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UNIPOTENT CHARACTERS OF $SO_{2n}^{\pm}$, Sp_{2n} AND SO_{2n+1} OVER F_q WITH SMALL q

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0. Introduction. Let G be a special orthogonal group or symplectic group over a finite field $\boldsymbol{F}_q$, F the Frobenius mapping and G^F the group of all F-stable points of G. G. Lusztig [7], [8] has obtained explicit formulas for the characters of the unipotent representations of G^F on any regular semisimple element of G^F provided that the order q of the defining field $\boldsymbol{F}_q$ is sufficiently large. Our purpose in this paper is to show that his formulas are valid for any q.

Let W be the Weyl group of G and m an odd positive integer. For $w\in W$, let $R_w^{(m)}$ be the Deligne-Lusztig virtual representation [2], [6, 3.4] of G^{F^m}. By [2, 7.9], to determine the values of the character of a unipotent representation ρ of G^{F^m} on regular semisimple elements, it suffices to determine the inner product

$$\langle R_w^{(m)}, \rho\rangle$$

for any $w\in W$. This has been done by G. Lusztig [7], [8] for a sufficiently large q^m. Let n be the rank of G and Ψ_n be the set of symbol classes (cf. [5, § 3]) that parameterizes the unipotent representations (up to equivalence) of G^F or G^{F^m}, i.e.

$$\Psi_n = \begin{cases} \Phi_n & \text{if } G = SO_{2n+1} \text{ or } Sp_{2n} \\ \Phi_n^{\pm} & \text{if } G = SO_{2n}^{\pm} \end{cases}$$

in the notations in [5, § 3]. For $\Lambda\in\Psi_n$, let $\rho_\Lambda^{(1)}$ and $\rho_\Lambda^{(m)}$ be the corresponding unipotent representations of G^F and G^{F^m} respectively. Our main result (Theorem 4.2, (iii)) is

$$(*)\qquad \langle R_w^{(m)}, \rho_\Lambda^{(m)}\rangle = \langle R_w^{(1)}, \rho_\Lambda^{(1)}\rangle$$

for any $\Lambda\in\Psi_n$ and $w\in W$ if m is any sufficiently large positive integer prime to $2p$ with p the characteristic of $\boldsymbol{F}_q$. Hence the required character formula is obtained for any q.

Our proof goes as follows. Firstly, we write the Frobenius mapping F

as $F=jF_0$ with F_0 a split Frobenius mapping and j an automorphism of G of finite order commuting with F_0, and let $\sigma_0=F_0|G^{F^m}$ and $\langle\sigma_0\rangle$ be the cyclic group generated by σ_0. Let $X_w^{(m)}$ ($w\in W$) be the Deligne-Lusztig varieties [2], [6] of G defined using the Frobenius mapping F^m. Then G^{F^m} and F_0 act naturally on $X_w^{(m)}$, hence on their l-adic cohomology spaces $H_c^i(X_w^{(m)})$.

Then we prove (Theorem 3.2) the relation

$$(**)\qquad Tr((xF_0)^*, \sum_{i\geq 0}(-1)^i H_c^i(X_w^{(m)})) = Tr((yF_0)^*, \sum_{i\geq 0}(-1)^i H_c^i(X_w^{(1)}))$$

for any odd integer m and any $x\in G^{F^m}$, where $y=N^{(m)}(x)$ and $N^{(m)}$ is the norm mapping defined by N. Kawanaka (see our definition preceding Theorem 3.2).

As a next step, we show that any unipotent representation of G^{F^m} is σ_0-invariant if m is odd. Then by applying N. Kawanaka's result on the lifting [3], [4], we prove (Theorem 4.2) that

$$(***)\qquad Tr(xj\sigma, \tilde\rho_\Lambda^{(m)}) = Tr(N^{(m)}(x)j, \tilde\rho_\Lambda^{(1)})$$

for any $x\in G^{F^m}$, any symbol class $\Lambda\in\Psi_n$ and any positive integer prime to $2p$, where $\tilde\rho_\Lambda^{(m)}$ and $\tilde\rho_\Lambda^{(1)}$ are the representations of the semi-direct product groups $G^{F^m}\langle\sigma_0\rangle$ and $G^F\langle j\rangle$ that extend $\rho_\Lambda^{(m)}$ and $\rho_\Lambda^{(1)}$ respectively in a normalized manner. Combining polynomial equations (in q) obtained from (**) and (***) with a result on Frobenius eigenvalues given in [1] (resp. [8]), we get the asserted relation (*) for $G=Sp_{2n}$, SO_{2n+1} (resp. $SO_{2n}^{\pm}$).

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1. First we need a generalization of Lusztig [6, 3.9]. Let G be a connected reductive group defined over a finite field $\boldsymbol{F}_q$ and F the Frobenius mapping. Let B be a fixed F-stable Borel subgroup, T a fixed F-stable maximal torus in B, U the unipotent radical of B and W the Weyl group of G relative to T. There exists an automorphism j of G of finite order δ defined over $\boldsymbol{F}_q$ such that j stabilizes B, T and induces the same action on W as that of F. For a positive integer m, we set

$$\sigma = F|G^{F^m}, \quad F_0=j^{-1}F, \quad \sigma_0=j^{-1}\sigma.$$

σ and σ_0 generate the cyclic groups $\langle\sigma\rangle$ of order m and $\langle\sigma_0\rangle$ of order $m\delta$ respectively. We denote by X the variety G/B of all Borel subgroups. For our purpose we have to borrow almost all the notations in [6, 3.3–3.9] such as

$$X_w,\ Y_{w,w',w_1},\ Z_{w,w',w_1},\ (w, w', w_1\in W).$$

But to specify the Frobenius mapping (either F or F^m), we write as follows (cf. [6, 3.3–3.4]).

$$X_w^{(1)}=\{B'\in X;\ B'\xrightarrow{w}F(B')\}\ ,$$

$$X_w^{(m)}=\{B'\in X;\ B'\xrightarrow{w}F^m(B')\}\ ,$$

$$Y_{w,w',w_1}^{(m)}=G^{F^m}\backslash((X_w^{(m)}\times X_{w'}^{(u)})\cap\mathcal{O}_{w_1})$$

$$Z_{w,w'w_1}^{(m)}=\{(B_1,B_2)\in X\times X;\ B\xrightarrow{w}B_1\xrightarrow{F^m(w_1)}B_2\xrightarrow{w'^{-1}}w_1'Bw_{w_1}'^{-1}\}\ .$$

The theorem 3.8 in [6] is generalized to the following.

