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q-DIFFERENCE ANALOGUE OF THE EULER-POISSON-DARBOUX EQUATION AND ITS LAPLACE SEQUENCE

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1. Introduction and main result

The Euler-Poisson-Darboux (EPD) equation is the second order hyperbolic equation

$$\bar{L}(\beta,\beta')u=\left\{\partial_x\partial_y-\frac{\beta-\beta'}{x-y}\partial_x+\frac{\beta(\beta'-1)}{(x-y)^2}\right\}u=0$$

which appears in various areas of mathematics and physics such as theory of surfaces [2], propagation of sounds [1] and collidings of gravitational waves [3], etc. By the conjugate transform of the differential operator $\bar{L}(\beta,\beta')$ with $(x-y)^{-\beta}$, we have the operator

$$(x-y)^{-\beta}\bar{L}(\beta,\beta')(x-y)^{\beta}=\bar{E}(\beta,\beta')=\partial_x\partial_y-\frac{\beta'}{x-y}\partial_x+\frac{\beta}{x-y}\partial_y.$$

In this note we consider a q-difference analogue of the operator

$$E(\beta,\beta')=(x-y)\bar{E}(\beta,\beta')=(x-y)\partial_x\partial_y-\beta'\partial_x+\beta\partial_y \tag{1}$$

and demonstrate that q-deformation of $E(\beta,\beta')$ is the q-difference operator (see section 2)

$$E_q(\beta,\beta')=[\theta_x+\beta]_q[\partial_y]_q-[\theta_y+\beta']_q[\partial_x]_q. \tag{2}$$

The EPD equation has very interesting properties, for example, Miller's symmetry, Laplace sequence and the relation to Toda molecule equation, etc. (see [2] and [6]). First we consider a q-deformation of Miller's symmetry explained below. Let $V(\beta,\beta')$ be the space of solutions of the differential equation $E(\beta,\beta')u=0$. Then $V(\beta,\beta')$ is invariant under the action of $SL(2,C)$ defined by

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$$u(x,y)\mapsto(bx+d)^{-\beta}(by+d)^{-\beta'}u\left(\frac{ax+c}{bx+d},\frac{ay+c}{by+d}\right),\ \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,C)$$

and hence infinitesimal generators of this symmetry are

$$E=-x^2\partial_x-y^2\partial_y-\beta x-\beta' y,$$
$$H=\partial_x+\partial_y,$$
$$F=2x\partial_x+2y\partial_y+\beta+\beta'.$$

We call this Miller's symmetry. Indeed, this Lie algebra is isomorphic to Lie algebra $sl(2,C)$. It shall be shown that its q-deformation is quantum group $U_q(sl(2,C))$ with generators

$$e=-\{q^{-\theta_x}y[\theta_y+\beta']_q+q^{\theta_y}x[\theta_x+\beta]_q\},$$
$$f=q^{-\theta_x-\beta}y^{-1}[\theta_y]_q+q^{\theta_y+\beta'}x^{-1}[\theta_x]_q,$$
$$q^h=q^{2\theta_x+2\theta_y+\beta+\beta'}.$$

If the parameter q tends to unit, obviously we get Miller's symmetry.

Theorem 1.1. *The difference operators e, f and q^h are symmetries of the q-difference EPD equation and are generators of the quantum group $U_q(sl(2,C))$.*

REMARK 1.1. *This kind of representation of quuantum group can be seen in* [4] *and* [5].

The second aim of our research is to find a q-deformation of the so-called Laplace sequence. We give a brief explanation of the Laplace sequence for the EPD equation. Let us consider a family of differential operators parametrized by an integer n

$$E_n(\beta,\beta')=(x-y)\partial_x\partial_y-(\beta'+n)\partial_x+(\beta-n)\partial_y. \tag{3}$$

This is a typical example of Laplace sequence for the second order hyperbolic equation with two independent variables (also see [2], [6]). Define two operators H_n and B_n by

$$H_n=(x-y)\partial_y-(\beta'+n),\quad B_n=(x-y)\partial_x+(\beta-n).$$

Then we have

$$H_{n+1}E_n=E_{n+1}H_n,\quad B_{n-1}E_n=E_{n-1}B_n$$

for any integer n. These equations mean that if u_n is a solution of the equation

$E_n(\beta,\beta')u=0$, then $u_{n+1}=H_n u_n$ or $u_{n-1}=B_n u_n$ is a solution of the equation $E_{n+1}u=0$ or $E_{n-1}u=0$, respectively. Therefore we may think that H_n and B_n are a kind of increasing or decreasing operators. We shall show that q-analogues of H_n and B_n are

$$H_{q,n}=-q^{-\theta_x}[\theta_y+\beta'+n]_q+q^{-\theta_x-(\beta-n-1)}xy^{-1}[\theta_y]_q,$$
$$B_{q,n}=q^{\theta_y}[\theta_x+\beta-n]_q-q^{\theta_y+(\beta'+n-1)}yx^{-1}[\theta_x]_q.$$

