

| | |
|--------------|---|
| Title | A characterization of some partial geometric spaces |
| Author(s) | Kimura, Hiroshi; Ohmori, Hiroyuki |
| Citation | Osaka Journal of Mathematics. 1985, 22(3), p. 401-410 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/7131 |
| rights | |
| Note | |

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

A CHARACTERIZATION OF SOME PARTIAL GEOMETRIC SPACES

Dedicated to Professor Hiroshi Nagao on his 60th birthday

HIROSHI KIMURA AND HIROYUKI OHMORI

(Received April 17, 1984)

1. Introduction

A partial geometric space S of dimension $m \geq 2$ defined in [2, 6] consists of the sets $\{A_i\}_{i=-1}^m$ and the set T such that the following eight axioms are satisfied:

- (1) $A_i \cap A_j = \phi$ whenever $i \neq j$ and $-1 \leq i, j \leq m$.
- (2) $|A_{-1}| = |A_m| = 1$.
- (3) $T \subset \prod_{i=-1}^m A_i$.

The elements of A_i , $-1 \leq i \leq m$, are called i elements of S . The elements of T are called flags of S . There is a property called incidence which is a relation between the elements of S based on the flags.

(4) For each i element x_i there is a flag $(t_{-1}, \dots, t_m) \in T$ such that $x_i = t_i$, where $-1 \leq i \leq m$.

(5) Whenever $(y_{-1}, \dots, y_m) \in T$ and $(z_{-1}, \dots, z_m) \in T$ and $y_k = z_k$ for some k , $-1 \leq k \leq m$, then there exists a flag $(t_{-1}, \dots, t_m) \in T$, where $t_i = y_i$ for $-1 \leq i \leq k$, and $t_j = z_j$ for $k \leq j \leq m$.

(6) If $x_i \in A_i$ and $x_j \in A_j$, then x_i and x_j have an l intersection $x_l \in A_l$ and an s join $x_s \in A_s$. Here x_i and x_j are said to have an l intersection x_l (s join x_s), where $-1 \leq l \leq \min\{i, j\}$ ($\max\{i, j\} \leq s \leq m$) if and only if x_l (x_s) is incident with x_i and x_j such that whenever x_n is an n element of S for $-1 \leq n \leq \min\{i, j\}$ ($\max\{i, j\} \leq n \leq m$) which is incident with x_i and x_j , then x_n is incident with x_l (x_s) and $-1 \leq n \leq l$ ($s \leq n \leq m$). By the definition, x_i and x_j have unique intersection and unique join.

(7) If $x_{i-1} \in A_{i-1}$ and $x_{i+1} \in A_{i+1}$ are incident, then there are $k(i)$ i elements which are incident with x_{i-1} and x_{i+1} , where $2 \leq k(i) < \infty$, for $0 \leq i \leq m-1$. The number $k(i)$ is independent of the choice of x_{i-1} and x_{i+1} , and depends only on i . $k(0), k(1), \dots, k(m-1)$ are called the configuration parameters of S .

(8) Let $m \geq 2$. If $x_i \in A_i$ and $x_{i+1} \in A_{i+1}$ have an $(i-1)$ intersection x_{i-1} and an s join x_s , where $0 \leq i \leq m-2$ and $i+2 \leq s \leq m$, then there are $t(i, s, k)$ i

elements y_i , which are incident with x_{i-1} and x_{i+1} such that y_i and x_i have an $(i+k)$ join for $1 \leq k \leq s-i-1$. Also $\sum_{k=1}^{s-i-1} t(i, s, k) \geq 1$ for $0 \leq i \leq m-2$ and $i+2 \leq s \leq m$. The numbers $t(i, s, k)$ are called the geometric parameters of S .

The concept of a partial geometric space of dimension m is an extension of the concept of a partial geometry introduced by R.C. Bose [1]. A partial geometry of dimension three introduced by R. Lasker and J. Dunbar [5] is called an L.D. partial geometric space of dimension three in [6].

We have two examples of partial geometric spaces of dimension m .

EXAMPLE 1 [6]. Let A be a set consisting $(m+1)$ distinct symbols, where $m \geq 2$. Let $A_{-1} = \{\phi\}$. For $0 \leq j \leq m$, $A_j = \{B \subset A \mid |B| = j+1\}$. Note that $A_m = \{A\}$. Let $T = \{(t_{-1}, \dots, t_m) \in \prod_{i=-1}^m A_i \mid t_i \subset t_{i+1} \text{ for } -1 \leq i \leq m-1\}$. Then $S_1 = (\{A_i\}_{i=-1}^m, T)$ is a partial geometric space of dimension m . The configuration parameters are $k(i) = 2$ for $0 \leq i \leq m-1$. The geometric parameters are $t(i, i+2, 1) = 2$ for $0 \leq i \leq m-2$ and the rest geometric parameters need not be defined.

