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ON THE PIECEWISE GEODESIC APPROXIMATION OF ANDERSSON AND DRIVER

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Abstract

In 1999 Andersson and Driver proved a piecewise geodesic approximation formula for path integrals on compact Riemannian manifolds. Here, geodesics are those of the Levi-Civita connection. In this paper we will generalize their result for H^1-type metric to the case of general metric connections.

1. Introduction and main result

Let M be a compact Riemannian manifold and $W(M)$ be the continuous path space over M. For any partition $\mathcal{P}$ the space $H_{\mathcal{P}}(M)$ of piecewise geodesics is defined and equipped with two inner products, namely H^1-type and L^2-type metrics. With these metrics $H_{\mathcal{P}}(M)$ becomes a finite dimensional Riemannian manifold. $H_{\mathcal{P}}(M)$ is equipped with probability measures which have "Gaussian-like" densities with respect to the Riemannian volume measures. In [1] Andersson and Driver considered the Levi-Civita connection and proved that $H_{\mathcal{P}}(M)$ with these probability measures converges in an appropriate sense (i) (in H^1-case) to $W(M)$ with the Wiener measure and (ii) (in L^2-case) to $W(M)$ with a probability measure whose density with respect to the Wiener measure is written in terms of the scalar curvature of M.

In this paper we generalize Andersson and Driver's approximation theorem for H^1-metric to the case of a general metric connection ∇. Metric connections other than the Levi-Civita connection naturally appear in stochastic analysis on Lie groups and homogeneous spaces. (For example, see Example 8.1 in Driver [2]). Our method is based on Andersson and Driver [1].

Before we state our main theorem we introduce notations. Here, we only give a brief explanation. The precise definitions will be given in later sections. Let (M, o) be a compact Riemannian manifold with fixed initial point $o \in M$. We consider a metric connection ∇ on TM. Let $(O(M), u_o)$ be the orthonormal frame bundle with initial point $u_o \in O(M)$. We choose u_o in the fiber of o. The path space $W(M)$ is the space of continuous maps from $[0, 1]$ to M which start at o. We define $W(O(M))$ in a similar way. For a partition $\mathcal{P}$ of $[0, 1]$, $|\mathcal{P}|$ denotes the mesh of the partition and $H_{\mathcal{P}}(M)$ denotes the space of piecewise geodesics in M which change directions only

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at $\mathcal{P}$ and start at o. For $\sigma \in H_{\mathcal{P}}(M)$, we define the energy of a path $E(\sigma)$ in the usual way. $H_{\mathcal{P}}(M)$ is a finite dimensional manifold. On $H_{\mathcal{P}}(M)$ we consider a natural "H^1-type" Riemannian metric (see Definition 4.3 for the precise definition). We equip $H_{\mathcal{P}}(M)$ with the probability measure

$$\nu^1_{\mathcal{P}} = (\text{normalizing const.}) \times \exp\left(\frac{-E(\sigma)}{2}\right) \mathrm{Vol}_{G^1_{\mathcal{P}}},$$

where $\mathrm{Vol}_{G^1_{\mathcal{P}}}$ is the Riemannian volume measure on $H_{\mathcal{P}}(M)$. We denote by ν the diffusion measure on $W(M)$ which corresponds to the generator $\mathrm{trace}(\nabla^2)/2 = \triangle_M/2 + Z$. Here, $\triangle_M$ is the Laplace-Beltrami operator and Z is a vector field given by (4.8).

Now we state our main theorem. This is a natural generalization of Theorem 4.17 in Andersson and Driver [1].

Theorem 1.1. *Let the notations be as above. If G is a bounded continuous function on $W(M)$ then,*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} G(\sigma)\nu^1_{\mathcal{P}}(d\sigma) = \int_{W(M)} G(\sigma)\nu(d\sigma).$$

In the final section we give a simple remark for the case of non-compact complete manifolds.

2. Preliminaries from differential geometry

In this section we introduce notations and assumptions and give preliminary results from differential geometry, which will be in later use. Let M be a compact connected Riemannian manifold of dimension d and let o be an arbitrarily chosen fixed point in M. The Riemannian metric tensor is denoted by g. As usual we often write $\langle X, Y\rangle$ for $g(X, Y)$, where X and Y are tangent vectors. The tangent bundle of M is denoted by TM and the set of all the sections of TM is denoted by $\Gamma(TM)$.

Let ∇ be a metric connection on TM (i.e., $\nabla g = 0$). Its torsion tensor and curvature tensor are defined as follows;

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{2.1}$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.2}$$

Here, $X, Y, Z \in \Gamma(TM)$. Note that, in some other literatures, the definitions of T and R may be different by constant multiplication. Our definitions are the same as the ones in Driver [2].

Now we introduce the orthonormal bundle over M. For $x \in M$, set

$$O_x(M) = \{u : \mathbf{R}^d \to T_x M \mid \text{linear isometry}\} \quad \text{and} \quad O(M) = \bigcup_{x\in M} O_x(M).$$

$O(M)$ is called the orthonormal bundle over M and becomes a principal fiber bundle over M with its structure group $O(d)$. The natural projection is denoted by $\pi\colon O(M) \to M$. We arbitrarily choose $u_o \in O_o(M)$ as the initial frame and fix it throughout this paper. For $u \in O(M)$, $\ker(\pi_*\colon T_uO(M) \to T_{\pi(u)}M)$ is called the vertical subspace of $T_uO(M)$ and is denoted by V_u.

