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ON THE CLASSIFICATION OF ESSENTIALLY EFFECTIVE $SL(n; C)$-ACTIONS ON ALGEBRAIC n-FOLDS

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0. Introduction

The main purpose of this paper is to prove the following:

Theorem. *A non-singular irreducible n-dimensional complete variety endowed with an essentially effective regular action of the algebraic group* $SL(n; \boldsymbol{C})$ $(n \geqq 2)$ *is isomorphic to one of the following*:

i) *First we assume* $n \neq 3, 4$. *Then*

(1) *The complex projective space* $\boldsymbol{P}^n(\boldsymbol{C})$.

(2) $\boldsymbol{P}^{n-1}(\boldsymbol{C}) \times K$, *where* K *is an arbitrary non-singular complete curve.*

(3) *The projective bundle* $Proj(\mathcal{O}_{\boldsymbol{P}^{n-1}}(m) \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(0))$, $m \in \boldsymbol{Z}_+$, *associated with the vector bundle* $\mathcal{O}_{\boldsymbol{P}^{n-1}}(m) \oplus \mathcal{O}_{\boldsymbol{P}^{n-1}}(0)$ *over* $\boldsymbol{P}^{n-1}(\boldsymbol{C})$.

ii) *If* $n=3$, *then in addition to* (1), (2), *and* (3) *above, one more case is possible*:

(4) *The projective bundle* $Proj(T(\boldsymbol{P}^2(\boldsymbol{C})))$ *associated with the tangent bundle* $T(\boldsymbol{P}^2(\boldsymbol{C}))$ *of* $\boldsymbol{P}^2(\boldsymbol{C})$.

iii) *If* $n=4$, *then in addition to* (1), (2), *and* (3) *above, one more case is again possible*:

(4)′ *The complex Grassmannian* $G_2(\boldsymbol{C}^4)$ *of* 2-*planes in* $\boldsymbol{C}^4$.

(See Theorem (5.1) for the corresponding $SL(n; \boldsymbol{C})$-actions and more details.)

The proof is essentially reduced to classifying the closed subgroups of codimension $\leqq n$ of the group $SL(n; \boldsymbol{C})$, (cf. §2), whereas the main point of the reduction is the following elementary observation, (cf. §1).

OBSERVATION. *Let* V *be an irreducible variety endowed with a regular action of a connected linear algebraic group* G. *If there exists a* G-*equivariant completion* $\tilde{V}$ *of* V *satisfying the conditions*

(α) $\tilde{V}$ *is a normal variety*

and

(β) $\tilde{V} - V$ *is a finite union of* 1-*codimensional* G-*orbits in* $\tilde{V}$,

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then such a completion $\tilde{V}$ *is unique up to G-equivariant isomorphism, and furthermore every G-equivariant completion* $\tilde{\tilde{V}}$ *of V is dominated by* $\tilde{V}$*, i.e., there exists a G-equivariant birational surjective regular map:* $\tilde{V}\to\tilde{\tilde{V}}$ *which extends the identity mapping* id_V: *V (as a subset of* $\tilde{V}$*)*$\to$*V (as a subset of* $\tilde{\tilde{V}}$*).*

NOTATIONS AND CONVENTIONS.

(0.1) $\boldsymbol{Z}$ = the set of all integers,
$\boldsymbol{Z}_+$ = the set of all positive integers,
$\boldsymbol{C}$ = the complex number field,
$\boldsymbol{C}^*$ = the set of all non-zero complex numbers.

(0.2) All varieties and algebraic groups are defined over $\boldsymbol{C}$.

(0.3) Assume that an algebraic group G acts on varieties V and V' regularly. A regular mapping f: $V\to V'$ is said to be G-equivariant, if the equality $f(g\cdot p)=g\cdot f(p)$ holds for every pair $(g, p)\in G\times V$.

(0.4) A closed subgroup of an algebraic group G is always understood to be an algebraic subgroup of G, ("closed" means "Zariski closed").

(0.5) An algebraic group G is said to act essentially effectively on a variety V if the group of the elements in G which act identically on V is finite.

In concluding this introduction, I wish to thank all those people who encouraged me and gave me suggestions, and in particular Professors S. Kobayashi, S.S. Roan, and I. Satake who helped me again and again during the preparation of this paper.

1. Basic theorems

In this section, we shall quote three basic theorems (cf. [3], [4]) which turn out to be very useful later.

(1.1) Here, we briefly discuss the notion of "dominant G-equivariant completion."

DEFINITION 1.1.1. Let U be an irreducible variety on which a connected linear algebraic group G acts regularly. Then a variety V with a regular G-action is said to be a *G-equivariant completion* of U if the following two conditions are satisfied:

i) U is (embedded as) a G-invariant open dense subset of V.

ii) V is a complete variety.

A G-equivariant completion V of U is said to be *dominant* if the following two conditions are satisfied:

i) V is a normal variety.

ii) $V-U$ is a disjoint union of (a finite number of) 1-codimensional G-orbits in V.

The importance of this notion comes from the following:

Theorem 1.1.2 ([3; Corollary (1.1.3)]). *Let U be an irreducible variety on which a connected linear algebraic group G acts regularly. Assume that there exists a dominant G-equivariant completion V' of U. Then,*

i) *For any G-equivariant completion V of U, the identity mapping $id_U: U$ (as a subset of V')$\to U$ (as a subset of V) extends to a G-equivariant birational surjective regular map: $V' \to V$.*

ii) *In particular, any other dominant G-equivariant completion V'' of U is G-equivariantly isomorphic to V', where the isomorphism between V' and V'' is a canonical extension of the identity automorphism of U.*

(1.2) We secondly quote the following theorem which is obtained as an immediate consequence of Zariski's Main Theorem.

