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<tr>
<th><strong>Title</strong></th>
<th>Unipotent characters of $\text{SO}^{\pm}<em>{2n}$, $\text{Sp}</em>{2n}$ and $\text{SO}_{2n+1}$ over $F_q$ with small $q$</th>
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0. Introduction. Let $G$ be a special orthogonal group or symplectic group over a finite field $F_q$, $F$ the Frobenius mapping and $G^F$ the group of all $F$-stable points of $G$. G. Lusztig [7], [8] has obtained explicit formulas for the characters of the unipotent representations of $G^F$ on any regular semisimple element of $G^F$ provided that the order $q$ of the defining field $F_q$ is sufficiently large. Our purpose in this paper is to show that his formulas are valid for any $q$.

Let $W$ be the Weyl group of $G$ and $m$ an odd positive integer. For $w \in W$, let $R_w^{(m)}$ be the Deligne-Lusztig virtual representation [2], [6, 3.4] of $G_{pm}$. By [2, 7.9], to determine the values of the character of a unipotent representation $\rho$ of $G^F$ on regular semisimple elements, it suffices to determine the inner product

$$\langle R_w^{(m)}, \rho \rangle$$

for any $w \in W$. This has been done by G. Lusztig [7], [8] for a sufficiently large $q^m$. Let $n$ be the rank of $G$ and $\Psi_n$ be the set of symbol classes (cf. [5, §3]) that parameterizes the unipotent representations (up to equivalence) of $G^F$ or $G^F_m$, i.e.

$$\Psi_n = \begin{cases} \Phi_n & \text{if } G = SO_{2n+1} \text{ or } Sp_{2n} \\ \Phi_n^\pm & \text{if } G = SO_{2n}^- \end{cases}$$

in the notations in [5, §3]. For $\Lambda \in \Psi_n$, let $\rho^{(1)}_\Lambda$ and $\rho^{(m)}_\Lambda$ be the corresponding unipotent representations of $G^F$ and $G^F_m$ respectively. Our main result (Theorem 4.2, (iii)) is

$$(*) \quad \langle R_w^{(m)}, \rho^{(m)}_\Lambda \rangle = \langle R_w^{(1)}, \rho^{(1)}_\Lambda \rangle$$

for any $\Lambda \in \Psi_n$ and $w \in W$ if $m$ is any sufficiently large positive integer prime to $2p$ with $p$ the characteristic of $F_q$. Hence the required character formula is obtained for any $q$.

Our proof goes as follows. Firstly, we write the Frobenius mapping $F$
as $F=jF_0$ with $F_0$ a split Frobenius mapping and $j$ an automorphism of $G$ of finite order commuting with $F_0$, and let $\sigma_0=\sigma_0|G_{F^m}$ and $\langle \sigma_0 \rangle$ be the cyclic group generated by $\sigma_0$. Let $X_w^{(m)} (w \in W)$ be the Deligne-Lusztig varieties [2], [6] of $G$ defined using the Frobenius mapping $F^m$. Then $G_{F^m}$ and $F_0$ act naturally on $X_w^{(m)}$, hence on their $\ell$-adic cohomology spaces $H^i(X_w^{(m)})$.

Then we prove (Theorem 3.2) the relation

$$\text{Tr}(xF^*_0, \sum_{i=0}^m (-1)^i H^i(X_w^{(m)})) = \text{Tr}(yF^*_0, \sum_{i=0}^m (-1)^i H^i(X_w^{(1)}))$$

for any odd integer $m$ and any $x \in G_{F^m}$, where $y=N^{(m)}(x)$ and $N^{(m)}$ is the norm mapping defined by N. Kawanaka (see our definition preceding Theorem 3.2).

As a next step, we show that any unipotent representation of $G_{F^m}$ is $\sigma_0$-invariant if $m$ is odd. Then by applying N. Kawanaka's result on the lifting [3], [4], we prove (Theorem 4.2) that

$$\text{Tr}(xj\sigma, \hat{\rho}_\Lambda^{(m)}) = \text{Tr}(N^{(m)}(x)j, \hat{\rho}_\Lambda^{(1)})$$

for any $x \in G_{F^m}$, any symbol class $\Lambda \in \Psi_n$ and any positive integer prime to $2p$, where $\hat{\rho}_\Lambda^{(m)}$ and $\hat{\rho}_\Lambda^{(1)}$ are the representations of the semi-direct product groups $G_{F^m}\langle \sigma_0 \rangle$ and $G_{F}\langle j \rangle$ that extend $\rho_\Lambda^{(m)}$ and $\rho_\Lambda^{(1)}$ respectively in a normalized manner. Combining polynomial equations (in $q$) obtained from (***) and (****) with a result on Frobenius eigenvalues given in [1] (resp. [8]), we get the asserted relation (*) for $G=Sp_{2n}$, $SO_{2n+1}$ (resp. $SO_{2n}$).

Finally the author is very grateful to Professor N. Kawanaka for his kind conversations, through which a perspective on the lifting theory was shown to the author.

1. First we need a generalization of Lusztig [6, 3.9]. Let $G$ be a connected reductive group defined over a finite field $F_q$ and $F$ the Frobenius mapping. Let $B$ be a fixed $F$-stable Borel subgroup, $T$ a fixed $F$-stable maximal torus in $B$, $U$ the unipotent radical of $B$ and $W$ the Weyl group of $G$ relative to $T$. There exists an automorphism $j$ of $G$ of finite order $m$ defined over $F_q$ such that $j$ stabilizes $B$, $T$ and induces the same action on $W$ as that of $F$. For a positive integer $m$, we set

$$\sigma = F|G_{F^m}, \quad F_0 = j^{-1} F, \quad \sigma_0 = j^{-1} \sigma .$$

$\sigma$ and $\sigma_0$ generate the cyclic groups $\langle \sigma \rangle$ of order $m$ and $\langle \sigma_0 \rangle$ of order $m\delta$ respectively. We denote by $X$ the variety $G/B$ of all Borel subgroups. For our purpose we have to borrow almost all the notations in [6, 3.3–3.9] such as

$$X_w, Y_{w, w', w_1}, Z_{w, w', w_1}, (w, w', w_1 \in W).$$

But to specify the Frobenius mapping (either $F$ or $F^m$), we write as follows (cf. [6, 3.3–3.4]).
Theorem 3.8 in [6] is generalized to the following.