These q-difference operators are found by quantizing some solution of the EPD equation. The EPD equation has a formal solution

$$\varphi(\lambda,\mu;\beta,\beta';x,y)=\sum_{n\in Z}\frac{[\lambda+\beta;n][\mu-n+1;n]}{[\mu-n+\beta';n][\lambda+1;n]}x^{\lambda+n}y^{\mu-n}$$

where $[\alpha;n]=\Gamma(\alpha+n)/\Gamma(\alpha)$ and $\Gamma(\alpha)$ is the gamma function. We may think that its q-deformation is

$$\varphi_q(\lambda,\mu;\beta,\beta';x,y)=\sum_{n\in Z}\frac{[\lambda+\beta;n]_q[\mu-n+1;n]_q}{[\mu-n+\beta';n]_q[\lambda+1;n]_q}x^{\lambda+n}y^{\mu-n}$$

where $[\alpha;n]_q=\Gamma_q(\alpha+n)/\Gamma_q(\alpha)$ and $\Gamma_q(\alpha)$ is the basic gamma function (see section 2). We use the notations φ_q^λ and $\varphi_{q,\lambda}$ to denote contiguous functions of φ_q, such as $\varphi_q^\lambda=\varphi_q(\lambda+1,\mu;\beta,\beta';x,y)$ and $\varphi_{q,\lambda}=\varphi_q(\lambda-1,\mu;\beta,\beta';x,y)$, etc. To describe the action of e, f, $q^{\pm h}$, $H_{q,n}$ and $B_{q,n}$ in a simple form, it is convenient to introduce the function

$$\Phi_q=\frac{\Gamma_q(\lambda+\beta)\Gamma_q(\mu+\beta')}{\Gamma_q(\lambda)\Gamma_q(\mu+1)}\varphi_q.$$

By using this function we can get the next expression of the action of $U_q(sl(2,C))$ and Laplace sequence

$$e\Phi_q=-[\lambda+\mu+1]_q\Phi_q^\lambda,$$
$$f\Phi_q=[\lambda+\mu+\beta+\beta'-1]_q\Phi_{q,\mu},$$
$$q^h\Phi_q=q^{2(\lambda+\mu)+\beta+\beta'}\Phi_q,$$
$$H_{q,0}\Phi_q=-[\beta-1]_q\Phi_{q,\beta}^{\beta'},$$
$$B_{q,0}\Phi_q=[\beta'-1]_q\Phi_{q,\beta'}.$$

Finally we give the explanation of the organization of this paper. In the next section, we introduce and fix our notations appeared in the q-analogue calcules. In section 3, we define a q-difference analogue of the EPD equation and give a proof of theorem 1.1 and we shall find its q-Laplace sequence $H_{q,n}$ and $B_{q,n}$ in section 4. The classical results about the EPD equation are stated in Appendix A. A part of the proof of Theorem 1.1 is given in Appendix B. Finally we express the Casimir operator of $U_q(sl(2,C))$ by means of the operator $E_{q,0}$ in Appendix C.

2. q-difference calculus

In this section, a few elementary results involving basic differentiation are obtained. For any number A, we define basic number $[A]_q$ by the relation

$$[A]_q=\frac{q^A-q^{-A}}{q-q^{-1}}$$

where q may be real or complex. Then we can easily verify the formula

$$\begin{aligned}[A+B]_q&=q^A[B]_q+q^{-B}[A]_q\\&=q^{-A}[B]_q+q^B[A]_q\end{aligned}\tag{4}$$

and

$$[A+1]_q[B+1]_q-[A]_q[B]_q=[A+B+1]_q.\tag{5}$$

In the following sections, we need q-difference operator (q-differentiation or basic differentiation). First we introduce q-shift operator T by

$$(Tf)(x)=f(qx),$$

then q-difference operator $[\partial]_q$ is defined by

$$\begin{aligned}([\partial]_qf)(x)&=\frac{f(qx)-f(q^{-1}x)}{(q-q^{-1})x}\\&=\frac{1}{x}\left(\frac{T-T^{-1}}{q-q^{-1}}f\right)(x).\end{aligned}$$

Further we need q-difference Euler operator $[\theta]_q$

$$[\theta]_q=\frac{T-T^{-1}}{q-q^{-1}}.$$

Because of this definiton, we may identify T and q^θ, namely, $q^\theta\overset{\text{def}}{=}T$. One of the important properties of the operator $[\theta]_q$ is that it behaves just as the ordinary Euler differential, i.e.

$$[\theta]_qx^n=[n]_qx^n$$

We shall often use the following relations

$$x^n q^{-\theta} = q^{-\theta+n} x^n, \quad x^n q^{\theta} = q^{\theta-n} x^n, \quad x^n[\theta+\alpha]_q = [\theta+\alpha-n]_q x^n,$$

where these all relations are considered as operators. Finally we define basic gamma function by

$$\Gamma_q(x) = q^{(x^2-3x)/2} \frac{(q^2)_\infty}{(q^{2x})_\infty} (1-q^2)^{1-x}, \quad (a)_\infty = \prod_{j=0}^{\infty} (1-q^j a).$$

