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# BEURLING'S THEOREM FOR NILPOTENT LIE GROUPS

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**Abstract**

In this paper, we prove an analogue of Beurling's theorem for an arbitrary simply connected nilpotent Lie group extending then earlier cases.

## 1. Introduction

It is a well-known fact in classical Fourier analysis that a function $f$ and its Fourier transform $\hat{f}$, defined on $\mathbb{R}$ by:

$$\hat{f}(x) = \int_{\mathbb{R}} f(y) e^{2i\pi xy}\, dy, \quad x \in \mathbb{R},$$

cannot simultaneously decay very rapidly. As illustrations of this, one has Beurling's theorem, the Gelfand–Shilov theorem, the Cowling–Price theorem, Hardy's theorem, etc. (see [7], [9], [11], [17] and references therein). The Beurling's theorem, for the real line, can be stated as follows:

**Theorem 1.1** (Beurling). *Let* $f \in L^2(\mathbb{R})$, *such that*:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)|\, |\hat{f}(y)| e^{2\pi|xy|}\, dx\, dy < +\infty. \tag{1.1}$$

*Then,* $f = 0$ *almost everywhere.*

This result is actually generalized by Bonami et al. [1]:

**Theorem 1.2** (Bonami, Demange, Jaming). *Let* $f \in L^2(\mathbb{R})$ *and* $N \geq 0$. *Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)|\, |\hat{f}(y)|}{(1 + |x| + |y|)^N} e^{2\pi|xy|}\, dx\, dy < +\infty \tag{1.2}$$

*implies* $f(x) = P(x)e^{-tx^2}$, *where* $t > 0$ *and* $P$ *is polynomial function with deg* $P < (N-1)/2$.

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Considerable attention has been paid to prove analogues of previous theorems in setup of non-commutative Lie groups. Specifically, analogue of Theorem 1.2 have been established for semi-simple Lie groups (see [16]), for the $n$-dimensional Euclidean motion group (see [13]) and for Riemannian symmetric spaces (see [14]). The perfect symmetry of conditions (1.1) and (1.2) is a serious obstacle in establishing analogues of Theorem 1.1 and Theorem 1.2 for an arbitrary nilpotent Lie groups. However, some attempts to generalize these theorems to special classes of nilpotent Lie groups have already been made (see [2], [3] and [15]). The aim of this paper is to prove the following analogue of Theorem 1.1 for an arbitrary connected and simply connected nilpotent Lie group.

**Theorem 1.3.** *Let $G$ be a connected simply connected nilpotent Lie group. Let $f$ be a function on $L^2(G)$ such that*:

$$\int_{\mathcal{W}} \int_G |f(x)| \, \|\pi_l(f)\|_{HS} \, e^{2\pi \|x\| \, \|l\|} |Pf(l)| \, dx \, dl < +\infty. \tag{1.3}$$

*Then, $f = 0$ almost everywhere.*

Here $\mathcal{W}$ is a suitable cross-section for the generic coadjoint orbits in $\mathfrak{g}^*$, the vector space dual of $\mathfrak{g}$. To prove our main result, we need to compute the Hilbert–Schmidt norm and the matrix coefficients of the group Fourier transform $\pi_l(f)$. The third section of the paper is devoted to these computations. In Section 4 we present an explicit proof of our main result. In Section 5 we indicate how the other uncertainty principles follow from our main result.

## 2. Preliminaries and notations

In this section we are going to review some useful facts and notations for a nilpotent Lie group. We refer the reader to [10] for details.

**2.1. Coadjoint orbits.** Let $G = \exp \mathfrak{g}$ be a connected simply connected nilpotent Lie group. Let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$, i.e.,

$$\mathrm{ad}_{\mathfrak{g}}(X)Y = \mathrm{ad}(X)Y = [X, Y], \quad \forall X, Y \in \mathfrak{g}.$$

The group $G$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{Ad}_G$, i.e.,

$$\mathrm{Ad}_G(g)Y = \mathrm{Ad}(g)Y = e^{\mathrm{ad}(X)}Y, \quad g = \exp X \in G, \ Y \in \mathfrak{g},$$

and on $\mathfrak{g}^*$ by the coadjoint representation $\mathrm{Ad}_G^*$, i.e.,

$$\langle \mathrm{Ad}_G^*(g)l, X\rangle = \langle g \cdot l, X\rangle = \langle l, \mathrm{Ad}_G(g^{-1})X\rangle, \quad g \in G, \ l \in \mathfrak{g}^*, \ X \in \mathfrak{g}.$$

The set $G \cdot l = \{g \cdot l,\ g \in G\} =: \mathcal{O}_l$ is called the coadjoint orbit of $l$. Let $\mathfrak{g}(l) = \{X \in \mathfrak{g},\ \langle l, [X, \mathfrak{g}]\rangle = \{0\}\}$ be the stabilizer of $l \in \mathfrak{g}^*$ in $\mathfrak{g}$, it's also the Lie algebra of $G(l) = \{g \in G,\ g \cdot l = l\}$. For $\mathfrak{p} \subset \mathfrak{g}$, define $\mathfrak{p}^\perp = \{f \in \mathfrak{g}^*,\ f|_\mathfrak{p} = 0\}$, the annihilator of $\mathfrak{p}$ in $\mathfrak{g}^*$. We say that the coadjoint orbit $\mathcal{O}_l$ of $l \in \mathfrak{g}^*$ is saturated with respect to a one codimensional ideal $\mathfrak{g}_0 = \operatorname{Lie} G_0$ in $\mathfrak{g}$, if $\mathfrak{g}(l) \subset \mathfrak{g}_0$. In such case, we have that $G \cdot l = G \cdot l + \mathfrak{g}_0^\perp$ and $\dim(G_0 \cdot l_0) = \dim(G \cdot l) - 2$, $l_0 = l|_{\mathfrak{g}_0}$.

**2.2. Induced representations.** The irreducible unitary representations of the group $G$ are obtained in the following way: Let $l \in \mathfrak{g}^*$ (dual of $\mathfrak{g}$) and let $\mathfrak{h} = \mathfrak{h}(l)$ be a polarization for $l$ in $\mathfrak{g}$, i.e., a subalgebra $\mathfrak{h}$ of maximal dimension such that:

$$l([\mathfrak{h}, \mathfrak{h}]) = 0.$$

So we can consider the unitary character $\chi_l$ of $H = \exp \mathfrak{h}$ associated to $l$ defined by:

$$\chi_l(\exp X) = e^{2i\pi\langle l, X\rangle}, \quad X \in \mathfrak{h}.$$

We consider the space:

$$K_{\chi_l}(G) = \{F : G \to \mathbb{C},\ \text{continuous and with compact support modulo } H \\ \text{such that}: F(hg) = \chi_l(h)F(g),\ \forall (g, h) \in G \times H\}.$$

If $F$ is in $K_{\chi_l}(G)$, the mapping $g \mapsto |F(g)|^2$ belongs to $C_c(H/G)$. This relation allows us to define an $L^2$-norm on $K_{\chi_l}(G)$ in the following way:

$$\|F\|_2 = \left(\int_{H/G} |F(g)|^2\, d\nu(\dot{g})\right)^{1/2},$$

where $\nu$ is the unique $G$-invariant measure on $H/G$. The induced representation $\pi_l = \pi_{l,\mathfrak{h}} = \operatorname{Ind}_H^G \chi_l$ is defined by letting $G$ act on the right on the completion $L^2(H/G, \chi_l)$ of $C_c(H/G)$ with respect to the norm $\|\ .\ \|$ defined above, i.e.,

$$(\operatorname{Ind}_H^G \chi_l)(x)(\xi)(y) = \xi(yx), \quad \forall x, y \in G,\ \xi \in L^2(H/G, \chi_l).$$

**2.3. The orbit theory.** The induced representation $\pi_{l,\mathfrak{h}}$ is irreducible and unitary. Different polarizations for the same $l$ give equivalent representations. In addition, two different linear forms $l$ an $l'$ give equivalent representations if and only if they belong to the same coadjoint orbit. Let $\mathfrak{g}^*/G$ denote the orbit space with the quotient topology and let $\hat{G}$ denote the set of equivalence classes of unitary irreducible representations of $G$. If $\hat{G}$ is endowed with an appropriate topology (Fell topology), then $\mathfrak{g}^*/G$ and $\hat{G}$ are homeomorphic under the map $\mathcal{O}_l \mapsto [\pi_l]$, where $[\pi_l]$ denotes the equivalent class of $\pi_l$ (see [10] and [12]).