Theorem 1.2.1 (*cf.* [3; Theorem (1.2.1)]). *Let U^* (resp. U, U') be a non-empty open subset of a complete irreducible variety V^* (resp. V, V'). Assume that there exist regular mappings $\sigma: V^* \to V$ and $\sigma': V^* \to V'$ such that*

(1) *$\sigma|_{U^*}$ maps U^* isomorphically onto U.*

(2) *$\sigma'|_{U^*}$ maps U^* isomorphically onto U'.*

(3) *For any point $q \in V' - U'$, $\sigma(\sigma'^{-1}(q))$ is a finite set.*

Furthermore, we assume that:

(4) *V' is a normal variety.*

Then the mapping $(\sigma|_{U^*})\circ(\sigma'|_{U^*})^{-1}: U' \to U$ extends to a birational surjective regular mapping $\tau: V' \to V$ uniquely, and this τ satisfies $\sigma = \tau\circ\sigma'$.

(1.3) We finally consider algebraic group actions with equidimensional orbits.

Theorem 1.3.1 ([4; Theorem (1.2.1)]). *Let V be an n-dimensional irreducible complete normal variety on which a connected linear algebraic group G acts regularly, satisfying the following two conditions:*

(1) *All orbits in V have the same dimension r.*

(2) *There exists a finite subset $\{p_i; i=1, 2, \cdots, k\}$ of V such that, for every $p \in V$, the isotropy subgroup G_p of G at p is conjugate to some G_{p_i} in G.*

Then, it follows that:

(3) *$G_{p_1}, G_{p_2}, \cdots, G_{p_k}$ are all conjugate.*

(4) *The quotient V/G exists as an $(n$-$r)$-dimensional complete normal variety.*

(5) *V is G-equivariantly isomorphic to $G/G_{p_1} \times V/G$.*

2. Closed subgroups of codimension$\leqq n$ of the group $SL(n; \boldsymbol{C})$

In this section, we shall classify all closed subgroups of codimension$\leqq n$ of the algebraic group $SL(n; \boldsymbol{C})$.

NOTATION. For any linear algebraic group G, its identity component (resp. the group of algebraic group automorphisms of G) is denoted by G^0 (resp. $\mathrm{Aut}(G)$).

Definition 2.1. Fix an arbitrary integer n such that $n \geqq 2$. For each $m \in \boldsymbol{Z}_+$, we define a closed subgroup $F(m; n)$ of $SL(n; \boldsymbol{C})$ by

$$F(m; n) = \{f = (f_{ij}) \in SL(n; \boldsymbol{C}); f_{21} = f_{31} = \cdots = f_{n1} = 0,\ (f_{11})^m = 1\} .$$

We also define:

$$\begin{aligned} D(n) &= \{f = (f_{ij}) \in SL(n; \boldsymbol{C}); f_{ij} = 0 \quad \text{for} \quad i \neq j\} , \\ B(n) &= \{f = (f_{ij}) \in SL(n; \boldsymbol{C}); f_{ij} = 0 \quad \text{for} \quad i > j\} , \\ P(n; k) &= \{f = (f_{ij}) \in SL(n; \boldsymbol{C}); f_{ij} = 0 \quad \text{whenever} \quad i > k \geqq j.\} , \end{aligned}$$

where $k = 1, 2, \cdots, n-1$. Note that the normalizer $N_{SL(2; C)}(D(2))$ of $D(2)$ in $SL(2; \boldsymbol{C})$ is expressible as

$$N_{SL(2; C)}(D(2)) = J \cdot D(2) , \quad \text{where} \quad J = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} .$$

(2.2) In terms of the notation defined above, we list here all closed subgroups of codimension $\leqq n$ of the algebraic group $SL(n; \boldsymbol{C})$, $n \geqq 2$.

Theorem 2.2.1. i) *Every algebraic group automorphism of $SL(n; \boldsymbol{C})$ coincides, up to inner automorphisms, with one of the following*:

(1) $id_{SL(n; C)}: SL(n; \boldsymbol{C}) \to SL(n; \boldsymbol{C})$
$\qquad f \longmapsto f$,

(2) *tran. inv.*: $SL(n; \boldsymbol{C}) \to SL(n; \boldsymbol{C})$
$\qquad f \longmapsto {}^t f^{-1}$.

ii) $SL(n; \boldsymbol{C})$, $n \geqq 2$, *contains no proper closed subgroups of codimension* $\leqq n-2$.
iii) *Every* $(n-1)$*-codimensional closed subgroup of* $SL(n; \boldsymbol{C})$, $n \geqq 2$, *is mapped* (*isomorphically*) *onto* $P(n; 1)$ *by some algebraic group automorphism of* $SL(n; \boldsymbol{C})$.
iv) *Every n-codimensional closed subgroup of* $SL(n; \boldsymbol{C})$ *is mapped* (*isomorphically*) *onto one of the following by some algebraic group automorphism of* $SL(n; \boldsymbol{C})$:

(1) (*In the case* $n=2$): $D(2)$, $J \cdot D(2)$, $F(m; 2)$ *where* $m = 1, 2, \cdots$.
(2) (*In the case* $n=3$): $B(3)$, $F(m; 3)$ *where* $m = 1, 2, \cdots$.
(3) (*In the case* $n=4$): $P(4; 2)$, $F(m; 4)$ *where* $m = 1, 2, \cdots$,
(4) (*In the case* $n \geqq 5$): $F(m; n)$ *where* $m = 1, 2, \cdots$.

Proof of i) of (2.2.1): i) is a standard fact.

Proof of ii), iii), and iv) of (2.2.1): ii), iii), and iv) are a straightforward consequence of the following theorem of Dynkin ([2; Chapter 1]):

Theorem. *Every maximal proper connected closed subgroup of* $SL(n; \boldsymbol{C})$ *is* *conjugate to one of the following three types of subgroups*:

(1) $P(n; k)$, $k=1, 2, \cdots, n-1$.