EXAMPLE 2. Let $PG(m, q)$ be the finite projective geometry of dimension m and of order q , where $m \geq 2$ and q is a prime power. Let $A_{-1} = \{\phi\}$. For $0 \leq j \leq m$, $A_j = \{B \mid B \text{ is a } j \text{ dimensional subspace of } PG(m, q)\}$. Let $T = \{(t_{-1}, \dots, t_m) \in \prod_{i=-1}^m A_i \mid t_i \subset t_{i+1} \text{ for } 0 \leq i \leq m-1\}$. Then $S_2 = (\{A_i\}_{i=-1}^m, T)$ is a partial geometric space of dimension m . The configuration parameters are $k(i) = q+1$ for $0 \leq i \leq m-1$. The geometric parameters are $t(i, i+2, 1) = q+1$ for $0 \leq i \leq m-2$ and the rest geometric parameters need not be defined.

Two partial geometric spaces S_1 and S_2 of dimension m have common property:

$$(\#) \begin{cases} \text{(i)} & k(i) \text{ is constant for } 0 \leq i \leq m-1 \\ \text{(ii)} & t(i, s, k) = k(i) \text{ for } 0 \leq i \leq m-2, \text{ where } s = i+2 \text{ and } k=1, \\ & \text{and the rest geometric parameters need not be defined.} \end{cases}$$

From (ii) of the property, we note that for any i element and $i+1$ element which have an $(i-1)$ intersection and are not incident, they have an $(i+2)$ join.

In section 2, we shall prove the following theorem.

Theorem. Let $S = (\{A_i\}_{i=-1}^m, T)$ be a partial geometric space of dimension $m \geq 2$ satisfying property $(\#)$. Then $S = S_1$ if $k(i) = 2$, and $S = S_2$ if $k(i) = \alpha + 1 > 2$ and $m \geq 3$.

In section 3, we shall give an another example of partial geometric space of dimension $m \geq 3$.

2. Proof of Theorem

Let $S = (\{A_i\}_{i=-1}^m, T)$ be a partial geometric space of dimension $m \geq 2$. Let $x_i \in A_i$ and $x_j \in A_j$, where $-1 \leq i, j \leq m$. x_i is said to be incident with x_j if and only if there exists a flag $(t_{-1}, \dots, t_m) \in T$ such that $x_i = t_i$ and $x_j = t_j$. Let $x_j \in A_j$ and $x_k \in A_k$ such that x_j and x_k are incident, where $-1 \leq j < k \leq m$. $\phi(i, x_j, x_k)$ is the number of i elements of S which are incident with x_j and x_k , where $-1 \leq i \leq m$. The number $\phi(i, x_j, x_k)$ is a finite positive integer which is independent of the choice of the j element x_j and the k element x_k [2]. Therefore put $\phi(i, j, k) = \phi(i, x_j, x_k)$.

From now on in this section, we assume that S satisfies the property (#).

Lemma 1. *Let x_i and y_i be two distinct i elements such that they have an $(i-1)$ intersection x_{i-1} for $0 \leq i \leq m-1$. Then x_i and y_i have an $(i+1)$ join.*

Proof. Let x_{i+1} be a join of x_i and y_i , where $l > 1$. Then there exists an $(i+1)$ element y_{i+1} which is incident with x_{i-1} and x_{i+1} and is not incident with x_i . From the property (#), we have $l=2$ and there are $k(i)$ i elements z_i , which are incident with x_{i-1} and y_{i+1} , such that z_i and x_i have an $(i+1)$ join. Those $k(i)$ i elements are distinct from y_i . Consequently, there are $(k(i)+1)$ i elements which are incident with x_{i-1} and y_{i+1} . This is a contradiction. Therefore $l=1$, i.e. x_i and y_i have an $(i+1)$ join.

Lemma 2. $\phi(i, x_{i-1}, x_k) = \phi(i, i-1, k) = k(i)(k(i)-1)^{k-i-1} + (k(i)-1)^{k-i-2} + \dots + (k(i)-1) + 1$, where $0 \leq i < k \leq m$, and $x_{i-1} \in A_{i-1}$ and $x_k \in A_k$.