To a metric connection ∇ on TM corresponds a connection 1-form $\omega = \omega^\nabla$. ω is an $\mathfrak{so}(d)$-valued 1-form on $O(M)$ with some appropriate properties. (See textbooks of differential geometry for basic facts of connection forms on $O(M)$.) For $u \in O(M)$, $\ker(\omega\colon T_uO(M) \to \mathfrak{so}(d))$ is called the horizontal subspace of $T_uO(M)$ and is denoted by H_u. Note that $V_u \oplus H_u = T_uO(M)$. There exists a unique isomorphism $\mathcal{H}_u\colon T_{\pi(u)}M \to H_u$ such that $\pi_* \circ \mathcal{H}_u = \mathrm{Id}_{T_{\pi(u)}M}$, which is called the horizontal lift. Let $\{e_i\}_{i=1}^d$ be the canonical orthonormal basis of $\mathbf{R}^d$. Vector fields $A_i \in \Gamma(TO(M))$ $(1 \le i \le d)$ defined by $A_i(u) = \mathcal{H}_u u e_i$ are called the canonical horizontal vector fields. Let θ be an $\mathbf{R}^d$-valued 1-form on $O(M)$ defined by $\theta(X) = u^{-1}\pi_*(X)$ for $X \in T_uO(M)$. Then,

$$(\omega, \theta)\colon T_uO(M) = V_u \oplus H_u \cong \mathfrak{so}(d) \oplus \mathbf{R}^d$$

is a linear isomorphism. In other words, given a connection, $TO(M)$ can be trivialized. Hence, $O(M)$ can be considered as a Riemannian manifold.

Let ω be as above. The curvature form Ω of ω is an $\mathfrak{so}(d)$-valued 2-form on $O(M)$ defined by

$$\Omega(X, Y) = d\omega(X^H, Y^H), \quad X, Y \in T_uO(M),\ u \in O(M),$$

where X^H denotes the horizontal component of X. The torsion form Θ of ω is an $\mathbf{R}^d$-valued 2-form on $O(M)$ defined by

$$\Theta(X, Y) = d\theta(X^H, Y^H), \quad X, Y \in T_uO(M),\ u \in O(M).$$

For all $u \in O(M)$ and $a, b \in \mathbf{R}^d$, set $\Omega_u(a, b) = \Omega(\mathcal{H}_u ua, \mathcal{H}_u ub) \in \mathfrak{so}(d)$ and $\Theta_u(a, b) = \Theta(\mathcal{H}_u ua, \mathcal{H}_u ub) \in \mathbf{R}^d$.

The following is a well-known lemma. So we omit a proof. The first and the second equations are called structure equations. The third and the fourth equations state that Ω_u and Θ_u defined as above are, in fact, the scalarizations of R and T, respectively.

Lemma 2.1. *Let the notations be as above. Then, the following propositions hold.*

$$\Theta = d\theta + \omega \wedge \theta \quad (\textit{first structure equation}),$$

$$\Omega = d\omega + \omega \wedge \omega \quad (\textit{second structure equation}),$$

$$\Omega_u(a,b) = u^{-1}R(ua,ub)u \quad \textit{for all } u \in O(M) \textit{ and } a,b \in \mathbf{R}^d,$$
$$\Theta_u(a,b) = u^{-1}T(ua,ub) \quad \textit{for all } u \in O(M) \textit{ and } a,b \in \mathbf{R}^d.$$

Proof. See pp.280–281 in Driver [2] or Section III in Kobayashi and Nomizu [4], Vol. I. □

For an absolutely continuous path $\sigma\colon [0,1] \to M$, its energy is defined by $E(\sigma) = \int_0^1 |\sigma'(s)|^2\, ds$. We set

$$W(M) = \{\sigma\colon [0,1] \to M \mid \text{continuous and } \sigma(0) = o\}$$

and

$$H(M) = \{\sigma \in W(M) \mid \text{absolutely continuous and } E(\sigma) < \infty\}.$$

We define $W(\mathbf{R}^d)$ and $H(\mathbf{R}^d)$ in a similar way with o replaced by $0 \in \mathbf{R}^d$. The inner product of $H(\mathbf{R}^d)$ is defined by $(h,k) = \int_0^1 \langle h'(s), k'(s)\rangle\, ds$.

For $\sigma \in H(M)$, its horizontal lift $s \to u(s)$ is defined by $(d/ds)u(s) = \mathcal{H}_{\sigma(s)}\sigma'(s)$ with $u(0) = u_o$. Let $//_s(\sigma)\colon T_o(M) \to T_{\sigma(s)}(M)$ denote the parallel transport along σ with respect to ∇. It is well-known that $//_s(\sigma) = u(s)u_o^{-1}$. Since we regard $T_o(M) = \mathbf{R}^d$ by fixing the initial frame u_o, we will simply write $u(s) = //_s(\sigma)$ in the following.

Let X be an absolutely continuous vector field along $\sigma \in H(M)$ with $X(0) = 0$. Then, its covariant derivative along σ can be written as follows;

$$\frac{\nabla}{ds}X(s) = //_s(\sigma)\frac{d}{ds}\left(//_s(\sigma)^{-1}X(s)\right).$$

For such X and Y, the inner product $G^1(X,Y)$ is defined by

$$G^1(X,Y) = \int_0^1 \left\langle \frac{\nabla X(s)}{ds}, \frac{\nabla Y(s)}{ds} \right\rangle ds.$$

For $\sigma \in H(M)$, the tangent space $T_\sigma H(M)$ of $H(M)$ is defined by

$$T_\sigma H(M) = \{X \mid \text{absolutely continuous vector field along } \sigma \in H(M) \text{ such that } X(0) = 0 \text{ and } G^1(X,X) < \infty\}.$$

We define the development map and the anti-development map. The development map $\phi\colon H(\mathbf{R}^d) \to H(M)$ is defined, for $b \in H(\mathbf{R}^d)$, by $\phi(b) = \sigma \in H(M)$ where σ is the solution of the following differential equation:

$$\sigma'(s) = //_s(\sigma)b'(s) \quad \text{with } \sigma(0) = o.$$

There is another way to obtain $\phi(b)$. Consider the solution of the following differential equation (for u):

$$u'(s) = \mathcal{H}_{u(s)}u(s)b'(s) \quad \text{with } u(0) = u_o. \tag{2.3}$$

Then, $\pi(u) = \phi(b)$. By using the canonical horizontal vector fields $\{A_i\}$, we can rewrite the equation (2.3) in the following form:

$$u'(s) = \sum_{i=1}^{d} A_i(u(s))b_i'(s) \quad \text{with } u(0) = u_o, \tag{2.4}$$

where $b(s) = \sum_i b_i(s)e_i$. It is well-known that (2.3) and (2.4) are equivalent.