(2) *Irreducible simple subgroups of* $SL(n; \boldsymbol{C})$.

(3) *The Kronecker product* $SL(r; \boldsymbol{C}) \otimes SL(t; \boldsymbol{C})$ *where* $r, t \in \boldsymbol{Z}_+$ *are such that* $2 \leqq r \leqq t$ *and* $r \cdot t = n$.

Thus, by enumerating all irreducible representations of simple algebraic groups, we infer from comparison of dimensions that $SL(n; \boldsymbol{C})$ contains no irreducible simple subgroups of codimension $\leqq n$. Since $SL(r; \boldsymbol{C}) \otimes SL(t; \boldsymbol{C})$ has codimension $(r^2-1)\cdot(t^2-1) > r \cdot t = n$ in $SL(n; \boldsymbol{C})$, it follows that every closed subgroup of codimension $\leqq n$ in $SL(n; \boldsymbol{C})$ is contained in some $P(n; k)$ with $k \in \{1, 2, \cdots, n-1\}$. Then ii), iii), and iv) are straightforward from this fact.

3. Examples of dominant $SL(2; C)$-equivariant completions

In this section, a couple of examples of dominant $SL(2; \boldsymbol{C})$-equivariant completions will be given for later purpose.

(3.1) EXAMPLE 1. We define an action of $G = SL(2; \boldsymbol{C})$ on $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$ by

$$\begin{array}{cccc} G = SL(2; \boldsymbol{C}) & \times\ (\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})) & \rightarrow & \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}) \\ g & (a, b) & \mapsto & (g \cdot a, g \cdot b), \end{array}$$

where $SL(2; \boldsymbol{C})$ acts on $\boldsymbol{P}^1(\boldsymbol{C})$ via the canonical homomorphism: $SL(2; \boldsymbol{C}) \rightarrow PGL(2; \boldsymbol{C})$. Let $q' = ((1:0), (0:1)) \in \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$ and let $q'' = ((1:0), (1:0)) \in \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$. Then

$$\begin{aligned} G \cdot q' &= \{(a, b) \in \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C});\ a \neq b\} \\ &= \text{an open dense orbit in } \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}), \\ G \cdot q'' &= \{(a, b) \in \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C});\ a = b\} \\ &= \text{a 1-codimensional orbit in } \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}), \\ \boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C}) &= (G \cdot q') \cup (G \cdot q''). \end{aligned}$$

Since the isotropy subgroup $G_{q'}$ of G at q' is $D(2)$ in terms of the notation in (2.1), we have:

(*) $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$ with the above action is a dominant $SL(2; \boldsymbol{C})$-equivariant completion of the homogeneous space $SL(2; \boldsymbol{C})/D(2)$.

(3.2) EXAMPLE 2. We define an action of $G = SL(2; \boldsymbol{C})$ on $\boldsymbol{P}^2(\boldsymbol{C})$ via the algebraic group homomorphism:

$$G = SL(2; \boldsymbol{C}) \rightarrow PGL(3; \boldsymbol{C})$$

$$\begin{pmatrix} r & t \\ s & u \end{pmatrix} \mapsto \begin{pmatrix} r^2, & t^2, & rt \\ s^2, & u^2, & su \\ 2rs, & 2tu, & ru+st \end{pmatrix}.$$

Since the 2-sheeted ramified covering

$$f\colon \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C}) \to \boldsymbol{P}^2(\boldsymbol{C})$$
$$(x\colon y),\ (v\colon w) \mapsto (xv\colon yw\colon xw+yv)$$

is G-equivariant in terms of the actions defined above, (see also (3.1)), it immediately follows that:

$$G\cdot f(q') = f(G\cdot q') = \text{an open dense orbit in } \boldsymbol{P}^2(\boldsymbol{C}),$$
$$G\cdot f(q'') = f(G\cdot q'') = \text{a 1-codimensional orbit in } \boldsymbol{P}^2(\boldsymbol{C}),$$
$$\boldsymbol{P}^2(\boldsymbol{C}) = (G\cdot f(q'))\cup(G\cdot f(q'')), \qquad \text{(cf. (3.1)).}$$

Furthermore, the isotropy subgroup $G_{f(q')}$ of G at $f(q')$ is $\{g\in G;\, g\cdot q'\in f^{-1}(f(q'))\} = J\cdot D(2)$, (cf. (2.1)), and hence
(**) $\boldsymbol{P}^2(\boldsymbol{C})$ with the above action is a dominant $SL(2;\boldsymbol{C})$-equivariant completion of the homogeneous space $SL(2;\boldsymbol{C})/(J\cdot D(2))$.

4. Canonical $SL(n;\boldsymbol{C})$-actions on the line bundles $\mathcal{O}_{P^{n-1}}(m)$; $m\in\boldsymbol{Z}$, and equivariant completions of the homogeneous space $SL(n;\boldsymbol{C})/F(m;n)$

For each $m\in\boldsymbol{Z}$, we denote by $\mathcal{O}_{P^{n-1}}(m)$ the m-fold tensor product of the hyperplane bundle on the complex projective space $\boldsymbol{P}^{n-1}(\boldsymbol{C})$, $n\geqq 2$. The beginning of this section is spent in defining a canonical $SL(n;\boldsymbol{C})$-action on $\mathcal{O}_{P^{n-1}}(m)$, with the help of which, we shall classify all $SL(n;\boldsymbol{C})$-equivariant non-singular completions of the homogeneous space $SL(n;\boldsymbol{C})/F(m;n)$, (cf. (2.1)).

(4.1) Definition of a canonical $SL(n;\boldsymbol{C})$-action on $\mathcal{O}_{P^{n-1}}(m)$.