Theorem 1.1. *For w, $w'\in W$, F_0 acts naturally on the variety $G^{F^m}\backslash(X_w^{(m)}\times X_{w'}^{(m)})$, and*

(i) *all the eigenvalues of F_0^* on $H_c^i(G^{F^m}\backslash(X_w^{(m)}\times X_{w'}^{(m)}))$ are integral powers of q,*

(ii) *for a positive integer e, the number of F_0^e-fixed points of the quotient variety $G^{F^m}\backslash(X_w^{(m)}\times X_{w'}^{(m)})$ is equal to the trace of the linear transformation $x\to t_wF_0(x)t_{w'^{-1}}$ of $\mathcal{H}(W,q^e)$.*

Proof. The proof of [6, 3.8] shows that it suffices to prove the following variation of [6, 3.5]:

There exists a natural isomorphism $H_c^i(Y_{w,w'w_1}^{(m)})\cong H_c^i(Z_{w,w',w_1}^{(m)})$ for any $i\geq 0$ which commutes with the action of F_0^*.

But this can be proved by almost the same argument as in the proof of [6, 3.5].

Let ρ be a unipotent representation of G^{F^m}. For $w\in W$ and $i\geq 0$, $H_c^i(X_w^{(m)})_\rho$ denotes the largest subspace of $H_c^i(X_w^{(m)})$ on which G^{F^m} acts by a multiple of ρ. We choose w and i in such a way that $H_c^i(X_w^{(m)})_\rho\neq 0$. Fix a decomposition

$$H_c^i(X_w^{(m)})_\rho=\underbrace{(\bar{\boldsymbol{Q}}_l\oplus\cdots\oplus\bar{\boldsymbol{Q}}_l)}_{r\text{-times}}\otimes\rho$$

as a G^{F^m}-module. Then the G^{F^m}-module endomorphism algebra of $H_c^i(X_w^{(m)})_\rho$ is identified with the matrix algebra $M_r(\bar{\boldsymbol{Q}}_l)$ of rank r. Assume that ρ is σ_0-invariant (up to equivalence). Then ρ is extended to an irreducible representation $\tilde{\rho}$ of the semi-direct product $G^{F^m}\langle\sigma_0\rangle$. There are $m\delta$-choices for such $\tilde{\rho}$. We fix $\tilde{\rho}$ to be one of them. We may regard $H_c^i(X_w^{(m)})_\rho$ as a $G^{F^m}\langle\sigma_0\rangle$-module by the identification

$$H_c^i(X_w^{(m)})_\rho=\underbrace{(\bar{\boldsymbol{Q}}_l\oplus\cdots\oplus\bar{\boldsymbol{Q}}_l)}_{r\text{-times}}\otimes\tilde{\rho}$$

Since ρ is σ_0-invariant, F_0^* stabilizes $H_c^i(X_w^{(m)})_\rho$ and F_0^* acts on $H_c^i(X_w^{(m)})_\rho$ by

$$\xi\otimes\tilde{\rho}(\sigma_0^{-1})$$

with $\xi\in M_r(\bar{\boldsymbol{Q}}_l)$.

Theorem 1.2. *Let ρ be a σ_0-invariant unipotent representation of G^{F^m} and $\tilde{\rho}$ be its extension to an irreducible representation of $G^{F^m}\langle\sigma_0\rangle$. Let μ be any eigenvalue of the matrix ξ defined as above for some i and w. Then μ is uniquely determined by $\tilde{\rho}$ up to a multiplicative factor q^a for an integer a and does not depend on the choice of i and w.*

Proof. We proceed quite identically with the proof of [6, 3.9]. Let $\bar{\tilde{\rho}}$ be the dual representation of $\tilde{\rho}$. Obviously the representation $\bar{\tilde{\rho}}$ restricted to G^{F^m} is the dual representation $\bar{\rho}$ of ρ. Take $w'\in W$, $i'\geq 0$ such that $\bar{\rho}$ is a subrepresentation of $H_c^{i'}(X_{w'}^{(m)})$. Fix an identification

$$H_c^i(X_w^{(m)})_{\bar{\rho}}=\underbrace{(\bar{\boldsymbol{Q}}_l\oplus\cdots\oplus\bar{\boldsymbol{Q}}_l)}_{r'\text{-times}}\otimes\bar{\rho}$$

and write $F_0^*=\xi'\otimes\bar{\tilde{\rho}}(\sigma_0^{-1})$ on $H_c^{i'}(X_{w'}^{(m)})_{\bar{\rho}}$ with $\xi'\in M_{r'}(\bar{\boldsymbol{Q}}_l)$. First we consider the orthogonal projection from the space $\tilde{\rho}\otimes\bar{\tilde{\rho}}$ to the G^{F^m}-invariant subspace $(\tilde{\rho}\otimes\bar{\tilde{\rho}})^{G^{F^m}}\cong\bar{\boldsymbol{Q}}_l$, which is defined by

$$v_1\otimes v_2\rightarrow|G^{F^m}|^{-1}\sum_{x\in G^{F^m}}\tilde{\rho}(x)v_1\otimes\bar{\tilde{\rho}}(x)v_2$$

Since $Tr(|G^{F^m}|^{-1}\sum_{x\in G^{F^m}}\tilde{\rho}(x\sigma_0)\otimes\bar{\tilde{\rho}}(x\sigma_0)=1$, the following diagram commutes.

$$\begin{array}{ccc}\tilde{\rho}\otimes\bar{\tilde{\rho}} & \xrightarrow{\text{proj.}} & (\tilde{\rho}\otimes\bar{\tilde{\rho}})^{G^{F^m}}\\ \Big\downarrow{\scriptstyle \tilde{\rho}(\sigma_0)\otimes\bar{\tilde{\rho}}(\sigma_0)} & & \Big\downarrow{\scriptstyle \text{id.}}\\ \tilde{\rho}\otimes\bar{\tilde{\rho}} & \xrightarrow{\text{proj.}} & (\tilde{\rho}\otimes\bar{\tilde{\rho}})^{G^{F^m}}\end{array}$$