The anti-development map $\phi^{-1}\colon H(M) \to H(\mathbf{R}^d)$ is defined, for $\sigma \in H(M)$, by $b = \phi^{-1}(\sigma) \in H(\mathbf{R}^d)$, where

$$b'(s) = //_s(\sigma)^{-1}\sigma'(s) \quad \text{with } b(0) = 0.$$

It is known that ϕ is a diffeomorphism of two Hilbert manifolds, $H(\mathbf{R}^d)$ and $H(M)$. However, ϕ_* is not an isometry in general as we will see in Theorem 3.1. See Palais [6] for definitions of Hilbert manifolds such as $H(M)$ and their basic properties such as the inverse function theorem. In Section 13 in [6] the differential structure of $H(M)$ is defined by embedding M into a Euclidean space. Hence, when we discuss the smoothness problems we may consider M as a submanifold of a Euclidean space if necessary.

For each $h \in C^\infty(H(M) \to H(\mathbf{R}^d))$, we define $X^h(\sigma) \in T_\sigma H(M)$ by

$$X_s^h(\sigma) = //_s(\sigma)h_s(\sigma), \quad s \in [0, 1],$$

where $h_s(\sigma) = h(\sigma)(s)$. The mapping $(\sigma, h) \mapsto X^h(\sigma)$ is an isometry of two vector bundles, $H(M) \times H(\mathbf{R}^d)$ and $TH(M)$.

We recall the definition of the volume form on an oriented Riemannian manifold for later use. For an oriented Riemannian manifold (N, g) with $\dim N = k$, its volume form is the k-form defined by the following;

$$\mathrm{Vol}(v_1, \ldots, v_k) = \det\left(\{g(v_i, v_j)\}_{i,j=1}^k\right)^{1/2},$$

where $\{v_1, \ldots, v_k\}$ is an oriented basis of the tangent space. Abusing the notation we write Vol again for the measure induced by the linear functional $f \in C_0^\infty(N) \mapsto \int_N f\,\mathrm{Vol}$.

3. Differentials of development map for metric connections

In this section we compute differentials of development map for the case of metric connections. First we fix notations. For $u \in O(M)$ and $v, w \in T_{\pi(u)}M$, let

$$R_u(v, w) = \Omega(\mathcal{H}_u v, \mathcal{H}_u w).$$

Then, it follows from the definitions of Ω and Ω_u and from Lemma 2.1 that $R_u(v, w) = u^{-1}R(v, w)u$ and $\Omega_u(a, b) = R_u(ua, ub)$ for $a, b \in \mathbf{R}^d$.

For $\sigma \in H(M)$, $u(s) = //_s(\sigma)$ and $X \in T_\sigma H(M)$, set

$$\begin{aligned} A_s(X) &= \int_0^s R_{u(s)}(\sigma'(r), X(r))\,dr, \\ q_s(X) &= A_s(X) + \Theta_{u(s)}(\,\cdot\,, u(s)^{-1}X(s)). \end{aligned} \tag{3.1}$$

Note that $A_s(X)$ is $\mathfrak{so}(d)$-valued functions defined on $[0, 1]$ and that $q_r(X) = 0$ for all $r \in [0, s]$ if $X(r) = 0$ for all $r \in [0, s]$.

In the following theorem (Theorems 3.1) we abuse some notations. For example, $(//_s^*\omega)(X)$ should be written $[(//^*\tilde{\omega})(X)](s)$, where $\tilde{\omega}$ is the $H(\mathfrak{so}(d))$-valued 1-form on $H(O(M))$, which is naturally defined by ω. However, this is lengthy and we write $(//_s^*\omega)(X)$ for simplicity. For $u \in O(M)$ and $A \in \mathfrak{so}(d)$, $u \cdot A \in V_u$ denotes the vertical vector tangent to the path $t \mapsto ue^{tA}$ at $t = 0$. Note that a theorem similar to Theorem 3.1 below was proven for the case of smooth path spaces (Theorem 2.2 in Driver [2]).

Theorem 3.1. *Let $\sigma \in H(M)$ and $X \in T_\sigma H(M)$. We write $u = //(\sigma)$ and $b = \phi^{-1}(\sigma)$ as above. Then, we have the following*;

$$(//_s^*\omega)(X) = A_s(X), \tag{3.2}$$

$$(//_s^*\theta)(X) = u(s)^{-1}X(s), \tag{3.3}$$

$$(//_*X)(s) = u(s) \cdot A_s(X) + \mathcal{H}_{u(s)}X(s), \tag{3.4}$$

$$(\phi^*X)(s) = u(s)^{-1}X(s) - \int_0^s q_r(X)b'(r)\,dr, \tag{3.5}$$

where $\phi^*X = \phi_*^{-1}X$.

Proof. By using (ii) in Lemma 3.2 below we can prove equations (3.2), (3.3) and (3.4) in the same as in Theorem 3.3 in [1]. Here we prove equation (3.5), in which the torsion term appears.

Let $t \mapsto \sigma_t$ be a smooth curve in $H(M)$ such that $\sigma_0 = \sigma$ and $\dot{\sigma}_0 = X$. In this proof we write $\dot{\sigma}_t(s)$ for $(d/dt)\sigma_t(s)$ and $\sigma_t'(s)$ for $(d/ds)\sigma_t(s)$. Let $u_t(s) = //_s(\sigma_t)$ and $b_t(s) = \phi^{-1}(\sigma_t)(s)$. We set $\dot{u}(s) = \dot{u}_0(s)$. Then, it follows immediately that $\dot{u}(s) = (//_*X)(s)$ and $\dot{b}(s) = (\phi_*^{-1}X)(s)$.