Let $\pi\colon \boldsymbol{C}^n-\{0\}\to\boldsymbol{P}^{n-1}(\boldsymbol{C})$ be the canonical projection, and let $\sigma\colon Q_0(\boldsymbol{C}^n)\to\boldsymbol{C}^n$ be the blowing-up of the origin 0 of $\boldsymbol{C}^n$. Then

$$Q_0(\boldsymbol{C}^n)-\sigma^{-1}(0) = \boldsymbol{C}^n-\{0\},$$

and under this identification, the mapping π extends to

$$\bar{\pi}\colon Q_0(\boldsymbol{C}^n) \to \boldsymbol{P}^{n-1}(\boldsymbol{C}).$$

In terms of this mapping, we can regard $Q_0(\boldsymbol{C}^n)$ as the line bundle $\mathcal{O}_{P^{n-1}}(-1)$ over $\boldsymbol{P}^{n-1}(\boldsymbol{C})$. Note that:
(1) The matrix $SL(n;\boldsymbol{C})$-action on $\boldsymbol{C}^n$ canonically induces an $SL(n;\boldsymbol{C})$-action on $Q_0(\boldsymbol{C}^n)$ $(=\mathcal{O}_{P^{n-1}}(-1))$, and under this action, $Q_0(\boldsymbol{C}^n)$ $(=\mathcal{O}_{P^{n-1}}(-1))$ decomposes into a disjoint union of two orbits $\sigma^{-1}(0)$ (=the zero section of $\mathcal{O}_{P^{n-1}}(-1)$) and $Q_0(\boldsymbol{C}^n)-\sigma^{-1}(0)$. Now, for each $p\in\boldsymbol{P}^{n-1}(\boldsymbol{C})$, let l_p denote the corresponding line through 0 in $\boldsymbol{C}^n$, (l_p is canonically identified with the fibre of $\mathcal{O}_{P^{n-1}}(-1)$ over p), and we fix a base $\boldsymbol{e}_p$ of this fibre l_p. For instance, if $p_0=(1\colon 0\colon 0\colon \cdots\colon 0)\in\boldsymbol{P}^{n-1}(\boldsymbol{C})$, we set:

$$e_{p_0} = (1, 0, 0, \cdots, 0) \in \ell_{p_0}.$$

In terms of this notation, the fibre of $\mathcal{O}_{P^{n-1}}(m)$ $(=(\mathcal{O}_{P^{n-1}}(-1))^{\otimes -m})$ over p is expressed as $(\ell_p)^{\otimes -m}$. Hence

(2) we can now define a canonical $SL(n; \boldsymbol{C})$-action on $\mathcal{O}_{P^{n-1}}(m)$ by setting $g\cdot(\lambda\cdot(e_p)^{\otimes -m}) \overset{\text{defn}}{=\!=} \lambda\cdot(g\cdot e_p)^{\otimes -m}$ for all $g \in SL(n; \boldsymbol{C})$ and $\lambda \in \boldsymbol{C}$.

From now, we assume $m \in \boldsymbol{Z}_+$. Then, in view of (1) above, we have:

(3) $\mathcal{O}_{P^{n-1}}(m)$ is a disjoint union of two orbits, one of which is the zero section of $\mathcal{O}_{P^{n-1}}(m)$, and the other is its complement $(=\mathcal{O}_{P^{n-1}}(m)$-(zero section)).

Recall that $e_{p_0} = (1, 0, \cdots, 0) \in \boldsymbol{C}^n - \{0\}$ $(=\mathcal{O}_{P^{n-1}}(-1)$-(zero section)). Hence,

$$(e_{p_0})^{\otimes -m} \in \mathcal{O}_{P^{n-1}}(m) - (\text{zero section}),$$

and the isotropy subgroup of $SL(n; \boldsymbol{C})$ at this point is, by a straightforward computation, shown to be $F(m; n)$, (cf. (2)). Thus,

(4) $\mathcal{O}_{P^{n-1}}(m)$-(zero section) is $SL(n; \boldsymbol{C})$-equivariantly isomorphic to the homogeneous space $SL(n; \boldsymbol{C})/F(m; n)$.

(4.2) Fix integers m and n such that $m \geqq 1$ and $n \geqq 2$. We now construct a dominant $SL(n; \boldsymbol{C})$-equivariant completion of the homogeneous space $SL(n; \boldsymbol{C})/F(m; n)$. Later, several properties of this completion will also be discussed.

(4.2.1) Dominant $SL(n; \boldsymbol{C})$-equivariant completion of the homogeneous space $SL(n; \boldsymbol{C})/F(m; n)$.

Note that, for every vector space E, $\text{Proj}(E \oplus \boldsymbol{C})$ $(=((E \oplus \boldsymbol{C}) - \{0\})/\boldsymbol{C}^*)$ is a disjoint union of

$$\text{Proj}(E \oplus 0) \cong \text{Proj}(E)$$

and

$$\{\boldsymbol{C}^* \cdot (e \oplus 1);\ e \in E\} \cong E.$$

Therefore, the projective bundle

$$V_{m;n} \overset{\text{defn}}{=\!=} \text{Proj}(\mathcal{O}_{P^{n-1}}(m) \oplus \mathcal{O}_{P^{n-1}}(0))$$

associated with the 2-dimensional vector bundle $\mathcal{O}_{P^{n-1}}(m) \oplus \mathcal{O}_{P^{n-1}}(0)$ over $\boldsymbol{P}^{n-1}(\boldsymbol{C})$ is a disjoint union of

(5) $X_- \overset{\text{defn}}{=\!=} \text{Proj}(\mathcal{O}_{P^{n-1}}(m) \oplus 0) \cong \text{Proj}(\mathcal{O}_{P^{n-1}}(m)) = \boldsymbol{P}^{n-1}(\boldsymbol{C})$

and

(6) $X_{+;0} \overset{\text{defn}}{=\!=} V_{m;n} - X_- \cong \mathcal{O}_{P^{n-1}}(m)$.