**2.4. The Plancherel formula.** Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ and $\mathfrak{g}_j = \mathbb{R}\text{-span}\{X_1, \ldots, X_j\}$. For $l \in \mathfrak{g}^*$, $\mathcal{O}_l$ denotes the coadjoint orbit of $l$. An index $j \in \{1, \ldots, n\}$ is a jump index for $l$ if

$$\mathfrak{g}(l) + \mathfrak{g}_j \neq \mathfrak{g}(l) + \mathfrak{g}_{j-1}.$$

We let

$$e(l) = \{j : j \text{ is a jump index for } l\}.$$

This set contains exactly $\dim(\mathcal{O}_l)$ indices, which is necessarily an even number. Even more, there are two disjoint sets of indices $S, T$ with $S \cup T = \{1, \ldots, n\}$, and a $G$-invariant Zariski open set $\mathcal{U}$ of $\mathfrak{g}^*$ (set of generic elements in the sense of Pukanszky) such that $e(l) = S$ for all $l \in \mathcal{U}$. Define the Pfaffian $Pf(l)$ of the skew-symmetric matrix $M_S(l) = (l([X_i, X_j]))_{i,j \in S}$. Then, one has that

$$|Pf(l)|^2 = \det M_S(l).$$

Let $V_T = \mathbb{R}\text{-span}\{X_i^*; i \in T\}$, $V_S = \mathbb{R}\text{-span}\{X_i^*; i \in S\}$ and $dl$ be the Lebesgue measure on $V_T$ such that the unit cube spanned by $\{X_i^*; i \in T\}$ has volume 1. Then $\mathfrak{g}^* = V_T \oplus V_S$, $V_T$ meets $\mathcal{U}$ and $\mathcal{W} = \mathcal{U} \cap V_T$ is a cross section for the coadjoint orbits through points in $\mathcal{U}$. So, every $G$-orbit in $\mathcal{U}$ related to a representation $\pi$ meets $\mathcal{W}$ in a single unique element. Furthermore, if $dl$ is the Lebesgue measure on $\mathcal{W}$, then $d\mu = |Pf(l)|\, dl$ is a Plancherel measure for $\hat{G}$. Let $dg$ be the Haar measure on $G$, then the Plancherel formula reads:

$$\|\varphi\|_2^2 = \int_G |\varphi(g)|^2\, dg = \int_{\mathcal{W}} \|\pi_l(\varphi)\|_{HS}^2\, d\mu(l), \quad \varphi \in L^1(G) \cap L^2(G)$$

where $\pi_l(\varphi) = \int_G \varphi(g)\pi_l(g)\, dg$ and $\|\pi_l(\varphi)\|_{HS}$ denotes the Hilbert–Schmidt norm of $\pi_l(\varphi)$.

**2.5. Euclidean norms on nilpotent Lie groups.** Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$. We introduce a *norm function* on $G$ by setting, for $x = \exp(x_1X_1 + \cdots + x_nX_n) \in G$, $x_j \in \mathbb{R}$:

$$\|x\| = \sqrt{(x_1^2 + \cdots + x_n^2)}.$$

The composed map:

$$\mathbb{R}^n \to \mathfrak{g} \to G, \quad (x_1, \ldots, x_n) \mapsto \sum_{j=1}^{n} x_j X_j \mapsto \exp\left(\sum_{j=1}^{n} x_j X_j\right)$$

is a diffeomorphism and maps the Lebesgue measure on $\mathbb{R}^n$ to the Haar measure on $G$. In this setup, we shall identify $\mathfrak{g}$ and $G$, as sets with $\mathbb{R}^n$. We shall also identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ via the map $\xi = (\xi_1, \ldots, \xi_n) \mapsto \sum_{j=1}^n \xi_j X_j^*$. We consider the euclidean norm of $\mathfrak{g}^*$ with respect to the basis $\mathcal{B}^*$, that is,

$$\left\| \sum_{j=1}^n \xi_j X_j^* \right\| = \sqrt{(\xi_1^2 + \cdots + \xi_n^2)} = \|\xi\|.$$

REMARK 2.5.1. The condition (1.3) in Theorem 1.3, implies that: for any strong Malcev bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $\mathfrak{g}$,

$$\int_{\mathcal{W}_{\mathcal{B}_2}} \int_G |f(x)| \, \|\pi_l(f)\|_{HS} \, e^{2\pi \|x\|_{\mathcal{B}_1} \|l\|_{\mathcal{B}_2}} |Pf(l)| \, dx \, dl < +\infty.$$

**2.6. The generalized Minkowski inequality.** We recall the generalized Minkowski inequality for integrals. For $r \geq 1$, for two measure spaces $(X, \mu)$, $(Y, \nu)$ we have for any measurable function $F\colon X \times Y \to \mathbb{C}$

$$\left( \int_X \left( \int_Y |F(x, y)| \, d\nu(y) \right)^r d\mu(x) \right)^{1/r} \leq \int_Y \left( \int_X |F(x, y)|^r \, d\mu(x) \right)^{1/r} d\nu(y).$$

## 3. Group Fourier transform

We consider two cases:

FIRST CASE: We suppose that $\mathfrak{g}(l) \subset [\mathfrak{g}, \mathfrak{g}]$ for all $l \in \mathcal{U}$. Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ passing through $[\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{g}_i = \mathbb{R}\text{-span}\{X_1, \ldots, X_i\}$, $1 \leq i \leq n$. Then, all the general position orbits are saturated with respect to $\mathfrak{g}_{n-1}$. So, for $l \in \mathcal{U}$, the coadjoint orbit $\mathcal{O}_l$ of $l$ is given by:

$$\mathcal{O}_l = \mathcal{O}_l + \mathfrak{g}_{n-1}^{\perp} \equiv \mathcal{O}_l|_{\mathfrak{g}_{n-1}} + \mathfrak{g}_{n-1}^{\perp}.$$

Let's define different stabilizers:

$$\begin{aligned} G(l) &= \exp \mathfrak{g}(l) = \{g \in G\colon \mathrm{Ad}^*(g)(l) = l\}, \\ \mathfrak{g}_{n-1}^l &= \{X \in \mathfrak{g}\colon \langle l, [X, \mathfrak{g}_{n-1}] \rangle \equiv 0\}, \\ G_{n-1}^l &= \exp(\mathfrak{g}_{n-1}^l) = \{g \in G\colon \mathrm{Ad}^*(g)(l)|_{\mathfrak{g}_{n-1}} = l|_{\mathfrak{g}_{n-1}}\}. \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{O}_l &= \{\mathrm{Ad}^*(g^{-1})(l)\colon g \in G(l)/G\} \\ &= \{\mathrm{Ad}^*(g^{-1})(l)|_{\mathfrak{g}_{n-1}} + q\colon g \in G_{n-1}^l/G, \ q \in \mathfrak{g}_{n-1}^{\perp}\}. \end{aligned}$$

Furthermore, the map $G^l_{n-1}/G \times \mathfrak{g}^\perp_{n-1} \to \mathcal{O}_l$: $(\dot{g}, q) \mapsto \mathrm{Ad}^*(g^{-1})(l)|_{\mathfrak{g}_{n-1}} + q$ is a parametrization of the orbit $\mathcal{O}_l$. We shall compute the Hilbert–Schmidt norm of the group Fourier transform $\pi_l(f)$. If $f \in L^1(G) \cap L^2(G)$, then $\pi_l(f)$ is a Hilbert–Schmidt operator and

$$\|\pi_l(f)\|^2_{HS} = \mathrm{tr}(\pi_l(f * f^*)) = \int_{\mathcal{O}_l} ((f * f^*) \circ \exp)^\wedge(q)\, d\theta_l(q), \tag{3.1}$$

where $\theta_l$ is the unique $G$-invariant measure on $\mathcal{O}_l$ (up to a constant). Identifying $\mathfrak{g}^\perp_{n-1}$ with $\mathbb{R}X^*_n$. As the measure $ds\, d\dot{g}$ is $G$-invariant on $G^l_{n-1}/G \times \mathbb{R}$,

$$\begin{aligned}
&\|\pi_l(f)\|^2_{HS}\\
&= \int_{G^l_{n-1}/G} \int_{\mathbb{R}} ((f * f^*) \circ \exp)^\wedge(\mathrm{Ad}^*(g^{-1})(l)|_{\mathfrak{g}_{n-1}} + sX^*_n)\, ds\, d\dot{g}\\
&= \int_{G^l_{n-1}/G} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathfrak{g}_{n-1}} f * f^*(\exp(Y + tX_n)) e^{2i\pi\langle \mathrm{Ad}^*(g^{-1})(l)|_{\mathfrak{g}_{n-1}}, Y\rangle} e^{2i\pi ts}\, dY\, dt\, ds\, d\dot{g}\\
&= \int_{G^l_{n-1}/G} \int_{\mathfrak{g}_{n-1}} f * f^*(\exp Y) e^{2i\pi\langle \mathrm{Ad}^*(g^{-1})(l)|_{\mathfrak{g}_{n-1}}, Y\rangle}\, dY\, d\dot{g}\\
&= \int_{G^l_{n-1}/G} \int_{\mathfrak{g}_{n-1}} \int_{\mathbb{R}} \int_{\mathfrak{g}_{n-1}} f(\exp(sX_n)\exp Z) f^*(\exp(-Z)\exp(-sX_n)\exp Y)\\
&\qquad\qquad \times e^{2i\pi\langle \mathrm{Ad}^*(g^{-1})(l)|_{\mathfrak{g}_{n-1}}, Y\rangle}\, dZ\, ds\, dY\, d\dot{g}\\
&= \int_{G^l_{n-1}/G} \int_{\mathfrak{g}_{n-1}} \int_{\mathbb{R}} \int_{\mathfrak{g}_{n-1}} f(\exp(sX_n)\exp Z) f^*(\exp(-Z)\exp Y \exp(-sX_n))\\
&\qquad\qquad \times e^{2i\pi\langle \mathrm{Ad}^*((g\exp(sX_n))^{-1})(l)|_{\mathfrak{g}_{n-1}}, Y\rangle}\, dZ\, ds\, dY\, d\dot{g}.
\end{aligned}$$

Using the right invariance of the measure $d\dot{g}$, we get that:

(3.2)

$$\begin{aligned}
\|\pi_l(f)\|^2_{HS} = \int_{G^l_{n-1}/G} \int_{\mathfrak{g}_{n-1}} \int_{\mathbb{R}} \int_{\mathfrak{g}_{n-1}} & f(\exp(sX_n)\exp Z) f^*(\exp(-Z)\exp Y \exp(-sX_n))\\
& \times e^{2i\pi\langle \mathrm{Ad}^*(g^{-1})(l)|_{\mathfrak{g}_{n-1}}, Y\rangle}\, dZ\, ds\, dY\, d\dot{g}.
\end{aligned}$$

Remark that, the Lebesgue measure $ds$ on $\mathfrak{g}^\perp_{n-1}$ and $d\dot{g}$ on $G^l_{n-1}/G$ are normalized in such a way that the equation (3.1) holds.

