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On Universal Implementability of Generalized Mechanisms I

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Abstract

In this paper, we treat a universal sense of implementability for a generalized class of mechanisms. The concept is a generalization of Sonnenschein (1974) axiomatic characterization. Our implementability framework includes the second fundamental theorem of welfare economics and is also opened to many kinds of social choice axiomatic characterization problems.

JEL classification: D50, D71, D82

Key words: Implementability, Message Mechanism, Axiomatic Characterization, Core Limit Theorem, Second Welfare Theorem, Category Theory

1 Introduction

In this paper, we treat a universal sense of implementability for generalized class of mechanisms. The concept is a generalization and integration of Sonnenschein (1974) axiomatic characterization and the resent game theoretic mechanism arguments (see, e.g., Mas-Colell et al. 1995; Ch.23). Our implementability framework includes the second fundamental theorem of welfare economics and is also opened to many kinds of social choice axiomatic characterization problems.

We reconstruct Sonnenschein (1974) argument as a *universal implementability* problem for generalized mechanisms with messages incorporating *economy-dependent* message structures.[1] The concept of generalized mechanisms with messages is defined as the class of mechanisms that use messages to restrict their strategy sets. To generalize Sonnenschein's argument for *economy-dependent* response functions is important since for the universal implementability argument, how we take the category of all message mechanisms together with the universal domain of all economies is essential.

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1 In Sonnenschein (1974), Propositions 3, 4, 5, and 6 are related to the argument on the minimal dimension of the information space like Hurwicz (1960) and Mount and Reiter (1974). His Propositions 1 and 7 were presented independently for arguments on the *dictionary* and the *unique up to isomorphism* properties, respectively, which we generalize here as the universal implementability.

2 The Mechanism and Implementability

We identify a *game* $\boldsymbol{G}$ with a list of agents (players) *characteristics* θ_1 , . . . , θ_n with their *strategy* sets S_1, . . . , S_n, i.e., $\boldsymbol{G} = (\theta_1, \ldots, \theta_n, S_1, \ldots, S_n)$. The set of all characteristics of agents is given by Θ. In the following, we consider many kinds of games and the number of agents n will be treated as a variable. We take, however, a common *outcome* space X for such games uniquely, i.e., X is taken largely enough to represent all agents' consequences. A *mechanism* $\Gamma = (S_1, \ldots, S_n, \mathfrak{g})$ is a collection of strategy sets and an outcome function $\mathfrak{g} : S_1 \times \cdots \times S_n \to X$. An *economy* $\mathcal{E}$ is a finite list of agents' characteristics, i.e., $\mathcal{E} = (\theta_1, \ldots, \theta_n)$. We also assume that it is always possible to define agent's strategy set S_i uniquely from his characteristics θ_i. The set of all economies is denoted by $\mathcal{E}con$. A *social choice function* g is a function on $\mathcal{E}con$ to X. Note that the list of agents' characteristics $(\theta_1, \ldots, \theta_n)$ is sufficient to define a game for each mechanism. Let π be an operation that derived from an *equilibrium concept* for games such that π defines a certain *equilibrium* strategy profile, $\pi(\boldsymbol{G}) = \pi(\theta_1, \ldots, \theta_n, S_1, \ldots, S_n) \in S_1 \times \cdots \times S_n$ for each game $\boldsymbol{G} = (\theta_1, \ldots, \theta_n, S_1, \ldots, S_n)$. Now fix the number of agents n and strategy sets $S_1, \ldots, S_n$. Mechanism $\Gamma = (S_1, \ldots, S_n, \mathfrak{g})$ is said to *implement* social choice function g under a game theoretic equilibrium concept π if $g(\theta_1, \ldots, \theta_n) = \mathfrak{g}(\pi(\boldsymbol{G})) = \mathfrak{g}(\pi(\theta_1, \ldots, \theta_n, S_1, \ldots, S_n))$, that is, the outcome can be obtained as the game theoretic equilibrium for any specification of n-agents' characteristics $(\theta_1, \ldots, \theta_n) \in \mathcal{E}con^*$, where $\mathcal{E}con^*$ is the set of all n-agent economies in $\mathcal{E}con$. (Our settings in this section are standard. See, e.g., Mas-Colell et al. 1995; Chapter 23, p. 866.)

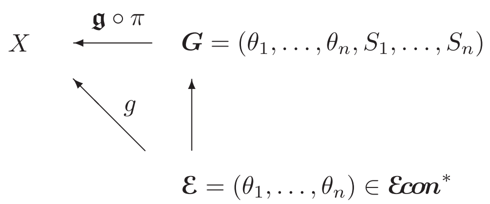

Figure 1 : Implementability by Mechanism $\Gamma = (S_1, \ldots, S_n, \mathfrak{g})$.