Furthermore, $X_{+;0}$ decomposes into

$$X_+ \overset{\text{defn}}{=\!=} \text{Proj}(0 \oplus \mathcal{O}_{P^{n-1}}(0)) \cong \boldsymbol{P}^{n-1}(\boldsymbol{C})$$

and

$$X_0 \overset{\text{defn}}{=\!=\!=} X_{+;0}-X_+ = V_{m;n}-(X_+\cup X_-)\,.$$

Note that, in terms of the isomorphism $X_{+;0}\cong\mathcal{O}_{P^{n-1}}(m)$, we identify X_+, X_0 with the corresponding subsets of $\mathcal{O}_{P^{n-1}}(m)$ as follows:

$$(7)\quad \begin{aligned} X_+ &= (\text{zero section of } \mathcal{O}_{P^{n-1}}(m))\,, \\ X_0 &= \mathcal{O}_{P^{n-1}}(m)-(\text{zero section}). \end{aligned}$$

Now, the $SL(n;\boldsymbol{C})$-actions on $\mathcal{O}_{P^{n-1}}(m)$ and $\mathcal{O}_{P^{n-1}}(0)$ defined in (4.1) induce the one on $\mathcal{O}_{P^{n-1}}(m)\oplus\mathcal{O}_{P^{n-1}}(0)$, and hence we can canonically define an $SL(n;\boldsymbol{C})$-action on $V_{m;n}=\mathrm{Proj}(\mathcal{O}_{P^{n-1}}(m)\oplus\mathcal{O}_{P^{n-1}}(0))$. By the naturality of this action, the isomorphisms in (5) and (6) are both $SL(n;\boldsymbol{C})$-equivariant. Hence, in view of (3), (4), (5), and (7), we obtain:

(8) $V_{m;n}=X_+\cup X_0\cup X_-$ (disjoint uinon) such that

(8-a): both X_+ and X_- are 1-codimensional orbits in $V_{m;n}$ and are $SL(n;\boldsymbol{C})$-equivariantly isomorphic to $\boldsymbol{P}^{n-1}(\boldsymbol{C})$,
(8-b): X_0 is open dense in $V_{m;n}$ and forms a single orbit which is $SL(n;\boldsymbol{C})$-equivariantly isomorphic to the homogeneous space $SL(n;\boldsymbol{C})/F(m;n)$. Thus,

(9) $V_{m;n}=\mathrm{Proj}(\mathcal{O}_{P^{n-1}}(m)\oplus\mathcal{O}_{P^{n-1}}(0))$ is a dominant $SL(n;\boldsymbol{C})$-equivariant completion of the homogeneous space $SL(n;\boldsymbol{C})/F(m;n)$.

(4.2.2) We shall now show that the normal bundles $N(V_{m;n}\colon X_+)$, $N(V_{m;n}\colon X_-)$ of X_+, X_- in $V_{m;n}$ are, under the identifications $X_+=\boldsymbol{P}^{n-1}(\boldsymbol{C})$ and $X_-=\boldsymbol{P}^{n-1}(\boldsymbol{C})$, expressed in the form

(10) $N(V_{m;n}\colon X_+)\cong\mathcal{O}_{P^{n-1}}(m)\,,$
(11) $N(V_{m;n}\colon X_-)\cong\mathcal{O}_{P^{n-1}}(-m)\,.$

Proof of (10): (10) is straightforward:

$$\begin{aligned} N(V_{m;n}\colon X_+) &\cong N(X_{+;0}\colon X_+)\cong N(\mathcal{O}_{P^{n-1}}(m)\colon(\text{zero section})),\quad(\text{cf. (6), (7)}), \\ &\cong \mathcal{O}_{P^{n-1}}(m)\,. \end{aligned}$$

Proof of (11): Recall that there is a canonical isomorphism

$$(12)\quad j\colon \underset{(=V_{m;n})}{\mathrm{Proj}(\mathcal{O}_{P^{n-1}}(m)\oplus\mathcal{O}_{P^{n-1}}(0))} \cong \underset{(=\mathrm{Proj}(\mathcal{O}_{P^{n-1}}(0)\oplus\mathcal{O}_{P^{n-1}}(-m)))\,.}{\mathrm{Proj}(\mathcal{O}_{P^{n-1}}(-m)\otimes(\mathcal{O}_{P^{n-1}}(m)\oplus\mathcal{O}_{P^{n-1}}(0)))}$$

The images of X_+, X_-, $X_-\cup X_0$ under this isomorphism j are

$$j(X_+)=\mathrm{Proj}(\mathcal{O}_{P^{n-1}}(-m)\otimes(0\oplus\mathcal{O}_{P^{n-1}}(0)))=\mathrm{Proj}(0\oplus\mathcal{O}_{P^{n-1}}(-m))\,.$$

$$j(X_-) = \mathrm{Proj}(\mathcal{O}_{P^{n-1}}(-m) \otimes (\mathcal{O}_{P^{n-1}}(m) \oplus 0)) = \mathrm{Proj}(\mathcal{O}_{P^{n-1}}(0) \oplus 0) .$$
$$j(X_- \cup X_0) = j(V_{m\,;\,n}) - j(X_+) .$$

Now we put $X_{-\,;\,0} = X_- \cup X_0$. Then the same argument as in deriving (6) and (7) shows that:

$j(X_{-\,;\,0})$ is canonically identified with $\mathcal{O}_{P^{n-1}}(-m)$, and under this identification, we have $j(X_-)$=(zero section of $\mathcal{O}_{P^{n-1}}(-m)$). Hence

$$N(V_{m\,;\,n}\colon X_-) \cong N(X_{-\,;\,0}\colon X_-) \cong N(j(X_{-\,;\,0})\colon j(X_-))$$
$$\cong N(\mathcal{O}_{P^{n-1}}(-m)\colon \text{(zero section)}) \cong \mathcal{O}_{P^{n-1}}(-m) .$$