The commutativity of this diagram in turn shows the commutativity of the following.

$$\begin{array}{ccc}H_c^i(X_w^{(m)})_\rho\otimes H_c^{i'}(X_{w'}^{(m)})_{\bar{\rho}} & \xrightarrow{\text{proj.}} & (H_c^i(X_w^{(m)})_\rho\otimes H_c^{i'}(X_{w'}^{(m)})_{\bar{\rho}})^{G^{F^m}}\\ \Big\downarrow{\scriptstyle \tilde{\rho}(\sigma_0)\otimes\bar{\tilde{\rho}}(\sigma_0)} & & \Big\downarrow{\scriptstyle \text{id.}}\\ H_c^i(X_w^{(m)})_\rho\otimes H_c^{i'}(X_{w'}^{(m)})_{\bar{\rho}} & \xrightarrow{\text{proj.}} & (H_c^i(X_w^{(m)})_\rho\otimes H_c^{i'}(X_{w'}^{(m)})_{\bar{\rho}})^{G^{F^m}}\end{array}$$

Thus the induced action of F_0^* on

$$(H_c^i(X_w^{(m)})_\rho\otimes H_c^{i'}(X_{w'}^{(m)})_{\bar{\rho}})^{G^{F^m}}\cong\underbrace{(\bar{\boldsymbol{Q}}_l\oplus\cdots\oplus\bar{\boldsymbol{Q}}_l)}_{r\text{-times}}\otimes\underbrace{(\bar{\boldsymbol{Q}}_l\oplus\cdots\oplus\bar{\boldsymbol{Q}}_l)}_{r'\text{-times}}$$

is identified with $\xi\otimes\xi'$. Now, the canonical inclusion

$$(H_c^i(X_w^{(m)})_\rho \otimes H_c^{i'}(X_{w'}^{(m)})_{\bar\rho})^{G^{F^m}} \hookrightarrow H_c^{i+i'}(G^{F^m}\backslash(X_w^{(m)}\times X_{w'}^{(m)}))$$

commutes with the action of F_0^*. Therefore, Theorem 1.1 shows that all the eigenvalues of $\xi\otimes\xi'$ have the form q^a for some integer a. Since another choice of i and w yields the same result, the required statement follows.

DEFINITION 1.3. Let $\tilde\rho$, μ be as in Theorem 1.2. We define $\mu_{\tilde\rho}$ by

$$1\leq|\mu_{\tilde\rho}|<q\,,\quad \mu_{\tilde\rho}=\mu q^a$$

for some integer a.

Corollary 1.4. *For $w\in W$, there exists a unique polynomial $f_{\rho,w}(X)$ such that*

(i) $Tr((xF_0^e)^*, \sum_{i\geq 0}(-1)^i H_c^i(X_w^{(m)})_\rho) = f_{\rho,w}(q^e)\mu_{\tilde\rho}^e Tr((x\sigma_0^e)^{-1}, \tilde\rho)$

for any $x\in G^{F^m}$ and positive integer e,

(ii) $f_{\rho,w}(1)=\langle\rho_w^{(m)}, R_w^{(m)}\rangle$,

where $R_w^{(m)}$ denotes the virtual G^{F^m}-module $\sum_{i\geq 0}(-1)^i H_c^i(X_w^{(m)})$.

Since $j^\delta=1$, $F_0^{m\delta}=F^{m\delta}$. Let λ_ρ be the normalized eigenvalue of $(F^{m\delta})^*$ associated with ρ, i.e. λ_ρ is equal to an eigenvalue of $(F^{m\delta})^*$ (acting on $H_c^i(X_w^{(m)})_\rho$ for some i and w) up to a multiplicative factor $q^{m\delta a}$ for some integer a, and satisfies

$$1\leq|\lambda_\rho|<q^{m\delta}$$

By [6, 3.9], λ_ρ is uniquely determined by ρ. Let $\tilde\rho$, $\mu_{\tilde\rho}$ be as in Definition 1.3. Obviously $\mu_{\tilde\rho}^{m\delta}=\lambda_\rho$. There are $m\delta$-extensions $\tilde\rho$ for the fixed σ_0-invariant ρ and there are $m\delta$-constants μ such that $\mu^{m\delta}=\lambda_\rho$. Therefore we have

Lemma 1.5. *Let ρ be a σ_0-invariant unipotent representation of G^{F^m}. Then the mapping $\tilde\rho\to\mu_{\tilde\rho}$ induces the bijection*

$$\{\tilde\rho\in(G^{F^m}\langle\sigma_0\rangle)^\wedge;\ \tilde\rho\,|\,G^{F^m}=\rho\}\to\{\mu;\ \mu^{m\delta}=\lambda_\rho\}$$

where $(G^{F^m}\langle\sigma_0\rangle)^\wedge$ denotes the set of irreducible representations of $G^{F^m}\langle\sigma_0\rangle$ (up to equivalence).

2. Henceforth we assume that the positive integer m is prime to the order δ of j. Let S be the set of simple reflections of W associated with the Borel subgroup B. For $I\subseteq S$, let P_I be the corresponding standard parabolic subgroup and L_I its standard Levi subgroup. Let I_0 be an F-stable subset of S. Let ρ_0 be a unipotent cuspidal representation of $L_{I_0}^{F^m}$. Let ρ be a unipotent representation of G^{F^m}. If ρ appears in the induced representation of G^{F^m} from the representation ρ_0 inflated to $P_{I_0}^F$, then we call ρ a unipotent representation of G^{F^m} in the series of ρ_0. Now, we assume that ρ_0 is σ_0-invariant, and we fix a representation $\tilde{\rho}_0$ of the semi-direct product $L_{I_0}^F\langle\sigma_0\rangle$ that extends ρ_0. Let J be any F-stable subset of S containing I_0. We further assume that any unipotent representation ρ of $L_J^{F^m}$ in the series of ρ_0 is σ_0-invariant (for any J). By [2, 8.2], the eigenvalues of $(F_0^{m\delta})^*$ associated with ρ and ρ_0 coincide with each other (up to a multiplicative factor $q^{m\delta a}$ for some integer a). Therefore we may fix a representation $\tilde{\rho}$ of $L_J^{F^m}\langle\sigma_0\rangle$ extending ρ by the condition

$$\mu_{\tilde{\rho}} = \mu_{\tilde{\rho}_0}$$

(cf. Lemma 1.5).