Noting that

$$b_t(s) = \int_0^s u_t(r)^{-1}\sigma_t'(r)\,dr = \int_0^s \theta(u_t'(r))\,dr,$$

we obtain that

$$\frac{d}{ds}\phi_*^{-1}X(s) = \frac{d}{ds}\left.\frac{d}{dt}\right|_0 b_t(s) = \left.\frac{d}{dt}\right|_0 \theta(u_t'(s)). \tag{3.6}$$

Here we changed the order of derivatives (for a proof, see Lemma 3.2 below). For any 1-form α and vector fields X and Y, the following Cartan identity holds;

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]). \tag{3.7}$$

Applying this with $X = \partial/\partial t$, $Y = \partial/\partial s$ and $\alpha = u^*\theta$ (for a more rigorous proof, see Lemma 3.2 below), we see that the right hand side of (3.6) is equal to

$$\begin{aligned}
&\frac{d}{ds}\theta(u_0'(s)) + d\theta(\dot u_0(s), u_0'(s))\\
&= \frac{d}{ds}(u^{-1}(s)X(s)) - (\omega\wedge\theta)(\dot u_0(s), u_0'(s)) + \Theta(\dot u_0(s), u_0'(s))\\
&= \frac{d}{ds}(u^{-1}(s)X(s)) - \omega(\dot u_0(s))\theta(u_0'(s)) + \Theta(\dot u_0(s), u_0'(s))\\
&= \frac{d}{ds}(u^{-1}(s)X(s)) - A_s(X)b'(s) + \Theta((//_*X)(s), u'(s)).
\end{aligned} \tag{3.8}$$

Here, we used (1) the first structure equation (Lemma 2.1) for the first equality, (2) the fact that $\omega(u_t'(s)) = 0$ for the second equality, (3) (3.2) and the fact that $\theta(u'(s)) = b(s)$ for the last equality. Noting that $(//_*X)(s)^H = \mathcal{H}_{u(s)}X(s)$ from (3.4) and that

$$\Theta((//_*X)(s), u'(s)) = \Theta(\mathcal{H}_{u(s)}X(s), \mathcal{H}_{u(s)}u(s)b'(s)) = \Theta_{u(s)}(u(s)^{-1}X(s), b'(s)),$$

we can easily deduce (3.5) from the right hand side of (3.8). □

Lemma 3.2. (i) *Let $t \mapsto f_t$ be a smooth curve in $H(\mathbf{R}^d)$. We write $f(t)(s) = f(t,s)$ for $(t,s) \in \mathbf{R}\times[0,1]$. Let df/dt denote the derivative of $t\mapsto f(t)$ in $H(\mathbf{R}^d)$. Note that $(df/dt)(t)(s) = (\partial f/\partial t)(t,s)$ for all (t,s) since $h\in H(\mathbf{R}^d)\mapsto h(s)\in\mathbf{R}^d$ is a continuous linear mapping. Then, it holds that*

$$\frac{d}{dt}\frac{df}{ds}(t_0,s) = \frac{d}{ds}\frac{df}{dt}(t_0,s) \quad \text{for a.a. } s\in[0,1].$$

Here, $(d/dt)(df/ds)$ in the left hand side denotes the derivative at $t = t_0$ of the smooth path $t\mapsto (df/ds)(t,\cdot)$ in $L^2([0,1]:\mathbf{R}^d)$.

(ii) *Let N be a Riemannian manifold and let $\alpha \in \Gamma(T^*N)$ and $X, Y \in \Gamma(TN)$ be smooth vector fields. Let $t \mapsto f_t$ be a smooth curve in $H(M)$. We write $f(t)(s) = f(t,s)$ for $(t,s) \in \mathbf{R} \times [0,1]$. Then, for each fixed t_0, it holds that*

$$(d\alpha)\left(f_*\frac{d}{dt}, f_*\frac{d}{ds}\right)\bigg|_{t=t_0} = \frac{d}{dt}\alpha\left(f_*\frac{d}{ds}\right)\bigg|_{t=t_0} - \frac{d}{ds}\alpha\left(f_*\frac{d}{dt}\right)\bigg|_{t=t_0} \quad \textit{for a.a. } s \in [0,1].$$

Proof. First we show (i). Note that the mapping defined by

$$h = \{h(s)\}_{s\in[0,1]} \in H(\mathbf{R}^d) \mapsto \frac{dh}{ds} \in L^2([0,1]\colon \mathbf{R}^d)$$

is continuous linear. Hence, the curve $t \mapsto df_t/ds$ is a smooth curve in $L^2([0,1]\colon \mathbf{R}^d)$ and thus we have shown (i).

Next we show (ii). As we stated before we may regard M as a submanifold of an Eucleadian space of high dimension. Hence we can apply (i). Note that in (i) the time interval $[0,1]$ is not essential and can easily be replaced with any subinterval $[s_1, s_2]$.

Let us take a local chart $(x^1, \dots, x^n)$ around $f(t_0, s_0)$. Then, the exist $[t_1, t_2]$ and $[s_1, s_2]$ such that $(t_0, s_0) \in (t_1, t_2) \times (s_1, s_2)$ and $[t_1, t_2] \times [s_1, s_2]$ is contained in the local chart. The local functions α^i and x^i can be extended as a global function defined on the Eucleadian space. Then, the function

$$s \in [0,1] \mapsto \alpha\left(f_*\frac{\partial}{\partial t}\right)\bigg|_{t=t_0} = (\alpha_i(f)dx^i)\left(\frac{\partial f^i}{\partial t}\frac{\partial}{\partial x^i}\right)\bigg|_{t=t_0} = \alpha_i(f)\frac{df^i}{dt}\bigg|_{t=t_0}$$

is an well-defined element in $H(\mathbf{R})$. (Here, the time interval in the definition of $H(\mathbf{R})$ should be appropriately replaced). Hence,

$$(3.9)\quad s \in [0,1] \mapsto \frac{d}{ds}\alpha\left(f_*\frac{d}{dt}\right)\bigg|_{t=t_0} = \alpha_i(f)\frac{d}{ds}\frac{df^i}{dt}\bigg|_{t=t_0} + \frac{\partial\alpha^i}{\partial x^j}(f)\frac{df^i}{dt}\frac{df^j}{ds}\bigg|_{t=t_0}$$

is an element in $L^2([s_1, s_2]\colon \mathbf{R})$. Similarly,

$$t \in \mathbf{R} \mapsto \alpha\left(f_*\frac{\partial}{\partial s}\right) = (\alpha_i(f)dx^i)\left(\frac{\partial f^i}{\partial s}\frac{\partial}{\partial x^i}\right) = \alpha_i(f)\frac{df^i}{ds}$$

is a smooth curve in $L^2([s_1, s_2]\colon \mathbf{R})$. Its derivative at $t = t_0$ in $L^2([s_1, s_2]\colon \mathbf{R})$ is

$$(3.10)\qquad \frac{d}{dt}\alpha\left(f_*\frac{d}{ds}\right)\bigg|_{t=t_0} = \alpha_i(f)\frac{d}{dt}\frac{df^i}{ds}\bigg|_{t=t_0} + \frac{\partial\alpha^i}{\partial x^j}(f)\frac{df^i}{ds}\frac{df^j}{dt}\bigg|_{t=t_0}.$$

From (3.9), (3.10) and (i) of this lemma we obtain (ii). □

4. Piecewise geodesic approximations

In this section we first construct, for a given partition $\mathcal{P} \subset [0,1]$, finite dimensional submanifold $H_{\mathcal{P}}(M) \subset H(M)$, which consists of piecewise geodesics. Then, we show that, in an appropriate sense, the probability measure ν_p^1 converges to the diffusion measure on $W(M)$ whose generator is $\pi_*\big(\sum_i A_i^2\big) = \mathrm{trace}(\nabla^2)$ as the mesh of $\mathcal{P}$ tends to zero.