For this basic gamma function we have fundamental difference relation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

3. q-difference analogue of the EPD equation

Let us prove that the q-deformed function φ_q satisfies Eq.(2). From the difference relation of the basic gamma function and the expression of the q-deformed function φ_q

$$\varphi_q(\lambda,\mu;\beta,\beta';x,y) = \sum_{n\in Z} \frac{[\lambda+\beta;n]_q [\mu-n+1;n]_q}{[\mu-n+\beta';n]_q [\lambda+1;n]_q} x^{\lambda+n} y^{\mu-n},$$

we can get the following contiguous relations of φ_q.

Proposition 3.1. *The function φ_q has the following contiguous relations*:

1. $x^{-1}[\theta_x]_q \varphi_q = [\lambda]_q \varphi^{\beta}_{q,\lambda}, \qquad y^{-1}[\theta_y]_q \varphi_q = [\mu]_q \varphi^{\beta'}_{q,\mu}.$
2. $[\theta_x+\beta]_q \varphi_q = [\lambda+\beta]_q \varphi^{\beta}_q, \qquad [\theta_y+\beta']_q \varphi_q = [\mu+\beta']_q \varphi^{\beta'}_q.$

By using these contiguous relations, we have

$$[\theta_x+\beta]_q [\partial_y]_q \varphi_q = [\mu]_q [\lambda+\beta]_q \varphi^{\beta,\beta'}_{q,\mu},$$

$$[\theta_y+\beta']_q [\partial_x]_q \varphi_q = [\lambda]_q [\mu+\beta']_q \varphi^{\beta,\beta'}_{q,\lambda},$$

and further we can easily verify

$$[\mu]_q [\lambda+\beta]_q \varphi^{\beta,\beta'}_{q,\mu} = [\lambda]_q [\mu+\beta']_q \varphi^{\beta,\beta'}_{q,\lambda},$$

by direct calculation. Hence we have proved

$$[\theta_x+\beta]_q [\partial_y]_q \varphi_q = [\theta_y+\beta']_q [\partial_x]_q \varphi_q,$$

which we call the q-difference EPD equation.

Now we will prove that the algebra generated by three q-defference operators

$$e=-\{q^{-\theta_x}y[\theta_y+\beta']_q+q^{\theta_y}x[\theta_x+\beta]_q\},$$
$$f=q^{-\theta_x-\beta}y^{-1}[\theta_y]_q+q^{\theta_y+\beta'}x^{-1}[\theta_x]_q,$$
$$q^h=q^{2\theta_x+2\theta_y+\beta+\beta'}$$

is a q-deformation of Miller's symmetry. First we show the next proposition.

Proposition 3.2. *Let $E_q(\beta,\beta')$ be the q-difference EPD operator defined by Eq.(2), then operators e, f and q^h satisfy the following relations*:

1. $E_q(\beta,\beta')e=-\{q^{-\theta_x-1}[\theta_y+\beta']_q y+q^{\theta_y+1}[\theta_x+\beta]_q x\}E_q(\beta,\beta')$.
2. $E_q(\beta,\beta')f=fE_q(\beta,\beta')$.
3. $E_q(\beta,\beta')q^{\pm h}=q^{\pm 2}q^{\pm h}E_q(\beta,\beta')$.

From this proposition we immediately have the next corollary.

Corollary 3.1. *The q-difference operators e, f and $q^{\pm h}$ are symmetries of the q-difference EPD equation.*

Proof of Proposition 3.2. Let us prove the first relation. From the difinition of the difference operator e, we have

$$\begin{aligned}&E_q(\beta,\beta')e\\&=[\theta_x+\beta']_q x^{-1}[\theta_x]_q q^{-\theta_x}y[\theta_y+\beta']_q+[\theta_y+\beta']_q x^{-1}[\theta_x]_q q^{\theta_y}x[\theta_x+\beta]_q\\&\quad-[\theta_x+\beta]_q y^{-1}[\theta_y]_q q^{-\theta_x}y[\theta_y+\beta']_q-[\theta_x+\beta]_q y^{-1}[\theta_y]_q q^{\theta_y}x[\theta_x+\beta]_q.\end{aligned}$$

By using the following relation

$$x^{-1}q^{-\theta_x}=q^{-\theta_x-1}x^{-1},\quad x^{-1}q^{\theta_x}=q^{\theta_x+1}x^{-1},\quad x^{-1}[\theta_x]_q x=[\theta_x+1]_q,$$

we see

$$\begin{aligned}&E_q(\beta,\beta')e\\&=q^{-\theta_x-1}[\theta_y+\beta']_q y[\partial_x]_q[\theta_y+\beta']_q+q^{\theta_y}[\theta_x+\beta]_q[\theta_x+1]_q[\theta_y+\beta']_q\\&\quad-q^{-\theta_x}[\theta_y+\beta']_q[\theta_y+1]_q[\theta_x+\beta]_q-q^{\theta_y+1}[\theta_x+\beta]_q x[\partial_y]_q[\theta_x+\beta]_q.\end{aligned}$$