Lemma 2.1. *Let the assumptions be as above. Let J be an F-stable subset of S such that $I_0\subseteq J\subseteq S$. Let ρ be a unipotent representation of $L_J^{F^m}$ in the series of ρ_0. Assume that*

$$\mathrm{Ind}_{P_J^{F^m}}^{G^{F^m}} \rho = \sum_{1\leq i\leq r} m_i\rho_i$$

with each ρ_i a unipotent representation of G^{F^m} in the series of ρ_0 and m_i a positive integer. Then

$$\mathrm{Ind}_{P_J^{F^m}\langle\sigma_0\rangle}^{G^{F^m}\langle\sigma_0\rangle} \tilde{\rho} = \sum_{1\leq i\leq r} m_i\tilde{\rho}_i$$

Proof. There are two methods in extending a unipotent representation of G^{F^m} in the series of ρ_0 to a representation of $G^{F^m}\langle\sigma_0\rangle$ in normalized manners:

One is by using the eigenvalues of the Frobenius mapping F_0^* (the method which we have adopted here). The other is simply inducing the action of σ_0 on the representation $\tilde{\rho}_0$.

To prove our lemma it suffices to show that these two methods yield the same extension for any ρ_i (or ρ). But this is apparent from the proof of [2, 8.2].

3. Let H be a finite group and α an automorphism of H. For h_1, $h_2\in H$, we define the equivalence relation $\tilde{\alpha}$ by

$$h_1\tilde{\alpha}h_2 \Leftrightarrow h_1 = h^{-1}h_2^{\alpha}h \qquad \text{for some } h\in H\,.$$

For $x\in G^{F^m}$, write $x=a^{-1F_0}a$ with $a\in G$ and put $y={}^{F^m}aa^{-1}$. Then $x\to y$ defines the bijection

$$G^{F^m}/\tilde{F}_0 \to G^{F_0}/\widetilde{F}^{-m}$$

which will be denoted by n_{F^m/F_0}. Quite analogously to Lemma 1.2.1 of [1], we obtain

Lemma 3.1. *For any* $x\in G^{F^m}$ *and* $w\in W$,

$$Tr((xF_0)^*, \sum_{i\geq 0}(-1)^i H_c^i(X_w^{(m)}))$$
$$=(|T^{F_0}|q^d)^{-1}\#\{h\in G^{F_0};\ h^{-1}n_{F^m/F_0}(x)^{-1F^m}h\in\dot{w}B\}\ ,$$

where $d=\dim(U\cap\dot{w}U\dot{w}^{-1})$, *and* $\dot{w}$ *is an* F_0*-stable representative of* w *in the normalizer* $N_G(T)$ *of* T *in* G.

Assume $m\equiv 1 \bmod \delta$. Then we may define the mapping

$$N^{(m)}=n_{F/F_0}^{-1}\circ n_{F^m/F_0}\colon G^{F^m}/\widetilde{F}_0\to G^F/\widetilde{F}_0$$

Thus by the relation in the lemma combined with that relation with $m=1$, we obtain

Theorem 3.2. *Assume* $m\equiv 1 \bmod \delta$. *For any* $x\in G^{F^m}$ *and* $w\in W$,

$$Tr((xF_0)^*, \sum_{i\geq 0}(-1)^i H_c^i(X_w^{(m)}))$$
$$=Tr((N^{(m)}(x)F_0)^*, \sum_{i\geq 0}(-1)^i H_c^i(X_w^{(1)}))\ .$$

4. We preserve the notations used until now. Assume $G=SO_{2n}^{\pm}$ Sp_{2n} or SO_{2n+1}. In some cases, G is also denoted by G_n to specify n. If $G\neq SO_{2n}^{-}$, we take j to be identify, and if $G=SO_{2n}^{-}$, we take j to be of order 2. Let $\tilde{G}$ be the semi-direct product $G\langle j\rangle$. If $m\equiv 1 \bmod \delta$, then $\tilde{G}^{F^m}\langle\sigma\rangle=G^{F^m}\langle\sigma_0\rangle$. First we need

Lemma 4.1. *Assume* $m\equiv 1 \bmod \delta$. *Then all the unipotent representations of* G^{F^m} *(resp.* G^F*) are* σ_0*-invariant.*

Proof. For an F-stable closed subgroup H of G, we denote by $H^{(m)}$ the group of all F^m-stable points of H. Let I_0 be a subset of S such that there exists a unipotent cuspidal representation ρ_0 of $L_{I_0}^{(m)}$. To prove the lemma it suffices to prove that any unipotent representation of $G^{(m)}$ in the series of ρ_0 is σ_0-invariant. We recall a result of Lusztig [5, § 5]. Let $\overline{W}=(N_G(L_{I_0})/L_{I_0})^{F^m}$, where $N_G(L_{I_0})$ is the normalizer of L_{I_0} in G. $\overline{W}$ has a natural structure as a Coxter group with the canonical set of generators $\overline{S}$. For a subset J of S with $I_0\subseteq J\subseteq S$, a subset $\overline{J}$ of $\overline{S}$ is associated in a natural manner and any subset of $\overline{S}$ is obtained in this form. We denote by $\overline{W}_{\overline{J}}$ the subgroup of $\overline{W}$ generated by $\overline{J}(\subseteq\overline{S})$. Then unipotent representations (up to equivalence) of $G^{(m)}$ (resp. $L_J^{(m)}$) in the series of ρ_0 are parameterized by the set of irreducible representations $\overline{W}^\wedge$ (resp. $(\overline{W}_{\overline{J}})^\wedge$) of $\overline{W}$ (resp. $\overline{W}_{\overline{J}}$). And this parameterization is compatible with the inductions:

$$\begin{array}{ccc} \chi\in R(\bar W_{\bar J}) & \xrightarrow{\sim} & \left\{\begin{array}{l} \boldsymbol{Z}\text{-linear combi. of} \\ \text{unip. char. of } L_J^{(m)} \\ \text{in the series of } \rho_0 \end{array}\right\} \ni \rho \\ \downarrow & & \downarrow \\ \operatorname{Ind}_{\bar W_{\bar J}}^{\bar W} \chi\in R(\bar W) & \xrightarrow{\sim} & \left\{\begin{array}{l} \boldsymbol{Z}\text{-linear combi. of} \\ \text{unip. char. of } G^{(m)} \\ \text{in the series of } \rho_0 \end{array}\right\} \ni \operatorname{Ind}_{P_J^{(m)}}^{G^{(m)}} \rho \end{array}$$

where $R(\bar W_{\bar J})$ and $R(\bar W)$ denote the group of all virtual characters of $\bar W_{\bar J}$ and $\bar W$ respectively, and irreducible characters are mapped to the irreducible characters by the horizontal isomorphisms. Now, $(\bar W, \bar S)$ is isomorphic to a classical Weyl group. Thus, if $\operatorname{rank}(\bar W, \bar S)\geq 2$, then we have:

For $\chi_1, \chi_2\in \bar W^\wedge$, if $\chi_1 | \bar W_{\bar J}=\chi_2 | \bar W_{\bar J}$ for any $\bar J\subsetneq \bar S$, then $\chi_1=\chi_2$.
Therefore to prove that any unipotent representation ρ in the series of ρ_0 is σ_0-invariant, it suffices to prove the statement only when ρ is a cuspidal (i.e. $I_0=S$) or subcuspidal (i.e. $|S\backslash I_0|=1$) representation (see [5]). Assume that ρ is cuspidal, i.e. $\rho=\rho_0$. Then ρ is the unique unipotent cuspidal representation. Therefore ρ is σ_0-invariant. Assume that ρ is subcuspidal. Let ρ' be another unipotent subcuspidal representation (see [5]). Since $\dim\rho\neq\dim\rho'$ (cf. [4]) and there is no other unipotent subcuspidal representation, ρ and ρ' are both σ_0-invariant.

Henceforth we assume that m is prime to $2p$ with p the characteristic of $\boldsymbol{F}_q$. Then by N. Kawanaka [3], [4], the following statement is true:

For any σ_0-invariant irreducible representation $\rho^{(m)}$ of G^{F^m}, there exists a σ_0-invariant (or j-invariant) irreducible representation $\rho^{(1)}$ of G^F such that

$$\operatorname{Tr}(xj\sigma, \tilde\rho^{(m)}) = c\operatorname{Tr}(N^{(m)}(x)j, \tilde\rho^{(1)})$$

for any $x\in G^{F^m}$, where $\tilde\rho^{(m)}$ (resp. $\tilde\rho^{(1)}$) is an irreducible representation of $\tilde G^{F^m}\langle\sigma\rangle$ (resp. $\tilde G^{F^m}$) that extends $\rho^{(m)}$ (resp. $\rho^{(1)}$), and c is a root of unity. We now assume that m is sufficiently large so that the main theorem in [7] (resp. [8]) holds for the group G^{F^m} if $G=SO_{2n+1}$ or Sp_{2n} (resp. $G=SO_{2n}^{\pm}$). Let Φ_n, $\Phi_n^{\pm}$ be the sets of symbol classes defined in [5, § 3]. We set

$$\Psi_n = \begin{cases} \Phi_n & \text{if } G = SO_{2n+1} \text{ or } Sp_{2n} \\ \Phi_n^+(\text{resp. } \Phi_n^-) & \text{if } G = SO_{2n}^+(\text{resp. } SO_{2n}^-) \end{cases}$$

By [5], the unipotent representations of G^{F^m} (resp. G^F) are parameterized by the symbol classes in Ψ_n. For $\Lambda\in\Psi_n$, we denote by $\rho_\Lambda^{(m)}$ (resp. $\rho_\Lambda^{(1)}$) the corresponding unipotent representation of G^{F^m} (resp. G^F), and by $\lambda_{\rho_\Lambda^{(m)}}$ (resp. $\lambda_{\rho_\Lambda^{(1)}}$) the normalized eigenvalue of $(F^{m\delta})^*$ (resp. $(F^\delta)^*$) associated with the unipotent representation $\rho_\Lambda^{(m)}$ (resp. $\rho_\Lambda^{(1)}$). By [1], $\lambda_{\rho_\Lambda^{(m)}}$ and $\lambda_{\rho_\Lambda^{(1)}}$ are 1 or -1 if $G=SO_{2n+1}$, Sp_{2n} or SO_{2n}^+. By [8, 3.4], $\lambda_{\rho_\Lambda^{(m)}}=\lambda_{\rho_\Lambda^{(1)}}=1$ for any $\Lambda\in\Psi_n$ if $G=SO_{2n}^-$.

Since m is odd, we may choose the extension $\tilde{\rho}_{\Lambda}^{(m)} \in (G^{F^m}\langle\sigma_0\rangle)^{\wedge}$ of $\rho_{\Lambda}^{(m)}$ by the condition

$$\mu_{\tilde{\rho}_{\Lambda}^{(m)}} = \lambda_{\rho_{\Lambda}^{(m)}}$$

(See Lemma 1.5). And we may choose the extension $\tilde{\rho}_{\Lambda}^{(1)} \in (G^E\langle j\rangle)^{\wedge}$ of $\rho_{\Lambda}^{(1)}$ by the condition

$$\mu_{\tilde{\rho}_{\Lambda}^{(1)}} = \lambda_{\rho_{\Lambda}^{(1)}};$$

Here we applied Lemma 1.5 with $m=1$. Let $(W\langle j\rangle)^{\wedge *}$ be the set of irreducible representations χ (up to equivalence) of the semi-direct product $W\langle j\rangle$ such that $\chi|W$ is irreducible. For any $\chi \in (W\langle j\rangle)^{\wedge *}$, let $R_{\chi}^{(m)}$ be the class function of G^{F^m} defined in [6, (3.17.1)], i.e.