(4.2.3) In concluding (4.2), we shall show that there exists a surjective $SL(n; \boldsymbol{C})$-equivariant regular mapping from $V_{1\,;\,n}$ to $\boldsymbol{P}^n(\boldsymbol{C})$: First note that $\mathcal{O}_{P^{n-1}}(0) \oplus \mathcal{O}_{P^{n-1}}(-1)$ is, as a variety, identified with $\boldsymbol{C} \times Q_0(\boldsymbol{C}^n)$ (cf. (4.1)). Therefore the canonical projection $id_{\boldsymbol{C}} \times \sigma\colon \boldsymbol{C} \times Q_0(\boldsymbol{C}^n) \to \boldsymbol{C} \times \boldsymbol{C}^n (= \boldsymbol{C}^{n+1})$ is regarded as a regular mapping from $\mathcal{O}_{P^{n-1}}(0) \oplus \mathcal{O}_{P^{n-1}}(-1)$ onto $\boldsymbol{C}^{n+1}$ and hence it induces a surjective regular map

$$\sigma'\colon \mathrm{Proj}(\mathcal{O}_{P^{n-1}}(0) \oplus \mathcal{O}_{P^{n-1}}(-1)) \to \mathrm{Proj}(\boldsymbol{C}^{n+1}) (= \boldsymbol{P}^n(\boldsymbol{C})) .$$

Thus, in view of the equality $j(V_{1\,;\,n}) = \mathrm{Proj}(\mathcal{O}_{P^{n-1}}(0) \oplus \mathcal{O}_{P^{n-1}}(-1))$, (cf.(12)), we obtain:

(13) $\sigma'' \overset{\text{defn}}{=\!=\!=} \sigma' \circ j\colon V_{1\,;\,n} \to \boldsymbol{P}^n(\boldsymbol{C})$ is a well-defined surjective regular mapping.
Here, one can immediately check the following properties of σ'':

i) σ'' is $SL(n; \boldsymbol{C})$-equivariant,
ii) $\sigma''(X_-)$=the origin 0 of $\boldsymbol{C}^n (\subseteq \boldsymbol{P}^n(\boldsymbol{C}))$,
iii) σ'' maps $V_{1\,;\,n} - X_-$ isomorphically onto $\boldsymbol{P}^n(\boldsymbol{C}) - \{0\}$,

where $\boldsymbol{P}^n(\boldsymbol{C})$ is endowed with the $SL(n; \boldsymbol{C})$-action which extends the standard $SL(n; \boldsymbol{C})$-action on $\boldsymbol{C}^n$ via the inclusion

$$\begin{array}{ccc} \boldsymbol{C}^n & \hookrightarrow & \boldsymbol{P}^n(\boldsymbol{C}) \\ (z_1, z_2, \cdots, z_n) & \mapsto & (1 : z_1 : z_2 : \cdots : z_n) . \end{array}$$

(In particular, our $SL(n; \boldsymbol{C})$-action has the only fixed point $0 \in \boldsymbol{C}^n \subseteq \boldsymbol{P}^n(\boldsymbol{C})$.)

We now state our main purpose in §4:

Theorem 4.3.1. *We fix $m, n \in \boldsymbol{Z}_+$ with $n \geq 2$, and let V be a non-singular irreducible variety which is, at the same time, an $SL(n; \boldsymbol{C})$-equivariant completion of the homogeneous space $SL(n; \boldsymbol{C})/F(m; n)$. Then*

(a) *If $m \geq 2$, V is $SL(n; \boldsymbol{C})$-equivariantly isomorphic to $V_{m\,;\,n} = Proj(\mathcal{O}_{P^{n-1}}(m) \oplus \mathcal{O}_{P^{n-1}}(0))$.*

(b) *If $m = 1$, V is $SL(n; \boldsymbol{C})$-equivariantly isomorphic to either $V_{m\,;\,n} = Proj(\mathcal{O}_{P^{n-1}}(m) \oplus \mathcal{O}_{P^{n-1}}(0))$ or $\boldsymbol{P}^n(\boldsymbol{C})$.*

Here, the $SL(n; \boldsymbol{C})$*-action on* $V_{m;n}$ *(resp.* $\boldsymbol{P}^n(\boldsymbol{C})$*) is defined in* (4.2.1) *(resp.* (4.2.3)*).*

Proof of (4.3.1): Let U be the open dense subset of V which is identified with the homogeneous space $SL(n; \boldsymbol{C})/F(m; n)$. Since $V_{m;n}$ is a dominant $SL(n; \boldsymbol{C})$-equivariant completion of the open dense subset $X_0 \cong SL(n; \boldsymbol{C})/F(m; n)$, (cf. (8-b) and (9) of (4.2.1)), i) of Theorem (1.1.2) asserts that the canonical identification

$$id\colon X_0 \xrightarrow{\cong} U$$

extends to an $SL(n; \boldsymbol{C})$-equivariant birational surjective regular map

$$\tau\colon V_{m;n} \to V .$$

Since $V_{m;n}$ is a disjoint union of three orbits X_0, X_+, and X_-, Zariski's Main Theorem (cf. Mumford [6; p. 414-v]) applied to the birational proper regular map τ shows the disjointness of $\tau(X_0)\,(=U)$, $\tau(X_+)$, and $\tau(X_-)$. Hence,

(14) V is a disjoint union of three orbits U, $\tau(X_+)$, and $\tau(X_-)$.