**Theorem 1.1.** For \( w, w' \in W \), \( F_0 \) acts naturally on the variety \( G^{F_0} \setminus (X_w^{(m)} \times X_{w'}^{(m)}) \), and

(i) all the eigenvalues of \( F_0^* \) on \( H^i(\{ (X_w^{(m)} \times X_{w'}^{(m)}) \}) \) are integral powers of \( q \),

(ii) for a positive integer \( e \), the number of \( F_0^* \)-fixed points of the quotient variety \( G^{F_0} \setminus (X_w^{(m)} \times X_{w'}^{(m)}) \) is equal to the trace of the linear transformation \( x \mapsto t_wF_0(x)t_w^{-1} \) of \( \mathcal{A}(W, q') \).

**Proof.** The proof of [6, 3.8] shows that it suffices to prove the following variation of [6, 3.5]:

There exists a natural isomorphism \( H^i(Y_w^{(m)} \times Y_{w'}^{(m)}) \approx H^i(Z_w^{(m)} \times Z_{w'}^{(m)}) \) for any \( i \geq 0 \) which commutes with the action of \( F_0^* \).

But this can be proved by almost the same argument as in the proof of [6, 3.5].

Let \( \rho \) be a unipotent representation of \( G^{F_0} \). For \( w \in W \) and \( i \geq 0 \), \( H^i(X_w^{(m)}) \) denotes the largest subspace of \( H^i(X_w^{(m)}) \) on which \( G^{F_0} \) acts by a multiple of \( \rho \). We choose \( w \) and \( i \) in such a way that \( H^i(X_w^{(m)}) \) is non-zero. Fix a decomposition

\[
H^i(X_w^{(m)}) = (\mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_r) \otimes \rho
\]

as a \( G^{F_0} \)-module. Then the \( G^{F_0} \)-module endomorphism algebra of \( H^i(X_w^{(m)}) \) is identified with the matrix algebra \( M_r(\mathcal{Q}_1) \) of rank \( r \). Assume that \( \rho \) is \( \sigma_0 \)-invariant (up to equivalence). Then \( \rho \) is extended to an irreducible representation \( \hat{\rho} \) of the semi-direct product \( G^{F_0} \langle \sigma_0 \rangle \). There are \( m \delta \)-choices for such \( \hat{\rho} \). We fix \( \hat{\rho} \) to be one of them. We may regard \( H^i(X_w^{(m)}) \) as a \( G^{F_0} \langle \sigma_0 \rangle \)-module by the identification

\[
H^i(X_w^{(m)}) = (\mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_r) \otimes \hat{\rho}
\]

Since \( \rho \) is \( \sigma_0 \)-invariant, \( F_0^* \) stabilizes \( H^i(X_w^{(m)}) \) and \( F_0^* \) acts on \( H^i(X_w^{(m)}) \) by
\[ \xi \otimes \bar{\rho}(\sigma_0^{-1}) \]

with \( \xi \in M_r(\overline{Q}) \).

**Theorem 1.2.** Let \( \rho \) be a \( \sigma_0 \)-invariant unipotent representation of \( G^{p_m} \) and \( \bar{\rho} \) be its extension to an irreducible representation of \( G^{p_m} \langle \sigma_0 \rangle \). Let \( \mu \) be any eigenvalue of the matrix \( \xi \) defined as above for some \( i \) and \( w \). Then \( \mu \) is uniquely determined by \( \bar{\rho} \) up to a multiplicative factor \( q^a \) for an integer \( a \) and does not depend on the choice of \( i \) and \( w \).

**Proof.** We proceed quite identically with the proof of [6, 3.9]. Let \( \bar{\rho} \) be the dual representation of \( \rho \). Obviously the representation \( \bar{\rho} \) restricted to \( G^{p_m} \) is the dual representation \( \bar{\rho} \) of \( \rho \). Take \( w' \in W, i' \geq 0 \) such that \( \bar{\rho} \) is a subrepresentation of \( H_{i'}(X^{(m)}) \). Fix an identification

\[
H_{i'}(X^{(m)})_{\rho} = (\overline{Q} \oplus \cdots \oplus \overline{Q}) \otimes \rho \text{ \ Make \ r'-times} 
\]

and write \( F^* = \xi' \otimes \bar{\rho}(\sigma_0^{-1}) \) on \( H_{i'}(X^{(m)})_{\rho} \) with \( \xi' \in M_{r'}(\overline{Q}) \). First we consider the orthogonal projection from the space \( \bar{\rho} \otimes \bar{\rho} \) to the \( G^{p_m} \)-invariant subspace \( (\bar{\rho} \otimes \bar{\rho}) \otimes \rho \approx \overline{Q} \rho \), which is defined by

\[
v_1 \otimes v_2 \rightarrow |G^{p_m}|^{-1} \sum_{x \in G^{p_m}} \bar{\rho}(x)v_1 \otimes \bar{\rho}(x)v_2
\]

Since \( Tr(\rho) \otimes \bar{\rho}(x) = 1 \), the following diagram commutes.