$$R_{\chi}^{(m)} = |W|^{-1} \sum_{w \in W} \mathrm{Tr}(wj, \chi) R_w^{(m)}$$

where $R_w^{(m)}$ is the character of the virtual G^{F^m}-module $\sum_{i \geq 0} (-1)^i H_c^i(X_w^{(m)})$. We are to prove

Theorem 4.2. *Let* $\tilde{\rho}_{\Lambda}^{(m)}$ *and* $\tilde{\rho}_{\Lambda}^{(1)}$ $(\Lambda \in \Psi_n)$ *be the extensions of* $\rho_{\Lambda}^{(m)}$ *and* $\rho_{\Lambda}^{(1)}$ *chosen as above. Then we have*

(i) $\mathrm{Tr}(xj\,\sigma, \tilde{\rho}_{\Lambda}^{(m)}) = \mathrm{Tr}(N^{(m)}(x)j, \tilde{\rho}_{\Lambda}^{(1)})$ *for any* $x \in G^{F^m}$,

(ii) $\lambda_{\rho_{\Lambda}^{(m)}} = \lambda_{\rho_{\Lambda}^{(1)}}$,

(iii) $\langle \rho_{\Lambda}^{(m)}, R_{\chi}^{(m)} \rangle = \langle \rho_{\Lambda}^{(1)}, R_{\chi}^{(1)} \rangle$ *for any* $\chi \in (W\langle j\rangle)^{\wedge *}$,

(iv) $f_{\rho_{\Lambda}^{(m)}, w}(X) = f_{\rho_{\Lambda}^{(1)}, w}(X)$ *for any* $w \in W$.

Corollary 4.3 *The main theorems in* G. Lusztig [7], [8] *are true for any finite field.*

Lemma 4.4. *Let* $\Lambda_1, \Lambda_2 \in \Psi_n$. *Assume*

$$(*) \qquad \mathrm{Tr}(xj\sigma, \tilde{\rho}_{\Lambda_1}^{(m)}) = c\,\mathrm{Tr}(N^{(m)}(x)j, \tilde{\rho}_{\Lambda_2}^{(1)})$$

for any $x \in G^{F^m}$ *with some root* c *of* 1. *Then*

(i) $\lambda_{\rho_{\Lambda_1}^{(m)}} = c\lambda_{\rho_{\Lambda_2}^{(1)}}$,

(ii) $\dim \rho_{\Lambda_1}^{(1)} = \dim \rho_{\Lambda_2}^{(1)}$,

(iii) $\langle \rho_{\Lambda_1}^{(1)}, R_{\chi}^{(m)} \rangle = \langle \rho_{\Lambda_2}^{(1)}, R_{\chi}^{(1)} \rangle$ *for any* $\chi \in (W\langle j\rangle)^{\wedge *}$,

(iv) $f_{\rho_{\Lambda_1}^{(m)}, w}(X) = f_{\rho_{\Lambda_2}^{(m)}, w}(X)$ *for any* $w \in W$.

To prove the lemma we need some preparations. Let $H(W)$ be the generalized Hecke algebra of the Coxeter group (W, S) over the polynomial ring $\boldsymbol{Q}[X]$ that yields by the specialization $(X \to q)$ the G^{F_0}-module endomorphism algebra of the induced representation of G^{F_0} from the trivial representation of

B^{F_0}. Let $\{a_w;\ w\in W\}$ be the canonical basis of $H(W)$. $H(W)$ is a subalgebra of an algebra $H(W\langle j\rangle)$ defined as follows.

$$\begin{aligned} &H(W\langle j\rangle) = H(W)\oplus a_j H(W) && \text{as linear spaces,}\\ &a_j a_w a_i^{-1} = a_{jwj^{-1}} && \text{for } w\in W,\\ &a_j^{\delta} = 1 && \end{aligned}$$

We put $a_w a_j = a_{wj}$ $(w\in W)$. Let $H^{(m)}(W\langle j\rangle)$ (resp. $H^{(1)}(W\langle j\rangle)$) denote the algebra obtained by specializing $X\to q^m$ (resp. $X\to q$) in the defining relations of $H(W\langle j\rangle)$. For $w\in W\langle j\rangle$, let $a_w^{(m)}$ (resp. $a_w^{(1)}$) denote the specialized element of a_w in $H^{(m)}(W\langle j\rangle)$ (resp. $H^{(1)}(W\langle j\rangle)$). For $\chi\in(W\langle j\rangle^\wedge)$, let ν_χ be the corresponding irreducible representation of $H(W\langle j\rangle)\otimes\Phi(X)$ and $\nu_\chi^{(m)}$ (resp. $\nu_\chi^{(1)}$) its specialized representation of $H^{(m)}(W\langle j\rangle)$ (resp. $H^{(1)}(W\langle j\rangle)$).

Proof of Lemma 4.4. By Corollary 1.4 and Lemma 3.1 we have

$$(1)\qquad \begin{aligned} &\sum_{\Lambda\in\Psi_n} f_{\rho_\Lambda^{(m)},w}(q)\lambda_{\rho_\Lambda^{(m)}} \operatorname{Tr}((xj\sigma)^{-1}, \tilde\rho_\Lambda^{(m)})\\ &\quad= \sum_{\Lambda\in\Psi_n} f_{\rho_\Lambda^{(m)},w}(q)\lambda_{\rho_\Lambda^{(1)}} \operatorname{Tr}((N^{(m)}(x)j)^{-1}, \tilde\rho_\Lambda^{(1)}) \end{aligned}$$

for any $w\in W$ and $x\in G^{F^m}$. The relation (1) and the relation $(*)$ in the lemma together with the orthogonality relations (cf. [1]) imply

$$(2)\qquad f_{\rho_{\Lambda_1}^{(m)},w}(q)\lambda_{\rho_{\Lambda_1}^{(m)}} = f_{\rho_{\Lambda_2}^{(1)},w}(q)\lambda_{\rho_{\Lambda_2}^{(1)}}c$$

for any $w\in W$. By [1, 2.4.7] and by [8, 3.5], we have

$$(3)\qquad f_{\rho_\Lambda^{(a)},w}(X) = \delta^{-1}\sum_{\chi\in(W\langle j\rangle)^{\wedge*}} \operatorname{Tr}(a_{wj}, \nu_\chi)\langle R_\chi^{(a)}, \rho_\Lambda^{(a)}\rangle$$