Proof. It shall be proved by induction on $k-i+1$, say t . When $t=2$, from the definition, $\phi(i, x_{i-1}, x_{i+1}) = \phi(i, i-1, i+1) = k(i)$. Therefore the lemma holds when $t=2$. Suppose that $t > 2$ and assume that the lemma holds whenever $k-i+1 < t$, where $0 \leq i < k \leq m$, and $2 < t \leq m+1$. Let x_{i-1} be an $(i-1)$ element and x_k be a k element such that x_{i-1} and x_k are incident in S , where $0 \leq i \leq m-2$, $i+2 \leq k \leq m$ and $k-i+1 = t$. Count triples (x_i, x_{i+1}, x_{i+2}) , where x_l ($i \leq l \leq i+2$) is an l element such that $x_{l'}$ and $x_{l'+1}$ ($i-1 \leq l' \leq i+1$), and x_{i+2} and x_k are incident in S .

Given a fixed i element x_i which is incident with x_{i-1} and x_k , there are $(k(i)(k(i)-1)^{k-i-2} + (k(i)-1)^{k-i-3} + \dots + (k(i)-1) + 1)$ $(i+1)$ elements x_{i+1} , which are incident with x_i and x_k , by the induction hypothesis.

Similarly, given a fixed pair (x_i, x_{i+1}) , where x_{i-1} and x_i , x_i and x_{i+1} , and x_{i+1} and x_k are incident in S , there are $(k(i)(k(i)-1)^{k-i-3} + (k(i)-1)^{k-i-4} + \dots + (k(i)-1) + 1)$ $(i+2)$ elements, which are incident with both x_{i+1} and x_k .

Therefore the number of triples is

$$\begin{aligned} & (k(i)(k(i)-1)^{k-i-2} + (k(i)-1)^{k-i-3} + \dots + (k(i)-1) + 1) \times \\ & (k(i)(k(i)-1)^{k-i-3} + (k(i)-1)^{k-i-4} + \dots + (k(i)-1) + 1) \phi(i, x_{i-1}, x_k). \end{aligned}$$

On the other hand, count pairs (x_i, x_{i+1}) , where x_l ($i \leq l \leq i+1$) is an l element such that $x_{l'}$ and $x_{l'+1}$ ($i-1 \leq l' \leq i$) are incident in S . Let x_i and y_i be distinct two i elements which are incident with x_{i-1} and x_k , then x_i and y_i have an $(i+1)$ join, say y_{i+1} . For y_{i+1} , there are $\binom{k(i)}{2}$ pairs (x'_i, y'_i) such that an i elements x'_i and y'_i have an $(i+1)$ join y_{i+1} and an intersection x_{i-1} , by the definition of $k(i)$. Consequently there are $\binom{\phi(i, x_{i-1}, x_k)}{2} k(i) / \binom{k(i)}{2}$ pairs (x_i, x_{i+1}) such that x_{i-1} and x_i , x_i and x_{i+1} , and x_{i+1} and x_k are incident in S . The contribution to triples of such a pair (x_i, x_{i+1}) is $(k(i)(k(i)-1)^{k-i-3} + \dots + (k(i)-1)+1)$ by the induction hypothesis. Therefore we get

$$\begin{aligned} & (k(i)(k(i)-1)^{k-i-2} + (k(i)-1)^{k-i-3} + \dots + (k(i)-1)+1) \times \\ & (k(i)(k(i)-1)^{k-i-3} + (k(i)-1)^{k-i-4} + \dots + (k(i)-1)+1) \phi(i, x_{i-1}, x_k) \\ & = (k(i)(k(i)-1)^{k-i-3} + (k(i)-1)^{k-i-4} + \dots + (k(i)-1)+1) \times \\ & \left(\phi(i, x_{i-1}, x_k) \right) k(i) / \binom{k(i)}{2}. \end{aligned}$$

Consequently we have the lemma.

REMARK. This lemma can be obtained from Theorem 7.1 in [6].

Lemma 3. *If $k(i) = 2$, $\phi(i, i-1, k) = k-i+1$, and if $k(i) = \alpha+1 > 2$, $\phi(i, i-1, k) = (\alpha^{k-i+1} - 1) / (\alpha - 1)$, for $0 \leq i < k \leq m$.*

Proof. It is obvious from Lemma 2.

