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Author(s)	Nagatomo, Kiyokazu; Koga, Yoshiyuki
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q-DIFFERENCE ANALOGUE OF THE EULER-POISSON-DARBOUX EQUATION AND ITS LAPLACE SEQUENCE

KIYOKAZU NAGATOMO¹ and YOSHIYUKI KOGA

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1. Introduction and main result

The Euler-Poisson-Darboux (EPD) equation is the second order hyperbolic equation

$$\bar{L}(\beta,\beta')u = \left\{\partial_x\partial_y - \frac{\beta - \beta'}{x - y}\partial_x + \frac{\beta(\beta' - 1)}{(x - y)^2}\right\}u = 0$$

which appears in various areas of mathematics and physics such as theory of surfaces [2], propagation of sounds [1] and collidings of gravitational waves [3], etc. By the conjugate transform of the differential operator $\bar{L}(\beta,\beta')$ with $(x-y)^{-\beta}$, we have the operator

$$(x-y)^{-\beta}\bar{L}(\beta,\beta')(x-y)^{\beta} = \bar{E}(\beta,\beta') = \partial_x \partial_y - \frac{\beta'}{x-y} \partial_x + \frac{\beta}{x-y} \partial_y.$$

In this note we consider a q-difference analogue of the operator

$$E(\beta,\beta') = (x-y)\overline{E}(\beta,\beta') = (x-y)\partial_x\partial_y - \beta'\partial_x + \beta\partial_y$$
(1)

and demonstrate that q-deformation of $E(\beta,\beta')$ is the q-difference operator (see section 2)

$$E_{q}(\beta,\beta') = [\theta_{x} + \beta]_{q}[\partial_{y}]_{q} - [\theta_{y} + \beta']_{q}[\partial_{x}]_{q}.$$
(2)

The EPD equation has very interesting properties, for example, Miller's symmetry, Laplace sequence and the relation to Toda molecule equation, etc. (see [2] and [6]). First we consider a q-deformation of Miller's symmetry explained below. Let $V(\beta,\beta')$ be the space of solutions of the differential equation $E(\beta,\beta')u=0$. Then $V(\beta,\beta')$ is invariant under the action of SL(2, C) defined by

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$$u(x,y) \mapsto (bx+d)^{-\beta}(by+d)^{-\beta'} u\left(\frac{ax+c}{bx+d}, \frac{ay+c}{by+d}\right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C)$$

and hence infinitesimal generators of this symmetry are

$$E = -x^{2}\partial_{x} - y^{2}\partial_{y} - \beta x - \beta' y,$$

$$H = \partial_{x} + \partial_{y},$$

$$F = 2x\partial_{x} + 2y\partial_{y} + \beta + \beta'.$$

We call this Miller's symmetry. Indeed, this Lie algebra is isomorphic to Lie algebra sl(2,C). It shall be shown that its q-deformation is quantum group $U_a(sl(2,C))$ with generators

$$e = -\{q^{-\theta_x}y[\theta_y + \beta']_q + q^{\theta_y}x[\theta_x + \beta]_q\},\$$

$$f = q^{-\theta_x - \beta}y^{-1}[\theta_y]_q + q^{\theta_y + \beta'}x^{-1}[\theta_x]_q,\$$

$$q^h = q^{2\theta_x + 2\theta_y + \beta + \beta'}.$$

If the parameter q tends to unit, obviously we get Miller's symmetry.

Theorem 1.1. The difference operators e, f and q^h are symmetries of the q-difference EPD equation and are generators of the quantum group $U_a(sl(2,C))$.

REMARK 1.1. This kind of representation of quuantum group can be seen in [4] and [5].

The second aim of our research is to find a q-deformation of the so-called Laplace sequence. We give a brief explanation of the Laplace sequence for the EPD equation. Let us consider a family of differential operators parametrized by an integer n

$$E_n(\beta,\beta') = (x-y)\partial_x\partial_y - (\beta'+n)\partial_x + (\beta-n)\partial_y.$$
(3)

This is a typical example of Laplace sequence for the second order hyperbolic equation with two independent variables (also see [2], [6]). Define two operators H_n and B_n by

$$H_n = (x-y)\partial_y - (\beta'+n), \quad B_n = (x-y)\partial_x + (\beta-n).$$

Then we have

$$H_{n+1}E_n = E_{n+1}H_n, \quad B_{n-1}E_n = E_{n-1}B_n$$

for any integer n. These equations mean that if u_n is a solution of the equation

 $E_n(\beta,\beta')u=0$, then $u_{n+1}=H_nu_n$ or $u_{n-1}=B_nu_n$ is a solution of the equation $E_{n+1}u=0$ or $E_{n-1}u=0$, respectively. Therefore we may think that H_n and B_n are a kind of increasing or decreasing operators. We shall show that q-analogues of H_n and B_n are

$$H_{q,n} = -q^{-\theta_x} [\theta_y + \beta' + n]_q + q^{-\theta_x - (\beta - n - 1)} x y^{-1} [\theta_y]_q,$$

$$B_{q,n} = q^{\theta_y} [\theta_x + \beta - n]_q - q^{\theta_y + (\beta' + n - 1)} y x^{-1} [\theta_x]_q.$$