The commutativity of this diagram in turn shows the commutativity of the following.

\[
H_{i'}(X^{(m)})_{\rho} \otimes H_{i'}(X^{(m)})_{\rho} \xrightarrow{\text{proj.}} (H_{i'}(X^{(m)})_{\rho} \otimes H_{i'}(X^{(m)})_{\rho})_{G^{p_m}}
\]

Thus the induced action of \( F^*_0 \) on

\[
(H_{i'}(X^{(m)})_{\rho} \otimes H_{i'}(X^{(m)})_{\rho})_{G^{p_m}} \approx (\overline{Q} \oplus \cdots \oplus \overline{Q}) \otimes (\overline{Q} \oplus \cdots \oplus \overline{Q}) \text{ \ \ \ \ Make \ r-times, \ r'-times}
\]
is identified with $\xi \otimes \xi'$. Now, the canonical inclusion

$$(H^i_\varepsilon(X_w^{(m)}_p) \otimes H^i_\varepsilon(X_w^{(m')}_{p'}^\dagger))^{G_F^m} \hookrightarrow H^i_\varepsilon(G^m_p(X_w^{(m)}_p) \times X_w^{(m')}_{p'})$$

commutes with the action of $F^\delta_0$. Therefore, Theorem 1.1 shows that all the eigenvalues of $\xi \otimes \xi'$ have the form $q^a$ for some integer $a$. Since another choice of $i$ and $w$ yields the same result, the required statement follows.

**Definition 1.3.** Let $\rho, \mu$ be as in Theorem 1.2. We define $\mu_3$ by

$$1 \leq |\mu_3| < q, \quad \mu_3 = q^a$$

for some integer $a$.

**Corollary 1.4.** For $w \in W$, there exists a unique polynomial $f_{\rho, w}(X)$ such that

1. $\text{Tr}((xF_0)^{*}, \sum_{i \geq 0} (-1)^i H^i_\varepsilon(X_w^{(m)})) = f_{\rho, w}(q)\mu_3 \text{Tr}((x\sigma_0)^{-1}, \rho)$

for any $x \in G_F^m$ and positive integer $e$,

2. $f_{\rho, w}(1) = \langle \rho^{(m)}, R^{(m)}_w \rangle$,

where $R^{(m)}_w$ denotes the virtual $G_F^m$-module $\sum_{i \geq 0} (-1)^i H^i_\varepsilon(X_w^{(m)}).$

Since $j^i = 1$, $F_0^{m_3} = F^{m_3}$. Let $\lambda_\rho$ be the normalized eigenvalue of $(F^{m_3})^*$ associated with $\rho$, i.e. $\lambda_\rho$ is equal to an eigenvalue of $(F^{m_3})^*$ (acting on $H^i_\varepsilon(X_w^{(m)}_p)$ for some $i$ and $w$) up to a multiplicative factor $q^{m_3 a}$ for some integer $a$, and satisfies

$$1 \leq |\lambda_\rho| < q^{m_3}$$

By [6, 3.9], $\lambda_\rho$ is uniquely determined by $\rho$. Let $\bar{\rho}$, $\mu_3$ be as in Definition 1.3. Obviously $\mu_3^{m_3} = \lambda_\rho$. There are $m_3 \delta$-extensions $\bar{\rho}$ for the fixed $\sigma_0$-invariant $\rho$ and there are $m_3 \delta$-constants $\mu$ such that $\mu^{m_3} = \lambda_\rho$. Therefore we have

**Lemma 1.5.** Let $\rho$ be a $\sigma_0$-invariant unipotent representation of $G_F^m$. Then the mapping $\bar{\rho} \mapsto \mu_3$ induces the bijection

$$\{\bar{\rho} \in (G_F^m)^{<\sigma_0>}^\vee; \bar{\rho} | G_F^m = \rho\} \rightarrow \{\mu; \mu^{m_3} = \lambda_\rho\}$$

where $(G_F^m)^{<\sigma_0>}^\vee$ denotes the set of irreducible representations of $G_F^m^{<\sigma_0>}$ (up to equivalence).

2. Henceforth we assume that the positive integer $m$ is prime to the order $\delta$ of $j$. Let $S$ be the set of simple reflections of $W$ associated with the Borel subgroup $B$. For $I \subseteq S$, let $P_I$ be the corresponding standard parabolic subgroup and $L_I$ its standard Levi subgroup. Let $I_0$ be an $F$-stable subset of $S$. Let $\rho_0$ be a unipotent cuspidal representation of $L_{I_0}^{F_m}$. Let $\rho$ be a unipotent representation of $G_F^m$. If $\rho$ appears in the induced representation of $G_F^m$ from
the representation \( \rho_0 \) inflated to \( P^F_{\rho_0} \), then we call \( \rho \) a unipotent representation of \( G^{p^m} \) in the series of \( \rho_0 \). Now, we assume that \( \rho_0 \) is \( \sigma_0 \)-invariant, and we fix a representation \( \rho_0 \) of the semi-direct product \( L^F_{\rho_0} \langle \sigma_0 \rangle \) that extends \( \rho_0 \). Let \( J \) be any \( F \)-stable subset of \( S \) containing \( I_0 \). We further assume that any unipotent representation \( \rho \) of \( L^J_{\rho_0} \) in the series of \( \rho_0 \) is \( \sigma_0 \)-invariant (for any \( J \)). By [2, 8.2], the eigenvalues of \((F^p)^*\) associated with \( \rho \) and \( \rho_0 \) coincide with each other (up to a multiplicative factor \( q^{m_{\sigma_0}} \) for some integer \( a \)). Therefore we may fix a representation \( \tilde{\rho} \) of \( L^J_{\rho_0} \langle \sigma_0 \rangle \) extending \( \rho \) by the condition

\[
\mu_\rho = \mu_{\rho_0}
\]

(cf. Lemma 1.5).

