Yang pure space is a GRW spacetime. Next we consider perfect fluid spacetimes with a generalized curvature tensor $K_{jkl}{}^m$ with the property

$$\nabla_m K_{jkl}{}^m = A\, \nabla_m R_{jkl}{}^m + B\,(g_{kl}\nabla_j R - g_{jl}\nabla_k R).$$

We prove that if $\nabla_m K_{jkl}{}^m = 0$ and either (i) $A \neq 2B(n-1)$ or (ii) $A = 2B(n-1)$ and $\nabla_k u_j = \nabla_j u_k$, then M is a GRW spacetime with Einstein fiber and with constant scalar curvature in the case (i). As particular cases, we consider perfect fluids with vanishing divergence of the projective, concircular or conharmonic curvature tensor.

Lemma 2.1. *Let (M, g) be an n-dimensional manifold ($n > 3$) whose Ricci tensor is $R_{kl} = \alpha g_{kl} + \beta u_k u_l$, where u_k is a unit timelike vector, and $\beta \neq 0$. If the Weyl tensor is divergence-free, $\nabla_m C_{jkl}{}^m = 0$, and if $\nabla_k R = 0$, then u_j is irrotational (i.e. $\nabla_k u_j = \nabla_j u_k$).*

Proof. The divergence of the conformal curvature tensor is [21, 24]:

$$\nabla_m C_{jkl}{}^m = \frac{n-3}{n-2}\left[\nabla_k R_{jl} - \nabla_j R_{kl} + \frac{1}{2(n-1)}(g_{kl}\nabla_j R - g_{jl}\nabla_k R)\right]. \tag{2.1}$$

The conditions $\nabla_m C_{jkl}{}^m = 0$ and $\nabla_k R = 0$, give $\nabla_k R_{jl} - \nabla_l R_{jk} = 0$. Now we write the covariant derivative of the Ricci tensor, and use the fact that $R = n\alpha - \beta$, so that $n\nabla_k \alpha = \nabla_k \beta$. We obtain

$$\nabla_j(\beta u_k u_l) - \nabla_k(\beta u_j u_l) = -\frac{1}{n}(g_{kl}\nabla_j \beta - g_{jl}\nabla_k \beta). \tag{2.2}$$

Transvecting it with u^l and using $u^l\nabla_k u_l = 0$, we get

$$\beta(\nabla_j u_k - \nabla_k u_j) = -\frac{n-1}{n}(u_k\nabla_j\beta - u_j\nabla_k\beta). \tag{2.3}$$

We show that the right hand side of (2.3) is zero. Transvecting (2.2) with g^{jl} we obtain

$$u^l\nabla_l(\beta u_k) + \beta u_k\nabla_l u^l = -\frac{1}{n}\nabla_k\beta.$$

Further transvecting with u^k gives $-\beta\nabla_l u^l = \frac{n-1}{n}u^l\nabla_l\beta$. The previous equation becomes:

$$\beta u^l\nabla_l u_k = -\frac{1}{n}(\nabla_k\beta + u_k u^l\nabla_l\beta). \tag{2.4}$$

On the other hand transvecting (2.2) with $u^j u^l$ we obtain

$$-\beta u^l\nabla_l u_k = \frac{n-1}{n}(\nabla_k\beta + u_k u^l\nabla_l\beta). \tag{2.5}$$

If $\beta \neq 0$, the equations (2.5) and (2.4) imply $u^l\nabla_l u_k = 0$ and

$$\nabla_k\beta + u_k u^l\nabla_l\beta = 0. \tag{2.6}$$

Multiply this equation by u_j and take the antisymmetric part to obtain: $u_j\nabla_k\beta - u_k\nabla_j\beta = 0$. Therefore, by eq.(2.3) we conclude that u_k is irrotational. □

As a consequence we can state the following:

Theorem 2.1. *Let (M, g) be a perfect fluid spacetime. If $\nabla_m C_{jkl}{}^m = 0$ and $\nabla_k R = 0$, then (M, g) is a GRW spacetime whose fiber is Einstein and the conditions $\nabla_k X_j = \rho g_{kj}$, $\nabla_k\rho = \frac{\alpha-\beta}{1-n}X_k$ and $u^m C_{jklm} = 0$ hold.*

Proof. We give here an alternative proof with respect to [23]. The properties obtained in Lemma 2.1, $u_k\nabla_j\beta = u_j\nabla_k\beta$ and $\nabla_k u_j = \nabla_j u_k$, simplify eq. (2.2):

$$\beta(u_k\nabla_j u_l - u_j\nabla_k u_l) = \frac{1}{n}(g_{jl}\nabla_k\beta - g_{kl}\nabla_j\beta). \tag{2.7}$$

Transvecting this with u^j, using the closedness condition obtained in Lemma 2.1 (that gives $u^j\nabla_j u_l = u^j\nabla_l u_j = 0$) and (2.6) we obtain:

$$\nabla_k u_l = -\frac{u^j\nabla_j\beta}{n\beta}(g_{kl} + u_l u_k). \tag{2.8}$$

Thus u_j is a torse-forming vector.

Let $\omega_k = -u_k\frac{u^j\nabla_j\beta}{n\beta}$. Since $u_k u^j\nabla_j\beta = -\nabla_k\beta$, it is: $\omega_k = \frac{\nabla_k\beta}{n\beta}$. A covariant derivative shows that $\nabla_j\omega_k = \nabla_k\omega_j$. Then ω_j is locally the gradient of a scalar function: $\omega_j = \nabla_j\sigma$. As in [23], setting $X_l = u_l e^{-\sigma}$ we have:

$$\nabla_k X_j = \rho g_{kj}, \tag{2.9}$$

being $\rho = -\frac{u^k\nabla_k\beta}{n\beta}e^{-\sigma}$ a scalar function and $X_j X^j = -e^{-2\sigma} < 0$ a timelike vector. From Chen's Theorem we infer that M is a GRW spacetime.

The condition $\nabla_m C_{jkl}{}^m = 0$ assures that the fiber is Einstein by Gębarowski's lemma [14, 15], that is, $R^*_{\alpha\beta} = \frac{R^*}{n-1}g^*_{\alpha\beta}$.

The integrability conditions of (2.9) are $R_{jkl}{}^m X_m = g_{kl}\nabla_j\rho - g_{jl}\nabla_k\rho$; transvecting this with

X^l gives $X_k \nabla_j \rho = X_j \nabla_k \rho$ and consequently $\nabla_k \rho = \varepsilon X_k$ for some scalar ε [3]. On the other hand, multiplying the integrability condition by g^{kl} gives $R_j{}^m X_m = (1-n)\nabla_j \rho = \varepsilon(1-n) X_j$ and X_j is an eigenvector of the Ricci tensor. However for perfect fluids it is $R_j{}^m X_m = (\alpha - \beta) X_j$, then

$$\nabla_k \rho = \frac{\alpha - \beta}{1-n} X_k. \tag{2.10}$$

Moreover we have

$$R_{jkl}{}^m X_m = \frac{\alpha - \beta}{1-n} (X_j g_{kl} - X_k g_{jl}). \tag{2.11}$$

Now we insert the previous expression in the local form of the Weyl tensor and, after some algebra, we infer $C_{jklm} X^m = 0$, so that $C_{jklm} u^m = 0$. □

From $C_{jklm} u^m = 0$ it follows that the Weyl tensor is purely electric [18]. In dimension $n = 4$ the condition is equivalent to $u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm} = 0$ (see for example Lovelock and Rund [20] page 128). Multiplying this by u^i gives $C_{jklm} = 0$; in this way M^* is a 3-dimensional Einstein Riemannian manifold and thus it is a manifold of constant curvature: the spacetime is an ordinary Robertson-Walker spacetime.