Lemma 4. *If $k(i) = 2$, $\phi(i, j, k) = \binom{k-j}{i-j}$ for $-1 \leq j < i < k \leq m$.*

Proof. Let x_j be a j element and x_k be a k element such that x_j and x_k be incident in S . Count $(k-j+1)$ -tuples $(x_j, \dots, x_i, \dots, x_k)$, where x_l ($j \leq l \leq k-1$) is an l element such that x_l and x_{l+1} are incident. By Lemma 2, there are $(k-j)(j+1)$ elements x_{j+1} which are incident with x_j and x_k . For such x_{j+1} , there are $(k-j-1)(j+2)$ elements which are incident with x_{j+1} and x_k , and so on. Consequently there are $(k-j)!(k-j+1)$ -tuples. On the other hand, given a fixed i element x_i which is incident with x_j and x_k , there are $(i-j)!(i-j+1)$ -tuples (x_j, \dots, x_i) where an l element x_l and an $(l+1)$ element x_{l+1} are incident ($j \leq l \leq i-1$), and there are $(k-i)!(k-i+1)$ -tuples (x_i, \dots, x_k) where an l' element $x_{l'}$ and an $(l'+1)$ element $x_{l'+1}$ are incident in S , for $i \leq l' \leq k-1$. Therefore we get $\phi(i, x_j, x_k)(k-i)!(i-j)! = (k-j)!$. Thus the proof is complete.

Lemma 5. *If $k(i) = \alpha+1 > 2$, $\phi(i, j, k) = \prod_{l=1}^{i-j} (\alpha^{k-i+l} - 1) / (\alpha^l - 1)$ for $-1 \leq j < i < k \leq m$.*

Proof. It is similar to the proof of Lemma 4. So, we shall omit a proof.

We note that $\phi(i, j, k) = \phi(i, -1, j)$ when $i < j$, and $\phi(i, j, k) = \phi(i, k, m)$ when $k < i$. So, $\phi(i, j, k)$ is defined for i, j and k such that $-1 \leq j < k \leq m$, $-1 \leq i \leq m$ and $i \neq j, k$.

By the incidence structure in S , an i element x_i can be corresponded to a subset $b(x_i)$ of A_0 consisting of 0 elements which are incident with x_i , where $0 \leq i \leq m$.

Lemma 6. *The above correspondence of A_i to a family consisting of subsets of A_0 is injective.*

Proof. Assume that $b(x_i) = b(y_i)$ for an i element y_i ($\neq x_i$). Let x_l be an l intersection of x_i and y_i . Then $l < i$ and $b(x_l) \supseteq b(x_i)$. On the other hand, $|b(x_j)| = \phi(0, -1, j)$ for every j element x_j . This contradicts $\phi(0, -1, l) < \phi(0, -1, i)$.

REMARK. Similarly we can prove that $b(x_i) \neq b(x_j)$ for $x_i \in A_i$ and $x_j \in A_j$, where $i \neq j$.

Lemma 7. *If $k(i) = 2$, then $S = S_1$.*

Proof. $|A_0| = \phi(0, -1, m) = m + 1$. Since $\phi(0, -1, i) = i + 1$, every element of A_i is a subset of A_0 consisting of $i + 1$ elements. By Lemma 4 and Lemma 6, A_i is a family of all subsets of A_0 containing $i + 1$ elements. By the definition, for $i < j$, $x_i \in A_i$ and $x_j \in A_j$ are incident if and only if $b(x_i) \subset b(x_j)$. Thus the proof is complete.

Next we assume that $k(i) = \alpha + 1 \geq 3$ for $0 \leq i \leq m - 1$. By Lemma 6, an i element x_i is identified with a subset of A_0 .

Lemma 8. *A incidence structure $D = (A_0, A_{m-1})$ is a symmetric $2-(v, k, \lambda)$ design, where $v = (\alpha^{m+1} - 1)/(\alpha - 1)$, $k = (\alpha^m - 1)/(\alpha - 1)$ and $\lambda = (\alpha^{m-1} - 1)/(\alpha - 1)$.*

Proof. By the definition, $v = \phi(0, -1, m)$ and $k = \phi(0, -1, m - 1)$. Let x_0 and y_0 be two elements of A_0 . Then there exists a 1 element x_1 by Lemma 1 which is a join of x_0 and y_0 . But every element of A_{m-1} containing x_0 and y_0 has to contain x_1 . Thus we have $\lambda = \phi(m - 1, 1, m)$. By Lemma 5, we have the lemma.