Now recall the following fact: Let Y be a variety such that there exists a surjective regular mapping $f\colon \boldsymbol{P}^r(\boldsymbol{C}) \to Y$, $(r \in \boldsymbol{Z}_+)$. Then either dim $Y=r$ or Y is a singleton.
Therefore, in view of $X_+ \cong \boldsymbol{P}^{n-1}(\boldsymbol{C})$ and $X_- \cong \boldsymbol{P}^{n-1}(\boldsymbol{C})$, (cf. (8-a) of (4.2.1)), the following four cases are possible:

Case i) dim $\tau(X_+)=$dim $\tau(X_-)=n-1$.
Case ii) dim $\tau(X_+)=n-1$ and $\tau(X_-)$ is a singleton.
Case iii) dim $\tau(X_-)=n-1$ and $\tau(X_+)$ is a singleton.
Case iv) Both $\tau(X_+)$ and $\tau(X_-)$ are a singleton.

Since $N(V_{m;n}\colon X_+)=\mathcal{O}_{P^{n-1}}(m)$ and $N(V_{m;n}\colon X_-)=\mathcal{O}_{P^{n-1}}(-m)$, (cf. (10) and (11) of (4.2.2)), a theorem of Moišezon [5; Chapter III, Corollary 2] immediately implies
(α) If $m \geqq 2$, then only Case i) can happen.
(β) If $m=1$, then only Cases i) and ii) can happen.

First, we consider Case i): In this case, V as well as $V_{m;n}$ is a dominant $SL(n; \boldsymbol{C})$-equivariant completion of $SL(n; \boldsymbol{C})/F(m; n)$. Hence by ii) of Theorem (1.1.2), V is $SL(n; \boldsymbol{C})$-equivariantly isomorphic to $V_{m;n}$.

Secondly, we consider Case ii) under the assumption $m=1$: In this case, we have the following two regular mappings:

$$\sigma''\colon V_{1;n} \to \boldsymbol{P}^n(\boldsymbol{C}), \qquad \text{(cf. (13) of (4.2.3))},$$
$$\tau\;\colon V_{1;n} \to V .$$

Let p denote the singleton $\tau(X_-)$. Then the restriction

$$\tau|_{V_{1;n}-X_-}\colon V_{1;n}-X_- \to V-\{p\}$$

is a birational surjective regular map with finite fibres, which is, by Zariski's Main Theorem, an isomorphism. On the other hand, by iii) of (4.2.3),

$$\sigma''|_{V_{1;n}-X_-}\colon V_{1;n}-X_- \to \boldsymbol{P}^n(\boldsymbol{C})-\{0\}$$

is also an isomorphism. Hence, by Theorem (1.2.1),

$$(\sigma''|_{V_{1;n}-X_-})\circ(\tau|_{V_{1;n}-X_-})^{-1}\colon V-\{p\} \xrightarrow{\cong} \boldsymbol{P}^n(\boldsymbol{C})-\{0\}$$

canonically extends to an $SL(n; \boldsymbol{C})$-equivariant isomorphism of V with $\boldsymbol{P}^n(\boldsymbol{C})$.

Thus, in view of (α) and (β) above, we obtain:

(a) If $m \geqq 2$, then V is $SL(n; \boldsymbol{C})$-equivariantly isomorphic to $V_{m;n}$.

(b) If $m=1$, then V is $SL(n; \boldsymbol{C})$-equivariantly isomorphic to either $V_{m;n}$ or $\boldsymbol{P}^n(\boldsymbol{C})$.

REMARK 4.3.2. With a little more work, we can obtain the classification of all normal $SL(n; \boldsymbol{C})$-equivariant completions of the homogeneous space $SL(n; \boldsymbol{C})/F(m; n)$.

5. Classification of essentially effective $SL(n; C)$-actions on algebraic n-folds

Let V be a variety endowed with a regular action $\gamma\colon G\times V\to V$ of an algebraic group G. (We denote such a V by the pair $(V; \gamma)$.) Then, to every algebraic group automorphism h of G, we associate a regular G-action $\gamma^h\colon G\times V\to V$ by

$$\gamma^h(g, y) = \gamma(h(g), y)\,, \qquad \text{for all} \quad (g, y)\in G\times V\,.$$

Before stating the main theorem, we first list seven types of n-dimensional varieties which admit an essentially effective action of $SL(n; \boldsymbol{C})$.

(1) $\boldsymbol{P}^n(\boldsymbol{C})$ with the $SL(n; \boldsymbol{C})$-action which is induced from the homomorphism $g\mapsto\begin{pmatrix}1&0\\0&g\end{pmatrix}$ from $SL(n; \boldsymbol{C})$ to $PGL(n+1; \boldsymbol{C})$, (cf. (4.2.3), [3; Theorem 4.1.2]).

(2) $\mathrm{Proj}(\mathcal{O}_{\boldsymbol{P}^{n-1}}(m)\oplus\mathcal{O}_{\boldsymbol{P}^{n-1}}(0))$, $m\in\boldsymbol{Z}_+$, with the $SL(n; \boldsymbol{C})$-action defined in (4.2.1).

(3) $\boldsymbol{P}^{n-1}(\boldsymbol{C})\times K$, (where K is an arbitrary non-singular complete curve), endowed with the $SL(n; \boldsymbol{C})$-action which factors to the product of the standard homogeneous one on $\boldsymbol{P}^{n-1}(\boldsymbol{C})$ and the trivial one on K.

(4) $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ with the $SL(2; \boldsymbol{C})$-action defined in (3.1).

(5) $\boldsymbol{P}^2(\boldsymbol{C})$ with the $SL(2; \boldsymbol{C})$-action defined in (3.2).