Remark 1. It is worth noting that for an n-dimensional Lorentzian manifold (without perfect fluid hypothesis) equipped with a timelike vector such that $\nabla_k X_j = \rho g_{kj}$ the integrability conditions are $R_{jkl}{}^m X_m = \varepsilon(g_{kl} X_j - g_{jl} X_k)$. Then X_m is an eigenvector of the Ricci tensor, and it is easy to show that $X_i X^m C_{jklm} + X_j X^m C_{kilm} + X_k X^m C_{ijlm} = 0$; the spacetime is thus purely electric [18]. The decomposition of the Weyl tensor in electric and magnetic parts in n=4 is suggested by the decomposition of the Faraday 2-form describing the electromagnetic field (eqs. 3.62 and 5.7 in [38]).

By Proposition 4.10 in [18] the algebraic types of the Weyl tensor can be only G, I_i, $D(d)$ or O. As a matter of fact, in [18] it was shown that a spacetime with line element $ds^2 = -V^2(x^\gamma, t) dt^2 + f^2(x^\gamma, t) g^*_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta$, of which the GRW metric is a particular case, is purely electric.

Since an n-dimensional spacetime with $\nabla_m C_{jkl}{}^m = 0$ and $\nabla_k R = 0$ is a Yang pure space [16], the previous result is read as follows:

Proposition 2.1. *Any $n(\geq 3)$-dimensional perfect fluid Yang pure space with $\beta \neq 0$ is a GRW spacetime whose fiber is Einstein. Moreover, conditions* (1.1) *and* (1.2) *hold.*

Now we consider generalized curvature tensors. The following Lemma is an extension of a result concerning harmonic generalized curvature tensors, that is, generalized curvature tensors with the property $\nabla_m K_{jkl}{}^m = 0$ (see [22] Prop. 4.6, [24] Theorem 2.2, [25] Theorem 3.7).

Lemma 2.2. *Let (M, g) be an n-dimensional pseudo-Riemannian manifold having a generalized curvature tensor which satisfies*

$$\nabla_m K_{jkl}{}^m = A \nabla_m R_{jkl}{}^m + B (g_{kl} \nabla_j R - g_{jl} \nabla_k R), \tag{2.12}$$

where A and B are functions and $A \neq 0$ at any point of M. If $\nabla_m K_{jkl}{}^m = 0$, then $\nabla_m C_{jkl}{}^m = 0$. Moreover, if $A \neq 2B(n-1)$, then $\nabla_j R = 0$.

Proof. If $\nabla_m K_{jkl}{}^m = 0$, transvecting (2.12) by g^{kl} and using $2\nabla_m R_j{}^m = \nabla_j R$, we obtain $[2B(n-1)-A]\nabla_j R = 0$. If $A \neq 2B(n-1)$ we have $\nabla_j R = 0$ and by eq.(2.12) it is $\nabla_m R_{jkl}{}^m = 0$ provided that $A \neq 0$. Finally, eq. (2.1) gives $\nabla_m C_{jkl}{}^m = 0$. If $A = 2B(n-1)$ eq.(2.1) gives $\nabla_m K_{jkl}{}^m = A\frac{n-2}{n-3}\nabla_m C_{jkl}{}^m$ from which $\nabla_m C_{jkl}{}^m = 0$, provided $A \neq 0$. □

In view of Theorems 1.2 and 2.1, and Lemma 2.2 we can assert:

Theorem 2.2. *Let (M,g) be a perfect fluid spacetime, and let $K_{jkl}{}^m$ be a generalized curvature tensor with the property* (2.12). *If $\nabla_m K_{jkl}{}^m = 0$ then (M,g) is a GRW spacetime whose fiber is Einstein, and conditions* (1.1) *and* (1.2) *hold. Moreover, if $A \neq 2B(n-1)$, then $\nabla_j R = 0$.*

There are some important generalized curvature tensors that display property (2.13). We consider some examples of them:

1) The projective curvature tensor (see [21, 22] and references therein):

$$P_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-1}(\delta^m_j R_{kl} - \delta^m_k R_{jl}). \tag{2.13}$$

Its divergence results to be:

$$\nabla_m P_{jkl}{}^m = \frac{n-2}{n-1}\nabla_m R_{jkl}{}^m. \tag{2.14}$$

Thus we are in the case $A = \frac{n-2}{n-1}$, $B = 0$ of (2.12).