for $a=1$, m and $\Lambda\in\Psi_n$. By (2) and (3),

$$(4)\qquad \begin{aligned} &\{\delta^{-1}\sum_{\chi\in(W\langle j\rangle)^{\wedge*}} \operatorname{Tr}(a_{wj}^{(1)}, \nu_\chi^{(1)})\langle R_\chi^{(m)}, \rho_{\Lambda_1}^{(m)}\rangle\}\lambda_{\rho_{\Lambda_1}^{(m)}}\\ &\quad= \{\delta^{-1}\sum_{\chi\in(W\langle j\rangle)^{\wedge*}} \operatorname{Tr}(a_{wj}^{(1)}, \nu_\chi^{(1)})\langle R_\chi^{(1)}, \rho_{\Lambda_2}^{(1)}\rangle\}\lambda_{\rho_{\Lambda_2}^{(1)}}c \end{aligned}$$

Let $\{a_w^*;\ w\in W\}$ be the dual basis of $\{a_w;\ w\in W\}$. We put $a_{jw}^* = a_j^{-1}a_w^*$ for $w\in W$. Then for χ, $\chi'\in(W\langle j\rangle)^{\wedge*}$,

$$\sum_{w\in W} \operatorname{Tr}(a_{jw}^{*(1)}, \nu_\chi^{(1)})\operatorname{Tr}(a_{wj}^{(1)}, \nu_{\chi'}^{(1)}) \neq 0$$

if and only if $\chi|W=\chi'|W$, where $a_{jw}^{*(1)}$ is the specialized element of a_{jw}^*. Thus by (4),

$$(5)\qquad \langle R_\chi^{(m)}, \rho_{\Lambda_1}^{(m)}\rangle\lambda_{\rho_{\Lambda_1}^{(m)}} = \langle R_\chi^{(1)}, \rho_{\Lambda_2}^{(1)}\rangle\lambda_{\rho_{\Lambda_2}^{(1)}}c$$

for any $\chi\in(W\langle j\rangle)^{\wedge*}$. By [6, 3.12],

$$\dim \rho_{\Lambda_1}^{(m)} = \delta^{-1} \sum_{\chi \in (W\langle j\rangle)^{\wedge *}} \langle R_\chi^{(m)}, \rho_{\Lambda_1}^{(m)} \rangle \dim R_\chi^{(m)} \tag{6}$$

By [4], $\dim \rho_{\Lambda_1}^{(m)}$ and $\dim R_\chi^{(m)}$ are expressed as polynomials in q^m. By Lusztig [7] and [8], $\langle R_\chi^{(m)}, \rho_{\Lambda_1}^{(m)}\rangle$ is independent of m, since we have assumed that m is a sufficiently large odd integer. Thus the relation (6) holds with each term regarded as polynomials in q^m. Hence by replacing q^m with q in (6) we have

$$\dim \rho_{\Lambda_1}^{(1)} = \delta^{-1} \sum_{\chi \in (W\langle j\rangle)^{\wedge *}} \langle R_\chi^{(m)}, \rho_{\Lambda_1}^{(m)} \rangle \dim R_\chi^{(1)} \tag{7}$$

By (5) and (7),

$$\begin{aligned} \dim \rho_{\Lambda_1}^{(1)} &= c\lambda_{\rho_{\Lambda_1}^{(m)}}^{-1} \lambda_{\rho_{\Lambda_2}^{(1)}} \delta^{-1} \sum_{\chi \in (W\langle j\rangle)^{\wedge *}} \langle R_\chi^{(1)}, \rho_{\Lambda_2}^{(1)} \rangle \dim R_\chi^{(1)} \\ &= c\lambda_{\rho_{\Lambda_1}^{(m)}}^{-1} \lambda_{\rho_{\Lambda_2}^{(1)}} \dim \rho_{\Lambda_2}^{(1)} \end{aligned}$$

Since c is of absolute value 1, $c\lambda_{\rho_{\Lambda_1}^{(m)}}^{-1} \lambda_{\rho_{\Lambda_2}^{(1)}}$ is also of absolute value 1. Considering that $\dim \rho_{\Lambda_1}^{(1)}$ and $\dim \rho_{\Lambda_2}^{(1)}$ are positive integers, we see that (i), (ii) of the lemma are true. (iii) is obtained by (5) and (i). (iv) is obtained by (3), (4) and (iii).

Lemma 4.5. *Let n_0 be a non-negative integer. We assume that there exists a symbol class $\Lambda_0 \in \Psi_{n_0}$ of defect d corresponding to the unipotent cuspidal representation. Let $\Lambda_1 \neq \Lambda_2 \in \Psi_{n_0+1}$ be the symbol classes of defect d corresponding to the subcuspidal representations.*

(i) *Assume* $\mathrm{Tr}(xj\sigma, \tilde{\rho}_{\Lambda_0}^{(m)}) = \mathrm{Tr}(N^{(m)}(x)j, \tilde{\rho}_{\Lambda_0}^{(1)})$ *for any* $x \in G_{n_0}^{F^m}$. *Then*

$$\mathrm{Tr}(xj\sigma, \tilde{\rho}_{\Lambda}^{(m)}) = \mathrm{Tr}(N^{(m)}(x)j, \tilde{\rho}_{\Lambda'}^{(1)})$$

for any $x \in G_{n_0+1}^{F^m}$ *with* (Λ, Λ') *one of the following conditions* (A) *and* (B):

$$\begin{aligned} &\text{(A)} \quad (\Lambda, \Lambda') = (\Lambda_1, \Lambda_1), (\Lambda_2, \Lambda_2) \\ &\text{(B)} \quad (\Lambda, \Lambda') = (\Lambda_1, \Lambda_2), (\Lambda_2, \Lambda_1) \end{aligned}$$

(ii) *Let* $n \geq n_0 + 1$ *and assume that the statement* (i) *with the condition* (A) *is true. Then*

$$\mathrm{Tr}(xj\sigma, \tilde{\rho}_{\Lambda}^{(m)}) = \mathrm{Tr}(N^{(m)}(x)j, \tilde{\rho}_{\Lambda}^{(1)})$$

for any $x \in G_n^{F^m}$ *and any* $\Lambda \in \Psi_n$ *of defect* d.