We set some notations. Let $I = [0,1]$ and $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$. $|\mathcal{P}| = \max_i |s_i - s_{i-1}|$ is called the mesh of the partition $\mathcal{P}$. We set $J_i = (s_{i-1}, s_i]$ and s for $s \in J_i$. For a function k, let $\Delta_i k = k(s_i) - k(s_{i-1})$ and $\Delta_i s = s_i - s_{i-1}$. For a piecewise continuous function k on $I = [0,1]$, we write $k(s+) = \lim_{\delta \searrow 0} k(s+\delta)$.

DEFINITION 4.1. For a partition $\mathcal{P}$, set

$$H_{\mathcal{P}} = \{b \in H(\mathbf{R}^d) \cap C^2(I \setminus \mathcal{P}) \mid b''(s) = 0 \text{ for } s \in I \setminus \mathcal{P}\}$$

and

$$H_{\mathcal{P}}(M) = \left\{\sigma \in H(M) \cap C^2(I \setminus \mathcal{P}) \;\middle|\; \left(\frac{\nabla}{ds}\right)\sigma'(s) = 0 \text{ for } s \in I \setminus \mathcal{P}\right\}.$$

$H_{\mathcal{P}}$ (resp. $H_{\mathcal{P}}(M)$) is the set of piecewise linear curves in $\mathbf{R}^d$ (resp. piecewise geodesics in M), which change directions only at the points in $\mathcal{P}$.

Note that $\phi(H_{\mathcal{P}}) = H_{\mathcal{P}}(M)$ since, for $\phi(b) = \sigma$,

$$\frac{\nabla}{ds}\sigma'(s) = \frac{\nabla}{ds}\big(//_s(\sigma)b'(s)\big) = //_s(\sigma)b''(s) = 0.$$

Since $H_{\mathcal{P}}$ is a subspace of $H(\mathbf{R}^d)$ and ϕ is a diffeomorphism of Hilbert manifolds, $H_{\mathcal{P}}(M)$ is an orientable submanifold of $H(M)$. Clearly, its dimension is $nd = (\#\mathcal{P}) \times \dim M$.

For $\sigma \in H_{\mathcal{P}}(M)$, the tangent space $T_\sigma H_{\mathcal{P}}(M)$ is a subspace of $T_\sigma H(M)$. In the next proposition, we give a necessary and sufficient condition for $X \in T_\sigma H(M)$ to be in $T_\sigma H_{\mathcal{P}}(M)$.

Proposition 4.2. *Let $\sigma \in H_{\mathcal{P}}(M)$ and $X \in T_\sigma H(M)$. We write $b = \phi^{-1}(\sigma)$ and $u = //(\sigma)$ as above.*
(i) *Then, $X \in T_\sigma H_{\mathcal{P}}(M)$ if and only if*

$$\frac{\nabla^2}{ds^2}X(s) = \frac{\nabla}{ds}\big(T(\sigma'(s), X(s))\big) + R(\sigma'(s), X(s))\sigma'(s)$$

$$
\begin{aligned}
&= T\left(\sigma'(s), \left(\frac{\nabla}{ds}\right) X(s)\right) \\
&\quad + (\nabla T)(\sigma'(s); \sigma'(s), X(s)) + R(\sigma'(s), X(s))\sigma'(s) \quad \textit{on } I \setminus \mathcal{P}.
\end{aligned}
\tag{4.1}
$$

Note that $\nabla_{\sigma'(s)} X(s) = (\nabla/ds)X(s)$, *etc.*
(ii) *Equivalently, for* $h \in H(\mathbf{R}^d)$, $X^h(\sigma) \in T_\sigma H_{\mathcal{P}}(M)$ *if and only if*

$$
\begin{aligned}
h''(s) &= \Theta_{u(s)}(b'(s), h'(s)) \\
&\quad + \Theta'_{u(s)}(b'(s); b'(s), h(s)) + \Omega_{u(s)}(b'(s), h(s))b'(s) \quad \textit{on } I \setminus \mathcal{P}.
\end{aligned}
\tag{4.2}
$$

Proof. By the definition of the Jacobi fields, we have the first equality (the Jacobi equation) in (4.1). The second equality can be easily shown by the Leibniz rule for ∇. For the Jacobi fields and equations of affine connections, see Section VIII in Kobayashi and Nomizu [4], Vol. II.

We can easily obtain (4.2), by scalarizing (4.1). Note that $u(s)b'(s) = \sigma'(s)$, $u(s)h(s) = X(s)$, $u(s)h'(s) = (\nabla/ds)X(s)$ and $u(s)h''(s) = (\nabla^2/ds^2)X(s)$. ☐

Now we define the Riemannian structure on $H_{\mathcal{P}}(M)$.

DEFINITION 4.3. Let $H_{\mathcal{P}}$ and $H_{\mathcal{P}}(M)$ be as above. For $h, k \in H_{\mathcal{P}}$, set

$$
\langle h, k \rangle_{H_{\mathcal{P}}} = \int_0^1 \langle h'(s), k'(s) \rangle \, ds = \sum_{i=1}^n \langle h'(s_{i-1}+), k'(s_{i-1}+) \rangle \Delta_i s.
$$

This is the restriction of the inner product on $H(\mathbf{R}^d)$. For $\sigma \in H_{\mathcal{P}}(M)$ and $X, Y \in T_\sigma H_{\mathcal{P}}(M)$, set

$$
G^1_{\mathcal{P}}(X, Y) = \sum_{i=1}^n \left\langle \frac{\nabla}{ds} X(s_{i-1}+), \frac{\nabla}{ds} Y(s_{i-1}+) \right\rangle \Delta_i s.
$$

By the same reason as in the Levi-Citita case (see Remark 4.5 in [1]), it follows from Proposition 4.2 that $G^1_{\mathcal{P}}$ is non-degenerate and hence defines a Riemannian structure on $H_{\mathcal{P}}(M)$.

