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On Attainable Set Compactness and Summation of Closed Sets in Debreu 1959 General Equilibrium Arguments without Using Asymptotic Cones

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Abstract

In this paper we present a simple and general condition for closed convex sets to assure the compactness of attainable set and closedness of their summation without using the asymptotic cone in Debreu (1959).

JEL classification: C60, D50

Key words: Asymptotic Cone, Salient Cone, Central Solution

1 Introduction

In general equilibrium existence argument, Debreu (1959) uses the concept of *asymptotic cone* to show the boundedness of attainable set and closedness of total production and/or consumption sets (see pp.41-42, p.53, pp.77-78). The arguments are important and repeatedly used in many parts of the proofs in chapters 5 (Existence) and 6 (Optimum). The mathematical prerequisites for this concept, however, is not straightforward in assuring such limit properties, and more direct and simple conditions would be desirable.

In this paper, we present elementary and general conditions (based on hyperplanes and salient cones) for closed convex sets to assure the compactness of attainable set and closedness of their summation without using the asymptotic cone.

2 The Model

The model in this paper is exactly the same with Debreu (1959). For the sake of convenience, we remark here several important concepts and notation.

The number ℓ of commodities is a given positive integer. An action of an agent, producer or

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consumer, is a point of R^ℓ, the commodity space. The set R^ℓ_+ is denoted by Ω. Denote by Δ the standard $\ell-1$ simplex and e^k be the element $(0,\ldots,0,1,0,\ldots,0)$, where 1 is the k-th coordinate. The number n of *producers* is a given positive integer. Each producer is indicated by an index $j=1,2,\ldots,n$. The j-th producer chooses a point, his production or his supply y_j, in a given non-empty subset of R^ℓ, his production set Y_j. Given a production y_j for each producer, $y=\sum_{j=1}^n y_j$ is called the total production or the total supply; the set $Y=\sum_{j=1}^n Y_j$ is called the total production set (Debreu p.39). The number m of *consumers* is a given positive integer. Each consumer is indicated by an index $i=1,2,\ldots,m$. The i-th consumer chooses a point, his consumption or his demand x_i, in a given non-empty subset of R^ℓ, his consumption set X_i. Given a consumption x_i for each consumer, $x=\sum_{i=1}^m x_i$ is called the total consumption or the total demand; the set $X=\sum_{i=1}^m X_i$ is called the total consumption set (Debreu p.52).

An *economy* E is defined by: for each $i=1,\ldots,m$, a non-empty subset X_i of R^ℓ completely preordered by $\precsim_i$; for each $j=1,\ldots,n$, a non-empty subset Y_j of R^ℓ; a point ω of R^ℓ. A *state* of E is an $(m+n)$-tuple of points of R^ℓ. Given a state $((x_i),(y_j))=((x_i)_{i=1}^m,(y_j)_{j=1}^n)$ of E, the point $x-y$ is the *net demand* and the point $z=x-y-\omega$ is called the *excess demand*. Set Z denotes $X-Y-\{\omega\}$ (Debreu p.75). A state $((x_i),(y_j))$ of E is said to be *attainable* if $x_i\in X_i$ for every i, $y_j\in Y_j$ for every j, $x-y=\omega$. The set of attainable states of E is denoted by A (Debreu p.76).

3 Theorems and Proofs

We use the following two conditions.

Condition 1: There exists $p\in R^\ell_{++}$ and $\alpha\in R$ such that $p\cdot y\leqq\alpha$ for all $y\in Y$. (See Figure 1.)

Condition 2: There exist a closed convex salient cone C with vertex 0, $C\cap\Omega=\{0\}$ and a semi-positive element $c\in\Omega\setminus\{0\}$ such that $Y\subset c+C$. (See Figure 2.)

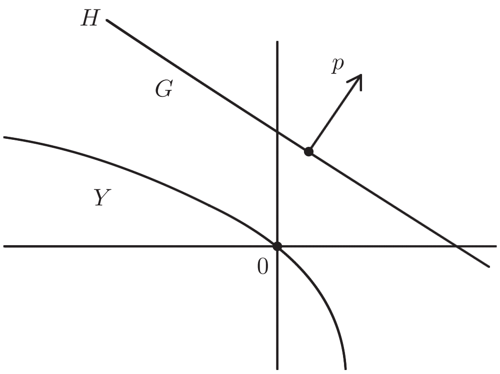

Figure 1: Condition 1

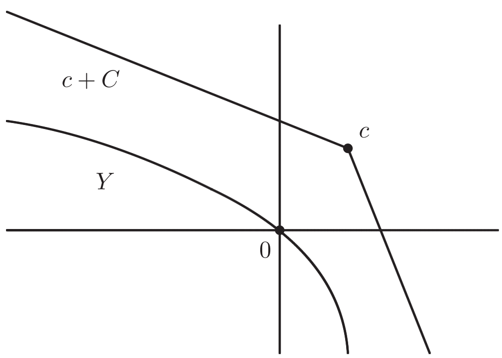

Figure 2: Condition 2

The closed half space determined by separating hyperplane H in Condition 1, $\{x \mid p \cdot x \leqq \alpha\}$, is denoted by G. Note that a cone C with vertex 0 is called *salient* if it contains no vector $x \neq 0$ such that $-x \in C$. Also consider the following two assumptions that are used in Debreu (1959) (see e.g., theorems in p.41, p.77, p.93).