These q-difference operators are found by quantizing some solution of the EPD equation. The EPD equation has a formal solution

$$\varphi(\lambda,\mu;\beta,\beta';x,y) = \sum_{n\in\mathbb{Z}} \frac{[\lambda+\beta;n][\mu-n+1;n]_q}{[\mu-n+\beta';n][\lambda+1;n]} x^{\lambda+n} y^{\mu-n}$$

where $[\alpha; n] = \Gamma(\alpha + n)/\Gamma(\alpha)$ and $\Gamma(\alpha)$ is the gamma function. We may think that its q-deformation is

$$\varphi_{q}(\lambda,\mu;\beta,\beta';x,y) = \sum_{n \in \mathbb{Z}} \frac{[\lambda+\beta;n]_{q}[\mu-n+1;n]_{q}}{[\mu-n+\beta';n]_{q}[\lambda+1;n]_{q}} x^{\lambda+n} y^{\mu-n}$$

where $[\alpha; n]_q = \Gamma_q(\alpha + n)/\Gamma_q(\alpha)$ and $\Gamma_q(\alpha)$ is the basic gamma function (see section 2). We use the notations φ_q^{λ} and $\varphi_{q,\lambda}$ to denote contiguous functions of φ_q , such as $\varphi_q^{\lambda} = \varphi_q(\lambda + 1, \mu; \beta, \beta'; x, y)$ and $\varphi_{q,\lambda} = \varphi_q(\lambda - 1, \mu; \beta, \beta'; x, y)$, etc. To describe the action of $e, f, q^{\pm h}, H_{q,n}$ and $B_{q,n}$ in a simple form, it is convenient to introduce the function

$$\Phi_{q} = \frac{\Gamma_{q}(\lambda + \beta)\Gamma_{q}(\mu + \beta')}{\Gamma_{q}(\lambda)\Gamma_{q}(\mu + 1)}\varphi_{q}.$$

By using this function we can get the next expression of the action of $U_q(sl(2, C))$ and Laplace sequence

$$e\Phi_{q} = -[\lambda + \mu + 1]_{q}\Phi_{q}^{\lambda},$$

$$f\Phi_{q} = [\lambda + \mu + \beta + \beta' - 1]_{q}\Phi_{q,\mu},$$

$$q^{h}\Phi_{q} = q^{2(\lambda + \mu) + \beta + \beta'}\Phi_{q},$$

$$H_{q,0}\Phi_{q} = -[\beta - 1]_{q}\Phi_{q,\beta}^{\beta'},$$

$$B_{q,0}\Phi_{q} = [\beta' - 1]_{q}\Phi_{q,\beta'}.$$

Finally we give the explanation of the organization of this paper. In the next section, we introduce and fix our notations appeared in the q-analogue calcules. In section 3, we define a q-difference analogue of the EPD equation and give a proof of theorem 1.1 and we shall find its q-Laplace sequence $H_{q,n}$ and $B_{q,n}$ in section 4. The classical results about the EPD equation are stated

in Appendix A. A part of the proof of Theorem 1.1 is given in Appendix B. Finally we express the Casimir operator of $U_q(sl(2, C))$ by means of the operator $E_{q,0}$ in Appendix C.

2. q-difference calculus

In this section, a few elementary results involving basic differentiation are obtained. For any number A, we define basic number $[A]_q$ by the relation

$$[A]_{q} = \frac{q^{A} - q^{-A}}{q - q^{-1}}$$

where q may be real or complex. Then we can easily verify the formula

$$[A+B]_{q} = q^{A}[B]_{q} + q^{-B}[A]_{q}$$

$$= q^{-A}[B]_{q} + q^{B}[A]_{q}$$
(4)

and

$$[A+1]_{q}[B+1]_{q} - [A]_{q}[B]_{q} = [A+B+1]_{q}.$$
(5)

In the following sections, we need q-difference operator (q-differentiation or basic differentiation). First we introduce q-shift operator T by

(Tf)(x) = f(qx),

then q-difference operator $[\partial]_q$ is defined by

$$([\partial]_q f)(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}$$
$$= \frac{1}{x} \left(\frac{T - T^{-1}}{q - q^{-1}} f\right)(x).$$

Further we need q-difference Euler operator $[\theta]_q$

$$\left[\theta\right]_{q} = \frac{T - T^{-1}}{q - q^{-1}}.$$

Because of this definiton, we may identify T and q^{θ} , namely, $q^{\theta = T}$. One of the important properties of the operator $[\theta]_q$ is that it behaves just as the ordinary Euler differential, i.e.

$$[\theta]_q x^n = [n]_q x^n$$

We shall often use the following relations

$$x^n q^{-\theta} = q^{-\theta+n} x^n, \quad x^n q^{\theta} = q^{\theta-n} x^n, \quad x^n [\theta+\alpha]_q = [\theta+\alpha-n]_q x^n,$$

where these all relations are considered as operators. Finally we define basic gamma function by

$$\Gamma_q(x) = q^{(x^2 - 3x)/2} \frac{(q^2)_{\infty}}{(q^{2x})_{\infty}} (1 - q^2)^{1 - x}, \quad (a)_{\infty} = \prod_{j=0}^{\infty} (1 - q^j a).$$

For this basic gamma function we have fundamental difference relation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

3. q-difference analogue of the EPD equation

Let us prove that the q-deformed function φ_q satisfies Eq.(2). From the difference relation of the basic gamma function and the expression of the q-deformed function φ_q

$$\varphi_q(\lambda,\mu;\beta,\beta';x,y) = \sum_{n\in\mathbb{Z}} \frac{[\lambda+\beta;n]_q[\mu-n+1;n]_q}{[\mu-n+\beta';n]_q[\lambda+1;n]_q} x^{\lambda+n} y^{\mu-n},$$

we can get the following contiguous relations of φ_q .