**Lemma 2.1.** Let the assumptions be as above. Let \( J \) be an \( F \)-stable subset of \( S \) such that \( I_0 \subseteq J \subseteq S \). Let \( \rho \) be a unipotent representation of \( L^J_{\rho_0} \) in the series of \( \rho_0 \). Assume that

\[
\text{Ind}_{P^F_{\rho_0}}^{G^{p^m}}(\rho) = \sum_{i \leq J} m_i \rho_i
\]

with each \( \rho_i \) a unipotent representation of \( G^{p^m} \) in the series of \( \rho_0 \) and \( m_i \) a positive integer. Then

\[
\text{Ind}_{P^F_{\rho_0}}^{G^{p^m} \langle \sigma_0 \rangle} \tilde{\rho} = \sum_{i \leq J} m_i \tilde{\rho_i}
\]

Proof. There are two methods in extending a unipotent representation of \( G^{p^m} \) in the series of \( \rho_0 \) to a representation of \( G^{p^m} \langle \sigma_0 \rangle \) in normalized manners:

One is by using the eigenvalues of the Frobenius mapping \( F^p \) (the method which we have adopted here). The other is simply inducing the action of \( \sigma_0 \) on the representation \( \tilde{\rho}_0 \).

To prove our lemma it suffices to show that these two methods yield the same extension for any \( \rho_i \) (or \( \rho \)). But this is apparent from the proof of [2, 8.2].

### 3. Let \( H \) be a finite group and \( \alpha \) an automorphism of \( H \). For \( h_1, h_2 \in H \), we define the equivalence relation \( \alpha \) by

\[
h_1 \alpha h_2 \Leftrightarrow h_1 = h^{-1} h_2 \alpha h
\]

for some \( h \in H \).

For \( x \in G^{p^m} \), write \( x = a^{-1} r^\alpha a \) with \( a \in G \) and put \( y = a^{-1} r^\alpha a^{-1} \). Then \( x \to y \) defines the bijection

\[
G^{p^m} / F_0 \to G^{p^m} / F_0^{p^m}
\]

which will be denoted by \( n_{F^m / F_0} \). Quite analogously to Lemma 1.2.1 of [1], we obtain
Lemma 3.1. For any $x \in G^m$ and $w \in W$,
\[
\text{Tr}((xF_0)^*, \sum_{i \in I_0} (-1)^i H_i^j(X^w)) = \left\{ \begin{array}{ll}
\{ h \in G^m; h^{-1}n_{F_0}(x)^{-1} h \in \dot{w}B \} ,
\end{array} \right.
\]
where $d = \dim(U \cap \dot{w}U\dot{w}^{-1})$, and $\dot{w}$ is an $F_0$-stable representative of $w$ in the normalizer $N_0(T)$ of $T$ in $G$.

Assume $m \equiv 1 \mod \delta$. Then we may define the mapping
\[
N^{(\sigma)} = n_{F_0}^1 \circ n_{F_0}^2 : G^m/F_0^0 \rightarrow G^F/F_0^0
\]
Thus by the relation in the lemma combined with that relation with $m = 1$, we obtain

Theorem 3.2. Assume $m \equiv 1 \mod \delta$. For any $x \in G^m$ and $w \in W$,
\[
\text{Tr}((xF_0)^*, \sum_{i \in I_0} (-1)^i H_i^j(X^w)) = \text{Tr}((N^{(\sigma)}(x)F_0)^*, \sum_{i \in I_0} (-1)^i H_i^j(X^w)).
\]

4. We preserve the notations used until now. Assume $G = SO_{2n}$, $Sp_{2n}$ or $SO_{2n+1}$. In some cases, $G$ is also denoted by $G_n$ to specify $n$. If $G \neq SO_{2n}$, we take $j$ to be identity, and if $G = SO_{2n}$, we take $j$ to be of order 2. Let $\bar{G}$ be the semi-direct product $G \langle j \rangle$. If $m \equiv 1 \mod \delta$, then $G^m(\langle \sigma \rangle) = G^m(\langle \sigma_0 \rangle)$. First we need

Lemma 4.1. Assume $m \equiv 1 \mod \delta$. Then all the unipotent representations of $G^m$ (resp. $G^r$) are $\sigma_0$-invariant.

Proof. For an $F$-stable closed subgroup $H$ of $G$, we denote by $H^{(m)}$ the group of all $F^m$-stable points of $H$. Let $I_0$ be a subset of $S$ such that there exists a unipotent cuspidal representation $\rho_0$ of $L_0^{(m)}$. To prove the lemma it suffices to prove that any unipotent representation of $G^{(m)}$ in the series of $\rho_0$ is $\sigma_0$-invariant. We recall a result of Lusztig [5, §5]. Let $\bar{W} = (N_0(L_{t_0})/L_{t_0})^{\rho_0}$, where $N_0(L_{t_0})$ is the normalizer of $L_{t_0}$ in $G$. $\bar{W}$ has a natural structure as a Coxeter group with the canonical set of generators $\bar{S}$. For a subset $J$ of $S$ with $I_0 \subseteq J \subseteq S$, a subset $J$ of $\bar{S}$ is associated in a natural manner and any subset of $\bar{S}$ is obtained in this form. We denote by $\bar{W}_J$ the subgroup of $\bar{W}$ generated by $\bar{J}(\subseteq \bar{S})$. Then unipotent representations (up to equivalence) of $G^{(m)}$ (resp. $L^{(m)}$) in the series of $\rho_0$ are parameterized by the set of irreducible representations $\bar{W}^\wedge$ (resp. $(\bar{W}_J)^\wedge$) of $\bar{W}$ (resp. $\bar{W}_J$). And this parameterization is compatible with the inductions:
\[ \chi \in R(\overline{W}_T) \sim \{ \text{Z-linear combi. of unip. char. of } L_{\rho_0}^{(m)} \} \equiv \rho \]

\[ \text{Ind}_{\overline{W}_T}^\overline{W} \chi \in R(\overline{W}) \sim \{ \text{Z-linear combi. of unip. char. of } G^{(m)} \} \equiv \text{Ind}_{\rho_0}^{G^{(m)}} \rho \]

where \( R(\overline{W}_T) \) and \( R(\overline{W}) \) denote the group of all virtual characters of \( \overline{W}_T \) and \( \overline{W} \) respectively, and irreducible characters are mapped to the irreducible characters by the horizontal isomorphisms. Now, \( (\overline{W}, \overline{S}) \) is isomorphic to a classical Weyl group. Thus, if \( \text{rank}(\overline{W}, \overline{S}) \geq 2 \), then we have:

For \( \chi_1, \chi_2 \in \overline{W} \), if \( \chi_1 | \overline{W}_T = \chi_2 | \overline{W}_T \) for any \( J \equiv \overline{S} \), then \( \chi_1 = \chi_2 \).