Further by applying the addition foumula Eq. (4)

$$[\theta_x+1]_q=q[\theta_x]_q+q^{-\theta_x}=q^{-1}[\theta_x]_q+q^{\theta_x},$$

in the second and third terms of the above equation, we get

$$
\begin{aligned}
&E_q(\beta,\beta')e \\
&= q^{-\theta_x-1}[\theta_y+\beta']_q y[\partial_x]_q[\theta_y+\beta']_q + q^{\theta_y+1}[\theta_x+\beta]_q[\theta_x]_q[\theta_y+\beta']_q \\
&\quad - q^{-\theta_x-1}[\theta_y+\beta']_q[\theta_y]_q[\theta_x+\beta]_q - q^{\theta_y+1}[\theta_x+\beta]_q x[\partial_y]_q[\theta_x+\beta]_q.
\end{aligned}
$$

Therefore we have

$$E_q(\beta,\beta')e = -\{q^{-\theta_x-1}[\theta_y+\beta']_q y + q^{\theta_y+1}[\theta_x+\beta]_q x\}E_q(\beta,\beta').$$

The second relation is proved just above by using the relation

$$x^{-1}q^{-\theta_x-\beta} = q^{-\theta_x-1-\beta}x^{-1}, \quad x^{-1}q^{\theta_x+\beta} = q^{\theta_x+\beta+1}x^{-1},$$
$$[\theta_x+\beta]_q x^{-1} = x^{-1}[\theta_x+\beta-1]_q,$$

and the addition formula

$$[\theta_x-1+\beta]_q = q^{-1}[\theta_x+\beta]_q - q^{-\theta_x-\beta} = q[\theta_x+\beta]_q - q^{\theta_x+\beta}.$$

Finally we prove the third relation. By the definition of q^h and the formula

$$x^{-1}q^{2\theta_x} = q^2 q^{2\theta_x}x^{-1}, \quad y^{-1}q^{2\theta_y} = q^2 q^{2\theta_y}y^{-1},$$

we get

$$E_q(\beta,\beta')q^h = q^2 q^h E_q(\beta,\beta').$$

q.e.d

Thus we have proved the first statement of Theorem 1. A proof of the second statement, that three operators e, f and $q^{\pm h}$ are generators of $U_q(sl(2,C))$, namely,

$$q^h e q^{-h} = q^2 e,$$
$$q^h f q^{-h} = q^{-2} f,$$
$$[e,f] = \frac{q^h - q^{-h}}{q - q^{-1}}.$$

is given in appendix B.

In the following we give a kind of representation of $U_q(sl(2,C))$ on the space of contiguous functions of φ_q.

Proposition 3.3. *The q-difference operators e, f and $q^{\pm h}$ act on the space of contiguous functions of φ_q as follows:*

$$e\varphi_q = -\frac{[\lambda+\beta]_q[\lambda+\mu+1]_q}{[\lambda+1]_q}\varphi_q^{\lambda},$$

$$f\varphi_q=\frac{[\mu]_q[\lambda+\mu+\beta+\beta'-1]_q}{[\mu+\beta'-1]_q}\varphi_{q,\mu},$$

$$q^h\varphi_q=q^{2(\lambda+\mu)+\beta+\beta'}\varphi_q,\quad q^{-h}\varphi_q=q^{-2(\lambda+\mu)-\beta-\beta'}\varphi_q.$$

Proof. By the definition of φ_q, we get

$$\begin{aligned}-e\varphi_q&=\sum_{n\in Z}q^{-\lambda-n}[\mu-n+\beta']_q\frac{[\mu-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[\mu-n+\beta';n]_q}x^\lambda y^{\mu+1}t^n\\&+\sum_{n\in Z}q^{\mu-n}[\lambda+n+\beta]_q\frac{[\mu-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n\\&=I_1+I_2,\end{aligned}$$

where we put $t=x/y$. Hence by replacing n by $n+1$ in the first term I_1, we have

$$\begin{aligned}I_1&=\sum_{n\in Z}q^{-\lambda-n-1}[\mu-n-1+\beta']_q\times\frac{[\mu-n;n+1]_q[\lambda+\beta;n+1]_q}{[\lambda+1;n+1]_q[\mu-n-1+\beta';n+1]_q}x^{\lambda+1}y^\mu t^n\\&=\frac{[\lambda+\beta]_q}{[\lambda+1]_q}\sum_{n\in Z}q^{-(\lambda+1)-n}[\mu-n]_q\times\frac{[\mu-n+1;n]_q[\lambda+1+\beta;n]_q}{[\lambda+2;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n.\end{aligned}$$