2) The concircular curvature tensor (see [21, 24] and references therein):

$$\tilde{C}_{jkl}{}^m = R_{jkl}{}^m + \frac{R}{n(n-1)}(\delta^m_j g_{kl} - \delta^m_k g_{jl}). \tag{2.15}$$

Its divergence results to be:

$$\nabla_m \tilde{C}_{jkl}{}^m = \nabla_m R_{jkl}{}^m + \frac{1}{n(n-1)}(g_{kl}\nabla_j R - g_{jl}\nabla_k R). \tag{2.16}$$

Thus we are in the case $A = 1$, $B = \frac{1}{n(n-1)}$ of (2.12).

3) The conharmonic curvature tensor (see [21, 24] and references therein):

$$N_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta^m_j R_{kl} - \delta^m_k R_{jl} + R^m_j g_{kl} - R^m_k g_{jl}). \tag{2.17}$$

Its divergence results to be:

$$\nabla_m N_{jkl}{}^m = \frac{n-3}{n-2}\nabla_m R_{jkl}{}^m + \frac{1}{2(n-2)}(g_{kl}\nabla_j R - g_{jl}\nabla_k R). \tag{2.18}$$

Thus we are in the case $A = \frac{n-3}{n-2}$, $B = \frac{1}{2(n-2)}$.

4) The quasi-conformal curvature tensor (Yano and Sawaki, 1968, [42]):

$$W_{jkl}{}^m = -(n-2)b\, C_{jkl}{}^m + [a + (n-2)b]\tilde{C}_{jkl}{}^m, \tag{2.19}$$

a, b are constants. Its divergence results to be:

$$\nabla_m W_{jkl}{}^m = (a+b)\nabla_m R_{jkl}{}^m + \frac{2a - b(n-1)(n-4)}{2n(n-1)}(g_{kl}\nabla_j R - g_{jl}\nabla_k R). \tag{2.20}$$

Thus we are in the case (2.12) with $A = a + b$, $B = \frac{2a-b(n-1)(n-4)}{2n(n-1)}$.

Note that the cases 1, 2, 3 satisfy the conditions $A \neq 0$ and $A \neq 2B(n-1)$ of (2.12). Case 4 satisfies the conditions if $a + b \neq 0$ and $a + b(n-2) \neq 0$.
For such cases, Theorem 2.2 can be rephrased as follows:

Proposition 2.2. *Let (M, g) be a perfect fluid spacetime, and let K_{jklm} be the 1) projective, 2) the concircular, 3) the conharmonic or 4) the quasi-conformal curvature tensor. If $\nabla_m K_{jkl}{}^m = 0$, $A \neq 0$, then (M, g) is a GRW spacetime whose fiber is Einstein and conditions* (1.1) *and* (1.2) *hold. Moreover, in cases 1, 2, 3: $\nabla_k R = 0$. In case 4: $\nabla_k R = 0$ if $a + b(n-2) \neq 0$.*

3. Physical consequences

In this section we outline some physical consequences of the above mentioned results. On a physical viewpoint, it is described the global structure of a perfect fluid spacetime, in several relevant cases.
Let (M, g) be an n-dimensional Lorentzian manifold where Einstein's field equations without cosmological term hold:

$$R_{ij} - \frac{R}{2} g_{ij} = \kappa T_{ij}, \tag{3.1}$$

$\kappa = 8\pi G$ is Einstein's gravitational constant (in units $c = 1$) and T_{ij} is the energy-momentum tensor (see for example [39]) describing the matter content of the spacetime. For a perfect fluid $T_{ij} = (\mu + p)u_i u_j + p g_{ij}$, where p is the isotropic pressure, μ is the energy density and u_j is the fluid flow velocity. The Ricci tensor gains the form

$$R_{ij} = \kappa(\mu + p)u_i u_j + \frac{\kappa}{2-n}(p - \mu) g_{ij}. \tag{3.2}$$