Proposition 3.1. The function φ_q has the following contiguous relations:

1.
$$x^{-1}[\theta_x]_q \varphi_q = [\lambda]_q \varphi_{q,\lambda}^{\beta}, \qquad y^{-1}[\theta_y]_q \varphi_q = [\mu]_q \varphi_{q,\mu}^{\beta'}.$$

2. $[\theta_x + \beta]_q \varphi_q = [\lambda + \beta]_q \varphi_q^{\beta}, \qquad [\theta_y + \beta']_q \varphi_q = [\mu + \beta']_q \varphi_q^{\beta'}.$

By using these contiguous relations, we have

$$[\theta_x + \beta]_q [\partial_y]_q \varphi_q = [\mu]_q [\lambda + \beta]_q \varphi_{q,\mu}^{\beta,\beta'}, [\theta_y + \beta']_q [\partial_x]_q \varphi_q = [\lambda]_q [\mu + \beta']_q \varphi_{q,\lambda}^{\beta,\beta'},$$

and further we can easily verify

 $[\mu]_q[\lambda+\beta]_q\varphi_{q,\mu}^{\beta,\beta'} = [\lambda]_q[\mu+\beta']_q\varphi_{q,\lambda}^{\beta,\beta'},$

by direct calculation. Hence we have proved

$$[\theta_x + \beta]_q [\partial_y]_q \varphi_q = [\theta_y + \beta']_q [\partial_x]_q \varphi_q,$$

which we call the q-difference EPD equation.

Now we will prove that the algebra generated by three q-defference operators

$$e = -\{q^{-\theta_x}y[\theta_y + \beta']_q + q^{\theta_y}x[\theta_x + \beta]_q\},\$$

$$f = q^{-\theta_x - \beta}y^{-1}[\theta_y]_q + q^{\theta_y + \beta'}x^{-1}[\theta_x]_q,\$$

$$g^h = q^{2\theta_x + 2\theta_y + \beta + \beta'}$$

is a q-deformation of Miller's symmetry. First we show the next proposition.

Proposition 3.2. Let $E_q(\beta, \beta')$ be the q-difference EPD operator defined by Eq.(2), then operators e, f and q^h satisfy the following relations:

1. $E_{q}(\beta,\beta')e = -\{q^{-\theta_{x}-1}[\theta_{y}+\beta']_{q}y+q^{\theta_{y}+1}[\theta_{x}+\beta]_{q}x\}E_{q}(\beta,\beta').$ 2. $E_{q}(\beta,\beta')f = fE_{q}(\beta,\beta').$ 3. $E_{d}(\beta,\beta')q^{\pm h} = q^{\pm 2}q^{\pm h}E_{d}(\beta,\beta').$

From this proposition we immediately have the next corollary.

Corollary 3.1. The q-difference operators e, f and $q^{\pm h}$ are symmetries of the q-difference EPD equation.

Proof of Proposition 3.2. Let us prove the first relation. From the difficience operator e, we have

$$\begin{split} E_{q}(\beta,\beta')e \\ &= \left[\theta_{x}+\beta'\right]_{q}x^{-1}\left[\theta_{x}\right]_{q}q^{-\theta_{x}}y\left[\theta_{y}+\beta'\right]_{q}+\left[\theta_{y}+\beta'\right]_{q}x^{-1}\left[\theta_{x}\right]_{q}q^{\theta_{y}}x\left[\theta_{x}+\beta\right]_{q} \\ &-\left[\theta_{x}+\beta\right]_{q}y^{-1}\left[\theta_{y}\right]_{q}q^{-\theta_{x}}y\left[\theta_{y}+\beta'\right]_{q}-\left[\theta_{x}+\beta\right]_{q}y^{-1}\left[\theta_{y}\right]_{q}q^{\theta_{y}}x\left[\theta_{x}+\beta\right]_{q}. \end{split}$$

By using the following relation

$$x^{-1}q^{-\theta_x} = q^{-\theta_x - 1}x^{-1}, \quad x^{-1}q^{\theta_x} = q^{\theta_x + 1}x^{-1}, \quad x^{-1}[\theta_x]_q x = [\theta_x + 1]_q,$$

we see

$$\begin{split} E_{q}(\beta,\beta')e \\ &= q^{-\theta_{x}-1}[\theta_{y}+\beta']_{q}y[\partial_{x}]_{q}[\theta_{y}+\beta']_{q} + q^{\theta_{y}}[\theta_{x}+\beta]_{q}[\theta_{x}+1]_{q}[\theta_{y}+\beta']_{q} \\ &- q^{-\theta_{x}}[\theta_{y}+\beta']_{q}[\theta_{y}+1]_{q}[\theta_{x}+\beta]_{q} - q^{\theta_{y}+1}[\theta_{x}+\beta]_{q}x[\partial_{y}]_{q}[\theta_{x}+\beta]_{q}. \end{split}$$

Further by applying the addition foumula Eq. (4)

$$[\theta_x+1]_q = q[\theta_x]_q + q^{-\theta_x} = q^{-1}[\theta_x]_q + q^{\theta_x},$$

in the second and third terms of the above equation, we get

$$\begin{split} E_{q}(\beta,\beta')e \\ &= q^{-\theta_{x}-1}[\theta_{y}+\beta']_{q}y[\partial_{x}]_{q}[\theta_{y}+\beta']_{q} + q^{\theta_{y}+1}[\theta_{x}+\beta]_{q}[\theta_{x}]_{q}[\theta_{y}+\beta']_{q} \\ &- q^{-\theta_{x}-1}[\theta_{y}+\beta']_{q}[\theta_{y}]_{q}[\theta_{x}+\beta]_{q} - q^{\theta_{y}+1}[\theta_{x}+\beta]_{q}x[\partial_{y}]_{q}[\theta_{x}+\beta]_{q} \end{split}$$

Therefore we have

$$E_q(\beta,\beta')e = -\{q^{-\theta_x-1}[\theta_y+\beta']_q y + q^{\theta_y+1}[\theta_x+\beta]_q x\}E_q(\beta,\beta').$$