3 A Generalized Mechanism and Implementability

Now we generalize the above definitions on the mechanism and implementability. In the above, a mechanism is defied with respect to a fixed number of agents n and their strategy sets, $S_1, \ldots, S_n$. In the following, we treat a mechanism defined on the universal domain $\mathcal{E}con$. To define a mechanism, we assume as before that for each agent's characteristics θ_i, a strategy set S_i is uniquely determined. Moreover, we suppose that there is a common space A of *messages* that is used to introduce a generalized game structure on those strategy sets. A *generalized mechanism* (with messages) $\hat{\Gamma}$ defines for each $\mathcal{E} = (\theta_i)_{i \in I} \in \mathcal{E}con$, a list $(A, (S_i)_{i \in I}, K, \mathfrak{g})$, where S_i is a strategy set derived from θ_i for each $i \in I$, $\mathfrak{g}$ is an outcome function $\mathfrak{g} : \prod_{i \in I} S_i \to X$ which gives for each list of strategies $(s_i)_{i \in I}$ an outcome $\mathfrak{g}((s_i)_{i \in I}) \in X$ so is not directly dependend on $\mathcal{E}$, and $K = \prod_{i \in I} K_i : A \ni a \to \prod_{i \in I} K_i(a) \subset \prod_{i \in I} S_i$ is a constraint structure for a generalized game such that for each message $a \in A$, a strategy profile $(s_i)_{i\in I}$ is required to be an element of $K(a)$. Set A is called a *message space* and K is called a *constraint correspondence*. Note that K may depend on $\mathcal{E} \in \mathcal{E}con$ or at least on each θ_i constructing $\mathcal{E}$ for each coordinate i.

Any specification of agents' characteristics $(\theta_i)_{i\in I} = \mathcal{E}$ provides for each $\hat{\Gamma}$ a game structure $((\theta_i)_{i\in I},\ (K_i(a))_{i\in I})$ for each $a \in A$. Here, we can also identify a parameterized game structure $\hat{\boldsymbol{G}}$ with the list, $((\theta_i)_{i\in I}, A, K)$, which gives a game structure $\hat{G}(a) = ((\theta_i)_{i\in I}, (K_i(a))_{i\in I})$ for each $a \in A$. An *equilibrium concept* $\hat{\pi}$ for $\hat{G}$ is a correspondence on $\mathcal{E}con$. For each $\mathcal{E} \in \mathcal{E}con$, $\hat{\pi}(\mathcal{E})$, the set of *equilibrium states* for $\mathcal{E}$, is a subset of $A \times \prod_{i\in I} S_i$, such that for each $a \in A$, $\hat{\pi}$ defines an equilibrium strategy profile $(s_i)_{i\in I}$, $(a, (s_i)_{i\in I}) \in \hat{\pi}(\mathcal{E})$. So $\hat{\pi}$ provides a certain equilibrium concept, π, for each $\hat{G}(a) = ((\theta_i)_{i\in I}, (K_i(a))_{i\in I})$. The first projection $\hat{\pi}_1(\mathcal{E})$ of $\hat{\pi}(\mathcal{E})$ is called the set of *equilibrium messages* for $\mathcal{E}$. The implementability triangle in section 2 is now generalized as follows.

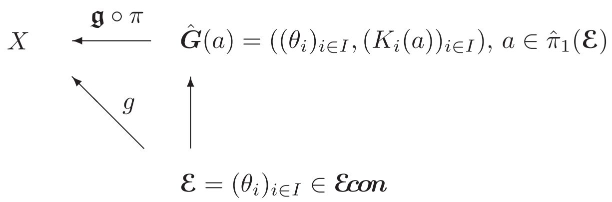

Figure 2: Implementability by Generalized Mechanism $\hat{\Gamma} = (A, (S_i)_{i\in I}, K, \mathfrak{g})$.

Let g be a *social choice correspondence* on $\mathcal{E}con$ to X. (Note that here g is generally taken to be a correspondence.) Mechanism $\hat{\Gamma} = (A, (S_i)_{i\in I}, K, \mathfrak{g})$ is said to *implement* social choice correspondence g under an equilibrium concept $\hat{\pi}$ for $\hat{\boldsymbol{G}}$ if $g((\theta_i)_{i\in I}) = \{\mathfrak{g}\left(\pi(\hat{\boldsymbol{G}}(a))\right) = \mathfrak{g} \circ (\pi((\theta_i)_{i\in I}, (K_i(a))_{i\in I})) \mid a \in \hat{\pi}_1(\mathcal{E})\}$. That is, the outcome can be obtained as the set of game theoretic equilibrium profiles for all equilibrium messages for each $\mathcal{E} \in \mathcal{E}con$.