Assumption 1: $Y \cap \Omega = \{0\}$.

Assumption 2: $Y \cap -Y \subset \{0\}$.

We have following two lemmas.

Lemma 1: *Assumption 1 implies Condition 1.*

Proof: Let us consider set $Y - \Delta$, which is convex, closed and contains $-\Delta$ by Assumption 1. Since Δ and $Y - \Delta$ has no common element, there exists a hyperplane H with normal vector $p \in R^{\ell} \setminus \{0\}$ strictly separating them (see Second Separation Theorem of Schaefer 1971, p.65). By considering the fact that we can take $p = (p_1, \ldots, p_\ell)$ as $-p_k = p \cdot (-e^k) < p \cdot e^k = p_k$, so p to be strictly positive. Let α be the value $\max\{p \cdot x \mid x \in \Delta\}$. ■

Lemma 2: *Assumptions 1 and 2 imply Condition 2.*

Proof: Let Λ be the closed convex hull of $\Delta \cup \{0\}$. Set $T = Y + (-\Lambda)$ is closed and convex. By Assumption 1, T includes Y and $-\Lambda$, and $T \cap \Omega = \{0\}$. Indeed, if there exists $z \neq 0$, $z \in T \cap \Omega$, then z can be written as $z = y - a$, where $y \in Y$ and $a \in \Lambda$. Since $z \in \Omega$ and $a \in \Lambda$, we have $y = z + a \in \Omega$ and $y \neq 0$, a contradiction.

Moreover, by Assumptions 1 and 2, $T \cap -T = \{0\}$. Indeed, if $x \neq 0$ satisfies $x \in T \cap -T$, x can be written as $x = y_1 - a_1$, $y_1 \in Y$ and $a_1 \in \Lambda$ and $x = -y_2 + a_2$, $y_2 \in Y$ and $a_2 \in \Lambda$. Hence we have $y_1 + y_2 = a_1 + a_2$ and $(y_1 + y_2)/2 = (a_1 + a_2)/2$. Since Y and Λ are convex, $(y_1 + y_2)/2 \in Y$ and $(a_1 + a_2)/2 \in \Lambda$. Note that $Y \cap \Lambda = \{0\}$, so we have $y_1 + y_2 = a_1 + a_2 = 0$. It follows that $a_1 = a_2 = 0$ and $x = y_1 = -y_2$. However, this implies that there exist $x \neq 0$ such that $x \in Y \cap -Y$, a contradiction.

Since $T \cap -T = \{0\}$ and T has an interior point in R^{ℓ}, there exists a hyperplane H with normal vector $p \in R^{\ell}$ that separates T and $-T$ (see First Separation Theorem of Schaefer 1971, p.64). Since T includes $\Lambda \ni 0$, p can be taken as non-negative and $H = \{x \mid p \cdot x = 0\}$.

Let S be the $1/\ell$-ball and consider set $\Gamma = \{x \mid x = \alpha z, z \in T, \|z\| \geqq 1, 0 \leqq \alpha \leqq 1\}$. The set Γ is a subset of T and hence the closed convex hull Γ^* of Γ is also a subset of T. Let C be the closed convex cone with vertex 0 generated by $\Gamma^* \cap S$. Note that $C \cap -C = \{0\}$ since for some $x \neq 0$, x and $-x$ belong to C implies that both $x/\|x\|$ and $-x/\|x\|$ belong to $\Gamma^* \cap S$, which is impossible since $(\Gamma^* \cap S) \subset T$. Hence C is closed convex salient cone with vertex 0 including all elements in $T \setminus S$.

Note that the intersection of $1/\ell$-ball and $-\Omega$ is a subset of $-\Lambda$, C necessarily includes $-\Omega$.

Then we take $c \in R^{\ell}$ as $c = (1, 1, \ldots, 1)$ and $c + C$ includes $T \supset Y$. Note that $C \cap \Omega = \{0\}$ follows from $C \supset -\Omega$ and $C \cap -C = \{0\}$. ■

Lemma 3: *Condition 2 implies Condition 1.*
Proof: Use Theorem 3.6 of Nikaido (1968; p.35). ■

As we see in Nikaido (1968; Theorem 3.7 and its Corollaries 1-3, pp.36-39), the existence of such a strictly positive vector is closely related to important results on linear inequalities like the central solution in Tucker (1956).