The second relation is proved just above by using the relation

$$\begin{aligned} x^{-1}q^{-\theta_{x}-\beta} = q^{-\theta_{x}-1-\beta}x^{-1}, \quad x^{-1}q^{\theta_{x}+\beta} = q^{\theta_{x}+\beta+1}x^{-1}, \\ [\theta_{x}+\beta]_{q}x^{-1} = x^{-1}[\theta_{x}+\beta-1]_{q}, \end{aligned}$$

and the addition formula

$$[\theta_x - 1 + \beta]_q = q^{-1} [\theta_x + \beta]_q - q^{-\theta_x - \beta} = q [\theta_x + \beta]_q - q^{\theta_x + \beta}.$$

Finally we prove the third relation. By the definition of q^h and the formula

$$x^{-1}q^{2\theta_x} = q^2 q^{2\theta_x} x^{-1}, \quad y^{-1}q^{2\theta_y} = q^2 q^{2\theta_y} y^{-1},$$

we get

$$E_q(\beta,\beta')q^h = q^2 q^h E_q(\beta,\beta').$$
 q.e.d

Thus we have proved the first statement of Theorem 1. A proof of the second statement, that three operators e, f and $q^{\pm h}$ are generators of $U_q(sl(2, C))$, namely,

$$q^{h}eq^{-h} = q^{2}e,$$

$$q^{h}fq^{-h} = q^{-2}f,$$

$$[e,f] = \frac{q^{h} - q^{-h}}{q - q^{-1}}.$$

is given in appendix B.

In the following we give a kind of representation of $U_q(sl(2,C))$ on the space of contiguous functions of φ_{q^*}

Proposition 3.3. The q-difference operators e, f and $q^{\pm h}$ act on the space of contiguous functions of φ_q as follows:

$$e\varphi_{q} = -\frac{[\lambda + \beta]_{q}[\lambda + \mu + 1]_{q}}{[\lambda + 1]_{q}}\varphi_{q}^{\lambda},$$

$$f\varphi_{q} = \frac{[\mu]_{q}[\lambda + \mu + \beta + \beta' - 1]_{q}}{[\mu + \beta' - 1]_{q}}\varphi_{q,\mu},$$
$$q^{h}\varphi_{q} = q^{2(\lambda + \mu) + \beta + \beta'}\varphi_{q}, \quad q^{-h}\varphi_{q} = q^{-2(\lambda + \mu) - \beta - \beta'}\varphi_{q}.$$

Proof. By the definition of φ_q , we get

$$\begin{split} -e\varphi_{q} &= \sum_{n\in\mathbb{Z}} q^{-\lambda-n} [\mu-n+\beta']_{q} \frac{[\mu-n+1;n]_{q} [\lambda+\beta;n]_{q}}{[\lambda+1;n]_{q} [\mu-n+\beta';n]_{q}} x^{\lambda} y^{\mu+1} t^{n} \\ &+ \sum_{n\in\mathbb{Z}} q^{\mu-n} [\lambda+n+\beta]_{q} \frac{[\mu-n+1;n]_{q} [\lambda+\beta;n]_{q}}{[\lambda+1;n]_{q} [\mu-n+\beta';n]_{q}} x^{\lambda+1} y^{\mu} t^{n} \\ &= I_{1} + I_{2}, \end{split}$$

where we put t = x/y. Hence by replacing n by n + 1 in the first term I_1 , we have

$$I_{1} = \sum_{n \in \mathbb{Z}} q^{-\lambda - n - 1} [\mu - n - 1 + \beta']_{q} \times \frac{[\mu - n; n + 1]_{q} [\lambda + \beta; n + 1]_{q}}{[\lambda + 1; n + 1]_{q} [\mu - n - 1 + \beta'; n + 1]_{q}} x^{\lambda + 1} y^{\mu} t^{n}$$
$$= \frac{[\lambda + \beta]_{q}}{[\lambda + 1]_{q} n \in \mathbb{Z}} q^{-(\lambda + 1) - n} [\mu - n]_{q} \times \frac{[\mu - n + 1; n]_{q} [\lambda + 1 + \beta; n]_{q}}{[\lambda + 2; n]_{q} [\mu - n + \beta'; n]_{q}} x^{\lambda + 1} y^{\mu} t^{n}.$$

On the other hand,

$$I_{2} = \frac{[\lambda + \beta]_{q}}{[\lambda + 1]_{q}n \in \mathbb{Z}} q^{\mu - n} [\lambda + n + 1]_{q}$$
$$\times \frac{[\mu - n + 1; n]_{q} [\lambda + 1 + \beta; n]_{q}}{[\lambda + 2; n]_{q} [\mu - n + \beta'; n]_{q}} x^{\lambda + 1} y^{\mu} t^{n}.$$

Therefore we get

$$\begin{split} e\varphi_{q} &= -I_{1} - I_{2} \\ &= -\frac{[\lambda + \beta]_{q}}{[\lambda + 1]_{q}neZ} \{q^{-(\lambda + 1) - n}[\mu - n]_{q} + q^{\mu - n}[\lambda + n + 1]_{q} \} \\ &\times \frac{[\mu - n + 1; n]_{q}[\lambda + 1 + \beta; n]_{q}}{[\lambda + 2; n]_{q}[\mu - n + \beta'; n]_{q}} x^{\lambda + 1} y^{\mu} t^{n} \\ &= -\frac{[\lambda + \beta]_{q}[\lambda + \mu + 1]_{q}}{[\lambda + 1]_{q}} \varphi_{q}^{\lambda}, \end{split}$$