Therefore to prove that any unipotent representation \( \rho \) in the series of \( \rho_0 \) is \( \sigma_0 \)-invariant, it suffices to prove the statement only when \( \rho \) is a cuspidal (i.e. \( I_0 = S \)) or subcuspidal (i.e. \( |S \setminus I_0| = 1 \)) representation (see [5]). Assume that \( \rho \) is cuspidal, i.e. \( \rho = \rho_0 \). Then \( \rho \) is the unique unipotent cuspidal representation. Therefore \( \rho \) is \( \sigma_0 \)-invariant. Assume that \( \rho \) is subcuspidal. Let \( \rho' \) be another unipotent subcuspidal representation (see [5]). Since \( \text{dim } \rho = \text{dim } \rho' \) (cf. [4]) and there is no other unipotent subcuspidal representation, \( \rho \) and \( \rho' \) are both \( \sigma_0 \)-invariant.

Henceforth we assume that \( m \) is prime to \( 2p \) with \( p \) the characteristic of \( F_q \). Then by N. Kawanaka [3], [4], the following statement is true:

For any \( \sigma_0 \)-invariant irreducible representation \( \rho^{(m)} \) of \( G^{(m)} \), there exists a \( \sigma_0 \)-invariant (or \( j \)-invariant) irreducible representation \( \rho^{(i)} \) of \( G^f \) such that

\[ \text{Tr}(xj|\sigma, \rho^{(i)}) = c \text{Tr}(N^{(m)}(xj), \rho^{(i)}) \]

for any \( x \in G^{f(m)} \), where \( \rho^{(m)} \) (resp. \( \rho^{(i)} \)) is an irreducible representation of \( G^{f(m)} \) (resp. \( G^f \)) that extends \( \rho^{(m)} \) (resp. \( \rho^{(i)} \)), and \( c \) is a root of unity. We now assume that \( m \) is sufficiently large so that the main theorem in [7] (resp. [8]) holds for the group \( G^f \) if \( G = SO_{2n+1} \) or \( Sp_{2n} \) (resp. \( G = SO_{2n}^2 \)). Let \( \Phi_n, \Phi_n^* \) be the sets of symbol classes defined in [5, §3]. We set

\[ \Psi_n = \begin{cases} \Phi_n & \text{if } G = SO_{2n+1} \text{ or } Sp_{2n} \\ \Phi_n^* (\text{resp. } \Phi_n^*) & \text{if } G = SO_{2n}^2 (\text{resp. } SO_{2n}^2) \end{cases} \]

By [5], the unipotent representations of \( G^{f(m)} \) (resp. \( G^f \)) are parameterized by the symbol classes in \( \Psi_n \). For \( \Lambda \in \Psi_n \), we denote by \( \rho^{(m)}_{\Lambda} \) (resp. \( \rho^{(i)}_{\Lambda} \)) the corresponding unipotent representation of \( G^{f(m)} \) (resp. \( G^f \)), and by \( \lambda_{\rho^{(m)}_{\Lambda}} \) (resp. \( \lambda_{\rho^{(i)}_{\Lambda}} \)) the normalized eigenvalue of \( (F^{m})^* \) (resp. \( (F^f)^* \)) associated with the unipotent representation \( \rho^{(m)}_\Lambda \) (resp. \( \rho^{(i)}_\Lambda \)). By [1], \( \lambda_{\rho^{(m)}_{\Lambda}} \) and \( \lambda_{\rho^{(i)}_{\Lambda}} \) are 1 or \(-1\) if \( G = SO_{2n+1}, Sp_{2n} \) or \( SO_{2n}^2 \). By [8, 3.4], \( \lambda_{\rho^{(m)}_{\Lambda}} = \lambda_{\rho^{(i)}_{\Lambda}} = 1 \) for any \( \Lambda \in \Psi_n \) if \( G = SO_{2n}^2 \).
Since \( m \) is odd, we may choose the extension \( \tilde{\rho}_\Lambda^{(m)} \in (G^{\rho_{\text{m}}}_\Lambda \langle \sigma \rangle)^\wedge \) of \( \rho_\Lambda^{(m)} \) by the condition
\[
\mu_{\tilde{\rho}_\Lambda^{(m)}} = \lambda_{\tilde{\rho}_\Lambda^{(m)}}
\]
(See Lemma 1.5). And we may choose the extension \( \tilde{\rho}_\Lambda^{(1)} \in (G^{\rho}_{\text{1}}j)^\wedge \) of \( \rho_\Lambda^{(1)} \) by the condition
\[
\mu_{\tilde{\rho}_\Lambda^{(1)}} = \lambda_{\tilde{\rho}_\Lambda^{(1)}};
\]
Here we applied Lemma 1.5 with \( m=1 \). Let \( (W \langle j \rangle)^* \) be the set of irreducible representations \( \chi \) (up to equivalence) of the semi-direct product \( W \langle j \rangle \) such that \( \chi|W \) is irreducible. For any \( \chi \in (W \langle j \rangle)^* \), let \( R_\chi^{(m)} \) be the class function of \( G^{\rho_{\text{m}}} \) defined in [6, (3.17.1)], i.e.
\[
R_\chi^{(m)} = |W|^{-1} \sum_{w \in W} \text{Tr}(w j, \chi) R_w^{(m)}
\]
where \( R_w^{(m)} \) is the character of the virtual \( G^{\rho_{\text{m}}} \)-module \( \sum_{i \geq 0} (-1)^i H_i^j(X_w^{(m)}) \).