DEFINITION 4.4. For a partition $\mathcal{P}$ we set $\mathrm{Vol}_{\mathcal{P}}$ and $\mathrm{Vol}_{G^1_{\mathcal{P}}}$ be the volume form on $H_{\mathcal{P}}$ and $H_{\mathcal{P}}(M)$, respectively. We also set

$$
\mu^1_{\mathcal{P}}(b) = (2\pi)^{-dn/2} \exp\left(-\frac{E(b)}{2}\right) \mathrm{Vol}_{\mathcal{P}}(b)
$$

and

$$
\nu^1_{\mathcal{P}}(\sigma) = (2\pi)^{-dn/2} \exp\left(-\frac{E(\sigma)}{2}\right) \mathrm{Vol}_{G^1_{\mathcal{P}}}(\sigma).
$$

Here $E(b)$ and $E(\sigma)$ denote the energy of the paths b and σ, respectively.

We introduce some notations. For a partition $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$ and $s \in (s_{i-1}, s_i]$, we set

$$\hat{q}_s(X) = q_{s_{i-1}}(X) \quad \text{and} \quad \tilde{q}_s(X) = q_s(X) - q_{s_{i-1}}(X).$$

Clearly, $q = \hat{q} + \tilde{q}$. From (3.5) in Theorem 3.1,

$$(\phi^* X^h)'(s) = h'(s) - \hat{q}_s(X^h)b'(s) - \tilde{q}_s(X^h)b'(s) \tag{4.3}$$

for $h \in H(\mathbf{R}^d)$.

The following is the key theorem in this section. It states that ϕ^* preserves the volume forms.

Theorem 4.5. *Let the notations be as above. Then,* $\phi^*(\mathrm{Vol}_{G^1_{\mathcal{P}}}) = \mathrm{Vol}_{\mathcal{P}}$ *and* $\phi^*(\nu^1_{\mathcal{P}}) = \mu^1_{\mathcal{P}}$.

Proof. This proof is based on Theorem 4.8 in [1]. The second assertion immediately follows from the first one, since $E(b) = E(\sigma)$ if $\phi(b) = \sigma$.

We show the first assertion. Let $\sigma \in H_{\mathcal{P}}(M)$, $b = \phi^{-1}(\sigma) \in H_{\mathcal{P}}$ and $\{h_k\}$ is an orthonormal basis of $H_{\mathcal{P}}$. It is sufficient to show that

$$\det\left(\left\{G^1_{\mathcal{P}}(\phi_* h_k, \phi_* h_j)\right\}_{k,j}\right) = 1. \tag{4.4}$$

We set $H_k(s) = u^{-1}(s)(\phi_* h_k)(s)$ and $\langle H_k, H_j\rangle_{\mathcal{P}} = \sum_i \langle H_k'(s_{i-1}+), H_j'(s_{i-1}+)\rangle \Delta_i s$. Restricted on $H_{\mathcal{P}}$, $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{P}}$ coincides with $\langle\,\cdot\,,\,\cdot\,\rangle_{H_{\mathcal{P}}}$. Then, $X^{H_k} = \phi_* h_k$ and

$$\det\left(\left\{G^1_{\mathcal{P}}(\phi_* h_k, \phi_* h_j)\right\}_{k,j}\right) = \det\left(\left\{\langle H_k, H_j\rangle_{\mathcal{P}}\right\}_{k,j}\right).$$

By (4.3) we have $h_k' = (\phi^*(X^{H_k}))' = H_k' - \hat{q}(X^{H_k})b' - \tilde{q}(X^{H_k})b'$ and, hence,

$$h_k' + \hat{q}(X^{H_k})b' = H_k' - \tilde{q}(X^{H_k})b'. \tag{4.5}$$

Note that h_k', $\hat{q}$ and b' are constant on (s_{i-1}, s_i) and that $\tilde{q}_{s_{i-1}}(X^{H_k}) = 0$. Hence, the both sides of (4.5) are constant and equal to $H_k'(s_{i-1}+)$. Therefore,

$$\langle H_k, H_j\rangle_{\mathcal{P}} = \int_0^1 \langle h_k'(r) + \hat{q}_r(X^{H_k})b'(r), h_j'(r) + \hat{q}_r(X^{H_j})b'(r)\rangle\, dr.$$

Set

$$(Th)(s) = \int_0^s \hat{q}_r(\phi_* h)b'(r)\, ds.$$

Note that $T \in \mathrm{End}(H_{\mathcal{P}})$ since $\hat{q}_r(\phi_* h)$ and $b'(r)$ are piecewise constant functions. Since we have just obtained

$$\det\left(\left\{G^1_{\mathcal{P}}(\phi_* h_k, \phi_* h_j)\right\}_{k,j}\right) = \det\left(\left\{\left\langle (I+T)h_k, (I+T)h_j \right\rangle_{H_{\mathcal{P}}}\right\}_{k,j}\right) = [\det(I+T)]^2,$$

it is sufficient to show that $\det(I+T) = 1$.