With respect to the basis $\mathcal{B}$, we will let $S = \{j_1 < \cdots < j_d\}$, $T = \{t_1 < \cdots < t_r\}$ denote the collection of jump and non-jump indices respectively. As $\mathfrak{g}(l) \subset [\mathfrak{g}, \mathfrak{g}]$ for all $l \in \mathcal{U}$, we have $j_d = n$ and $j_{d-1} = n - 1$. The index set $S_1$ for $G_{n-1}$ is equal to $S \setminus \{j_1, j_d\}$. Furthermore, $\mathfrak{g}(l)$ is a codimension one ideal in $\mathfrak{g}_{n-1}(l|_{\mathfrak{g}_{n-1}}) = \{X \in \mathfrak{g}_{n-1}\colon \langle l, [X, \mathfrak{g}_{n-1}]\rangle \equiv 0\}$ and $[X_{j_1}, \mathfrak{g}_{n-1}]$ lies in $\ker l$ for all $l \in \mathcal{U}$. Finally, $\mathfrak{g}_{n-1}(l|_{\mathfrak{g}_{n-1}}) = \mathfrak{g}(l) + \mathbb{R}X_{j_1}$ and $Pf(l) = l([X_{j_1}, X_n])\, Pf(l|_{\mathfrak{g}_{n-1}})$ (for more details, see [8]). Considering the coadjoint action of $G$ on $\mathfrak{g}^*$, we get parametrization of generic orbits in $\mathcal{U}$. From

Theorem 3.1.9 of [10], there is a diffeomorphism $\psi\colon \mathcal{U} \cap V_T \times V_S \to \mathcal{U}$ such that the Jacobian determinant is identically 1. If we identify $(l, \lambda) = \left(\sum_{i=1}^{r} l_i X_{t_i}^*, \sum_{i=1}^{d} \lambda_i X_{j_i}^*\right)$ with $(l_1, \ldots, l_r, \lambda_1, \ldots, \lambda_d) \in \mathbb{R}^r \times \mathbb{R}^d$, we have $\psi(l, \lambda) = \sum_{j=1}^{n} P_j(l, \lambda) X_j^*$, where:
(i) The $P_j$ are rational, non singular on $\mathcal{U} \cap V_T \times \mathbb{R}^d$.
(ii) If $j = t_i$, $P_j(l, \lambda) = l_i + R_j(l_1, \ldots, l_{i-1}, \lambda_1, \ldots, \lambda_k)$ where $k$ is the largest index such that $j_k < t_i$. Moreover, $P_1(l, \lambda) = l_1$.
(iii) $P_{j_i}(l, \lambda) = \lambda_i$, $1 \leq i \leq d$.
Let $S' = \{j_1, \ldots, j_{d-1}\}$, $V_{S'} = \mathbb{R}\text{-span}\{X_i^*\colon i \in S'\}$ and $\mathcal{U}' = \{l|_{\mathfrak{g}_{n-1}}\colon l \in \mathcal{U}\}$, then $\mathfrak{g}_{n-1}^* = V_T \oplus V_{S'}$. Since $P_j(l, \lambda)$, $1 \leq j \leq n-1$, does not depend on $\lambda_d$, the map $\mathcal{U} \cap V_T \times V_{S'} \to \mathcal{U}'\colon (l, \lambda_1, \ldots, \lambda_{d-1}) \mapsto \psi(l, \lambda_1, \ldots, \lambda_{d-1}, 0)$ is a diffeomorphism. We have the following lemma:

**Lemma 3.1.** *Let $\phi \in L^1(G_{n-1})$. For $q \in \mathfrak{g}_{n-1}^*$, let*

$$\hat{\phi}(q) = \int_{\mathfrak{g}_{n-1}} \phi(\exp X) e^{2i\pi q(X)}\, dX.$$

*Then,*

$$\int_{V_{S'}} |\hat{\phi}(\psi(l, \lambda))|^2\, d\lambda = \frac{|Pf(l)|^2}{|l([X_{j_1}, X_n])|} \int_{G_{n-1}^l/G} |\hat{\phi}(\mathrm{Ad}^*(g^{-1})(l|_{\mathfrak{g}_{n-1}}))|^2\, d\dot{g},$$

*where $d\lambda$ is the Lebesgue measure on $V_{S'}$.*

Proof. Let

$$S_0 = \{1 \leq i \leq n\colon \mathfrak{g}_{i-1} + \mathfrak{g}_{n-1}^l \neq \mathfrak{g}_i + \mathfrak{g}_{n-1}^l\}.$$

As $\mathfrak{g}_{n-1}^l = \mathfrak{g}_{n-1}(l|_{\mathfrak{g}_{n-1}})$ we have

$$\begin{aligned} S_0 &= \{1 \leq i \leq n\colon \mathfrak{g}_{i-1} + \mathfrak{g}_{n-1}(l|_{\mathfrak{g}_{n-1}}) \neq \mathfrak{g}_i + \mathfrak{g}_{n-1}(l|_{\mathfrak{g}_{n-1}})\} \\ &= \{1 \leq i \leq n-1\colon \mathfrak{g}_{i-1} + \mathfrak{g}_{n-1}(l|_{\mathfrak{g}_{n-1}}) \neq \mathfrak{g}_i + \mathfrak{g}_{n-1}(l|_{\mathfrak{g}_{n-1}})\} \cup \{j_d\} \\ &= S_1 \cup \{j_d\} = \{j_2 < \cdots < j_d\}. \end{aligned}$$

Let $C(l) = (l([X_i, X_j]))_{(i,j)\in S'\times S_0}$. Recall that $X_{j_1}$ is a central vector in $\mathfrak{g}_{n-1}$, then

$$C(l) = \begin{pmatrix} 0 & & & \\ 0 & & C'(l) & \\ \vdots & & & \\ l([X_{j_1}, X_{j_d}]) & * & \cdots & * \end{pmatrix}$$

where $C'(l) = (l([X_i, X_j]))_{(i,j)\in S_1\times S_1}$. From these computations one deduces that the Jacobian determinant of the map $G/G^l_{n-1} \to \{l\}\times V_{S'} : \dot{g} \mapsto \psi^{-1}(\mathrm{Ad}^*(g)(l|_{\mathfrak{g}_{n-1}}))$ is

$$\begin{aligned}|\det(\mathrm{Ad}^*(g)l([X_i, X_j]))_{(i,j)\in S'\times S_0}| &= |\det(l([X_i, X_j]))_{(i,j)\in S'\times S_0}| \\ &= |\det C(l)| = |l([X_{j_1}, X_{j_d}])|\,|\det C'(l)|.\end{aligned}$$

Then,

$$\int_{V_{S'}} |\hat{\phi}(\psi(l,\lambda))|^2\, d\lambda = |l([X_{j_1}, X_{j_d}])|\,|\det C'(l)| \int_{G/G^l_{n-1}} |\hat{\phi}(\mathrm{Ad}^*(g)(l|_{\mathfrak{g}_{n-1}}))|^2\, d\dot{g}.$$

As $Pf(l)^2 = |l([X_{j_1}, X_n])|^2\, Pf(l|_{\mathfrak{g}_{n-1}})^2 = |l([X_{j_1}, X_n])|^2\,|\det C'(l)|$, we get that

$$\int_{V_{S'}} |\hat{\phi}(\psi(l,\lambda))|^2\, d\lambda = \frac{|Pf(l)|^2}{|l([X_{j_1}, X_n])|} \int_{G/G^l_{n-1}} |\hat{\phi}(\mathrm{Ad}^*(g)(l|_{\mathfrak{g}_{n-1}}))|^2\, d\dot{g},$$

which is the desired formula. □

Thus from equation (3.2) and Lemma 3.1,

$$\begin{aligned}&\|\pi_l(f)\|^2_{HS} \\ &= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}}\int_{\mathfrak{g}_{n-1}}\int_{\mathbb{R}}\int_{\mathfrak{g}_{n-1}} f(\exp(sX_n)\exp Z)\, f^*(\exp(-Z)\exp Y\exp(-sX_n)) \\ &\qquad \times e^{2i\pi\langle\psi(l,\lambda),Y\rangle}\, dZ\, ds\, dY\, d\lambda.\end{aligned}$$

For $q \in \mathfrak{g}^*_{n-1}$, let $\chi_q$ be the function defined on $G_{n-1}$ by:

$$\chi_q(\exp X) = e^{2i\pi\langle q,X\rangle}, \quad X \in \mathfrak{g}_{n-1}.$$