We are to prove

**Theorem 4.2.** Let \( \tilde{\rho}_\Lambda^{(m)} \) and \( \tilde{\rho}_\Lambda^{(1)} \) \((\Lambda \in \Psi_*^n)\) be the extensions of \( \rho_\Lambda^{(m)} \) and \( \rho_\Lambda^{(1)} \) chosen as above. Then we have
(i) \( \lambda_{\tilde{\rho}_\Lambda^{(m)}} = \lambda_{\tilde{\rho}_\Lambda^{(1)}} \)
(ii) \( \langle \rho_\Lambda^{(m)}, R_\chi^{(m)} \rangle = \langle \rho_\Lambda^{(1)}, R_\chi^{(1)} \rangle \) for any \( \chi \in (W \langle j \rangle)^* \),
(iii) \( f_{\rho_\Lambda^{(m)}}, w(X) = f_{\rho_\Lambda^{(1)}}, w(X) \) for any \( w \in W \).

**Corollary 4.3** The main theorems in G. Lusztig [7], [8] are true for any finite field.

**Lemma 4.4.** Let \( \Lambda_1, \Lambda_2 \in \Psi_*^n \). Assume
\[
(*) \quad \text{Tr}(xj \sigma, \tilde{\rho}_\Lambda^{(m)}) = c \text{Tr}(N^{(m)}(x) j, \tilde{\rho}_{\Lambda_2}^{(1)})
\]
for any \( x \in G^{\rho_{\text{m}}} \) with some root \( c \) of \( 1 \). Then
(i) \( \lambda_{\rho_{\Lambda_1}^{(m)}} = c \lambda_{\rho_{\Lambda_2}^{(1)}} \)
(ii) \( \dim \rho_{\Lambda_1}^{(1)} = \dim \rho_{\Lambda_2}^{(1)} \)
(iii) \( \langle \rho_{\Lambda_1}^{(1)}, R_\chi^{(m)} \rangle = \langle \rho_{\Lambda_2}^{(1)}, R_\chi^{(1)} \rangle \) for any \( \chi \in (W \langle j \rangle)^* \),
(iv) \( f_{\rho_{\Lambda_1}^{(m)}}, w(X) = f_{\rho_{\Lambda_2}^{(1)}}, w(X) \) for any \( w \in W \).

To prove the lemma we need some preparations. Let \( H(W) \) be the generalized Hecke algebra of the Coxeter group \( (W, S) \) over the polynomial ring \( \mathbb{Q}[X] \) that yields by the specialization \( (X \rightarrow q) \) the \( G^{\rho_{\text{0}}} \)-module endomorphism algebra of the induced representation of \( G^{\rho_{\text{0}}} \) from the trivial representation of
Let $\{a_w; w \in W\}$ be the canonical basis of $H(W)$. $H(W)$ is a subalgebra of an algebra $H(W^\langle j \rangle)$ defined as follows.

$$H(W^\langle j \rangle) = H(W) \oplus a_j H(W)$$
as linear spaces,

$$a_j a_w a_j^{-1} = a_{w^j w^{-1}}$$
for $w \in W$,

$$a_j^2 = 1$$

We put $a_j w = a_{w^j}$ (w $\in W$). Let $H^{(m)}(W^\langle j \rangle)$ (resp. $H^{(1)}(W^\langle j \rangle)$) denote the algebra obtained by specializing $X \to q^m$ (resp. $X \to q$) in the defining relations of $H(W^\langle j \rangle)$. For $w \in W^\langle j \rangle$, let $a_j^{(m)}$ (resp. $a_j^{(1)}$) denote the specialized element of $a_w$ in $H^{(m)}(W^\langle j \rangle)$ (resp. $H^{(1)}(W^\langle j \rangle)$). For $\chi \in (W^\langle j \rangle)$, let $\nu_\chi$ be the corresponding irreducible representation of $H(W^\langle j \rangle) \otimes \Phi(\chi)$ and $\nu_\chi^{(m)}$ (resp. $\nu_\chi^{(1)}$) its specialized representation of $H^{(m)}(W^\langle j \rangle)$ (resp. $H^{(1)}(W^\langle j \rangle)$).

Proof of Lemma 4.4. By Corollary 1.4 and Lemma 3.1 we have

$$\sum_{\Lambda \in \Psi_m} f_{\rho_\Lambda^{(m)}, w_0}(\rho_\Lambda^{(m)}) \left< X \rho_\Lambda^{-1} \right|, \rho_\Lambda^{(m)} \right> = \sum_{\Lambda \in \Psi_m} f_{\rho_\Lambda^{(1)}, w_0}(\rho_\Lambda^{(1)}) \left< (N^{(m)}(w)^{-1} \rho_\Lambda^{(1)} \right|, \rho_\Lambda^{(1)}$$
for any $w \in W$ and $x \in G^m$. The relation (1) and the relation $(\ast)$ in the lemma together with the orthogonality relations (cf. [1]) imply

$$f_{\rho_\Lambda^{(m)}, w_0}(\rho_\Lambda^{(m)}) = f_{\rho_\Lambda^{(1)}, w_0}(\rho_\Lambda^{(1)})$$
for any $w \in W$. By [1, 2.4.7] and by [8, 3.5], we have

$$f_{\rho_\Lambda^{(1)}, w_0}(X) = \delta^{-1} \sum_{\chi \in (W^\langle j \rangle)^{\ast}} \left< a_{w_0}, \nu_\chi \right| X \rho_\Lambda^{(1)} \right>$$
for $a = 1, m$ and $\Lambda \in \Psi_m$. By (2) and (3),

$$\{ \delta^{-1} \sum_{\chi \in (W^\langle j \rangle)^{\ast}} \left< a_{w^j w}, \nu_\chi \right| X \rho_\Lambda^{(1)} \right> \} = \{ \delta^{-1} \sum_{\chi \in (W^\langle j \rangle)^{\ast}} \left< a_{w}, \nu_\chi \right| X \rho_\Lambda^{(1)} \right> \}$$