Proof. By Lemma 2.1, we can apply the arguments employed in [1, 2.2.3]. (See Lemma 4.1)

Proof of Theorem 4.2. By Lemma 4.4, to prove the theorem it suffices to prove (i) of the theorem for any $\Lambda \in \Psi_n$. And Lemma 4.5 shows that it suffices to prove (i) of the theorem only when $\rho_{\Lambda}^{(m)}$ is cuspidal or subcuspidal.

Let n_0, Λ_0, Λ_1, Λ_2 be as in Lemma 4.5.

Assume $n=n_0$. $\rho_{\Lambda_0}^{(m)}$ (resp. $\rho_{\Lambda_0}^{(1)}$) is the unique unipotent cuspidal representation of G^{F^m} (resp. G^F) and there is no unipotent subcuspidal representation of G^{F^m} (resp. G^F). By the induction, the statements of the theorem are true if $\Lambda\neq\Lambda_0$. In particular, the lifting of a non-cuspidal unipotent representation is a non-cuspidal unipotent representation, whereas the relation (1) in the proof of Lemma 4.4 shows that the lifting of $\tilde{\rho}_{\Lambda_0}^{(1)}$ is unipotent (or its restriction to G^{F^m} is unipotent if $G=SO_{2n}^-$), and therefore must be $\tilde{\rho}_{\Lambda_0}^{(m)}|G^{F^m}$. Thus

$$\mathrm{Tr}(xj\sigma,\ \tilde{\rho}_{\Lambda_0}^{(m)}) = c\,\mathrm{Tr}(N^{(m)}(x)j,\ \tilde{\rho}_{\Lambda_0}^{(1)})$$

for any $x\in G^{F^m}$ with a constant c. Assume $G=SO_{2n}^-$. Then $\lambda_{\rho_{\Lambda_0}^{(m)}}=\lambda_{\rho_{\Lambda_0}^{(1)}}=1$. Thus $c=1$ by Lemma 4.4, (i). Assume $G\neq SO_{2n}^-$ (, hence $j=\mathrm{id}$.). We are to prove $c=1$. By [1, 2.4.6], for any $\chi\in W^\wedge$,

$$\dim\rho_\chi^{(m)} = \sum_{\Lambda\in\Psi_n}\langle R_\chi^{(m)},\ \rho_\Lambda^{(m)}\rangle\lambda_{\rho_\Lambda^{(m)}}\dim\rho_\Lambda^{(m)}\,, \tag{1}$$

$$\dim\rho_\chi^{(1)} = \sum_{\Lambda\in\Psi_n}\langle R_\chi^{(1)},\ \rho_\Lambda^{(1)}\rangle\lambda_{\rho_\Lambda^{(1)}}\dim\rho_\Lambda^{(1)}\,, \tag{2}$$

where $\rho_\chi^{(m)}$ (resp. $\rho_\chi^{(1)}$) denotes the unipotent representation of G^{F^m} (resp. G^F) in the principal series corresponding with χ (cf. [1]). Since $\langle R_\chi^{(m)},\ \rho_\Lambda^{(m)}\rangle$ is independent of the odd integer m (m sufficiently large), the relation (1) holds with each term regarded as a polynomial in q^m. Thus by replacing q^m with q in (1),

$$\dim\rho_\chi^{(1)} = \sum_{\Lambda\in\Psi_n}\langle R_\chi^{(m)},\ \rho_\Lambda^{(m)}\rangle\lambda_{\rho_\Lambda^{(m)}}\dim\rho_\Lambda^{(1)} \tag{3}$$

If $\Lambda\neq\Lambda_0$, we have already $\langle R_\chi^{(m)},\ \rho_\Lambda^{(m)}\rangle=\langle R_\chi^{(1)},\ \rho_\Lambda^{(1)}\rangle$ and $\lambda_{\rho_\Lambda^{(m)}}=\lambda_{\rho_\Lambda^{(1)}}$. Thus, by comparing the relation (2) and the relation (3), we obtain

$$\langle R_\chi^{(m)},\ \rho_{\Lambda_0}^{(m)}\rangle\lambda_{\rho_{\Lambda_0}^{(m)}} = \langle R_\chi^{(1)},\ \rho_{\Lambda_0}^{(1)}\rangle\lambda_{\rho_{\Lambda_0}^{(1)}}$$

for any $\chi\in W^\wedge$. Thus by (iii) of Lemma 4.4, we have $\lambda_{\rho_{\Lambda_0}^{(u)}}=\lambda_{\rho_{\Lambda_0}^{(1)}}$. (Note that there exists $\chi\in W^\wedge$ such that $\langle R_\chi^{(1)},\ \rho_{\Lambda_0}^{(1)}\rangle\neq 0$.) Hence by (i) of Lemma 4.4, we have $c=1$. Therefore we have proved the theorem for $\Lambda=\Lambda_0$.

Assume $n=n_0+1$. $\rho_{\Lambda_i}^{(m)}$ (resp. $\rho_{\Lambda_i}^{(1)}$) ($i=1, 2$) are subcuspidal representations of G^{F^m} (resp. G^F) and the other unipotent representations of G^{F^m} (resp. G^F) are neither cuspidal nor subcuspidal. Let $i=1$ or 2. By Lemma 4.5, there exists $i'=1$ or 2 such that

$$\mathrm{Tr}(xj\sigma,\ \tilde{\rho}_{\Lambda_i}^{(m)}) = \mathrm{Tr}(N^{(m)}(x)j,\ \tilde{\rho}_{\Lambda_{i'}}^{(1)})$$

for any $x\in G^{F^m}$. Then by Lemma 4.4, $\dim\rho_{\Lambda_i}^{(1)}=\dim\rho_{\Lambda_{i'}}^{(1)}$. Since $\dim\rho_{\Lambda_1}^{(1)}\neq\dim\rho_{\Lambda_2}^{(1)}$, we must have $i=i'$. This proves the theorem for $\Lambda=\Lambda_1$, Λ_2.

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