Elements of A_0 and elements of A_{m-1} are called points and blocks in D , respectively. For $x_i \in A_i$ and $y_j \in A_j$, where $0 \leq i \leq j \leq m - 1$, we define $\langle x_i, y_j \rangle$

be an intersection of all blocks of D containing x_i and y_j . Especially $\langle x_0, y_0 \rangle$ is called a line spanned by x_0 and y_0 , where $x_0 \in A_0$ and $y_0 \in A_0$.

Lemma 9. *Let x_1 and y_1 be two elements of A_1 . Then there is an element of A_{m-1} which is incident with x_1 and not incident with y_1 .*

Proof. Let x_l be an l join of x_1 and y_1 . Then $l > 1$. By the property of x_l , the number of elements of A_{m-1} which are incident with x_1 and y_1 equals to the number of elements of A_{m-1} which are incident with x_l . This number is $\phi(m-1, l, m)$ which is smaller than $\phi(m-1, 1, m)$ by Lemma 5. This proves the lemma.

Lemma 10. *D is a design such that its points and blocks are points and hyperplanes of a finite projective geometry P of dimension m , respectively.*

Proof. Let x_1 be a 1 join of x_0 and y_0 , where $x_0, y_0 \in A_0$. By Lemma 1, x_1 is contained in every block of D which is incident with x_0 and y_0 . Therefore $\langle x_0, y_0 \rangle \supseteq x_1$. If $\langle x_0, y_0 \rangle \neq x_1$, then there is an element z_0 of $\langle x_0, y_0 \rangle$ which is not incident with x_1 . Let x_l be an l join of z_0 and x_1 , where $l > 1$. Let z_1 be an element of A_1 which is incident with x_l and z_0 . Then $z_1 \neq x_1$ and z_1 is contained in all blocks which contain x_0 and y_0 . But by Lemma 9, there exists a block of D which is incident with x_1 and not incident with z_1 , and hence z_1 is not contained in $\langle x_0, y_0 \rangle$. Hence $\langle x_0, y_0 \rangle = x_1$. Therefore $(v-\lambda)/(k-\lambda) = \alpha+1 = |x_1|$. By using a result in [4], we have the lemma.

Lemma 11. *An i element x_i is a subspace of P of dimension i for $1 \leq i \leq m$.*

Proof. We shall prove the lemma by the induction on i . By Lemma 10, the case of $i=1$ is true. Let $i \geq 2$. Then there exist elements x_{i-1} and y_{i-1} of A_{i-1} , and an element x_{i-2} of A_{i-2} such that they are incident with x_i , and that x_{i-2} is incident with x_{i-1} and y_{i-1} . By Lemma 6, there exists an element y_0 of y_{i-1} which is not contained in x_{i-1} . By the induction hypothesis, $y_{i-2} = \langle x_{i-2}, y_0 \rangle$ which is a subspace of P spanned by y_0 and all elements of x_{i-2} . Therefore we have $\langle x_{i-1}, y_0 \rangle = \langle x_{i-1}, y_{i-1} \rangle$. Since A_m is a projective space and x_{i-1} is an $i-1$ dimensional subspace, $\langle x_{i-1}, y_0 \rangle$ is an i dimensional subspace, and hence $|\langle x_{i-1}, y_0 \rangle| = (\alpha^{i+1} - 1)/(\alpha - 1)$. On the other hand, we have $\langle x_{i-1}, y_{i-1} \rangle \supset x_i$, because x_i is contained in every elements of A_{m-1} containing x_{i-1} and y_{i-1} . By Lemma 3, $|x_i| = |\langle x_{i-1}, y_0 \rangle|$. Therefore we have $x_i = \langle x_{i-1}, y_0 \rangle$. Thus the proof is complete.

By Lemma 7 and Lemma 11, a proof of Theorem completes.

3. Another example

EXAMPLE 3. Let V be an m dimensional vector space over $GF(2)$ ($m \geq 3$),

and H the set consisting of all $m-1$ dimensional subspaces of V . Put $A_{-1} = \{\phi\}$, $A_m = \{V - \{0\}\}$ and $A_i = \{M_i^c \cap \dots \cap M_{m-i}^c \mid M_1 \supseteq M_1 \cap M_2 \supseteq \dots \supseteq \bigcap_{u=1}^{m-i} M_u, M_u \in H\}$ for $0 \leq i \leq m-1$, where $M_u^c = V - M_u$. We say that $x_i \in A_i$ is incident with $x_j \in A_j$ if and only if $x_i \subset x_j$ ($i \leq j$). We shall show that $S_3 = (\{A_i\}_{i=-1}^m, T)$ is a partial geometric space of dimension m , where $T = \{(x_{-1}, \dots, x_i, \dots, x_j, \dots, x_m) \in \prod_{i=-1}^m A_i \mid x_i \text{ is incident with } x_j \ (-1 \leq i < j \leq m)\}$.