Let $\{e_a\}_{a=1}^d$ be the canonical orthonormal basis of $\mathbf{R}^d$ and set

$$h_{i,a} = \left(\frac{1}{\sqrt{\Delta_i s}} \int_0^s 1_{(s_{i-1}, s_i]}(r)\, dr\right) e_a$$

for $a = 1, \dots, d$ and $i = 1, \dots, n$. It is easy to see that $\{h_{i,a}\}$ is an orthonormal basis of $H_{\mathcal{P}}$. Note that $h_{i,a}(s) = 0$ on $[0, s_{i-1}]$ and $(\phi_* h_{i,a})(s) = 0$ on $[0, s_{i-1}]$, because ϕ is obtained by solving the ordinary differential equation (2.3) or (2.4). Hence, $q_s(\phi_* h_{i,a}) = 0$ on $[0, s_{i-1}]$ from (3.1) and $\hat{q}_s(\phi_* h_{i,a}) = 0 = (T h_{i,a})(s)$ on $[0, s_i)$. Thus, if $j \leq i$, $\langle T h_{i,a}, h_{j,b} \rangle_{\mathcal{P}} = 0$ for any a, b, since $h'_{j,b}(s_l+) = 0$ if $l \neq j-1$ and $(T h_{i,a})'(s_l+) = 0$ if $l = 1, 2, \dots, i-1$. Hence, T is represented by an upper-triangular matrix with vanishing diagonals. In particular, all the eigenvalues of T are 0 and $\det(I+T) = 1$. □

For a partition $\mathcal{P}$ Let $\pi_{\mathcal{P}} \colon W(\mathbf{R}^d) \to (\mathbf{R}^d)^n$ be given by $w \mapsto (w(s_1), \dots, w(s_n))$. Note that $\pi_{\mathcal{P}}|_{H_{\mathcal{P}}}$ is a linear isomorphism of vector spaces. We write $i_{\mathcal{P}} = \pi_{\mathcal{P}}|_{H_{\mathcal{P}}}^{-1}$. Then, we have the following (Lemma 4.11 in [1]);

$$i^*_{\mathcal{P}}(\mu^1_{\mathcal{P}}) = \left(\prod_{i=1}^n p_{\Delta_i s}(y_{i-1}, y_i)\right) \prod_{i=1}^n dy_i,$$

where $y_0 = 0$ by convention, $\prod_{i=1}^n dy_i$ is the standard volume form on $(\mathbf{R}^d)^n$ and $p_s(x, y) = (2\pi s)^{-d/2} \exp(|x-y|^2/(2s))$ is the heat kernel on $\mathbf{R}^d$. In particular, $i^*_{\mathcal{P}}(\mu^1_{\mathcal{P}})$ corresponds to the measure $\pi_{\mathcal{P}*}(\mu)$, where μ is the Wiener measure on $W(\mathbf{R}^d)$.

For a standard d-dimensional Brownian motion $\{B(s)\}_{0 \leq s \leq 1}$ starting at 0, we set $B_{\mathcal{P}} = i_{\mathcal{P}} \circ \pi_{\mathcal{P}}(B)$, or explicitly,

$$B_{\mathcal{P}}(s) = B(s_{i-1}) + (s - s_{i-1}) \frac{\Delta_i B}{\Delta_i s} \quad \text{if } s \in (s_{i-1}, s_i],$$

where $\Delta_i B = B(s_i) - B(s_{i-1})$.

Now, before stating Corollary 4.6, we give a simple fact. Let N_1, N_2 be two oriented manifolds of the same dimension k and $\psi \colon N_1 \to N_2$ be a diffeomorphism which preserves orientation. For $\gamma \in \Gamma\left(\bigwedge^k T^* N_2\right)$ and $\psi^*\gamma \in \Gamma\left(\bigwedge^k T^* N_1\right)$, we denote by $\tilde{\gamma}$ and $\widetilde{\psi^*\gamma}$ the (signed) measures on N_2 and N_1, which correspond to μ and $\psi^*\gamma$, respectively. Then, the law of $\widetilde{\psi^*\gamma}$ induced by the diffeomorphism ψ is $\tilde{\gamma}$.

Thus, we have obtained the following corollary.

Corollary 4.6. *Let μ be the Wiener measure on $W(\mathbf{R}^d)$ and B be the Brownian motion defined as above. Then, the law of $B_{\mathcal{P}}$ and the law of $\phi(B_{\mathcal{P}})$ is $\mu^1_{\mathcal{P}}$ and $\nu^1_{\mathcal{P}}$, respectively.*

We introduce some probabilistic notions. For the existence of such notions and basic properties of them, see Section 3 in Driver [2], for example. Let $\{B(s)\}_{0\le s\le 1}$ be the d-dimensional Brownian motion as above. Let $\{u(s)\}_{0\le s\le 1}$ be the solution of the following stochastic differential equation:

$$du(s) = \mathcal{H}_{u(s)}u(s) \circ dB(s) \quad \text{with } u(0) = u_o, \tag{4.6}$$

or equivalently:

$$du(s) = \sum_{i=1}^{d} A_i(u(s)) \circ dB^i(s) \quad \text{with } u(0) = u_o, \tag{4.7}$$

where A_i ($i = 1, \ldots, d$) is the canonical horizontal vector field on $O(M)$ and $B = (B^1, \ldots, B^d)$. By the existence and pathwise uniqueness of the solutions of (4.7), u defines a measurable mapping from $(W(\mathbf{R}^d), \mu)$ to $W(O(M))$ (which is denoted by u, again). The mapping $\tilde{\phi} = \pi \circ u\colon (W(\mathbf{R}^d), \mu) \to W(M)$ is called stochastic development map. The law of $\tilde{\phi}$ is denoted by ν. It is well-known that ν is a diffusion measure on $W(M)$ with its generator trace$(\nabla^2)/2 = \triangle_M/2 + Z$, where $\triangle_M$ is the Laplace-Beltrami operator on M and

$$Z^i = \frac{1}{2} g^{mk} \left(\hat{\Gamma}^i_{mk} - \Gamma^i_{mk}\right) \tag{4.8}$$

in a local coordinate $(x^1, \ldots, x^d)$. Here, $\hat{\Gamma}^i_{mk}$ and Γ^i_{mk} are the Cristoffel symbols of the Levi-Civita connection and ∇, respectively. (See equations (4.31) and (4.33) in Section 4, Chapter V, Ikeda and Watanabe [5].) If ∇ is torsion skew symmetric, then Z vanishes i.e., ν is the Wiener measure on M (see Section 8 in [2]). However, Z does not vanish in general.

On the probability space $(W(M), \nu)$, two measurable mappings are defined. $\widetilde{/\!/}(\sigma)$ denotes the stochastic horizontal lift of σ. $\{\widetilde{/\!/}_s(\sigma)\}_{0\le s\le 1}$ is an $O(M)$-valued process and its law is the same as the one induced by u. $\tilde{b}(s) = \int_0^s \widetilde{/\!/}_r^{-1}(\sigma) \circ d\sigma(r)$ is called the stochastic anti-development map. It is known that $\{\tilde{b}(s)\}$ is a standard d-dimensional Brownian motion defined on $(W(M), \nu)$.