Let $\{a_w^m; w \in W\}$ be the dual basis of $\{a_w; w \in W\}$. We put $a_w^{m(1)} = a_{w^j} a_w^m$ for $w \in W$. Then for $\chi, \chi' \in (W^\langle j \rangle)^{\ast},$

$$\sum_{w \in W} \left< a_{w^j w}^{m(1)}, \nu_\chi \right| X \rho_\Lambda^{(1)} \right> = 0$$
if and only if $\chi | W = \chi' | W$, where $a_{w}^{m(1)}$ is the specialized element of $a_w^m$. Thus by (4),

$$\left< R_\chi^{(m)}, \rho_\Lambda^{(m)} \right| X \rho_\Lambda^{(1)} = \left< R_\chi^{(1)}, \rho_\Lambda^{(1)} \right| X \rho_\Lambda^{(1)}$$
for any $\chi \in (W^\langle j \rangle)^{\ast}$. By [6, 3.12],
UNIPOTENT CHARACTERS OF $SO_{2n}$, $Sp_{2n}$ AND $SO_{2n+1}$

(6) \[ \dim \rho_{\Lambda_1}^{(m)} = \delta^{-1} \sum_{x \in (W_{\Lambda})^*} \langle R_{\chi}^{(m)}, \rho_{\Lambda_1}^{(m)} \rangle \dim R_{\chi}^{(m)} \]

By [4], $\dim \rho_{\Lambda_1}^{(m)}$ and $\dim R_{\chi}^{(m)}$ are expressed as polynomials in $q^m$. By Lusztig [7] and [8], $\langle R_{\chi}^{(m)}, \rho_{\Lambda_1}^{(m)} \rangle$ is independent of $m$, since we have assumed that $m$ is a sufficiently large odd integer. Thus the relation (6) holds with each term regarded as polynomials in $q^m$. Hence by replacing $q^m$ with $q$ in (6) we have

(7) \[ \dim \rho_{\Lambda_1}^{(1)} = \delta^{-1} \sum_{x \in (W_{\Lambda})^*} \langle R_{\chi}^{(1)}, \rho_{\Lambda_1}^{(1)} \rangle \dim R_{\chi}^{(1)} \]

By (5) and (7),

\[
\dim \rho_{\Lambda_1}^{(1)} = c_\lambda^{-1} \lambda_{\rho_{\Lambda_1}^{(1)}} \dim \rho_{\Lambda_2}^{(1)}
\]

Since $c$ is of absolute value 1, $c_\lambda^{-1} \lambda_{\rho_{\Lambda_1}^{(1)}}$ is also of absolute value 1. Considering that $\dim \rho_{\Lambda_1}^{(1)}$ and $\dim \rho_{\Lambda_2}^{(1)}$ are positive integers, we see that (i), (ii) of the lemma are true. (iii) is obtained by (5) and (i). (iv) is obtained by (3), (4) and (iii).

**Lemma 4.5.** Let $n_0$ be a non-negative integer. We assume that there exists a symbol class $\Lambda_0 \in \Psi_{n_0}$ of defect $d$ corresponding to the unipotent cuspidal representation. Let $\Lambda_1 \neq \Lambda_2 \in \Psi_{n_0+1}$ be the symbol classes of defect $d$ corresponding to the subcuspidal representations.

(i) Assume $\text{Tr}(x^j \sigma, \rho_{\Lambda_0}^{(m)}) = \text{Tr}(N(x)^j, \rho_{\Lambda_1}^{(1)})$ for any $x \in G_{n_0}^{E_m}$. Then

$$
\text{Tr}(x^j \sigma, \rho_{\Lambda_0}^{(m)}) = \text{Tr}(N(x)^j, \rho_{\Lambda_1}^{(1)})
$$

for any $x \in G_{n_0+1}^{E_m}$ with $(\Lambda, \Lambda')$ one of the following conditions (A) and (B):

(A) $(\Lambda, \Lambda') = (\Lambda_1, \Lambda_1), (\Lambda_2, \Lambda_2)$

(B) $(\Lambda, \Lambda') = (\Lambda_1, \Lambda_2), (\Lambda_2, \Lambda_1)$

(ii) Let $n \geq n_0+1$ and assume that the statement (i) with the condition (A) is true. Then

$$
\text{Tr}(x^j \sigma, \rho_{\Lambda_0}^{(m)}) = \text{Tr}(N(x)^j, \rho_{\Lambda_1}^{(1)})
$$

for any $x \in G_{n_0+1}^{E_m}$ and any $\Lambda \in \Psi_n$ of defect $d$.

**Proof.** By Lemma 2.1, we can apply the arguments employed in [1, 2.2.3]. (See Lemma 4.1)

Proof of Theorem 4.2. By Lemma 4.4, to prove the theorem it suffices to prove (i) of the theorem for any $\Lambda \in \Psi_n$. And Lemma 4.5 shows that it suffices to prove (i) of the theorem only when $\rho_{\Lambda}^{(m)}$ is cuspidal or subcuspidal.
Let \( n_0, \Lambda_0, \Lambda_1, \Lambda_2 \) be as in Lemma 4.5.