On the other hand,

$$\begin{aligned}I_2=&\frac{[\lambda+\beta]_q}{[\lambda+1]_q}\sum_{n\in Z}q^{\mu-n}[\lambda+n+1]_q\\&\times\frac{[\mu-n+1;n]_q[\lambda+1+\beta;n]_q}{[\lambda+2;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n.\end{aligned}$$

Therefore we get

$$\begin{aligned}e\varphi_q&=-I_1-I_2\\&=-\frac{[\lambda+\beta]_q}{[\lambda+1]_q}\sum_{n\in Z}\{q^{-(\lambda+1)-n}[\mu-n]_q+q^{\mu-n}[\lambda+n+1]_q\}\\&\quad\times\frac{[\mu-n+1;n]_q[\lambda+1+\beta;n]_q}{[\lambda+2;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n\\&=-\frac{[\lambda+\beta]_q[\lambda+\mu+1]_q}{[\lambda+1]_q}\varphi_q^\lambda,\end{aligned}$$

where we use the addition formula Eq. (4). Similarly as above, we have

$$f\varphi_q=\frac{[\mu]_q}{[\mu-1+\beta']_q}\sum_{n\in Z}\{q^{-\lambda-n-\beta}[\mu-n-1+\beta']_q+q^{(\mu-1)-n+\beta'}[\lambda+\beta+n]_q\}$$
$$\times\frac{[(\mu-1)-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[(\mu-1)-n+\beta';n]_q}x^\lambda y^{\mu-1}t^n$$
$$=\frac{[\mu]_q}{[\mu-1+\beta']_q}\sum_{n\in Z}[\lambda+\mu+\beta+\beta'-1]_q$$
$$\times\frac{[(\mu-1)-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[(\mu-1)-n+\beta';n]_q}x^\lambda y^{\mu-1}t^n$$
$$=\frac{[\mu]_q[\lambda+\mu+\beta+\beta'-1]_q}{[\mu-1+\beta']_q}\varphi_{q,\mu}.$$

The last statement is easily proved by direct calculation. q.e.d

By using the funciton Φ_q, we get a simple expression of the action of operators e, f and q^h.

Corollary 3.2. *The action of operators e, f and q^h on the function Φ_q is*

$$e\Phi_q=-[\lambda+\mu+1]_q\Phi_q^\lambda,$$
$$f\Phi_q=[\lambda+\mu+\beta+\beta'-1]_q\Phi_{q,\mu},$$
$$q^h\Phi_q=q^{2(\lambda+\mu)+\beta+\beta'}\Phi_q.$$

4. q-Laplace sequence

Here we consider a family of the difference operators

$$E_{q,n}(\beta,\beta')=[\theta_x+\beta-n]_q[\partial_y]_q-[\theta_y+\beta'+n]_q[\partial_x]_q,\qquad n\in \boldsymbol{Z} \tag{6}$$

which may be thought as a q-difference analogue of the operator E_n defined by Eq. (2). Our purpose is to find a kind of increasing or decreasing operators. Let us denote two types of q-difference operators $H_{q,n}$ and $B_{q,n}$ by

$$H_{q,n}=-q^{-\theta_x}[\theta_y+\beta'+n]_q+q^{-\theta_x-(\beta-n-1)}xy^{-1}[\theta_y]_q,$$
$$B_{q,n}=q^{\theta_y}[\theta_x+\beta-n]_q-q^{\theta_y+(\beta'+n-1)}yx^{-1}[\theta_x]_q.$$

Then the next theorem can be proved by direct calculation.

Theorem 4.1.

1. $H_{q,n+1}E_{q,n}=qE_{q,n+1}H_{q,n}$

2. $B_{q,n-1}E_{q,n}=q^{-1}E_{q,n-1}B_{q,n}$

Proof. By replacing β and β' by $\beta+n$ or $\beta'-n$, it is enough to prove when $n=0$. From the difiniton, we see

$$H_{q,1}E_{q,0}$$
$$=-q^{-\theta_x}[\theta_y+\beta'+1]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q+q^{-\theta_x}[\theta_y+\beta'+1]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q$$
$$+q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q-q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q.$$

By using the relations

$$[\theta_x+\beta]_q=q[\theta_x+\beta-1]_q+q^{-\theta_x-\beta+1},\quad [\theta_x]_q=q^{-1}[\theta_x+1]_q-q^{-\theta_x-1},$$