where we use the addition formula Eq. (4). Similarly as above, we have

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$$\begin{split} f \varphi_{q} &= \frac{\left[\mu\right]_{q}}{\left[\mu - 1 + \beta'\right]_{q} n \epsilon z} \{q^{-\lambda - n - \beta} \left[\mu - n - 1 + \beta'\right]_{q} + q^{(\mu - 1) - n + \beta'} \left[\lambda + \beta + n\right]_{q}\} \\ &\times \frac{\left[(\mu - 1) - n + 1; n\right]_{q} \left[\lambda + \beta; n\right]_{q}}{\left[\lambda + 1; n\right]_{q} \left[(\mu - 1) - n + \beta'; n\right]_{q}} x^{\lambda} y^{\mu - 1} t^{n} \\ &= \frac{\left[\mu\right]_{q}}{\left[\mu - 1 + \beta'\right]_{q} n \epsilon z} \left[\lambda + \mu + \beta + \beta' - 1\right]_{q} \\ &\times \frac{\left[(\mu - 1) - n + 1; n\right]_{q} \left[\lambda + \mu + \beta + \beta' - 1\right]_{q}}{\left[\lambda + 1; n\right]_{q} \left[(\mu - 1) - n + \beta'; n\right]_{q}} x^{\lambda} y^{\mu - 1} t^{n} \\ &= \frac{\left[\mu\right]_{q} \left[\lambda + \mu + \beta + \beta' - 1\right]_{q}}{\left[\mu - 1 + \beta'\right]_{q}} \varphi_{q,\mu}. \end{split}$$

The last statement is easily proved by direct calculation.

q.e.d

By using the function Φ_q , we get a simple expression of the action of operators e, f and q^h .

Corollary 3.2. The action of operators e, f and q^{h} on the function Φ_{q} is

$$e\Phi_{q} = -[\lambda + \mu + 1]_{q}\Phi_{q}^{\lambda},$$

$$f\Phi_{q} = [\lambda + \mu + \beta + \beta' - 1]_{q}\Phi_{q,\mu},$$

$$q^{h}\Phi_{q} = q^{2(\lambda + \mu) + \beta + \beta'}\Phi_{q}.$$

4. q-Laplace sequence

Here we consider a family of the difference operators

$$E_{q,n}(\beta,\beta') = [\theta_x + \beta - n]_q [\partial_y]_q - [\theta_y + \beta' + n]_q [\partial_x]_q, \qquad n \in \mathbb{Z}$$
(6)

which may be thought as a q-difference analogue of the operator E_n defined by Eq. (2). Our purpose is to find a kind of increasing or decreasing operators. Let us denote two types of q-difference operators $H_{q,n}$ and $B_{q,n}$ by

$$H_{q,n} = -q^{-\theta_{x}} [\theta_{y} + \beta' + n]_{q} + q^{-\theta_{x} - (\beta - n - 1)} x y^{-1} [\theta_{y}]_{q},$$

$$B_{q,n} = q^{\theta_{y}} [\theta_{x} + \beta - n]_{q} - q^{\theta_{y} + (\beta' + n - 1)} y x^{-1} [\theta_{x}]_{q}.$$

Then the next theorem can be proved by direct calculation.

Theorem 4.1.

1.
$$H_{q,n+1}E_{q,n} = qE_{q,n+1}H_{q,n}$$

2.
$$B_{q,n-1}E_{q,n} = q^{-1}E_{q,n-1}B_{q,n}$$

Proof. By replacing β and β' by $\beta+n$ or $\beta'-n$, it is enough to prove when n=0. From the difiniton, we see

$$\begin{split} H_{q,1}E_{q,0} \\ &= -q^{-\theta_x}[\theta_y + \beta' + 1]_q[\theta_x + \beta]_q y^{-1}[\theta_y]_q + q^{-\theta_x}[\theta_y + \beta' + 1]_q[\theta_y + \beta']_q x^{-1}[\theta_x]_q \\ &+ q^{-\theta_x - (\beta - 2)} x y^{-1}[\theta_y]_q[\theta_x + \beta]_q y^{-1}[\theta_y]_q - q^{-\theta_x - (\beta - 2)} x y^{-1}[\theta_y]_q[\theta_y + \beta']_q x^{-1}[\theta_x]_q. \end{split}$$

By using the relations

$$[\theta_x + \beta]_q = q[\theta_x + \beta - 1]_q + q^{-\theta_x - \beta + 1}, \quad [\theta_x]_q = q^{-1}[\theta_x + 1]_q - q^{-\theta_x - 1},$$