Assume \( n=n_0 \). \( \rho_{\Lambda_0}^{(m)} \) (resp. \( \rho_{\Lambda_0}^{(2)} \)) is the unique unipotent cuspidal representation of \( G^F \) (resp. \( G^F \)) and there is no unipotent subcuspidal representation of \( G^F \) (resp. \( G^F \)). By the induction, the statements of the theorem are true if \( \Lambda=\Lambda_0 \). In particular, the lifting of a non-cuspidal unipotent representation is a non-cuspidal unipotent representation, whereas the relation (1) in the proof of Lemma 4.4 shows that the lifting of \( \tilde{\rho}_{\Lambda_0}^{(1)} \) is unipotent (or its restriction to \( G^F \) is unipotent if \( G=SO_{2n} \)), and therefore must be \( \tilde{\rho}_{\Lambda_0}^{(1)} \mid G^F \).

Thus

\[
\text{Tr}(xj\sigma, \tilde{\rho}_{\Lambda_0}^{(m)}) = c \cdot \text{Tr}(N^{(m)}(x)j, \tilde{\rho}_{\Lambda_0}^{(1)})
\]

for any \( x \in G^F \) with a constant \( c \). Assume \( G=SO_{2n} \). Then \( \lambda_{\rho_{\Lambda_0}^{(m)}} = \lambda_{\rho_{\Lambda_0}^{(1)}} = 1 \).

Thus \( c=1 \) by Lemma 4.4, (i). Assume \( G=SO_{2n} \). We are to prove \( c=1 \).

By [1, 2.4.6], for any \( \chi \in W^\wedge \),

\[
\begin{align*}
1) & \quad \dim \rho_{\Lambda}^{(m)} = \sum_{\Lambda \in \mathcal{W}_n} \langle R_{\Lambda}^{(m)}, \rho_{\Lambda}^{(m)} \rangle \lambda_{\rho_{\Lambda}^{(m)}} \dim \rho_{\Lambda}^{(m)}, \\
2) & \quad \dim \rho_{\Lambda}^{(1)} = \sum_{\Lambda \in \mathcal{W}_n} \langle R_{\Lambda}^{(1)}, \rho_{\Lambda}^{(1)} \rangle \lambda_{\rho_{\Lambda}^{(1)}} \dim \rho_{\Lambda}^{(1)},
\end{align*}
\]

where \( \rho_{\Lambda}^{(m)} \) (resp. \( \rho_{\Lambda}^{(1)} \)) denotes the unipotent representation of \( \rho_{\Lambda}^{(m)} \) (resp. \( \rho_{\Lambda}^{(1)} \)) in the principal series corresponding with \( \chi \) (cf. [1]). Since \( \langle R_{\Lambda}^{(m)}, \rho_{\Lambda}^{(m)} \rangle \) is independent of the odd integer \( m \) (\( m \) sufficiently large), the relation (1) holds with each term regarded as a polynomial in \( q^m \). Thus by replacing \( q^m \) with \( q \) in (1),

\[
\begin{align*}
3) & \quad \dim \rho_{\Lambda}^{(1)} = \sum_{\Lambda \in \mathcal{W}_n} \langle R_{\Lambda}^{(m)}, \rho_{\Lambda}^{(1)} \rangle \lambda_{\rho_{\Lambda}^{(m)}} \dim \rho_{\Lambda}^{(1)}.
\end{align*}
\]

If \( \Lambda=\Lambda_0 \), we have already \( \langle R_{\Lambda}^{(m)}, \rho_{\Lambda}^{(m)} \rangle = \langle R_{\Lambda}^{(1)}, \rho_{\Lambda}^{(1)} \rangle \) and \( \lambda_{\rho_{\Lambda}^{(m)}} = \lambda_{\rho_{\Lambda}^{(1)}} \). Thus, by comparing the relation (2) and the relation (3), we obtain

\[
\langle R_{\Lambda}^{(m)}, \rho_{\Lambda_0}^{(m)} \rangle \lambda_{\rho_{\Lambda_0}^{(m)}} = \langle R_{\Lambda}^{(1)}, \rho_{\Lambda_0}^{(1)} \rangle \lambda_{\rho_{\Lambda_0}^{(1)}}
\]

for any \( \chi \in W^\wedge \). Thus by (iii) of Lemma 4.4, we have \( \lambda_{\rho_{\Lambda_0}^{(m)}} = \lambda_{\rho_{\Lambda_0}^{(1)}} \). (Note that there exists \( \chi \in W^\wedge \) such that \( \langle R_{\Lambda}^{(1)}, \rho_{\Lambda_0}^{(1)} \rangle \neq 0 \). Hence by (i) of Lemma 4.4, we have \( c=1 \). Therefore we have proved the theorem for \( \Lambda=\Lambda_0 \).

Assume \( n=n_0+1 \). \( \rho_{\Lambda_i}^{(m)} \) (resp. \( \rho_{\Lambda_i}^{(1)} \)) \((i=1, 2)\) are subcuspidal representations of \( G^F \) (resp. \( G^F \)) and the other unipotent representations of \( G^F \) (resp. \( G^F \)) are neither cuspidal nor subcuspidal. Let \( i=1 \) or 2. By Lemma 4.5, there exists \( i'=1 \) or 2 such that

\[
\text{Tr}(xj\sigma, \tilde{\rho}_{\Lambda_i}^{(m)}) = \text{Tr}(N^{(m)}(x)j, \tilde{\rho}_{\Lambda_i}^{(1)})
\]

for any \( x \in G^F \). Then by Lemma 4.4, \( \dim \rho_{\Lambda_i}^{(1)} = \dim \rho_{\Lambda_i}^{(1)} \). Since \( \dim \rho_{\Lambda_i}^{(1)} \neq \dim \rho_{\Lambda_i}^{(3)} \), we must have \( i=i' \). This proves the theorem for \( \Lambda=\Lambda_1, \Lambda_2 \).
References


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