at the first and the second terms, we have

$$H_{q,1}E_{q,0}$$
$$=-q^{-\theta_x+1}[\theta_y+\beta'+1]_q[\theta_x+\beta-1]_q y^{-1}[\theta_y]_q-q^{-\theta_x}[\theta_y+\beta'+1]_q q^{-\theta_x-\beta+1}y^{-1}[\theta_y]_q$$
$$+q^{-\theta_x}[\theta_y+\beta'+1]_q[\theta_x+\beta']_q x^{-1}[\theta_x]_q+q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q$$
$$-q^{-\theta_x-(\beta-1)}xy^{-1}[\theta_y]_q[\theta_y+\beta']_q x^{-1}[\theta_x+1]_q$$
$$+q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_y+\beta']_q x^{-1}q^{-\theta_x-1}$$
$$=-q[\theta_x+\beta-1]_q y^{-1}[\theta_y]_q q^{-\theta_x}[\theta_y+\beta']_q-q^{-2\theta_x-(\beta-1)}y^{-1}[\theta_y+\beta']_q[\theta_y]_q$$
$$+q[\theta_y+\beta'+1]_q x^{-1}[\theta_x]q^{-\theta_x}[\theta_y+\beta']_q+q[\theta_x+\beta-1]_q y^{-1}[\theta_y]_q q^{-\theta_x-(\beta-1)}xy^{-1}[\theta_y]_q$$
$$-q[\theta_y+\beta'+1]_q x^{-1}[\theta_x]_q q^{-\theta_x-(\beta-1)}xy^{-1}[\theta_y]_q+q^{-2\theta_x-(\beta-1)}y^{-1}[\theta_y]_q[\theta_y+\beta']_q$$
$$=qE_{q,1}H_{q,0}.$$

Thus the first statement is proved. We will show the second statement.

$$B_{q,-1}E_{q,0}$$
$$=q^{\theta_y}[\theta_x+\beta+1]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q-q^{\theta_y}[\theta_x+\beta+1]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q$$
$$-q^{\theta_y+(\beta'-2)}yx^{-1}[\theta_x]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q+q^{\theta_y+(\beta'-2)}yx^{-1}[\theta_x]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q.$$

Substituting

$$[\theta_y+\beta']_q=q^{-1}[\theta_y+\beta'-1]_q+q^{\theta_y+\beta'-1},\quad [\theta_y]_q=q[\theta_y+1]_q-q^{\theta_y+1}$$

into the second and third terms, we have

$$B_{q,-1}E_{q,0}$$
$$=q^{\theta_y}[\theta_x+\beta+1]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q-q^{\theta_y-1}[\theta_x+\beta+1]_q[\theta_y+\beta'-1]_q x^{-1}[\theta_x]_q$$

$$-q^{\theta_y}[\theta_x+\beta+1]_q q^{\theta_y+\beta'-1}x^{-1}[\theta_x]_q-q^{\theta_y+(\beta'-1)}yx^{-1}[\theta_x]_q[\theta_x+\beta]_q y^{-1}[\theta_y+1]_q$$
$$+q^{\theta_y+(\beta'-2)}yx^{-1}[\theta_x]_q[\theta_x+\beta]_q y^{-1}q^{\theta_y+1}+q^{\theta_y+(\beta'-2)}xy^{-1}[\theta_x]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q$$
$$=q^{-1}[\theta_x+\beta+1]_q y^{-1}[\theta_y]_q q^{\theta_y}[\theta_x+\beta]_q-q^{-1}[\theta_y+\beta'-1]_q x^{-1}[\theta_x]_q q^{\theta_y}[\theta_x+\beta]_q$$
$$-q^{2\theta_y+(\beta'-1)}x^{-1}[\theta_x+\beta]_q[\theta_x]_q-q^{-1}[\theta_x+\beta+1]_q y^{-1}[\theta_y]_q q^{\theta_y+(\beta'-1)}yx^{-1}[\theta_x]_q$$
$$+q^{2\theta_y+(\beta'-1)}x^{-1}[\theta_x]_q[\theta_x+\beta]_q+q^{-1}[\theta_y+\beta'-1]_q x^{-1}[\theta_x]_q q^{\theta_y+(\beta'-1)}yx^{-1}[\theta_x]_q$$
$$=q^{-1}E_{q,-1}B_{q,0}.$$
q.e.d

REMARK 4.1. *The above theorem implies that if u_n is a solution of the equation $E_{q,n}u_n=0$, then $u_{n+1}=H_{q,n}u_n$ or $u_{n-1}=B_{q,n}u_n$ is a solution of $E_{q,n+1}u=0$ or $E_{q,n-1}u=0$, respectively.*

We have more infomation about the action of $H_{q,n}$ and $B_{q,n}$.

Proposition 4.1. *The action of operators $H_{q,0}$ and $B_{q,0}$ on the space of contiguous functions of φ_q is*

$$H_{q,0}\varphi_q=-\frac{[\mu+\beta']_q[\beta-1]_q}{[\lambda+\beta-1]_q}\varphi^{\beta'}_{q,\beta},\quad B_{q,0}\varphi_q=\frac{[\lambda+\beta]_q[\beta'-1]_q}{[\mu+\beta'-1]_q}\varphi^{\beta}_{q,\beta'}. \tag{7}$$