Now we prove our main theorem (Theorem 1.1) in a similar way to Andersson and Driver [1].

Proof of Theorem 1.1. We can prove this in the same way as in Andersson and Driver [1]. So we only give a sketch of proof.

Since we have shown Corollary 4.6, it is sufficient to show that

$$\lim_{|\mathcal{P}|\to 0} \mathbb{E}[G(\phi(B_{\mathcal{P}}))] = \mathbb{E}[G(\tilde{\phi}(B))]. \tag{4.9}$$

For a bounded continuous function F on $W(O(M))$, set $f(\sigma) = F(//(\sigma))$ and $\tilde{f}(\sigma) = F(\widetilde{//}(\sigma))$. As we mentioned above, it holds that

$$\int_{W(M)} \tilde{f}(\sigma)\nu(d\sigma) = \mathbb{E}[F(u)]$$

and

$$\int_{H_{\mathcal{P}}(M)} f(\sigma)\nu^1_{\mathcal{P}}(d\sigma) = \mathbb{E}[F(u_{\mathcal{P}})].$$

Here, $\mathbb{E}$ denotes the integration with respect to μ and $u_{\mathcal{P}}$ is defined from $B_{\mathcal{P}}$ by (2.3) or (2.4). Note that we used the fact that the law of $u_{\mathcal{P}}$ is equal to the law of $//$ under $\nu^1_{\mathcal{P}}$. In order to verify $\mathbb{E}[F(u_{\mathcal{P}})]$ converges to $\mathbb{E}[F(u)]$, we embed $O(M)$ in a Euclidean space of high dimension using Nash's embedding theorem and apply the approximation theorem of Wong-Zakai type for SDE (4.7) and ODE (2.4) with b replaced with $B_{\mathcal{P}}$. (See Theorem 4.14 in [1].) More precisely, the convergence is in the following sense; for any $p > 1$,

$$\lim_{|\mathcal{P}|\to 0} \mathbb{E}\left[\sup_{0\le s\le 1} |u_{\mathcal{P}}(s) - u(s)|^p\right] = 0.$$

Here, $u_{\mathcal{P}}$ and u are regarded as the Euclidean space valued process by using Nash's embedding (see p.450 in [1]). Thus, we have shown $\lim_{|\mathcal{P}|\to 0} \mathbb{E}[F(u_{\mathcal{P}})] = \mathbb{E}[F(u)]$. By setting $F = G \circ \pi$, we obtain (4.9). □

5. A remark for non-compact case

In the previous sections we assumed the compactness of M for simplicity. In this section we give a simple remark for non-compact case. We assume the Riemannian manifold M is complete. Note that the solutions of ordinary differential equation (2.3) or (2.4) does not explode in finite time.

In the previous sections compactness is used only to prove that (i) the solutions of stochastic differential equation (4.6) or (4.7) does not explode in finite time and (ii) the approximation (4.9) holds.

Hence, on a complete Riemmanian manifold M with a metric connection ∇ such that the above conditions (i) and (ii) hold, Theorem 1.1 can be extended in a natural way.

Now we give an example, which is essentially taken from Driver [3]. In this example, the manifold is a (non-compact) matrix Lie group with the left invariant structure. (The right invariant case can be done in the same way.)

EXAMPLE 5.1. Let G be a matrix Lie group in $\mathrm{GL}(n, \mathbf{R})$ for some $n \in \mathbf{N}$. For $X, Y \in \mathfrak{gl}(n, \mathbf{R})$, we set $\langle X, Y\rangle = \mathrm{trace}(X^*Y)$. We consider the left invariant structure as follows. A tangent vector at the identity element is identified with a left invariant vector field over G as usual. We consider the restriction of the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ to $\mathfrak{g} = T_e(G)$. Here, e denotes the unit element of G. By using the left translation we can define a Riemannian structure on G. We consider a left invariant connection ∇, that is, $\nabla_X Y \in \mathfrak{g}$ for all $X, Y \in \mathfrak{g}$. We assume metric compatibility of ∇, which is equivalent to the property that $\langle \nabla_X Y, Z\rangle = -\langle Y, \nabla_X Z\rangle$ for all $X, Y, Z \in \mathfrak{g}$. Hence, ∇_X can be regarded as an element in $\mathfrak{so}(\mathfrak{g})$.

Now we consider horizontal vector fields on $O(G)$. By using the left translation we may identify TG and $O(G)$ with $G \times \mathfrak{g}$ and $G \times O(\mathfrak{g})$, respectively. With this identification the horizontal subspace can be written explicitly as follows:

$$\{(X, -\nabla_X) \in \mathfrak{g} \times \mathfrak{so}(\mathfrak{g}) \mid X \in \mathfrak{g}\}\,.$$

Fix an orthonormal basis $\{X_1, \dots, X_d\}$ of $\mathfrak{g}$, where $d = \dim G$. This is equivalent to fixing an initial frame over $e \in G$. Now we may identify $\mathfrak{g}$ with $\mathbf{R}^d$. In this case the canonical horizontal vector fields are given by $\{(X_i, -\nabla_{X_i})\}_{i=1}^d$.

Let $\{u(s)\} = \{(g(s), r(s))\}$ be the solution of the stochastic differential equation (4.7). Note that g and r are G-valued and $O(d)$-valued processes, respectively. Then, the equation (4.7) can be written in matrix form as follows:

$$dg(s) = g(s) \circ \left(\sum_{i=1}^d X_i dB^i(s)\right),$$
$$dr(s) = -r(s) \circ \left(\sum_{i=1}^d \nabla_{X_i} dB^i(s)\right).$$

Here, $\{B^i(s)\}_{i=1}^d$ is the standard d-dimensional Brownian motion. Note that $\tilde{\phi}(B) = g$ in this case. (Similarly, ordinary differential equation (2.4) can be written in matrix form and its first component is $\phi(b)$.)

Driver showed in Appendix A, [3] that condition (ii) above (the convergence in (4.9)) holds in this case. Hence, Theorem 1.1 holds in this case, too.

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