(6) $\mathrm{Proj}(T(\boldsymbol{P}^2(\boldsymbol{C})))$ (=the associated projective bundle of the tangent bundle $T(\boldsymbol{P}^2(\boldsymbol{C}))$ of $\boldsymbol{P}^2(\boldsymbol{C})$) endowed with the $SL(3; \boldsymbol{C})$-action which is canonically induced from the standard homogeneous one on $\boldsymbol{P}^2(\boldsymbol{C})$.

(7) $G_2(\boldsymbol{C}^4)$ (=the complex Grassmannian of 2-planes in $\boldsymbol{C}^4$) endowed with the canonical $SL(4; \boldsymbol{C})$-action.

We now prove the following main theorem:

Theorem 5.1. *Let V be a non-singular irreducible n-dimensional complete variety endowed with an essentially effective regular action γ of the algebraic group $G=SL(n; \boldsymbol{C})$, $n\geqq 2$. Then, for some algebraic group automorphism h of G, the space $(V; \gamma^h)$ is G-equivariantly isomorphic to one of the following:*

i) *(In the case $n=2$): The varieties in* (1), (2), (3), (4), *and* (5) *above.*

ii) *(In the case $n=3$): The varieties in* (1), (2), (3), *and* (6) *above.*

iii) *(In the case $n=4$): The varieties in* (1), (2), (3), *and* (7) *above.*

iv) *(In the case $n\geqq 5$): The varieties in* (1), (2), *and* (3) *above.*

Proof of (5.1). Let r be the minimal dimension of the G-orbits in V and r' be the maximal dimension of the G-orbits in V. Since $SL(n; \boldsymbol{C})$ contains no proper closed subgroups of codimension $\leqq n-2$, (cf. (ii) of Theorem (2.2.1)), either $r=0$ or $r>n-2$. Hence the following four cases are possible:

Case A: $r=0$, (i.e., $V^G\neq\phi$).

Case B: $r=r'=n-1$.

Case C: $r=n-1$ and $r'=n$.

Case D: $r=r'=n$.

First we consider Case A: Since $V^G\neq\phi$, a theorem in [3; cf. (4.1.2)] immediately implies that:

(#) In Case A, for some $h\in \mathrm{Aut}(G)$, the space $(V; \gamma^h)$ is G-equivariantly isomorphic to $\boldsymbol{P}^n(\boldsymbol{C})$ in (1) above.

Secondly, we consider Case B: Since $r=r'=n-1$, all orbits in V have the same dimension $n-1$. Hence, by Theorem (1.3.1) applied to $k=2$, (cf. (i) and (iii) of Theorem (2.2.1)), there exists an $h\in\mathrm{Aut}(G)$ such that $(V; \gamma^h)$ is G-equivariantly isomorphic to $(G/P(n; 1))\times(V/G)$, where the quotient V/G exists as a 1-dimensional normal (and hence non-singular) complete variety. Since $G/P(n; 1)$ is regarded as $\boldsymbol{P}^{n-1}(\boldsymbol{C})$ with the standard G-action, we obtain:

(##) In Case B, for some $h\in\mathrm{Aut}(G)$, the space $(V; \gamma^h)$ is G-equivariantly isomorphic to some $\boldsymbol{P}^{n-1}(\boldsymbol{C})\times K$ in (3) above.

Thirdly, we consider Case C: Since $r'=n=\dim V$, V contains a unique open dense (n-dimensional) G-orbit (which we denote by $U=G\cdot p$), (cf. Borel [1; p.98]). Then, by $r<n$, the isotropy subgroup of G at p is non-parabolic and of codimension n. Hence, in view of iv) of Theorem (2.2.1), we immediately infer that, for some $h\in\mathrm{Aut}(G)$, our U endowed with (the restriction to U of) the G-action γ^h is G-equivariantly isomorphic to one of the following:

(In the case $n=2$): $G/D(2)$, $G/(J\cdot D(2))$, $G/F(m; 2)$ where $m=1, 2, \cdots$.
(In the case $n\geqq 3$): $G/F(m; n)$ where $m=1, 2, \cdots$.

Now, by the equality $r=n-1$, $(V; \gamma^h)$ is a dominant G-equivariant completion of the homogeneous space $(U; \gamma^h)$. Therefore, in view of (*) of (3.1), (**) of (3.2), and (9) of (4.2.1), the uniqueness theorem of dominant G-equivariant completions (cf. (ii) of (1.1.2)) enables us to conclude that:

(###) In Case C, for some $h\in\mathrm{Aut}(G)$, the space $(V; \gamma^h)$ is G-equivariantly isomorphic to one of the following:

$$\begin{cases} \text{The varieties in (2), (4), and (5) above,} & \text{if } n=2. \\ \text{The varieties in (2),} & \text{if } n\geqq 3. \end{cases}$$

Finally, we consider Case D: Since, in this case, $G=SL(n; C)$ acts homogeneously on the complete variety V, we can regard V as the quotient space of $SL(n; C)$ by some n-codimensional parabolic subgroup. But then, by iv) of Theorem (2.2.1), such subgroups exist only when $n=3, 4$. Noting that $\mathrm{Proj}(T(P^2(C)))$ (resp. $G_2(C^4)$) endowed with the standard $SL(3; C)$-action (resp. $SL(4; C)$-action) is naturally identified with the homogeneous space $SL(3; C)/B(3)$ (resp. $SL(4; C)/P(4; 2)$), we now conclude the following, (cf. (iv) of (2.2.1)):

(####) In Case D, n is either 3 or 4, and for some $h\in\mathrm{Aut}(G)$, the space $(V; \gamma^h)$ is G-equivariantly isomorphic to

$$\begin{cases} \mathrm{Proj}(T(P^2(C))) \text{ in (6) above,} & \text{if } n=3. \\ G_2(C^4) \qquad \text{in (7) above,} & \text{if } n=4. \end{cases}$$

Thus, (#), (##), (###), and (####) above complete the proof of Theorem (5.1).

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