Hence,

$$\begin{aligned}&\|\pi_l(f)\|^2_{HS} \\ &= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} f\left(\exp(sX_n)\exp\left(\sum_{i=1}^{n-1} z_i X_i\right)\right) \\ &\qquad \times f^*\left(\exp\left(\sum_{i=1}^{n-1} y_i X_i\right)\exp(-sX_n)\right) \\ &\qquad \times \chi_{\psi(l,\lambda)}\left(\exp\left(\sum_{i=1}^{n-1} z_i X_i\right)\exp\left(\sum_{i=1}^{n-1} y_i X_i\right)\right) dz\, ds\, dy\, d\lambda.\end{aligned}$$

Now, remark that

$$\exp\left(\sum_{i=1}^{n-1} z_i X_i\right) \exp\left(\sum_{i=1}^{n-1} y_i X_i\right) = \exp\left(\sum_{i=1}^{n-1} Q_i(z, y)X_i\right),$$

where, for $1 \leq i \leq n-1$, $Q_i(z, y)$ is a polynomial function depending on $z_i, \ldots, z_{n-1}, y_i, \ldots, y_{n-1}$. Furthermore, one can write

$$Q_i(z, y) = z_i + y_i + Q'_i(z_{n-1}, \ldots, z_{i+1}, y_{n-1}, \ldots, y_{i+1}), \tag{3.3}$$

where $Q'_i$ is a polynomial function depending on $z_{i+1}, \ldots, z_{n-1}, y_{i+1}, \ldots, y_{n-1}$. Moreover,

$$Q_{n-1}(z, y) = z_{n-1} + y_{n-1}.$$

It results that,

$$\begin{aligned}\|\pi_l(f)\|^2_{HS} &= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f\left(\exp(sX_n) \exp\left(\sum_{i=1}^{n-1} z_i X_i\right)\right) \\ &\quad \times f^*\left(\exp\left(\sum_{i=1}^{n-1} y_i X_i\right) \exp(-sX_n)\right) \\ &\quad \times e^{2i\pi\left(\sum_{j=1}^{n-1} Q_j(z,y)P_j(l,\lambda)\right)}\, dz\, ds\, dy\, d\lambda.\end{aligned}$$

Recall that $j_{d-1} = n-1$, so in view of equations (3.3), (ii) and (iii),

$$\begin{aligned}\|\pi_l(f)\|^2_{HS} &= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f\left(\exp(sX_n) \exp\left(\sum_{i=1}^{n-1} z_i X_i\right)\right) \\ &\quad \times f^*\left(\exp\left(\sum_{i=1}^{n-1} y_i X_i\right) \exp(-sX_n)\right) \\ &\quad \times e^{2i\pi\left(\sum_{k=1}^{r}(z_{t_k}+y_{t_k})l_k + \sum_{k=1}^{d-2}(z_{j_k}+y_{j_k})\lambda_k\right)} \\ &\quad \times e^{2i\pi((z_{n-1}+y_{n-1})\lambda_{d-1} + A(z,y,l,\lambda))}\, dz\, ds\, dy\, d\lambda,\end{aligned}$$

where $A(z, y, l, \lambda)$ is a real function depending on $z_2, \ldots, z_{n-1}, y_2, \ldots, y_{n-1}, l_1, \ldots, zl_r, \lambda_1, \ldots, \lambda_{d-2}$. On the other hand, we have

$$\exp(sX_n) \exp\left(\sum_{i=1}^{n-1} z_i X_i\right) = \exp\left(\sum_{i=1}^{n-2} H_i(s, z)X_i + z_{n-1}X_{n-1} + sX_n\right),$$

where, for $1 \le i \le n-2$, $H_i(s,z)$ is a real polynomial function depending on $z_i, \ldots, z_{n-1}, s$. In addition, one can write

$$H_i(s, z) = z_i + H_i'(s, z), \tag{3.4}$$

where $H_i'$ is a polynomial function depending only on the variables $z_{i+1}, \ldots, z_{n-1}, s$. It follows that,

$$\begin{aligned}
&\|\pi_l(f)\|_{HS}^2 \\
&= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(H_1(s, z), \ldots, H_{n-2}(s, z), z_{n-1}, s) \\
&\qquad \times f^*(-H_1(s, -y), \ldots, -H_{n-2}(s, -y), y_{n-1}, -s) \\
&\qquad \times e^{2i\pi\left(\sum_{k=1}^{r}(z_{t_k}+y_{t_k})l_k + \sum_{k=1}^{d-2}(z_{j_k}+y_{j_k})\lambda_k\right)} \\
&\qquad \times e^{2i\pi((z_{n-1}+y_{n-1})\lambda_{d-1}+A(z,y,l,\lambda))}\, dz\, ds\, dy\, d\lambda,
\end{aligned}$$

where $-y = (-y_1, \ldots, -y_{n-1})$. Then, by substituting $H_i(s,z)$ for $z_i$ (respectively $H_i(s,-y)$ for $-y_i$), by means of equation (3.4), $1 \le i \le n-2$, we get that:

$$\begin{aligned}
&\|\pi_l(f)\|_{HS}^2 \\
&= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(z_1, \ldots, z_{n-1}, s) f^*(y_1, \ldots, y_{n-1}, -s) \\
&\qquad \times e^{2i\pi\left(\sum_{k=1}^{r}(z_{t_k}+y_{t_k})l_k + \sum_{k=1}^{d-2}(z_{j_k}+y_{j_k})\lambda_k\right)} \\
&\qquad \times e^{2i\pi(A'(s,z,y,l,\lambda)+(z_{n-1}+y_{n-1})\lambda_{d-1})}\, dz\, ds\, dy\, d\lambda,
\end{aligned}$$

where $A'(s, z, y, l, \lambda)$ is a function depending only on the variables $s, z_2, \ldots, z_{n-1}, y_2, \ldots, y_{n-1}, l_1, \ldots, l_r, \lambda_1, \ldots, \lambda_{d-2}$. It results that,

$$\begin{aligned}
&\|\pi_l(f)\|_{HS}^2 \\
&= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(z_1, \ldots, z_{n-2}, z_{n-1} - y_{n-1}, s) \\
&\qquad \times f^*(y_1, \ldots, y_{n-1}, -s) \\
&\qquad \times e^{2i\pi\left(\sum_{k=1}^{r}(z_{t_k}+y_{t_k})l_k + \sum_{k=1}^{d-2}(z_{j_k}+y_{j_k})\lambda_k\right)} \\
&\qquad \times e^{2i\pi\left(\lambda_{d-1}z_{n-1}+A'(s,z_2,\ldots,z_{n-2},z_{n-1}-y_{n-1},y,l,\lambda)\right)}\, dz\, ds\, dy\, d\lambda \\
&= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(z_1, \ldots, z_{n-2}, z_{n-1} - y_{n-1}, s) \\
&\qquad \times f^*(y_1, \ldots, y_{n-1}, -s) \\
&\qquad \times e^{2i\pi\left(\sum_{k=1}^{r}(z_{t_k}+y_{t_k})l_k + \sum_{k=1}^{d-2}(z_{j_k}+y_{j_k})\lambda_k\right)} \\
&\qquad \times e^{2i\pi\lambda_{d-1}z_{n-1}}\, dz\, ds\, dy\, d\lambda
\end{aligned}$$

(by substituting $\lambda_{d-1}z_{n-1} + A'(s, z_2, \ldots, z_{n-2}, z_{n-1} - y_{n-1}, y, l, \lambda)$ for $\lambda_{d-1}z_{n-1}$).

Note that the Jacobian determinant of the previous substitution is equal to one, since the real function $A'(s, z_2, \ldots, z_{n-2}, z_{n-1} - y_{n-1}, y, l, \lambda)$ does not depend on $\lambda_{d-1}$. Now, by substituting $z_{n-1} - y_{n-1}$ for $z_{n-1}$, we have

$$\begin{aligned}\|\pi_l(f)\|_{HS}^2 &= \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(z_1, \ldots, z_{n-1}, s) f^*(y_1, \ldots, y_{n-1}, -s) \\ &\qquad \times e^{2i\pi\left(\sum_{k=1}^{r}(z_{t_k}+y_{t_k})l_k + \sum_{k=1}^{d-2}(z_{j_k}+y_{j_k})\lambda_k\right)} \\ &\qquad \times e^{2i\pi(\lambda_{d-1}(z_{n-1}+y_{n-1}))}\, dz\, ds\, dy\, d\lambda,\end{aligned}$$

and therefore

$$\|\pi_l(f)\|_{HS}^2 = \frac{|l([X_{j_1}, X_n])|}{Pf(l)^2} \int_{V_{S'}} \int_{\mathbb{R}} |\hat{f}_s(l, \lambda)|^2\, ds\, d\lambda, \tag{3.5}$$

where $f_s$ is the function defined on $G_{n-1}$ by:

$$f_s(\exp m) = f(\exp(m + sX_n)), \quad m \in \mathfrak{g}_{n-1}.$$

SECOND CASE: We suppose that $\mathfrak{g}(l) \not\subset [\mathfrak{g}, \mathfrak{g}]$ for all $l \in \mathcal{U}$. Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ passing through $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\text{-span}\{X_1, \ldots, X_a\}$ and through $\mathfrak{z}(\mathfrak{g}) + [\mathfrak{g}, \mathfrak{g}] = \mathbb{R}\text{-span}\{X_1, \ldots, X_m\}$. Let $\mathfrak{g}_j = \mathbb{R}\text{-span}\{X_1, \ldots, X_j\}$, $1 \le j \le n$. For $l$ in $\mathcal{W}$, let $\tilde{l}_j = l|_{\mathfrak{g}_j}$ and $\mathfrak{g}_j(\tilde{l}_j) = \{X \in \mathfrak{g}_j : \tilde{l}_j[X, \mathfrak{g}_j] \equiv 0\}$. Then, $\mathfrak{h}^l = \sum_{j=1}^{n} \mathfrak{g}_j(\tilde{l}_j)$ is a polarizing subalgebra for $l$ called Vergne's polarization. We shall compute the matrix coefficients of $\pi_l(f)$. For this we need to construct a weak Malcev basis $\mathcal{B}^l$ of $\mathfrak{g}$ passing through $\mathfrak{h}^l$. We proceed as in [5]. Let $\{X_1, \ldots, X_a\}$ the first $a$ vectors of the basis $\mathcal{B}^l$. For $a+1 \le j \le n$, construct the $j$-th basis vectors as follows: if $\mathfrak{g}_j(\tilde{l}_j) \subset \mathfrak{g}_{j-1}(\tilde{l}_{j-1})$, set $s_i = j$ and call $X_{s_i}$ the $i$-th external basis vector. We note that $X_{s_i}$ is not in $\mathfrak{h}^l$. Now, if $\mathfrak{g}_{j-1}(\tilde{l}_{j-1}) \subset \mathfrak{g}_j(\tilde{l}_j)$, set $t_r = j$. In this case, there is a vector in $\mathfrak{g}_j(\tilde{l}_j)$ of the form

$$Y_r^l = X_{t_r} + \sum_{k=1}^{t_r - 1} w_{t_r,k}(l) X_k,$$

where the coefficients $w_{t_r,k}(l)$ are rational functions depending on the components of $l|_{[\mathfrak{g},\mathfrak{g}]}$. We may further triangularize this basis by assuming that:

$$Y_r^l = X_{t_r} + \sum_{s_i < t_r} w_{t_r,s_i}(l) X_{s_i},$$

for a distinct indices $t_1, \ldots, t_b$. Remark that, the set of $l$ in $\mathfrak{g}^*$ such that the rational functions $w_{t_r,s_i}$ are not singular is a Zariski open set. Furthermore,

$$\{X_1, \dots, X_a, Y_1^l, \dots, Y_b^l, X_{s_1}, \dots, X_{s_c}\}$$

is a weak Malcev basis for $\mathfrak{g}$ passing through $\mathfrak{h}^l = \mathbb{R}\text{-span}\{X_1, \dots, X_a, Y_1^l, \dots, Y_b^l\}$. In fact, for each $1 \le r \le b$, the subspace $\mathbb{R}\text{-span}\{X_1, \dots, X_a, Y_1^l, \dots, Y_r^l\}$ is a subalgebra. On the other hand, as $\mathfrak{g}_{s_i}$ is an ideal, $\mathbb{R}\text{-span}\{X_1, \dots, X_{s_i}\} = \mathfrak{h}^l + \mathfrak{g}_{s_i}$ is also a subalgebra.

Now, we are going to calculate the action of the representation $\pi_{l,\mathfrak{h}^l} \simeq \pi_l$. We need some preliminary lemmas.

**Lemma 3.2** (see [5]). *The map $\gamma : \mathbb{R}^c \to G$ given by*:

$$\gamma(s) = \gamma(s_1, \dots, s_c) = \exp\left(\sum_{i=1}^{c} s_i X_{s_i}\right),$$

*is a cross-section for $H \setminus G$, $H = H(l) = \exp \mathfrak{h}^l$.*

**Lemma 3.3.** *There exists $r_0 \in T$ such that $X_{r_0} \notin [\mathfrak{g}, \mathfrak{g}]$.*

Proof. We have to treat two cases:

i) $\mathfrak{z}(\mathfrak{g}) \not\subset [\mathfrak{g}, \mathfrak{g}]$: it is easy to remark that there exists $r_0 \in \{1, \dots, a\} \subset T$ such that $X_{r_0} \notin [\mathfrak{g}, \mathfrak{g}]$.

ii) $\mathfrak{z}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$: since $\mathfrak{g}(l) \not\subset [\mathfrak{g}, \mathfrak{g}]$, there exists $r_0 > m$ such that

$$L = X_{r_0} + \sum_{k<r_0} c_k(l) X_k \in \mathfrak{g}(l),$$

where the coefficients $c_k(l)$ are rational non singular functions on $l$. Then, for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\lambda X_{r_0} + \mathfrak{g}_{r_0-1} + \mathfrak{g}(l) \\ &= \lambda\left(X_{r_0} + \sum_{k<r_0} c_k(l) X_k\right) - \lambda \sum_{k<r_0} c_k(l) X_k + \mathfrak{g}_{r_0-1} + \mathfrak{g}(l) \subset \mathfrak{g}_{r_0-1} + \mathfrak{g}(l), \end{aligned}$$

and so $r_0 \in T$. □

Let $\tilde{T} = \{1, \dots, a, t_1, \dots, t_b\}$ and $\tilde{S} = \{1, \dots, n\} \setminus \tilde{T} = \{s_1, \dots, s_c\}$. Then, $T \subset \tilde{T}$ and $\tilde{S} \subset S$. Using Lemma 3.3 and the Campbell–Baker–Hausdorff formula, we get the following two lemmas:

**Lemma 3.4.** *Let $x = \exp\left(\sum_{i=1}^{n} x_i X_i\right) \in G$ and $g = \gamma(g_{s_1}, \dots, g_{s_c})$ in the cross-section of $H \setminus G$. The product $gx$ is given by*:

$$gx = \exp\left(\sum_{k=1}^{n} Q_k(x, g) X_k\right),$$

*where the polynomials* $Q_k$ *satisfy*:

$$Q_{s_i}(x, y) = x_{s_i} + g_{s_i} + q_{s_i}(x_{s_i+1}, \dots, x_n, g_{s_{i+1}}, \dots, g_{s_c}),$$
$$Q_k(x, y) = x_k + q_k(x_{k+1}, \dots, x_n; g_{s_i} : s_i > k), \quad \text{if} \quad k \in \tilde{T}.$$

*Moreover,* $q_{r_0}(x_{r_0+1}, \dots, x_n; g_{s_i} : s_i > r_0) = 0$ *and* $Q_{r_0}(x, y) = x_{r_0}$.

**Lemma 3.5.** *Let* $\alpha = \exp\left(\sum_{k=1}^{a} \alpha_k X_k + \sum_{k=1}^{b} \alpha_{t_k} Y_k^l\right) \in \mathfrak{h}^l$ *and* $\beta = \exp\left(\sum_{i=1}^{c} \beta_{s_i} X_{s_i}\right)$ *in cross-section of* $H \setminus G$. *The product* $\alpha\beta$ *has polynomial coordinates on* $\alpha_i$, $\beta_{s_i}$ *and rational coordinates on the components of* $l|_{[\mathfrak{g},\mathfrak{g}]}$. *In addition,*

$$\alpha\beta = \exp\left(\sum_{k=1}^{n} P_k(\alpha, \beta, l) X_k\right),$$

*where*

$$P_{s_i}(\alpha, \beta, l) = \beta_{s_i} + p_{s_i}(\alpha_{t_r} : t_r > s_i; \beta_{s_{i+1}}, \dots, \beta_{s_c}; l),$$
$$P_k(\alpha, \beta, l) = \alpha_k + p_k(\alpha_i : i \in \tilde{T} \text{ and } i > k; \beta_{s_i} : s_i > k; l), \quad \text{if} \quad k \in \tilde{T} \setminus \{r_0\}$$

*and*

$$P_{r_0}(\alpha, \beta, l) = \alpha_{r_0}.$$

For $x = \exp\left(\sum_{i=1}^{n} x_i X_i\right) \in G$ and $g = \gamma(g_{s_1}, \dots, g_{s_c})$ in the cross-section of $H \setminus G$, we choose the unique $\alpha = \exp\left(\sum_{k=1}^{a} \alpha_k X_k + \sum_{k=1}^{b} \alpha_{t_k} Y_k^l\right) \in \mathfrak{h}^l$ and $\beta = \exp\left(\sum_{i=1}^{c} \beta_{s_i} X_{s_i}\right)$ in cross-section of $H \setminus G$ such that $gx = \alpha\beta$. Then for $l \in \mathcal{W}$, the action of the representation $\pi_l$ is given by:

$$\pi_l(x)\xi(g) = \xi(gx) = \xi(\alpha\beta) = e^{2i\pi l(\log \alpha)} \xi(\beta), \quad \xi \in L^2(\mathbb{R}^c). \tag{3.6}$$