Lemma 12. For $x_i \in A_i$, $|x_i| = 2^i$ ($i \geq 0$).

Proof. Let $x_i = \bigcap_{u=1}^{m-i} M_u^c$, then $\bigcap_{u=1}^{m-i} M_u$ is a subspace of dimension i . Therefore we have that by the principle of inclusion and exclusion $|x_i| = 2^m + \sum_{u=1}^{m-i} \binom{m-i}{u} (-1)^u 2^{m-u} = 2^i (2-1)^{m-i} = 2^i$.

REMARK A. Let $x_0 \in A_0$ and $M \in H$ ($x_0 \notin M$). Since $V - \{0\}$ is a projective space, M^c is an affine space. Thus $M^c - \{x_0\}$ is a projective space over $GF(2)$.

At first, we define the intersection and the join. For $z_l \in A_l$ ($0 \leq l \leq m-1$), put $K(z_l) = \{M \in H \mid M^c \supset z_l\}$, and $K(z_{-1}) = H$ and $K(z_m) = \phi$, where $z_{-1} \in A_{-1}$ and $z_m \in A_m$. Let x_i and y_j ($-1 \leq i, j \leq m$) be elements of A_i and A_j , respectively. Then a set $\bigcap_{u=1}^{m-l} L_u^c$ is defined to be an l intersection of x_i and x_j where elements L_u ($1 \leq u \leq m-l$) of $K(x_i) \cup K(y_j)$ satisfy $L_1 \supseteq L_1 \cap L_2 \supseteq \dots \supseteq \bigcap_{u=1}^{m-l} L_u$ and $\bigcap_{u=1}^{m-l} L_u \subset L$ for any element L of $K(x_i) \cup K(y_j)$. We denote $\bigcap_{u=1}^{m-l} L_u^c$ by $x_i \wedge y_j$. We note that if there exists an element L_{m-l+1} of $K(x_i) \cup K(y_j)$ such that $\bigcap_{u=1}^{m-l} L_u = \bigcap_{u=1}^{m-l+1} L_u$ and $\bigcap_{u=1}^{m-l} L_u^c = \bigcap_{u=1}^{m-l+1} L_u^c$, then x_i and y_j have a -1 intersection. Because let $\bar{V} = V/L_1 \cap \dots \cap L_{m-l}$ and $\bar{L}_u = L_u/L_1 \cap \dots \cap L_{m-l}$ ($1 \leq u \leq m-l$). By Lemma 12, $|\bar{L}_1^c \cap \dots \cap \bar{L}_{m-l}^c| = 1$, and hence $\bar{L}_1^c \cap \dots \cap \bar{L}_{m-l+1}^c = \phi$. This implies $x_i \wedge y_j = \phi$.

Next, a set $\bigcap_{w=1}^{m-s} J_w^c$ is defined to be an s join of x_i and y_j where element J_w ($1 \leq w \leq m-s$) of $K(x_i) \cap K(y_j)$ ($\neq \phi$) are satisfy $J_1 \supseteq J_1 \cap J_2 \supseteq \dots \supseteq \bigcap_{w=1}^{m-s} J_w$ and $\bigcap_{w=1}^{m-s} J_w \subset J$ for any element J of $K(x_i) \cap K(y_j)$. We denote $\bigcap_{w=1}^{m-s} J_w^c$ or $V - \{0\}$ by $x_i \vee y_j$ according to $K(x_i) \cap K(y_j) \neq \phi$ or $= \phi$. It is obvious that the intersection and the join of x_i and y_j is well-defined.

By the above paragraph, we have the following lemma.

Lemma 13. Let K be a subset of H . Then $\bigcap_{N \in K} N^c$ is an element of A_l for some l .

Lemma 14. Let $x_i = \bigcap_{w=1}^{m-i} M_w^c$ and $x_j = \bigcap_{u=1}^{m-j} N_u^c$ be elements of A_i and A_j , respectively. If $x_i \subset x_j$, then $\bigcap_{w=1}^{m-i} M_w \subset \bigcap_{u=1}^{m-j} N_u$.