Proof. By the definition of φ_q, we get

$$H_{q,0}\varphi_q=-\sum_{n\in Z}q^{-\lambda-n}[\mu-n+\beta']_q\frac{[\lambda+\beta;n]_q[\mu-n+1;n]_q}{[\mu-n+\beta';n]_q[\lambda+1;n]_q}x^\lambda y^\mu t^n$$
$$+\sum_{n\in Z}q^{-\lambda-n-(\beta-1)}[\mu-n+1]_q\frac{[\lambda+\beta;n-1]_q[\mu-n+2;n-1]_q}{[\mu-n+1+\beta';n-1]_q[\lambda+1;n-1]_q}x^\lambda y^\mu t^n$$
$$=\frac{[\mu+\beta']_q}{[\lambda+\beta-1]_q}\sum_{n\in Z}\{-q^{-\lambda-n}[\lambda+n+\beta-1]_q+q^{-\lambda-n-(\beta-1)}[\lambda+n]_q\}$$
$$\times\frac{[\lambda+(\beta-1);n]_q[\mu-n+1;n]_q}{[\mu-n+(\beta'+1);n]_q[\lambda+1;n]_q}x^\lambda y^\mu t^n$$
$$=-\frac{[\mu+\beta']_q[\beta-1]_q}{[\lambda+\beta-1]_q}\varphi^{\beta'}_{q,\beta}.$$

Here we used the addition formula

$$-q^{-\lambda-n}[\lambda+n+\beta-1]_q+q^{-\lambda-n-(\beta-1)}[\lambda+n]_q=-[\beta-1]_q.$$

The second statement is proved just above by using addition formula

$$q^{\mu-n}[\mu-n+(\beta'-1)]_q-q^{\mu-n+(\beta'-1)}[\mu-n]_q=[\beta'-1]_q$$

as follows:

$$\begin{aligned}B_{q,0}\varphi_q&=\frac{[\lambda+\beta]_q}{[\mu+(\beta'-1)]_q}\sum_{n\in Z}\{q^{\mu-n}[\mu-n+(\beta'-1)]_q-q^{\mu-n+(\beta'-1)}[\mu-n]_q\}\\&\quad\times\frac{[\lambda+(\beta+1);n]_q[\mu-n+1;n]_q}{[\mu-n+(\beta'-1);n]_q[\lambda+1;n]_q}x^\lambda y^\mu t^n\\&=\frac{[\lambda+\beta]_q[\beta'-1]_q}{[\mu+(\beta'-1)]_q}\varphi^{\beta}_{q,\beta'}.\end{aligned}$$

q.e.d

Remark 4.2. *The action of $H_{q,0}$ and $B_{q,0}$ on Φ_q is*

$$H_{q,0}\Phi_q=-[\beta-1]_q\Phi^{\beta'}_{q,\beta},\quad B_{q,0}\Phi_q=[\beta'-1]_q\Phi^{\beta}_{q,\beta'}.$$

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A. The Euler-Poisson-Darboux Equation

Let us consider some analytic properties of the equation

$$E(\beta,\beta')u=\{(x-y)\partial_x\partial_y-\beta'\partial_x+\beta\partial_y\}u=0 \tag{8}$$

We would like to find a solution of the form

$$u=x^\lambda y^\mu\varphi\left(\frac{x}{y}\right)$$

where λ and μ are complex parameters. By substituting this expression into Eq. (8) we have

$$\begin{aligned}t^2(1-t)\varphi''(t)+t\{(\mu-\lambda-1-\beta)t-(\mu-\lambda-1+\beta')\}\varphi'(t)\\+\{(\lambda+\beta)\mu t-\lambda(\mu+\beta')\}\varphi(t)=0\end{aligned}$$

Especially in the case of $\lambda=0$ this equation is reduced to Gauss's hypergeometric euqation

$$t(1-t)\varphi''(t)+t\{(\mu-1-\beta)t-(\mu-1+\beta')\}\varphi'(t)+\beta\mu\varphi(t)=0$$

Hence Eq. (8) have special solutions related to hypergeometric series. For example, we have a solution

$$u(x,y)=y^{\mu}F\left(\mu,-\beta,1-\mu-\beta';\frac{y}{x}\right),$$

where

$$F(a,b,c;t)=\sum_{n=0}^{\infty}\frac{[a;n][b;n]}{[c;n][1;n]}t^{n},\quad [a;n]=\Gamma(a+n)/\Gamma(a),$$

is Gauss's hypergeometric series. Hence by using the action of $SL(2,C)$, we obtain Appell's formula

$$u(x,y)=(bx+d)^{-\beta}(by+d)^{-\beta'}(ay+c)^{\mu}(by+d)^{-\mu}F(\mu,-\beta,1-\mu+\beta';\sigma)$$

$$\sigma=\frac{(bx+d)(ay+c)}{(ax+c)(by+d)}.$$

B. A proof of Theorem 1

Here we will prove that three operators e, f and q^{h} are generators of the quantum group $U_q(sl(2,C))$. Namely, let us prove Serre's relations

$$q^{h}eq^{-h}=q^{2}e,\quad q^{h}fq^{-h}=q^{-2}f,\quad [e,f]=\frac{q^{h}-q^{-h}}{q-q^{-1}}$$

which characterize $U_q(sl(2,C))$. From the definition, we see

$$q^{h}eq^{-h}=-q^{2\theta_x+2\theta_y+\beta+\beta'}q^{-\theta_x}y[\theta_y+\beta']_q q^{-2\theta_x-2\theta_y-\beta-\beta'}$$
$$-q^{2\theta_x+2\theta_y+\beta+\beta'}q^{\theta_y}x[\theta_x+\beta]_q q^{-2\theta_x-2\theta_y-\beta-\beta'}$$