We need to calculate $\alpha_k$ and $\beta_{s_i}$ in terms of the coordinates $x_k$ and $g_{s_i}$. In view of Lemma 3.4 and Lemma 3.5, we obtain:

$$Q_k(x, g) = P_k(\alpha, \beta, l), \quad \text{for} \quad k = 1, \dots, n.$$

Use this to solve for $\alpha$ and $\beta$. By triangular dependencies of the polynomials $Q_k$ and $P_k$, there are functions $A_k$ and $B_{s_i}$ (polynomial on $x_k$, $g_k$ and rational on $l|_{[\mathfrak{g},\mathfrak{g}]}$) such that:

$$\beta_{s_i}(x, g, l) = x_{s_i} + g_{s_i} + B_{s_i}(x_{s_i+1}, \dots, x_n, g_{s_{i+1}}, \dots, g_{s_c}, l), \tag{3.7}$$
$$\alpha_k(x, g, l) = x_k + A_k(x_{k+1}, \dots, x_n; g_{s_i} : s_i > k; l), \quad \text{for} \quad k \in \tilde{T} \setminus \{r_0\} \tag{3.8}$$

and

$$\alpha_{r_0}(x, g, l) = x_{r_0}. \tag{3.9}$$

It follows using equation (3.6) that:

$$\pi_l(x)\xi(g) = e^{2i\pi\left(\sum_{k=1}^{a}\alpha_k(x,g,l)l(X_k)+\sum_{k=1}^{b}\alpha_{t_k}(x,g,l)l(Y_k^l)\right)}\xi(\beta_{s_1}(x,g,l),\dots,\beta_{s_c}(x,g,l)).$$

Now, since $\tilde{S} \subset S$,

$$l(Y_k^l) = l\left(X_{t_k} + \sum_{s_i<t_k} w_{t_k,s_i}(l)X_{s_i}\right) = l(X_{t_k}) = \begin{cases} 0 & \text{if} \quad t_k \in \tilde{T} \setminus T, \\ l_{t_k} & \text{if} \quad t_k \in T. \end{cases}$$

It results that,

$$\pi_l(x)\xi(g) = e^{2i\pi\left(x_{r_0}l_{r_0}+\sum_{k\in T\setminus\{r_0\}}\alpha_k(x,g,l)l_k\right)}\xi(\beta_{s_1}(x,g,l),\dots,\beta_{s_c}(x,g,l)). \tag{3.10}$$

The next step consists in integrating the representation $\pi_l$. The matrix coefficients of $\pi_l(f)$ is given by: for $\xi, \eta \in \mathcal{H}_{\pi_l}$,

$$\begin{aligned}\langle\pi_l(f)\xi,\eta\rangle &= \int_G f(x)\langle\pi_l(x)\xi,\eta\rangle\,dx \\ &= \int_G\int_{H/G} f(x)\pi_l(x)\xi(g)\bar{\eta}(g)\,dg\,dx.\end{aligned}$$

Using equation (3.10),

$$\begin{aligned}\langle\pi_l(f)\xi,\eta\rangle = \int_{\mathbb{R}^{n+c}} & f(x_1,\dots,x_n)\xi(\beta_{s_1}(x,g,l),\dots,\beta_{s_c}(x,g,l))\bar{\eta}(g) \\ & \times e^{2i\pi\left(x_{r_0}l_{r_0}+\sum_{k\in T\setminus\{r_0\}}\alpha_k(x,g,l)l_k\right)}\,dg\,dx.\end{aligned}$$

Now, by substituting $\alpha_k(x,g,l)$ for $x_k$, $k \in T\setminus\{r_0\}$, using equation (3.8),

$$\begin{aligned}\langle\pi_l(f)\xi,\eta\rangle = \int_{\mathbb{R}^{n+c}} & e^{2i\pi\left(\sum_{k\in T} l_k x_k\right)} f(R_1(x,g,l),\dots,R_n(x,g,l)) \\ & \times \xi(D_{s_1}(x,g,l),\dots,D_{s_c}(x,g,l))\bar{\eta}(g)\,dg\,dx,\end{aligned}$$

where

$$D_{s_i}(x,g,l) = x_{s_i} + g_{s_i} + d_{s_i}(x_{s_i+1},\dots,x_n,g_{s_{i+1}},\dots,g_{s_c},l)$$

($d_{s_i}$ is a polynomial function on $x_{s_i+1},\dots,x_n,g_{s_{i+1}},\dots,g_{s_c}$ and a rational function on $l|_{[\mathfrak{g},\mathfrak{g}]}$),

$$R_k(x,g,l) = x_k, \quad \text{if} \quad k \in S\cup\{r_0\}$$

and

$$R_k(x,g,l) = x_k - A_k(x_{k+1},\dots,x_n;\, g_{s_i} : s_i > k;\, l), \quad \text{if} \quad k \in T\setminus\{r_0\}. \tag{3.11}$$

Let $\psi_{\xi,\eta}$ be the function defined on $\mathbb{R}$ by:
(3.12)

$$\begin{aligned}\psi_{\xi,\eta}&(x_{r_0})\\ &= \int_{\mathbb{R}^{n+c-1}} e^{2i\pi\left(\sum_{k\in T\setminus\{r_0\}} l_k x_k\right)} f(R_1(x,g,l),\ldots,R_n(x,g,l))\\ &\qquad\times \xi(D_{s_1}(x,g,l),\ldots,D_{s_c}(x,g,l))\bar{\eta}(g)\,dg\,dx_1\cdots dx_{r_0-1}\,dx_{r_0+1}\cdots dx_n,\end{aligned}$$

for fixed $l_k\in\mathbb{R}$, $k\in T\setminus\{r_0\}$. Then obviously

$$\langle\pi_l(f)\xi,\eta\rangle = \hat{\psi}_{\xi,\eta}(l_{r_0}). \tag{3.13}$$

**Lemma 3.6.** *If $\xi$ and $\eta$ are part of an orthonormal basis for $L^2(\mathbb{R}^c)$, then the function $\psi_{\xi,\eta}$ belongs to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$.*

Proof. We have

$$\begin{aligned}&\int_{\mathbb{R}}|\psi_{\xi,\eta}(x_{r_0})|\,dx_{r_0}\\ &\le \int_{\mathbb{R}^{n+c}}|f(R_1(x,g,l),\ldots,R_n(x,g,l))\xi(D_{s_1}(x,g,l),\ldots,D_{s_c}(x,g,l))\bar{\eta}(g)|\,dg\,dx\\ &= \int_{\mathbb{R}^{n+c}}|f(x_1,\ldots,x_n)\xi(\beta_{s_1}(x,g,l),\ldots,\beta_{s_c}(x,g,l))\bar{\eta}(g)|\,dg\,dx\end{aligned}$$

(by substituting $R_k(x,g,l)$ for $x_k$, $k\in T\setminus\{r_0\}$, using equation (3.11))

$$\begin{aligned}&\le \|\eta\|_2\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^c}|f(x_1,\ldots,x_n)\xi(\beta_{s_1}(x,g,l),\ldots,\beta_{s_c}(x,g,l))|^2\,dg\right)^{1/2}dx\\ &= \|\eta\|_2\,\|\xi\|_2\,\|f\|_1\end{aligned}$$

(by substituting $\beta_{s_i}(x,g,l)$ for $g_{s_i}$, using equation (3.7)). It remains to show that $\psi_{\xi,\eta}\in L^2(\mathbb{R})$. We will show that $\hat{\psi}_{\xi,\eta}\in L^2(\mathbb{R})$. Using Plancherel formula, we get that:

$$\begin{aligned}\int_{\mathcal{W}}|\langle\pi_l(f)\xi,\eta\rangle|^2\,|Pf(l)|\,dl &\le \int_{\mathcal{W}}\|\pi_l(f)\|_{HS}^2\,|Pf(l)|\,dl\\ &= \|f\|_2^2 < +\infty.\end{aligned}$$

As $|Pf(l)|$ depends only on the components of $l|_{[\mathfrak{g},\mathfrak{g}]}$,

$$\int_{\mathbb{R}}|\hat{\psi}_{\xi,\eta}(l_{r_0})|^2\,dl_{r_0} = \int_{\mathbb{R}}|\langle\pi_l(f)\xi,\eta\rangle|^2\,dl_{r_0} < +\infty,$$

for almost all $l_k\in\mathbb{R}$, $k\in T\setminus\{r_0\}$. Hence, $\hat{\psi}_{\xi,\eta}\in L^2(\mathbb{R})$ and so $\psi_{\xi,\eta}\in L^2(\mathbb{R})$. □

## 4. Proof of the Theorem

We begin this section by proving a mild generalization of Theorem 1.1.

**Theorem 4.1.** *Let* $f \in L^2(\mathbb{R})$, *such that*:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)|\,|\hat{f}(y)|\,|P(y)|^{\alpha} e^{2\pi|xy|}\,dx\,dy < +\infty,$$

*where* $\alpha$ *is positive number and* $P$ *is a polynomial function. Then,* $f = 0$ *almost everywhere.*

Proof. First of all, we mention that the hypothesis implies obviously that $f$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Hence, $\hat{f}$ is continuous. We can choose $\beta > 0$ such that $|P(y)|^{\alpha} > 1$ for all $|y| > \beta$. It follows that,

$$\int_{|y|>\beta} \int_{\mathbb{R}} |f(x)|\,|\hat{f}(y)| e^{2\pi|xy|}\,dx\,dy < +\infty. \tag{4.1}$$

On the other hand, for $y \in [-\beta, \beta]$, let

$$\xi(y, x) = |f(x)|\,|\hat{f}(y)| e^{2\pi|xy|}.$$

The function $\xi$ is continuous on $y$, in addition

$$|\xi(y, x)| \le C|f(x)| e^{2\pi|xy_0|},$$

for some positive constant and $y_0 \in ]\beta, +\infty[$ chosen by (4.1) such that

$$\int_{\mathbb{R}} |f(x)| e^{2\pi|xy_0|}\,dx < +\infty.$$

It results that, the integrand is a continuous function on $y$, and then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)|\,|\hat{f}(y)| e^{2\pi|xy|}\,dx\,dy < +\infty.$$