4 The Message Mechanism and Universal Implementability

Given a specification of agents' characteristics $(\theta_i)_{i\in I}$ or $\mathcal{E} \in \mathcal{E}con$, a generalized mechanism $\hat{\Gamma}$ always defines a parameterized game structure $\hat{G}$. Also by specifying for each $\mathcal{E} \in \mathcal{E}con$, the set of equilibrium messages $\hat{\pi}_1(\mathcal{E})$ and a game theoretic equilibrium concept π, we obtain an *equilibrium system* of a generalized mechanism with messages. In the following, we call such a generalized mechanism $\hat{\Gamma}$ together with an equilibrium concept, $\hat{\pi} = (\hat{\pi}_1, \pi)$, a *message mechanism*. The implementability in the previous section can be restated that a social choice correspondence g is implemented by a message mechanism. See the diagram in Figure 3.

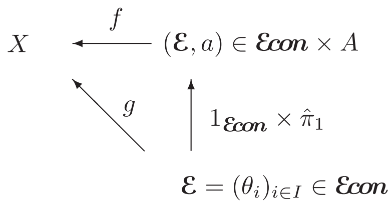

Figure 3: Implementability by Message Mechanism $(\hat{\Gamma}, \hat{\pi}) = ((A, (S_i)_{i \in I}, K, \mathfrak{g}), (\hat{\pi}_1, \pi))$.

In the above, f is a function defined from $\mathfrak{g}$, π and K as $f(\mathcal{E}, a) = \mathfrak{g} \circ \pi(\mathcal{E}, (K_i(a)_{i \in I})$. Also note that $\hat{\pi}_1$ is a correspondence, so $(\mathcal{E}, a)$ is taken as an element of set $\{(\mathcal{E}, a) \mid a \in \hat{\pi}_1(\mathcal{E})\}$.

As we see in the above diagram, the implementability property of message mechanism $(\hat{\Gamma}, \hat{\pi})$ is completely described through $(A, \hat{\pi}_1, f)$. In the following, we denote a message mechanism by a triple $(A, \hat{\pi}_1, f)$, a *message space* A, an *equilibrium message correspondence* $\hat{\pi}_1$ and an economy and message dependent *response function* f, instead of $(\hat{\Gamma}, \hat{\pi})$ as long as there is no fear of confusion.

It is possible that a social choice correspondence g can be implemented by several different message mechanisms. Now we introduce the concept of *universal implementability* that indicates a representative message mechansim for all message mechanisms implementing a certain social choice correspondence g together with some axioms. Consider that two message mechanisms (P, π, e) and (A, μ, f) implement a social choice correspondence g together with some axioms. See the next two diagrams.

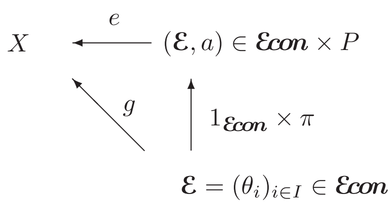

Figure 4: Implementability of g **by Message Mechanism** (P, π, e).

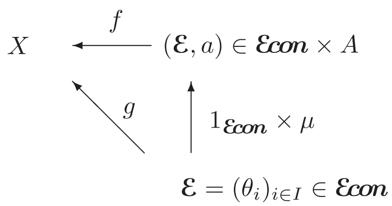

Figure 5: Implementability of g **by Message Mechanism** (A, μ, f).

We say that message mechanism (P, π, e) has a dictionary property to message mechanism (A, μ, f) if there exists a unique mapping ϕ on A to P such that the following diagram commutes.

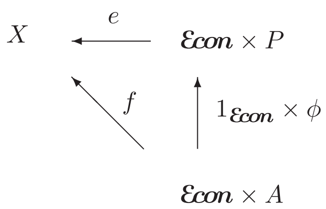

Figure 6: The Dictionary property of (P, π, e).

Mechanism (P, π, e) is said to have the *universal implementability* if (P, π, e) has the dictionary property for all message mechanism (A, μ, f) implementing social choice correspondence g together with some axioms.