The next theorem is a generalization of arguments (1) and (2) in Debreu (1959; p.77).

Theorem 1 (Attainable Set Compactness): *Assume that every X_i is closed and has a lower bound for $\leqq$, and every Y_j is closed. Then the attainable set A is compact if for $n = 1$, Condition 1 is satisfied or in general, Condition 2 is satisfied (cf. Debreu p.77).*
Proof: The closedness of attainable set A is immediately followed from the facts that the product of closed sets is closed and the intersection of two closed sets is closed (see Debreu p.77). Therefore, the problem is the boundedness of A. To this purpose, we will show that the set of all points (x, y) with the property $x = y + \omega$ is bounded in $R^{\ell} \times R^{\ell}$.

Since every X_i has a lower bound for $\leqq$, $X = \sum_{i=1}^{m} X_i$ is bounded from below. Then there exists $b \in R^{\ell}$ such that $X \subset B = \{x \mid x \geqq b\}$. If $((x_i), (y_j))$ is attainable, we have $x = y + \omega$, $x \in X \subset B$ and $y \in Y \subset c + C$ from Condition 2 (or when $n = 1$, $y \in Y \subset G$ from Condition 1). We can write $-\omega + x = y$ and the left side is bounded from below since $-\omega$ is constant and $x \in B$ for all x with $((x_i), (y_j)) \in A$, i.e., $\omega + b \leqq y$ for all y with $((x_i), (y_j)) \in A$. Therefore, $y \in (-\omega + b + \Omega) \cap (c + C)$ from Condition 2 (or when $n = 1$, $y \in (-\omega + b + \Omega) \cap G$ from Condition 1). Since $(-\omega + b + \Omega) \cap (c + C)$ is compact, all y with $((x_i), (y_j)) \in A$ is bounded (or when $n = 1$, replace $c + C$ with G). Hence the right side, $-\omega + x$ is also bounded and the boundedness of (x, y) with $((x_i), (y_j)) \in A$ is established.

Next, we will show the boundedness of every attainable $y_j \in Y_j$ (for $n = 1$, we have nothing to prove since $y = y_1$). Since $Y = \sum_{j=1}^{n} Y_j$, we have $c_j \in R^{\ell}$ such that $Y_j \subset c_j + C$ for each $j = 1, \ldots, n$ from Condition 2. Indeed, we have $Y \subset c + C$ and $Y = \sum_{j=1}^{n} Y_j$. Then, by taking $y_h \in Y_h$ for each $h = 1, 2, \ldots, n$, $h \neq j$, and denote the sum $\sum_{h \neq j, h=1}^{n} y_h$ by y_{-j}, we have $y_{-j} + Y_j \subset Y \subset C$. That is, $Y_j \subset -y_{-j} + C$. Since $C + (-\Omega)$ is also a closed convex salient cone whose intersection with Ω is $\{0\}$, it is always possible to assume that $C \supset -\Omega$. Moreover, if $C \supset -\Omega$, $c \leqq c^*$ always means that $c + C \subset c^* + C$. Hence we can assume without loss of generality that c in Condition 2 is sufficiently large so that $c + C$ includes all Y_j and Y.

Now we suppose that for some j, there exists unbounded sequence y_j^{ν} ($\nu = 1, 2, \ldots$) with $y_{-j}^{\nu} = \sum_{h=1}^{n} y_h^{\nu} - y_j^{\nu}$ such that $y^{\nu} = \sum_{h=1}^{n} y_h^{\nu} \in Y$ satisfies $((x_i^{\nu}), (y_j^{\nu})) \in A$ for some $x^{\nu} \in X$. Note that for $y_j^{\nu} + y_{-j}^{\nu} = y^{\nu}$, the two terms in the left side are not bounded and the right side is bounded. Let us compare $\|y_j^{\nu}\|$ and $\|y_{-j}^{\nu}\|$ for all ν. In the following, we prove the case that $\|y_j^\nu\| \geqq \|y_{-j}^\nu\|$ for infinitely many ν. (The proof of the case with $\|y_j^\nu\| \leqq \|y_{-j}^\nu\|$ for infinitely many ν is exactly the same.) By taking a subsequence, we can suppose that sequence y_j^ν satisfies $\|y_j^\nu\| \geqq \|y_{-j}^\nu\|$ for all ν without loss of generality. If we normalize $(y_j^\nu - c) + (y_{-j}^\nu - c) = y^\nu - 2c$ by dividing the both sides with $\|y_j^\nu - c\|$, the first term of left side, $(y_j^\nu - c)/\|y_j^\nu - c\|$, converges to an element of salient cone C whose norm is equal to 1. Moreover, by further taking a subsequence, the second term of left side, $(y_{-j}^\nu - c)/\|y_j^\nu - c\|$, converges to an element of C whose norm is less than or equal to 1. Since $y^\nu \in Y$ is bounded and the sequence y_j^ν is unbounded, the right side, $y^\nu - 2c/\|y_j^\nu - c\|$, converges to 0. Therefore, from the closedness of C, the two terms of left side converge in C, and it implies that the summation of non-zero two points of C is equal to C. This is a contradiction to the fact that cone C is salient, and hence y_j is bounded for all j.