The form $R_{kl} = \alpha g_{kl} + \beta u_k u_l$ is recovered with $\beta = \kappa(\mu + p)$ and $\alpha = \frac{\kappa}{2-n}(p - \mu)$. It is worth noting that $\nabla_k R = 0$ if and only if the state equation $p = \frac{\mu}{n-1} + c$ holds, being c a constant.

Proposition 2.1 can be rephrased as follows:

Proposition 3.1. *Let (M, g) be an n-dimensional perfect fluid Yang Pure space with $\mu + p \neq 0$. Then (M, g) is a GRW spacetime with state equation $p = \frac{\mu}{n-1} + c$ being c a constant; the fiber is Einstein and conditions* (1.1) *and* (1.2) *hold.*

Gębarowski's lemma (see [14, 15]) states that the condition $\nabla_m C_{jkl}{}^m = 0$ for the GRW metric is equivalent to having Einstein fibers, that is, $R^*_{\alpha\beta} = \frac{R^*}{(n-1)} g^*_{\alpha\beta}$. Moreover Corollary 2.3 in [35] ensures that the fiber of a GRW spacetime is Einstein if and only if the spacetime is a perfect fluid. As a consequence we have

Proposition 3.2. *An n-dimensional GRW spacetime satisfies the condition $\nabla_m C_{jkl}{}^m = 0$ if and only if the spacetime is a perfect fluid.*

We have thus

Proposition 3.3. *Any n-dimensional Yang pure space GRW spacetime is a perfect fluid spacetime, with state equation $p = \frac{\mu}{n-1} + c$, being c a constant.*

In view of Lemma 2.2 and equations (2.14), (2.16) and (2.18) we infer that n-dimensional spacetimes with vanishing divergence of the projective, concircular or conharmonic curvature tensors are Yang pure spaces. Proposition 2.2 can be rephrased as follows.

Proposition 3.4. *Let (M, g) be an n-dimensional perfect fluid space time with $\mu + p \neq 0$ and let $K_{jkl}{}^m$ be the projective, the concircular or the conharmonic curvature tensor. If $\nabla_m K_{jkl}{}^m = 0$ then (M, g) is a GRW spacetime with state equation $p = \frac{\mu}{n-1} + c$, being c a constant, the fiber is Einstein and conditions* (1.1) *and* (1.2) *hold.*

Conversely Proposition 3.2 becomes

Proposition 3.5. *Let (M, g) be an n-dimensional GRW spacetime and let $K_{jkl}{}^m$ be the projective, the concircular or the conharmonic curvature tensor. If $\nabla_m K_{jkl}{}^m = 0$ then (M, g) is a perfect fluid with state equation $p = \frac{\mu}{n-1} + c$, being c a constant.*

In [23], Proposition 3.1, the authors proved that if there exists an equation of state of the form $p = p(\mu)$ for a perfect fluid, then the covector u_j is closed, that is, $\nabla_k u_j = \nabla_j u_k$. Theorem 2.2 can be rephrased as follows.

Theorem 3.1. *Let (M, g) be an n-dimensional perfect fluid spacetime with $\mu + p \neq 0$ and let $K_{jkl}{}^m$ be a generalized curvature tensor with the property* (2.12)*, $A \neq 0$ and $\nabla_m K_{jkl}{}^m = 0$. Then (M, g) is a GRW spacetime with Einstein fiber, and conditions* (1.1)*,* (1.2) *hold. If $A \neq 2B(n-1)$, then the state equation is $p = \frac{\mu}{n-1} + c$, being c a constant.*

References

[1] L. Alías, A. Romero and M. Sánchez: *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **27** (1995), 71–84.