Therefore, $f$ satisfies the hypothesis of Theorem 1.1 and hence $f = 0$ almost everywhere. □

Now, we are going to prove our main result. The mechanism of our proof basically consists in bringing the study of the function $f$ defined on the group $G$ to the study of new function defined on $\mathbb{R}$ satisfying an equivalent condition. We consider two cases:

FIRST CASE: We suppose that $\mathfrak{g}(l) \subset [\mathfrak{g}, \mathfrak{g}]$ for all $l \in \mathcal{W}$. From equation (3.5), we have

$$\begin{aligned}+\infty > B(f) &= \int_{\mathcal{W}} \int_G |f(x)| \, \|\pi_l(f)\|_{HS} \, e^{2\pi \|x\| \, \|l\|} |Pf(l)| \, dx \, dl \\ &= \int_{\mathcal{W}} \int_G |f(x)| \left( \int_{\mathbb{R}} \int_{V_{S'}} |\hat{f}_s(l, \lambda)|^2 \, d\lambda \, ds \right)^{1/2} e^{2\pi \|x\| \, \|l\|} |l([X_{j_1}, X_n])|^{1/2} \, dx \, dl.\end{aligned}$$

Using the generalized Minkowski inequality, one gets:

$$B(f) \geq \left( \int_{\mathbb{R}} \int_{V_{S'}} \left( \int_{\mathbb{R}^r} \int_{\mathbb{R}^n} |f(x_1, \ldots, x_n)| \, |\hat{f}_s(l, \lambda)| e^{2\pi \|x\| \, \|l\|} |l([X_{j_1}, X_n])|^{1/2} dx \, dl \right)^2 d\lambda \, ds \right)^{1/2}.$$

It follows that,

$$\int_{\mathbb{R}^r} \int_{\mathbb{R}^n} \left| f_{x_n} \left( \exp \left( \sum_{i=1}^{n-1} x_i X_i \right) \right) \right| |\hat{f}_s(l, \lambda)| e^{2\pi |x_1 l_1|} |l([X_{j_1}, X_n])|^{1/2} \, dx \, dl < +\infty,$$

for almost all $s \in \mathbb{R}$ and $\lambda \in V_{S'}$. Hence, there exists a conull subset $U \subset \mathbb{R}$ such that:

$$\int_{\mathbb{R}^r} \int_{\mathbb{R}^{n-1}} \left| f_{x_n} \left( \exp \left( \sum_{i=1}^{n-1} x_i X_i \right) \right) \right| |\hat{f}_s(l, \lambda)| e^{2\pi |x_1 l_1|} |l([X_{j_1}, X_n])|^{1/2} \, dx_1 \cdots dx_{n-1} \, dl < +\infty,$$

for almost all $s \in \mathbb{R}$, $\lambda \in V_{S'}$ and for all $x_n \in U$. As the function $\left| f_{x_n}\left(\exp\left(\sum_{i=1}^{n-1} x_i X_i\right)\right)\right|$ depends only on the variables $x_1, \ldots, x_n$, the set $U$ does not depend on $s, \lambda$. Then,

$$\int_{\mathbb{R}^r} \int_{\mathbb{R}^{n-1}} \left| f_s \left( \exp \left( \sum_{i=1}^{n-1} x_i X_i \right) \right) \right| |\hat{f}_s(l, \lambda)| e^{2\pi |x_1 l_1|} |l([X_{j_1}, X_n])|^{1/2} \, dx_1 \cdots dx_{n-1} \, dl < +\infty,$$

for almost all $s \in \mathbb{R}$ and $\lambda \in V_{S'}$. This implies that:

(4.2)

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \left| f_s \left( \exp \left( \sum_{i=1}^{n-1} x_i X_i \right) \right) \right| |\hat{f}_s(l, \lambda)| e^{2\pi |x_1 l_1|} |l([X_{j_1}, X_n])|^{1/2} \, dx_1 \cdots dx_{n-1} \, dl_1 < +\infty,$$

for almost all $s \in \mathbb{R}$, $\lambda \in V_{S'}$ and $l_i \in \mathbb{R}$, $i = 2, \ldots, r$.

**Lemma 4.2.** *Let $F_s$ be the function defined on $\mathbb{R}$ by*:

$$F_s(x_1) = \int_{\mathbb{R}^{n-2}} f_s \left( \exp \left( \sum_{i=1}^{n-1} x_i X_i \right) \right) e^{2i\pi \left( \sum_{k=2}^{r} x_{t_k} l_k + \sum_{k=1}^{d-1} x_{j_k} \lambda_k \right)} \, dx_2 \cdots dx_{n-1},$$

*for almost all $s \in \mathbb{R}$, $\lambda \in V_{S'}$ and $l_i \in \mathbb{R}$, $i = 2, \ldots, r$. Then,*

a) $F_s \in L^2(\mathbb{R})$, *for almost all* $s \in \mathbb{R}$.
b) *The function* $F_s$ *is zero almost everywhere if and only if the function* $f_s$ *is.*

Proof. An easy computation shows that $F_s \in L^1(\mathbb{R})$ and $\hat{F}_s(l_1) = \hat{f}_s(l, \lambda)$. As $f_s \in L^1(G_{n-1}) \cap L^2(G_{n-1})$ for almost all $s \in \mathbb{R}$,

$$\int_{\mathbb{R}^{n-1}} |\hat{f}_s(l, \lambda)|^2 \, dl \, d\lambda = \| f_s\|_2^2 < +\infty. \tag{4.3}$$

It follows that,

$$\int_{\mathbb{R}} |\hat{F}_s(l_1)|^2 \, dl_1 = \int_{\mathbb{R}} |\hat{f}_s(l, \lambda)|^2 \, dl_1 < +\infty,$$

for almost all $l_i \in \mathbb{R}$, $i = 2, \dots, r$ and $\lambda \in V_{S'}$. Hence, $\hat{F}_s \in L^2(\mathbb{R})$ and then $F_s \in L^2(\mathbb{R})$.

On the other hand, it is easy to see that b) follows from the definition of the function $F_s$ and equation (4.3). □

Finally, remark that

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |F_s(x_1)| \, |\hat{F}_s(l_1)| e^{2\pi |x_1 l_1|} |l([X_{j_1}, X_n])|^{1/2} \, dx_1 \, dl_1 \\
&\le \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \left| f_s\left(\exp\left(\sum_{i=1}^{n-1} x_i X_i\right)\right)\right| |\hat{f}_s(l, \lambda)| e^{2\pi |x_1 l_1|} |l([X_{j_1}, X_n])|^{1/2} \, dx_1 \cdots dx_{n-1} \, dl_1
\end{aligned}$$

which is finite by (4.2). Hence by Theorem 4.1, $F_s = 0$ almost everywhere. Then, $f_s = 0$ almost everywhere and so $f = 0$ almost everywhere.

SECOND CASE: We suppose that $\mathfrak{g}(l) \not\subset [\mathfrak{g}, \mathfrak{g}]$ for all $l \in \mathcal{W}$. Let $\{\xi_i\}$ be an orthonormal basis of $L^2(\mathbb{R}^c)$. The hypothesis (1.3) implies that: for all $j, k \in \mathbb{N}^*$,

$$+\infty > \int_{\mathcal{W}} \int_{\mathbb{R}^n} |f(x_1, \dots, x_n)| \, |\langle \pi_l(f)\xi_j, \xi_k\rangle| e^{2\pi \|x\| \, \|l\|} |Pf(l)| \, dx \, dl$$

(using equation (3.13))

$$= \int_{\mathcal{W}} \int_{\mathbb{R}^n} |f(x_1, \dots, x_n) \hat{\psi}_{\xi_j, \xi_k}(l_{r_0})| e^{2\pi \|x\| \, \|l\|} |Pf(l)| \, dx \, dl.$$

As $|Pf(l)|$ does not depend on $l_{r_0}$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |f(x_1, \dots, x_n) \hat{\psi}_{\xi_j, \xi_k}(l_{r_0})| e^{2\pi \|x\| \, \|l\|} \, dx \, dl_{r_0} < +\infty,$$

for almost all $l_k \in \mathbb{R}$, $k \in T \setminus \{r_0\}$. Therefore,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |f(x_1, \dots, x_n) \hat{\psi}_{\xi_j, \xi_k}(l_{r_0})| e^{2\pi |x_{r_0} l_{r_0}|} \, dx \, dl_{r_0} < +\infty, \tag{4.4}$$

for almost all $l_k \in \mathbb{R},\ k \in T \setminus \{r_0\}$. Finally remark that,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |\psi_{\xi_j,\xi_k}(x_{r_0})|\,|\hat{\psi}_{\xi_j,\xi_k}(l_{r_0})| e^{2\pi|x_{r_0}l_{r_0}|}\,dx_{r_0}\,dl_{r_0}\\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}^{n+c}} |f(R_1(x,g,l),\dots,R_n(x,g,l))\xi_j(D_{s_1}(x,g,l),\dots,D_{s_c}(x,g,l))\bar{\xi}_k(g)\hat{\psi}_{\xi_j,\xi_k}(l_{r_0})|\\
&\qquad\qquad \times e^{2\pi|x_{r_0}l_{r_0}|}\,dg\,dx\,dl_{r_0}
\end{aligned}$$