The remaining problem to be shown is the boundedness of each X_i. Since all X_i are bounded from below, if X_i is not bounded for some i, it is not bounded from above. Then there is a sequenve $x_i^\nu \in X_i$, $\nu = 1, 2, \ldots$, such that some coordinate k of x_i^ν tends to infinity. Since b is a lower bound of X_i, the equation $x = y + \omega$ implies that there exists a sequence $y^\nu \in Y$ such that the k-th coordinate of y^ν tends to infinity. However, this contradicts to the boundedness of y with $((x_i), (y_j)) \in A$. ■

A condition for assuring the closedness of the summation of closed sets as in Debreu (1959) can also be obtained through Condition 2. See the following two theorems. (Note that the next theorem cannot be identified with a generalization of Debreu p.41 (1). The restrictions $C \cap \Omega = \{0\}$ and $c \in \Omega \setminus \{0\}$ in Condition 2 must be removed. In the proof of Theorem 2, conditions $C \cap \Omega = \{0\}$ and $c \in \Omega \setminus \{0\}$ are not used.)

Theorem 2: *If every Y_j is closed and convex, and if Condition 2 is satisfied, then Y is closed (cf. Debreu p.41 (1)).*

Proof: As wee see in the third paragraph of the proof of Theorem 1, we can assume without loss of generality that $Y_j \subset c + C$ for each $j = 1, \ldots, n$ under Condition 2. Suppose that there exists a sequence $y^\nu \in Y$, $\nu = 1, 2, \ldots$, converging to $y^* \in R^\ell$. If y_j^ν is unbounded for some j, as in the same way with the fourth paragraph of the previous proof, there exists an unbounded sequence y_j^ν ($\nu = 1, 2, \ldots$) with $y_{-j}^\nu = \sum_{h=1}^n y_h^\nu - y_j^\nu$. Note that for $y_j^\nu + y_{-j}^\nu = y^\nu$, the two terms in the left side are not bounded and the right side is bounded. Let us compare $\|y_j^\nu\|$ and $\|y_{-j}^\nu\|$ for all ν. In the following, we prove the case that $\|y_j^\nu\| \geqq \|y_{-j}^\nu\|$ for infinitely many ν. (The proof of the case with $\|y_j^\nu\| \leqq \|y_{-j}^\nu\|$ for infinitely many ν is exactly the same.) By taking a subsequence, we can suppose that sequence y_j^ν satisfies $\|y_j^\nu\| \geqq \|y_{-j}^\nu\|$ for all ν without loss of generality. If we normalize $(y_j^\nu - c) + (y_{-j}^\nu - c) = y^\nu - 2c$ by dividing the both sides with $\|y_j^\nu - c\|$, the first term of left side, $(y_j^\nu - c)/\|y_j^\nu - c\|$, converges to an element of salient cone C whose norm is equal to 1. Moreover, by further taking a subsequence, the second term of left side, $(y_{-j}^\nu - c)/\|y_j^\nu - c\|$, converges to an element of C whose norm is less than or equal to 1. Since $y^\nu \in Y$ is bounded and the sequence y_j^ν is unbounded, the right side, $y^\nu - 2c/\|y_j^\nu - c\|$, converges to 0. Therefore, from the closedness of C, the two terms of left side converge in C, and it implies that the summation of non-zero two points of C is equal to C. This is a contradiction to the fact that cone C is salient, and hence we can assume that y_j^ν is bounded for all j.

If y_j^ν is bounded for all j, each sequence y_j^ν belongs to a compact subset of Y_j for all j. By taking a subsequence repeatedly, it is a routine task to check that $y^* \in Y$.

Theorem 3: *If every X_i is closed and has a lower bound for $\leqq$, then X is closed (cf. Debreu p.53 (1)).*

Proof: From the assumption, there exists an upper bound b_i of $-X_i$ for each $i = 1, \ldots, m$. Then we take $b \in R_{++}^\ell$ such that $b_i \leqq b$ for each $i = 1, \ldots, m$ and $\sum_{i=1}^m b_i \leqq b$. Replace $c,\ Y_j,\ C$ with b, $X_i,\ b + (-\Omega)$, respectively in Theorem 2. Since $-\Omega$ is a salient cone, Theorem 2 shows the closedness of $-X = \sum_{i=1}^m (-X_i)$, and hence X is also closed. ■

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