[2] L. Alías, A. Romero and M. Sánchez: *Compact spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*; in Geometry and Topology of Submanifolds VII, River Edge NJ, USA, World Scientific, Publ., 1995, 67–70.

[3] B-Y.Chen: *A simple characterization of generalized Robertson-Walker spacetimes,* Gen. Relativity Gravitation **46** (2014), 1833, 5pp.

[4] J.K. Beem, P.E. Ehrlich and K.L. Easley: Global Lorentzian Geometry, Pure and Applied Mathematics **202**, 2nd ed., Marcel Dekker, New York, 1996.

[5] M.C. Chaki and R.K. Maity: *On quasi Einstein manifolds,* Publ. Math. Debrecen **57** (2000), 257–306.

[6] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović: *On warped product manifolds satisfying some curvature conditions*, J. Geom. Phys. **74** (2013), 328–341.

[7] A.A. Coley: *Fluid spacetimes admitting a conformal Killing vector parallel to the velocity vector*, Classical Quantum Gravity **8** (1991), 955–968.

[8] A. De, C. Özgür and U.C. De: *On conformally flat Pseudo-Ricci Symmetric Spacetimes*, Internat J. Theoret. Phys. **51** (2012), 2878–2887.

[9] F. Defever and R. Deszcz: *On semi Riemannian manifolds satisfying the condition R.R=Q(S,R)*; in Geometry and Topology of Submanifolds, III, World Scientific Publ., Singapore, 1991, 108–130.

[10] R. Deszcz: *On conformally flat Riemannian manifolds satisfying certain curvature conditions*, Tensor (N.S.) **49** (1990), 134–145.

[11] R. Deszcz, F. Dillen, L. Verstraelen and L. Vrancken: *Quasi-Einstein totally real submanifolds of the nearly Kahler 6-sphere*, Tohoku Math. J. **51** (1999), 461–478.

[12] R. Deszcz , M. Głogowska, M. Hotloś and Z. Sentürk: *On certain quasi-Einstein semisymmetric hypersurfaces*, Annu. Univ. Sci. Budapest Eötvös Sect. Math. **41** (1998), 151–164.

[13] R. Deszcz, M. Hotloś and Z. Sentürk: *Quasi-Einstein hypersurfaces in semi-Riemannian space forms*, Colloq. Math. **89** (2001), 81–97.

[14] A. Gębarowski: *On nearly conformally symmetric warped product spacetimes*, Soochow J. Math. **20** (1994), 61–75.

[15] A. Gębarowski: *Doubly warped products with harmonic Weyl conformal curvature tensor*, Colloq. Math. **67** (1994), 73–89.

[16] B.S. Guilfoyle and B.C. Nolan: *Yang's gravitational theory*, Gen. Relativity Gravitation **30** (1998), 473–495.

[17] M. Gutiérrez and B. Olea: *Global decomposition of a Lorentzian manifold as a generalized Robertson-Walker space*, Differ. Geom. Appl. **27** (2009), 146–156.

[18] S. Hervik, M. Ortaggio and L. Wylleman: *Minimal tensors and purely electric and magnetic spacetimes of arbitrary dimensions*, Classical Quantum Gravity **30** (2013), 165014, 50pp.

[19] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry, Vol 1, Interscience, New York, 1963.

[20] D. Lovelock and H. Rund: Tensors, Differential Forms and Variational Principles, Reprinted Edition, Dover, 1988.

[21] C.A. Mantica and L.G. Molinari: *A second-order identity for the Riemann tensor and applications*, Colloq. Math. **122** (2011), 69–82.

[22] C.A. Mantica and L.G. Molinari: *Weakly Z-Symmetric manifolds*, Acta Math. Hungar. **135** (2012), 80–96.