(by using equation (3.12))

$$\begin{aligned}
&\le \int_{\mathbb{R}}\int_{\mathbb{R}^{n+c}} |f(x_1,\dots,x_n)\xi_j(\beta_{s_1}(x,g,l),\dots,\beta_{s_c}(x,g,l))\bar{\xi}_k(g)\hat{\psi}_{\xi_j,\xi_k}(l_{r_0})|\\
&\qquad\qquad \times e^{2\pi|x_{r_0}l_{r_0}|}\,dg\,dx\,dl_{r_0}
\end{aligned}$$

(by substituting $R_k(x,g,l)$ for $x_k,\ k \in T \setminus \{r_0\}$, using equation (3.11))

$$\begin{aligned}
&\le \int_{\mathbb{R}}\int_{\mathbb{R}^{n}} \left(\int_{\mathbb{R}^c} |\xi_j(\beta_{s_1}(x,g,l),\dots,\beta_{s_c}(x,g,l))|^2\,dg\right)^{1/2}\left(\int_{\mathbb{R}^c}|\bar{\xi}_k(g)|^2\,dg\right)^{1/2}\\
&\qquad\qquad \times |f(x_1,\dots,x_n)\hat{\psi}_{\xi_j,\xi_k}(l_{r_0})| e^{2\pi|x_{r_0}l_{r_0}|}\,dx\,dl_{r_0}\\
&= \|\xi_j\|_2\,\|\xi_k\|_2 \int_{\mathbb{R}}\int_{\mathbb{R}^n} |f(x_1,\dots,x_n)\hat{\psi}_{\xi_j,\xi_k}(l_{r_0})| e^{2\pi|x_{r_0}l_{r_0}|}\,dx\,dl_{r_0}
\end{aligned}$$

which is finite by (4.4). Using Beurling's theorem on $\mathbb{R}$ we obtain $\psi_{\xi_j,\xi_k}(x_{r_0}) = 0$ for almost every $l_k,\ k \in T \setminus \{r_0\}$. It follows that, for all $j, k \in \mathbb{N}^*$, $\langle \pi_l(f)\xi_j, \xi_k\rangle = 0$ and then $\|\pi_l(f)\|_{HS} = 0$. Finally, the Plancherel formula gives us that $f = 0$ almost everywhere.

## 5. Consequences of Beurling's theorem

We have already mentioned that our theorem implies some other uncertainty principles. First we recall the most important results obtained for an arbitrary nilpotent Lie group. Recently, Kaniuth and Kumar [6] proved the analogue version of Hardy's theorem for nilpotent Lie groups. They proved the following:

**Theorem 5.1** (Hardy type). *Let $G$ be a connected simply connected nilpotent Lie group and $f$ be a measurable function on $G$ such that*:
(i) $|f(x)| \le ce^{-a\pi\|x\|^2}$,
(ii) $\|\pi_l(f)\|_{HS} \le ce^{-b\pi\|l\|^2}$ *for all* $l \in \mathcal{W}$.
*Then* $f = 0$ *almost everywhere if* $ab > 1$.

This theorem was generalized later by A. Baklouti and N. Ben Salah [4]:

**Theorem 5.2** (Cowling–Price type). *Let $G$ be a connected simply connected nilpotent Lie group and $f$ be a measurable function on $G$. Let $2 \leq p, q \leq +\infty$, and $a, b \in \mathbb{R}_+^*$ such that*:
(i) $\int_G e^{pa\pi\|x\|^2} |f(x)|^p \, dx < +\infty$,
(ii) $\int_{\mathcal{W}} e^{qb\pi\|l\|^2} \|\pi_l(f)\|_{HS}^q \, |Pf(l)| \, dl < +\infty$.
*Then $f = 0$ almost everywhere if $ab > 1$.*

Note this result is proved for any nilpotent Lie group with the restriction $2 \leq p, q \leq +\infty$. We can have the analogue version of the Cowling–Price theorem with the original condition $1 \leq p, q \leq +\infty$, as a consequence of Theorem 1.3. In fact, let $f$ be a measurable function on $G$ which satisfies the conditions (i) and (ii) of Theorem 5.2. Then we can choose $0 < a' < a$ such that $a'b > 1$. We have $e^{a'\pi\|x\|^2} f \in L^1(G)$. Let $b' = 1/a'$ then $0 < b' < b$ and $e^{b'\pi\|l\|^2} \|\pi_l(f)\|_{HS} \in L^1(\mathcal{W}, |Pf(l)|dl)$. It follows that,

$$\begin{aligned} &\int_{\mathcal{W}} \int_G |f(x)| \, \|\pi_l(f)\|_{HS} \, e^{2\pi\|x\| \, \|l\|} |Pf(l)| \, dx \, dl \\ &= \int_{\mathcal{W}} \int_G e^{a'\pi\|x\|^2} |f(x)| e^{b'\pi\|l\|^2} \|\pi_l(f)\|_{HS} \, e^{-\pi(\sqrt{a'}\|x\| - \sqrt{b'} \, \|l\|)^2} |Pf(l)| \, dx \, dl \\ &\leq \int_G e^{a'\pi\|x\|^2} |f(x)| \, dx \int_{\mathcal{W}} e^{b'\pi\|l\|^2} \|\pi_l(f)\|_{HS} \, |Pf(l)| \, dl < +\infty. \end{aligned}$$

So $f$ satisfies the Beurling condition and then $f = 0$ almost everywhere by Theorem 1.3. Another consequence of Theorem 1.3 is the following analogue version of the Gelfand–Shilov theorem: If $f \in L^2(G)$ satisfies the conditions:
(i) $\int_G |f(x)| e^{2\pi a^p \|x\|^p / p} \, dx < +\infty$,
(ii) $\int_{\mathcal{W}} \|\pi_l(f)\|_{HS} \, e^{2\pi b^q \|l\|^q / q} |Pf(l)| \, dl < +\infty$,
where $1 < p < +\infty$, $1/p + 1/q = 1$.

Then $f = 0$ almost everywhere, if $ab \geq 1$. In fact, the inequality

$$ab\|x\| \, \|l\| \leq \frac{a^p \|x\|^p}{p} + \frac{b^q \|l\|^q}{q}$$

and the previous conditions (i) and (ii) imply the condition of Theorem 1.3.

## References


[1] A. Bonami, B. Demange and P. Jaming: *Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms*, Rev. Mat. Iberoamericana **19** (2003), 23–55.

[2] A. Baklouti, N. Ben Salah and K. Smaoui: *Some uncertainty principles on nilpotent Lie groups*; in Banach Algebras and Their Applications, Contemp. Math. **363**, Amer. Math. Soc., Providence, RI, 2004, 39–52.

[3] A. Baklouti and N. Ben Salah: *On theorems of Beurling and Cowling–Price for certain nilpotent Lie groups*, Bull. Sci. Math. **132** (2008), 529–550.

[4] A. Baklouti and N. Ben Salah: *The $L^p$-$L^q$ version of Hardy's theorem on nilpotent Lie groups*, Forum Math. **18** (2006), 245–262.

[5] D.C. Cook: *Müntz–Szasz theorems for nilpotent Lie groups*, J. Funct. Anal. **157** (1998), 394–412.

[6] E. Kaniuth and A. Kumar: *Hardy's theorem for simply connected nilpotent Lie groups*, Math. Proc. Cambridge Philos. Soc. **131** (2001), 487–494.

[7] G.H. Hardy: *A theorem concerning Fourier transforms*, J. London Math. Soc. **8** (1933), 227–231.

[8] G. Garimella: *Un théorème de Paley-Wiener pour les groupes de Lie nilpotents*, J. Lie Theory **5** (1995), 165–172.

[9] L. Hörmander: *A uniqueness theorem of Beurling for Fourier transform pairs*, Ark. Mat. **29** (1991), 237–240.

[10] L.J. Corwin and F.P. Greenleaf: Representations of Nilpotent Lie Groups and Their Applications, Part I, Basic Theory and Examples, Cambridge Univ. Press, Cambridge, 1990.

[11] M. Cowling and J.F. Price: *Generalisations of Heisenberg's inequality*; in Harmonic Analysis (Cortona, 1982), Lecture Notes in Math. **992**, Springer, Berlin, 1983, 443–449.

[12] P. Bernat, N. Conze, M. Duflo, M. Lévy-Nahas, M. Raïs, P. Renouard and M. Vergne: Représentations des Groupes de Lie Résolubles, Dunod, Paris, 1972.

[13] R.P. Sarkar and S. Thangavelu: *On theorems of Beurling and Hardy for the Euclidean motion group*, Tohoku Math. J. (2) **57** (2005), 335–351.

[14] R.P. Sarkar and J. Sengupta: *Beurling's theorem and characterization of heat kernel for Riemannian symmetric spaces of noncompact type*, Canad. Math. Bull. **50** (2007), 291–312.

[15] S. Parui and R.P. Sarkar: *Beurling's theorem and $L^p$-$L^q$ Morgan's theorem for step two nilpotent Lie groups*, Publ. Res. Inst. Math. Sci. **44** (2008), 1027–1056.

[16] S. Thangavelu: *On theorems of Hardy, Gilfand–Shilov and Beurling for semisimple Lie groups*, Publ. Res. Inst. Math. Sci. **40** (2004), 311–344.

[17] V. Havin and B. Jöricke: The Uncertainty Principle in Harmonic Analysis, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **28**, Springer, Berlin, 1994.

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