Proof. Suppose that there exists N_z ($1 \leq z \leq m-j$) such that $\bigcap_{w=1}^{m-i} M_w \not\subset N_z$. Then $\bigcap_{w=1}^{m-i} M_w \supseteq \bigcap_{w=1}^{m-i} M_w \cap N_z$, so $x_{i-1} = \bigcap_{w=1}^{m-i} M_w^c \cap N_z^c$ is an element of A_{i-1} . Hence $\bigcup_{w=1}^{m-i} M_w \cup N_z \supseteq \bigcup_{w=1}^{m-i} M_w$ by Lemma 12. On the other hand, by the hypothesis $x_i \subset x_j$, $\bigcup_{w=1}^{m-i} M_w \supset \bigcup_{u=1}^{m-j} N_u$. Hence $\bigcup_{w=1}^{m-i} M_w \supset N_z$. This is a contradiction.

Lemma 15. Let W be an i dimensional subspace of V . Then $|\{x_i \in A_i | x_i = \bigcap_{w=1}^{m-i} M_w^c, \text{ where } \bigcap_{w=1}^{m-i} M_w = W\}| = 2^{m-i} - 1$.

Proof. Put $\bar{V} = V/W$. By Lemma 12, $|\bar{M}_1^c \cap \dots \cap \bar{M}_{m-i}^c| = 1$. Since $GL(m-i, 2)$ acts transitively on $\bar{V} - \{0\}$, we have the lemma.

By Lemmas 14 and 15, we have the following:

Lemma 16. $|A_0| = 2^m - 1$ and $|A_i| = \left(\prod_{u=1}^i \frac{2^{m+1-u} - 1}{2^u - 1} \right) (2^{m-i} - 1)$ for $m > i > 0$.

Lemma 17. Let $x_i = \bigcap_{u=1}^{m-i} M_u^c$ be an element of A_i ($0 \leq i \leq m-1$), then $|K(x_i)| = 2^{m-i-1}$.

Proof. Without loss of generality, we may assume $i=0$. By Lemma 12, put $M_1^c \cap \dots \cap M_m^c = \{a\}$, that is every elements of H contained in $\bigcup_{u=1}^m M_u$ does not contain $\{a\}$. Since the number of hyperplanes of $V/\langle a \rangle$ equals $2^{m-1} - 1$, the number in the lemma equals $(2^m - 1) - (2^{m-1} - 1) = 2^{m-1}$.

Lemma 18. $k(0) = k(m-1) = 2$ and $k(i) = 3$ for $0 < i < m-1$.

Proof. Let x_i be an element of A_i . Since $|A_0| = 2^m - 1$ by Lemma 16 and $|x_1| = 2$ by Lemma 12, we have $k(0) = 2$. For $k(m-1)$, consider a factor space. Then we have similarly that $k(m-1) = 2$. For $0 < i < m-1$, the lemma follows from Remark A and Example 2.

Lemma 19. Let a and x_i be elements of $V - \{0\}$ and A_i , respectively. Assume that there exist elements M and N of $K(x_i)$ such that $a \in M$ and $a \notin N$, where $i \geq 0$. Then $|\{L \in K(x_i) | a \in L\}| = |\{L \in K(x_i) | a \notin L\}|$.

Proof. Without loss of generality, we may assume $\bigcap_{L \in K(x_i)} L = \{0\}$, that is, $i=0$. Put $X = \{L \in K(x_0) | a \in L\}$ and $Y = K(x_0) - X$. Let $y_j = \bigcap_{L \in X} L^c$ and $z_l = \bigcap_{L \in Y} L^c$. Since $\bigcap_{L \in X} L \ni a$, $j > 0$. Since $z_l \ni a$, $z_l \neq x_0$, and hence $l > 0$. Since $|K(x_0)| > |$

$|K(y_j)|=2^{m-j-1} \geq |X|$ and $|K(x_0)| > |K(z_i)|=2^{m-l-1} \geq |Y|$, we have that $2^{m-1} = |X| + |Y| \leq 2^{m-j-1} + 2^{m-l-1}$. Hence $j=l=1$. This proves the lemma.

Lemma 20. *The geometric parameters are the following:*

- (1) $t(i, i+2, 1)=3$ for $2 < i+2 < m$,
- (2) $t(i, m, m-1)=1$ and $t(i, m, 1)=2$ for $0 < i \leq m-2$,
- (3) $t(0, 2, 1)=1$ if $\langle x_0, x_1 \rangle$ is a subspace of dimension 3 and $t(0, m, 1)=2$ if $x_0 \subset \langle x_1 \rangle$, where x_u ($u=0, 1$) are elements of A_u such that x_0 is not incident with x_1 . The rest geometric parameters need not be defined.