at the first and the second terms, we have

$$\begin{split} H_{q,1}E_{q,0} \\ &= -q^{-\theta_{x}+1}[\theta_{y}+\beta'+1]_{q}[\theta_{x}+\beta-1]_{q}y^{-1}[\theta_{y}]_{q} - q^{-\theta_{x}}[\theta_{y}+\beta'+1]_{q}q^{-\theta_{x}-\beta+1}y^{-1}[\theta_{y}]_{q} \\ &+ q^{-\theta_{x}}[\theta_{y}+\beta'+1]_{q}[\theta_{x}+\beta']_{q}x^{-1}[\theta_{x}]_{q} + q^{-\theta_{x}-(\beta-2)}xy^{-1}[\theta_{y}]_{q}[\theta_{x}+\beta]_{q}y^{-1}[\theta_{y}]_{q} \\ &- q^{-\theta_{x}-(\beta-1)}xy^{-1}[\theta_{y}]_{q}[\theta_{y}+\beta']_{q}x^{-1}[\theta_{x}+1]_{q} \\ &+ q^{-\theta_{x}-(\beta-2)}xy^{-1}[\theta_{y}]_{q}[\theta_{y}+\beta']_{q}x^{-1}q^{-\theta_{x}-1} \\ &= -q[\theta_{x}+\beta-1]_{q}y^{-1}[\theta_{y}]_{q}q^{-\theta_{x}}[\theta_{y}+\beta']_{q} - q^{-2\theta_{x}-(\beta-1)}y^{-1}[\theta_{y}+\beta']_{q}[\theta_{y}]_{q} \\ &+ q[\theta_{y}+\beta'+1]_{q}x^{-1}[\theta_{x}]q^{-\theta_{x}}[\theta_{y}+\beta']_{q} + q[\theta_{x}+\beta-1]_{q}y^{-1}[\theta_{y}]_{q}q^{-\theta_{x}-(\beta-1)}xy^{-1}[\theta_{y}]_{q} \\ &- q[\theta_{y}+\beta'+1]_{q}x^{-1}[\theta_{x}]_{q}q^{-\theta_{x}-(\beta-1)}xy^{-1}[\theta_{y}]_{q} + q^{-2\theta_{x}-(\beta-1)}y^{-1}[\theta_{y}]_{q}[\theta_{y}+\beta']_{q} \\ &= qE_{q,1}H_{q,0}. \end{split}$$

Thus the first statement is proved. We will show the second statement.

$$\begin{split} B_{q,-1}E_{q,0} \\ &= q^{\theta_{y}}[\theta_{x}+\beta+1]_{q}[\theta_{x}+\beta]_{q}y^{-1}[\theta_{y}]_{q} - q^{\theta_{y}}[\theta_{x}+\beta+1]_{q}[\theta_{y}+\beta']_{q}x^{-1}[\theta_{x}]_{q} \\ &- q^{\theta_{y}+(\beta'-2)}yx^{-1}[\theta_{x}]_{q}[\theta_{x}+\beta]_{q}y^{-1}[\theta_{y}]_{q} + q^{\theta_{y}+(\beta'-2)}yx^{-1}[\theta_{x}]_{q}[\theta_{y}+\beta']_{q}x^{-1}[\theta_{x}]_{q}. \end{split}$$

Substituting

$$[\theta_{y}+\beta']_{q} = q^{-1}[\theta_{y}+\beta'-1]_{q} + q^{\theta_{y}+\beta'-1}, \quad [\theta_{y}]_{q} = q[\theta_{y}+1]_{q} - q^{\theta_{y}+1}$$

into the second and third terms, we have

$$B_{q,-1}E_{q,0} = q^{\theta_{y}}[\theta_{x}+\beta+1]_{q}[\theta_{x}+\beta]_{q}y^{-1}[\theta_{y}]_{q} - q^{\theta_{y}-1}[\theta_{x}+\beta+1]_{q}[\theta_{y}+\beta'-1]_{q}x^{-1}[\theta_{x}]_{q}$$

q-Analogue of Euler-Poisson-Darboux Equation

$$\begin{split} &-q^{\theta_{y}}[\theta_{x}+\beta+1]_{q}q^{\theta_{y}+\beta'-1}x^{-1}[\theta_{x}]_{q}-q^{\theta_{y}+(\beta'-1)}yx^{-1}[\theta_{x}]_{q}[\theta_{x}+\beta]_{q}y^{-1}[\theta_{y}+1]_{q} \\ &+q^{\theta_{y}+(\beta'-2)}yx^{-1}[\theta_{x}]_{q}[\theta_{x}+\beta]_{q}y^{-1}q^{\theta_{y}+1}+q^{\theta_{y}+(\beta'-2)}xy^{-1}[\theta_{x}]_{q}[\theta_{y}+\beta']_{q}x^{-1}[\theta_{x}]_{q} \\ &=q^{-1}[\theta_{x}+\beta+1]_{q}y^{-1}[\theta_{y}]_{q}q^{\theta_{y}}[\theta_{x}+\beta]_{q}-q^{-1}[\theta_{y}+\beta'-1]_{q}x^{-1}[\theta_{x}]_{q}q^{\theta_{y}}[\theta_{x}+\beta]_{q} \\ &-q^{2\theta_{y}+(\beta'-1)}x^{-1}[\theta_{x}+\beta]_{q}[\theta_{x}]_{q}-q^{-1}[\theta_{x}+\beta+1]_{q}y^{-1}[\theta_{y}]_{q}q^{\theta_{y}+(\beta'-1)}yx^{-1}[\theta_{x}]_{q} \\ &+q^{2\theta_{y}+(\beta'-1)}x^{-1}[\theta_{x}]_{q}[\theta_{x}+\beta]_{q}+q^{-1}[\theta_{y}+\beta'-1]_{q}x^{-1}[\theta_{x}]_{q}q^{\theta_{y}+(\beta'-1)}yx^{-1}[\theta_{x}]_{q} \\ &=q^{-1}E_{q,-1}B_{q,0}. \end{split}$$

REMARK 4.1. The above theorem implies that if u_n is a solution of the equation $E_{q,n}u_n=0$, then $u_{n+1}=H_{q,n}u_n$ or $u_{n-1}=B_{q,n}u_n$ is a solution of $E_{q,n+1}u=0$ or $E_{q,n-1}u=0$, respectively.

We have more infomation about the action of $H_{q,n}$ and $B_{q,n}$.