By using the relations $xq^{-2\theta_x}=q^{-2\theta_x+2}x$ and $yq^{-2\theta_y}=q^{-2\theta_y+2}y$, we obtain

$$q^{h}eq^{-h}=-\{q^{-\theta_x+2}y[\theta_y+\beta']_q+q^{\theta_y+2}x[\theta_x+\beta]_q\}=q^{2}e$$

and just as the same above we can show $q^{h}fq^{-h}=q^{-2}f$.

Now we prove the relation

$$[e,f]=\frac{q^{h}-q^{-h}}{q-q^{-1}}.$$

From the definition of e and f, we have

$$\begin{aligned}
&[e,f] \\
&=-[q^{\theta_y}x[\theta_x+\beta]_q, q^{\theta_y+\beta'}x^{-1}[\theta_x]_q]-[q^{\theta_y}x[\theta_x+\beta]_q, q^{-\theta_x-\beta}y^{-1}[\theta_y]_q] \\
&\quad -[q^{-\theta_x}y[\theta_y+\beta']_q, q^{\theta_y+\beta'}x^{-1}[\theta_x]_q]-[q^{-\theta_x}y[\theta_y+\beta']_q, q^{-\theta x-\beta}y^{-1}[\theta_y]_q] \\
&=-C_1-C_2-C_3-C_4.
\end{aligned}$$

Now we calculate each term C_i $i=1,2,3,4$. We have

$$\begin{aligned}
C_1&=q^{2\theta_y+\beta'}[x[\theta_x+\beta]_q, x^{-1}[\theta_x]_q] \\
&=q^{2\theta_y+\beta'}\{x[\theta_x+\beta]_q x^{-1}[\theta_x]_q-x^{-1}[\theta_x]_q x[\theta_x+\beta]_q\} \\
&=q^{2\theta_y+\beta'}\{[\theta_x+\beta-1]_q[\theta_x]_q-[\theta_x+1]_q[\theta_x+\beta]_q\} \\
&=-q^{2\theta_y+\beta'}[2\theta_x+\beta]_q,
\end{aligned}$$

where we use Eq. (5). The second term is

$$\begin{aligned}
C_2&=q^{\theta_y}x[\theta_x+\beta]_q q^{-\theta_x-\beta}y^{-1}[\theta_y]_q \\
&\quad -q^{-\theta_x-\beta}y^{-1}[\theta_y]_q q^{\theta_y}x[\theta_x+\beta]_q \\
&=q^{\theta_y-\theta_x-\beta+1}x[\theta_x+\beta]_q y^{-1}[\theta_y]_q \\
&\quad -q^{-\theta_x-\beta+\theta_y+1}y^{-1}[\theta_y]_q x[\theta_x+\beta]_q \\
&=0.
\end{aligned}$$

Similary just above, we obtain $C_3=0$. Finally

$$\begin{aligned}
C_4&=q^{-2\theta_x-\beta}[y[\theta_y+\beta']_q, y^{-1}[\theta_y]_q] \\
&=q^{-2\theta_x-\beta}\{y[\theta_y+\beta']_q y^{-1}[\theta_y]_q-y^{-1}[\theta_y]_q y[\theta_y+\beta']_q\} \\
&=q^{-2\theta_x-\beta}\{[\theta_y+\beta'-1]_q[\theta_y]_q-[\theta_y+1]_q[\theta_y+\beta']_q\} \\
&=-q^{-2\theta_x-\beta}[2\theta_y+\beta']_q,
\end{aligned}$$

where we use the addition formula Eq. (5). Hence we have

$$\begin{aligned}
[e,f]&=q^{2\theta_y+\beta'}[2\theta_x+\beta]_q+q^{-2\theta_x-\beta}[2\theta_y+\beta']_q \\
&=[2\theta_x+\beta+2\theta_y+\beta']_q \\
&=\frac{q^h-q^{-h}}{q-q^{-1}}.
\end{aligned}$$

C. Casimir operator

Here we express Casimir operator by means of the operator $E_{q,0}$. It is well known that the Casimir element C of $U_q(sl(2,C))$ is

$$C=\frac{q^{-1}\cdot q^{h}-2+q\cdot q^{-h}}{(q-q^{-1})^{2}}+ef.$$

In our case, by the direct calculation, we have

$$C=-q^{\theta_y-\theta_x}(q^{-\beta+1}x-q^{\beta'-1}y)E_{q,0}+\left[\frac{\beta+\beta'-1}{2}\right]_q^2.$$

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