Proof. (1) follows from Example 2 and Remark A. Let x_i and x_{i+1} be elements of A_i and A_{i+1} , respectively, such that they have an $(i-1)$ intersection x_{i-1} and an m join x_m . Considering a factor space, we may assume $i=1$. Put $x_0 = \{a\}$, $x_1 = \{a, b\}$ and $x_2 = \{a, c, d, e\}$ by Lemma 12, where a, b, c, d and e are distinct elements of $V - \{0\}$. Since $x_m = x_1 \vee x_2$, there exist elements M and N of $K(x_1)$ and $K(x_2)$, respectively, such that M does not contain a and b , and that N contains b and does not contain a, c, d and e . Let $Y = K(x_0) - K(x_1)$. Then $|Y| = 2^{m-2}$ by Lemma 17 and $N (\in H)$ is contained in Y if and only if N contains b and does not contain a . Put $y_1 = \bigcap_{N \in Y} N^c$, then $x_0 \subset y_1 \subset x_2$ since $\bigcap_{N \in Y} N \ni b$. Thus y_1 is an element of A_1 and $K(y_1) \cap K(x_1) = \phi$, since $Y = K(y_1)$. Therefore $y_1 \vee x_1$ is contained in A_m and $t(1, m, m-1) \geq 1$. Let $z_1 = \{a, c\}$ and $w_1 = \{a, d\}$. Since $K(y_1) \cap K(x_1) = \phi$ and $|K(x_i)| = |K(y_1)| = |K(x_0)|/2$, $K(z_1) \cap K(x_1) \neq \phi$ and $K(w_1) \cap K(x_1) \neq \phi$. This implies that there are elements M and N of $K(x_1)$ such that $c \in M$ and $c \notin N$. By Lemma 19, $|K(x_1) \cap K(z_1)| = |K(x_1)|/2$, and hence $x_1 \vee z_1 \in A_2$. Similarly $x_1 \vee w_1 \in A_2$. Therefore $t(1, m, 1) \geq 2$. By the definition, $\sum_{u=1}^{m-1} t(1, m, u) = k(1) = 3$. This implies (2). Next assume that $i=0$. Put $x_0 = \{a\}$, $x_1 = \{b, c\}$ and let $x_s = x_0 \vee x_1$. Since $|A_1| = \binom{2^m-1}{2}$ by Lemma 16, $\{a, b\}$ and $\{a, c\}$ are contained in A_1 . Thus $t(0, s, 1) = 2$. If $\langle a, b \rangle \ni c$, then $|H| - 3|\{M \in H \mid a \in M\}| + 2|\{M \in H \mid M \supset \langle a, b \rangle\}| = (2^m-1) - 3(2^{m-1}-1) + 2(2^{m-2}-1) = 0$. Therefore $K(x_0) \cap K(x_1) = \phi$, so $s=m$. If $\langle a, b \rangle \not\ni c$, then

$$\begin{aligned} & |H| - 3|\{M \in H \mid a \in M \text{ and } b, c \notin M\}| + 3|\{M \in H \mid a, b \in M \\ & \text{and } c \notin M\}| - |\{M \in H \mid a, b, c \in M\}| \\ & = (2^{m-1}-1) - 3(2^{m-1}-1) + 3(2^{m-2}-1) - (2^{m-3}-1) = 2^{m-3}. \end{aligned}$$

Therefore $|\{M \in H \mid a, b, c \notin M\}| = |K(x_0 \vee x_1)| = 2^{m-3}$ and hence $x_0 \vee x_1$ is an element of A_2 . This completes a proof of the lemma.

References

- [1] R.C. Bose: *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.
- [2] R.C. Bose and R. Miskimins: *Partial geometric spaces of m dimensions*, Algebraic methods in graph theory Vol. I, II (Szeged, 1978), 37–45, Colloq. Math. Soc. Janos Bolyai, 25, North-Holland, Amsterdam, 1981.
- [3] P. Dembowski: *Finite geometries*, Springer-Verlag, Berlin-New York, 1968.
- [4] P. Dembowski and A. Wagner: *Some characterization of finite projective spaces*, Arch. Math. **11** (1960), 465–469.
- [5] L. Lasker and J. Dunbar: *Partial geometry of dimension three*, J. Combin. Theory **24** (1978), 187–201.
- [6] R. Miskimins: *Partial geometries of dimension m* , Ph. D. Thesis (Colorado State University: Summer, 1978).

Department of Mathematics
Ehime University
Matsuyama 790, Japan