5 Examples

1. Let us consider that $\mathcal{E}con$ is the class of all Arrow-Debreu economies with ℓ commodities. The number of agents, m for consumers and n for producers, are taken as variables. Consider that the social choice correspondence g is the correspondence of Pareto optimal allocations for each $\mathcal{E} \in \mathcal{E}con$ and the outcome space X is taken as a large space that can represent every consumption allocation for finite economies. Also consider message space $P \times \prod_{\mathcal{E} \in \mathcal{E}con} R^{\sharp\mathcal{E}}$, where P is a normalized space of prices of ℓ commodities and $\sharp\mathcal{E}$ denotes the number of consumers in $\mathcal{E}$, so a message $a \in P \times \prod_{\mathcal{E} \in \mathcal{E}con} R^{\sharp\mathcal{E}}$ defines ℓ-prices together with a wealth distribution for each $\mathcal{E} \in \mathcal{E}con$. Then, by identifying π with a solution concept that defines a unique utility maximization behavior for each consumer and a unique profit maximization behavior for each producers for each $\mathcal{E} \in \mathcal{E}con$, and e with the Marshallian demand or supply function, we obtain an economic *price-wealth* message mechanism.

The second welfare theorem is nothing but the statement that every Pareto-optimal social choice function can be implemented through such a (price-wealth) message mechanism.

By adding a condition that the utility and profit maximization points for each agent in each $\mathcal{E}$ are unique, (e.g., by assuming the strong convexity conditions for preferences and technologies of economies in $\mathcal{E}con$,) together with the following Axiom S (an economy-dependent extension of the axiom in Sonnenschein 1974), we can also obtain the universal implementability of the price-wealth message mechanism.

Axiom S: For each finite list of agents and economies, $(i_1, \mathcal{E}^1), (i_2, \mathcal{E}^2), \ldots, (i_m, \mathcal{E}^m)$, each message $a \in A$ and each list of responses $(f_{i_s}(\mathcal{E}^s, a))_{s=1}^m$, there exists an economy $\mathcal{E}_* = (\succsim_i^*, \omega_i^*)_{i \in I^*}$ including $\{i_1, i_2, \ldots, i_m\}$ such that a is an equilibrium message for $\mathcal{E}_*$ for which the equilibrium list $(f_i(\mathcal{E}_*, a))_{i \in I^*}$ is an extension of $(f_{i_s}(\mathcal{E}^s, a))_{s=1}^m$.

2. Our universal implementability framework is *economy-dependent* (with respect to messages) in the sense that we treat the difference between the names of economies $\mathcal{E}^1, \ldots, \mathcal{E}^m$ as essential. More precisely, in the definition of $\mathcal{E}$, (θ_1, θ_2) and (θ_3, θ_4) can be distinguished since their agent index sets $\{1, 2\}$ and $\{3, 4\}$ are different even when $\theta_1 = \theta_3$ and $\theta_2 = \theta_4$. In Sonnenschein (1974), an economy is completely identified with the finite list of agents' characteristics (all domains of the family are treated as the same as long as they have the same cardinality).

Sonnenschein (1974) uses the Axiom S (non-economy dependent version) to show the universal implementability of the price mechanism for the core compatible social choice correspondence. It is also possible to weaken Axiom S to a condition based merely on utility levels (see Shiraishi et al. 2017). As the second welfare theorem, a core limit result is also possible to be identified with a universal implementability theorem (see Urai and Murakami 2016b, Urai and Murakami 2016a, Murakami and Urai 2017a and Murakami and Urai 2017b).

3. Axiom S also provides a unified method to characterize the price and/or the price-money message mechanisms through the concept of universal implementability. Our result, Urai and Murakami (2017), treats the monotonicity and incentive compatibility to characterize the price-money mechanism.

From the purely mathematical viewpoint, Sonnenschein (1974) uses the concept of *universal element* instead of the universal implementability or the universal mapping problem. It is also possible to treat the relation between the universal element and universal mapping through the representable functor. For example, the universal mapping property can be related to the problem of representable functor as follows.

Let $\mathcal{Mech}$ be a category of mechanisms (A, μ, f), $f : \mathcal{Econ} \times A \to X$. Define subcategory $\mathcal{D}$ as all objects of $\mathcal{Mech}$ that satisfies the axioms necessary to a dictionary theorem. Then $(\mathcal{P} \times \mathcal{M}, \pi, e)$ of $\mathcal{Mech}$ and a family of morphisms α_ν constitute a *cone* (Goldblatt 1979; p.58). If $(\mathcal{P} \times \mathcal{M}, \pi, e)$ is in $\mathcal{D}$, the cone becomes a *limit*, $\varprojlim \mathcal{D}$ (Kato 2006; p.22). Hence, price-money mechanism is characterized as the *unique* mechanism that can represent all message mechanisms having the dictionary property with several axioms. It is also possible to express the above arguments in terms of the notion of a *representable* functor (Kato 2006; pp.21-24).

The representation problem will be treated in our succeeding paper.

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