[23] C.A. Mantica, L.G. Molinari and U.C. De: *A condition for a perfect fluid spacetime to be a generalized Robertson-Walker spacetime*, J. Math. Phys. **57** (2016), 022508, 6pp.

[24] C.A. Mantica and Y.J. Suh: *Pseudo Z-symmetric Riemannian manifolds with harmonic curvature tensors*, Int. J. Geom. Methods Mod. Phys. **9** (2012), 1250004, 21pp.

[25] C.A. Mantica and Y.J. Suh: *Recurrent Z-forms on Riemannian and Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys. **9** (2012),1250059, 26pp.

[26] C.A. Mantica, Y.J. Suh and U.C. De: *A note on generalized Robertson-Walker spacetimes*, Int. J. Geom. Methods Mod. Phys. **13** (2016), 1650079, 9pp.

[27] C.A. Mantica and L.G. Molinari: *Generalized Robertson-Walker spacetimes, a survey*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1730001, 27pp.

[28] J. Mikeš and L. Rachůnek: *Torse-forming vector fields in T-symmetric Riemannian spaces*; in Steps in Differential Geometry, Proceedings of the Colloquium in Differential Geometry, Debrecen, July 25–30, 2000 (Debrecen, Hungary), edited by L. Kozma, P.T. Nagy and L. Tamassy, 219–229.

[29] B. O'Neill: Semi Riemannian Geometry with applications to Relativity, Academic Press, New York, 1983.

[30] M.M. Postnikov: Geometry VI, Riemannian Geometry, Encyclopaedia of Mathematical Sciences, **91**, Springer-Verlag, Berlin, 2001, (translated from the 1998 Russian edition by S.A. Vakhrameev).

[31] L. Rachůnek and J. Mikeš: *On tensor fields semi-conjugated with torse-forming vector fields*, Acta. Univ. Palacki. Olomuc., Fac. Rerum. Natur. Math **44** (2005), 151–160.

[32] A. Romero, R.N. Rubio and J.J. Salamanca: *Uniqueness of complete maximal hypersurfaces in spatially parabolic generalized Robertson-Walker spacetimes*, Classical. Quantum Gravity. **30** (2013), 115007.

[33] A. Romero, R.N. Rubio and J.J. Salamanca: *Uniqueness of complete maximal hypersurfaces in spatially parabolic generalized Robertson-Walker spacetimes. Applications to uniqueness results*, Int. J. Geom. Meth. Mod. Phys. **10** (2013), 1360014.

[34] M. Sánchez: *On the geometry of generalized Robertson-Walker spacetimes: geodesics*, Gen. Relativit. Gravitation **30** (1998), 915–932.

[35] M. Sánchez: *On the geometry of generalized Robertson-Walker spacetimes: curvature and Killing fields*, Gen. Relativ. Grav. **31** (1999), 1–15.

[36] R. Sharma: *Proper conformal symmetries of spacetimes with divergence-free Weyl tensor*, J. Math. Phys. **34** (1993), 3582–3587.

[37] L.C. Shepley and A.H. Taub: *Spacetimes containing perfect fluids and having a vanishing conformal divergence*, Commun. Math. Phys. **5** (1967), 237–256.

[38] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Hertl: Exact solutions of Einstein's Field Equations, Cambridge Monographs on Mathematical Physics, Cambridge University Press 2nd ed. 2003.

[39] R.M. Wald: General Relativity, The University of Chicago Press, Chicago, 1984.

[40] K. Yano: *Concircular geometry I-IV,* Proc. Imp. Acad. Tokyo **16** (1940), 195–200, 354–360, 442–448, 505–511.

[41] K. Yano: *On the torse-forming direction in Riemannian spaces,* Proc. Imp. Acad. Tokyo, **20** (1944), 340–345.

[42] K. Yano and S. Sawaki: *Riemannian manifolds admitting a conformal transformation group*, J. Diff. Geom. **2** (1968), 161–184.

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