Proposition 4.1. The action of operators $H_{q,0}$ and $B_{q,0}$ on the space of contiguous functions of φ_q is

$$H_{q,0}\varphi_{q} = -\frac{[\mu + \beta']_{q}[\beta - 1]_{q}}{[\lambda + \beta - 1]_{q}}\varphi_{q,\beta}^{\beta'}, \quad B_{q,0}\varphi_{q} = \frac{[\lambda + \beta]_{q}[\beta' - 1]_{q}}{[\mu + \beta' - 1]_{q}}\varphi_{q,\beta'}^{\beta}.$$
 (7)

Proof. By the definition of φ_q , we get

$$\begin{split} H_{q,0}\varphi_{q} &= -\sum_{n\in\mathbb{Z}} q^{-\lambda-n} [\mu-n+\beta']_{q} \frac{[\lambda+\beta;n]_{q} [\mu-n+1;n]_{q}}{[\mu-n+\beta';n]_{q} [\lambda+1;n]_{q}} x^{\lambda} y^{\mu} t^{n} \\ &+ \sum_{n\in\mathbb{Z}} q^{-\lambda-n-(\beta-1)} [\mu-n+1]_{q} \frac{[\lambda+\beta;n-1]_{q} [\mu-n+2;n-1]_{q}}{[\mu-n+1+\beta';n-1]_{q} [\lambda+1;n-1]_{q}} x^{\lambda} y^{\mu} t^{n} \\ &= \frac{[\mu+\beta']_{q}}{[\lambda+\beta-1]_{q} n\in\mathbb{Z}} \{-q^{-\lambda-n} [\lambda+n+\beta-1]_{q} + q^{-\lambda-n-(\beta-1)} [\lambda+n]_{q} \} \\ &\times \frac{[\lambda+(\beta-1);n]_{q} [\mu-n+1;n]_{q}}{[\mu-n+(\beta'+1);n]_{q} [\lambda+1;n]_{q}} x^{\lambda} y^{\mu} t^{n} \\ &= -\frac{[\mu+\beta']_{q} [\beta-1]_{q}}{[\lambda+\beta-1]_{q}} \varphi_{q,\beta}^{\beta'}. \end{split}$$

Here we used the addition formula

$$-q^{-\lambda-n}[\lambda+n+\beta-1]_q+q^{-\lambda-n-(\beta-1)}[\lambda+n]_q=-[\beta-1]_q.$$

The second statement is proved just above by using addition formula

$$q^{\mu-n}[\mu-n+(\beta'-1)]_{q}-q^{\mu-n+(\beta'-1)}[\mu-n]_{q}=[\beta'-1]_{q}$$

as follows:

$$\begin{split} B_{q,0}\varphi_{q} &= \frac{[\lambda+\beta]_{q}}{[\mu+(\beta'-1)]_{q}n\epsilon z} \{q^{\mu-n}[\mu-n+(\beta'-1)]_{q} - q^{\mu-n+(\beta'-1)}[\mu-n]_{q} \} \\ &\times \frac{[\lambda+(\beta+1);n]_{q}[\mu-n+1;n]_{q}}{[\mu-n+(\beta'-1);n]_{q}[\lambda+1;n]_{q}} x^{\lambda} y^{\mu} t^{n} \\ &= \frac{[\lambda+\beta]_{q}[\beta'-1]_{q}}{[\mu+(\beta'-1)]_{q}} \varphi_{q,\beta'}^{\beta}. \end{split}$$
 q.e.d
Remark 4.2. The action of $H_{q,0}$ and $B_{q,0}$ on Φ_{q} is

$$H_{q,0}\Phi_{q} = -[\beta - 1]_{q}\Phi_{q,\beta}^{\beta'}, \quad B_{q,0}\Phi_{q} = [\beta' - 1]_{q}\Phi_{q,\beta'}^{\beta}.$$

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A. The Euler-Poisson-Darboux Equation

Let us consider some analytic properties of the equation

$$E(\beta,\beta')u = \{(x-y)\partial_x\partial_y - \beta'\partial_x + \beta\partial_y\}u = 0$$
(8)

We would like to find a solution of the form

$$u = x^{\lambda} y^{\mu} \varphi \left(\frac{x}{y} \right)$$

where λ and μ are complex parameters. By substituting this expression into Eq. (8) we have

$$t^{2}(1-t)\varphi''(t) + t\{(\mu-\lambda-1-\beta)t - (\mu-\lambda-1+\beta')\}\varphi'(t) + \{(\lambda+\beta)\mu t - \lambda(\mu+\beta')\}\varphi(t) = 0$$

Especially in the case of $\lambda = 0$ this equation is reduced to Gauss's hypergeometric euqation

$$t(1-t)\varphi''(t) + t\{(\mu-1-\beta)t - (\mu-1+\beta')\}\varphi'(t) + \beta\mu\varphi(t) = 0$$

Hence Eq. (8) have special solutions related to hypergeometric series. For example, we have a solution

$$u(x,y) = y^{\mu}F\left(\mu,-\beta,1-\mu-\beta';\frac{y}{x}\right),$$

where

$$F(a,b,c;t) = \sum_{n=0}^{\infty} \frac{[a;n][b;n]}{[c;n][1;n]} t^n, \quad [a;n] = \Gamma(a+n)/\Gamma(a),$$

is Gauss's hypergeometric series. Hence by using the action of SL(2, C), we obtain Appell's formula

$$u(x,y) = (bx+d)^{-\beta}(by+d)^{-\beta'}(ay+c)^{\mu}(by+d)^{-\mu}F(\mu,-\beta,1-\mu+\beta';\sigma)$$

$$\sigma = \frac{(bx+d)(ay+c)}{(ax+c)(by+d)}.$$

B. A proof of Theorem 1

Here we will prove that three operators e, f and q^h are generators of the quantum group $U_a(sl(2,C))$. Namely, let us prove Serre's relations

$$q^{h}eq^{-h} = q^{2}e, \quad q^{h}fq^{-h} = q^{-2}f, \quad [e,f] = \frac{q^{h} - q^{-h}}{q - q^{-1}}$$

which characterize $U_q(sl(2, C))$. From the definition, we see

$$q^{h}eq^{-h} = -q^{2\theta_{x}+2\theta_{y}+\beta+\beta'}q^{-\theta_{x}}y[\theta_{y}+\beta']_{q}q^{-2\theta_{x}-2\theta_{y}-\beta-\beta'}$$
$$-q^{2\theta_{x}+2\theta_{y}+\beta+\beta'}q^{\theta_{y}}x[\theta_{x}+\beta]_{q}q^{-2\theta_{x}-2\theta_{y}-\beta-\beta'}$$

By using the relations $xq^{-2\theta_x} = q^{-2\theta_x+2}x$ and $yq^{-2\theta_y} = q^{-2\theta_y+2}y$, we obtain

$$q^{h}eq^{-h} = -\left\{q^{-\theta_{x}+2}y[\theta_{y}+\beta']_{q}+q^{\theta_{y}+2}x[\theta_{x}+\beta]_{q}\right\} = q^{2}e$$

and just as the same above we can show $q^h f q^{-h} = q^{-2} f$.

Now we prove the relation

$$[e,f] = \frac{q^h - q^{-h}}{q - q^{-1}}.$$

From the definition of e and f, we have

$$[e,f] = -[q^{\theta_{y}}x[\theta_{x}+\beta]_{q}, q^{\theta_{y}+\beta'}x^{-1}[\theta_{x}]_{q}] - [q^{\theta_{y}}x[\theta_{x}+\beta]_{q}, q^{-\theta_{x}-\beta}y^{-1}[\theta_{y}]_{q}] -[q^{-\theta_{x}}y[\theta_{y}+\beta']_{q}, q^{\theta_{y}+\beta'}x^{-1}[\theta_{x}]_{q}] - [q^{-\theta_{x}}y[\theta_{y}+\beta']_{q}, q^{-\theta_{x}-\beta}y^{-1}[\theta_{y}]_{q}] = -C_{1} - C_{2} - C_{3} - C_{4}.$$

Now we calculate each term C_i i=1,2,3,4. We have

$$C_{1} = q^{2\theta_{y}+\beta'} [x[\theta_{x}+\beta]_{q}, x^{-1}[\theta_{x}]_{q}]$$

$$= q^{2\theta_{y}+\beta'} \{x[\theta_{x}+\beta]_{q}x^{-1}[\theta_{x}]_{q} - x^{-1}[\theta_{x}]_{q}x[\theta_{x}+\beta]_{q}\}$$

$$= q^{2\theta_{y}+\beta'} \{[\theta_{x}+\beta-1]_{q}[\theta_{x}]_{q} - [\theta_{x}+1]_{q}[\theta_{x}+\beta]_{q}\}$$

$$= -q^{2\theta_{y}+\beta'} [2\theta_{x}+\beta]_{q},$$

where we use Eq. (5). The second term is

$$C_{2} = q^{\theta_{y}} x [\theta_{x} + \beta]_{q} q^{-\theta_{x} - \beta} y^{-1} [\theta_{y}]_{q}$$
$$- q^{-\theta_{x} - \beta} y^{-1} [\theta_{y}]_{q} q^{\theta_{y}} x [\theta_{x} + \beta]_{q}$$
$$= q^{\theta_{y} - \theta_{x} - \beta + 1} x [\theta_{x} + \beta]_{q} y^{-1} [\theta_{y}]_{q}$$
$$- q^{-\theta_{x} - \beta + \theta_{y} + 1} y^{-1} [\theta_{y}]_{q} x [\theta_{x} + \beta]_{q}$$
$$= 0.$$

Similary just above, we obtain $C_3 = 0$. Finally

$$C_{4} = q^{-2\theta_{x}-\beta} [y[\theta_{y}+\beta']_{q}, y^{-1}[\theta_{y}]_{q}]$$

$$= q^{-2\theta_{x}-\beta} \{y[\theta_{y}+\beta']_{q}y^{-1}[\theta_{y}]_{q} - y^{-1}[\theta_{y}]_{q}y[\theta_{y}+\beta']_{q}\}$$

$$= q^{-2\theta_{x}-\beta} \{[\theta_{y}+\beta'-1]_{q}[\theta_{y}]_{q} - [\theta_{y}+1]_{q}[\theta_{y}+\beta']_{q}\}$$

$$= -q^{-2\theta_{x}-\beta} [2\theta_{y}+\beta']_{q},$$

where we use the addition formula Eq. (5). Hence we have

$$[e,f] = q^{2\theta_y + \beta'} [2\theta_x + \beta]_q + q^{-2\theta_x - \beta} [2\theta_y + \beta']_q$$
$$= [2\theta_x + \beta + 2\theta_y + \beta']_q$$
$$= \frac{q^h - q^{-h}}{q - q^{-1}}.$$

C. Casimir operator

Here we express Casimir operator by means of the operator $E_{q,0}$. It is well known that the Casimir element C of $U_q(sl(2,C))$ is

$$C = \frac{q^{-1} \cdot q^{h} - 2 + q \cdot q^{-h}}{(q - q^{-1})^{2}} + ef.$$

In our case, by the direct calculation, we have

$$C = -q^{\theta_{y}-\theta_{x}}(q^{-\beta+1}x-q^{\beta'-1}y)E_{q,0} + \left[\frac{\beta+\beta'-1}{2}\right]_{q}^{2}.$$

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Kiyokazu Nagatomo Department of Mathematics, Faculty of Science, Osaka University, Toyonaka, Osaka, 560, Japan

Yoshiyuki Koga Department of Mathematics, Faculty of Science, Osaka University, Toyonaka, Osaka, 560, Japan