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A REMARK ON ODD-PRIMARY COMPONENTS OF SPECIAL UNITARY GROUPS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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Let G be a compact connected Lie group of dimension $d > 0$, and let us assume that an orientation of G is chosen. Let \mathfrak{L} denote the left invariant framing of the tangent bundle of G . For the pair (G, \mathfrak{L}) we obtain by the Pontrjagin-Thom construction an element $[G, \mathfrak{L}]$ in π_d^S . Ossa[6] proved that $72[G, \mathfrak{L}] = 0$. Of course this implies that the p -primary component of $[G, \mathfrak{L}]$ is zero for any prime $p > 3$. As for information on the 3-primary part of general nature we have the following results of Becker-Schultz: For $G = SO(2n)$, $Spin(2n)$ or $U(n)$ the 3-primary component of $[G, \mathfrak{L}]$ is zero [2]. For the exceptional Lie groups Knapp[4] proved that the 3-primary component of $[F, \mathfrak{L}]$ is zero. In this note we give the following additional information:

*) For $n \equiv 0$ or $3 \pmod{4}$ the 3-primary component of $[SU(n), \mathfrak{L}]$ vanishes.

Let τ be an involutive automorphism of G and let K denote the closed subgroup of G consisting of all elements fixed by τ . Then using the equivariant stable homotopy theory for involutions, we have

Proposition 1. *If K is of odd codimension in G , then*

$$[G, \mathfrak{L}]_{(odd)} = 0$$

where $a_{(odd)}$ denotes the odd-primary part of a .

The assertion *) is an immediate corollary of this proposition. According to the classification theorem of irreducible Riemannian symmetric spaces, examples of Lie groups to which this proposition applies are $SU(4n)$, $SU(4n+3)$, $Spin(2n)$ and $SO(2n)$. For $SU(n)$ (resp. $Spin(n+1)$ and $SO(n+1)$) we adopt the involutive automorphism corresponding to the symmetric space of AI-type (resp. of BDII-type), whose fixed point set is of codimension $(n-1)(n+2)/2$ (resp. n).

In the final section we make a remark on the real Adams e -invariant e'_k and we show

$$e'_k[SO(16n+6), \mathbb{Z}] = e'_k[Spin(16n+6), \mathbb{Z}] = e'_k[SU(4n), \mathbb{Z}] = 0 \quad [1, 4].$$

1. Proposition 1 is generalized as follows. Let $R^{p,q}$ denote the euclidean space R^{p+q} with the linear involution which reverses the first p coordinates and fixes the last q . Let M be a closed smooth manifold with a smooth involution τ . If there exist integers $r, s \geq 0$ and an isomorphism of Z_2 -vector bundles

$$\Phi_\tau: TM \oplus (M \times R^{r,s}) \rightarrow M \times R^{p+r, q+s}$$

for r, s where TM is the tangent bundle of M , then we say that M is (p, q) -framed. (This terminology stands for Segal's $R^{p,q}$ -framed in [7].) Let $\pi_{p,q}^S$ be the (p, q) -th equivariant stable homotopy group of sphere with involution of Landweber [5].

When M is (p, q) -framed, the pair (M, Φ_τ) defines an element $[M, \Phi_\tau]^{Z_2}$ in $\pi_{p,q}^S$ via the equivariant Pontrjagin-Thom construction. Here forget the involution, then Φ_τ becomes the usual stable framing of M which we denote by Φ . We also denote as usual by $[M, \Phi]$ the element of π_{p+q}^S defined by the pair (M, Φ) similarly.

Proposition 2. *Suppose that M is (p, q) -framed and p is odd. Then*

$$[M, \Phi]_{(odd)} = 0.$$

Before we prove Proposition 2 we observe that G is $(d-\dim K, \dim K)$ -framed. Recall that identifying R^d with $T_e G$ in the orientation preserving way, the left invariant framing $\mathfrak{L}: TG \rightarrow G \times R^d$ of G is given by $\mathfrak{L}(v_g) = (g, (L_{g^{-1}})_* v_g)$ $v_g \in T_g G$, where $T_g G$ denotes the tangent space at $g \in G$ and $L_{g^{-1}}: G \rightarrow G$ the left multiplication by g^{-1} . If we consider here the action on TG induced by τ , then it is easy to check that \mathfrak{L} itself is an isomorphism of Z_2 -vector bundles $TG \rightarrow G \times R^{d-h, h}$ where h denotes the dimension of K . Thus we see that Proposition 1 is obtained as a corollary of Proposition 2.

2. Proof of Proposition 2. According to [5] there are different equivariant stable homotopy groups of spheres with involutions $\lambda_{p,q}^S$ and an exact sequence involving them

$$(1) \quad \rightarrow \lambda_{p,q}^S \rightarrow \pi_{p,q}^S \xrightarrow{\phi} \pi_q^S \rightarrow \lambda_{p,q-1}^S \rightarrow$$

where ϕ is the fixed-point homomorphism. Furthermore we have by [5], Proposition 6.1 and Corollary 6.3 an isomorphism

$$\lambda_{p,q}^S \cong \pi_{2^a+q}^S(P^\infty/P^{2^a-p-1})$$

for a suitably large a (depending on p and q) where P^h is the h -dimensional real projective space. Using the Atiyah-Hirzebruch spectral sequence we therefore obtain

$$(2) \quad \lambda_{p,q}^S_{\langle odd \rangle} = 0 \quad \text{for } p \text{ odd.}$$

(For a finite abelian group A , $A_{\langle odd \rangle}$ denotes the odd components of A .) From (1) and (2) we have an isomorphism

$$(3) \quad \phi: \pi_{p,q}^S_{\langle odd \rangle} \cong \pi_{q}^S_{\langle odd \rangle} \quad \text{for } p \text{ odd [3].}$$

We now prove Proposition 2. Let I be an involution of $R^{p+r,q+s}$ defined by $I(x_1, x_2, \dots, x_{p+q+r+s}) = (-x_1, x_2, \dots, x_{p+q+r+s})$. Assuming that M has the equivariant stable framing Φ_τ as above, we see by definition that $\phi[M, \Phi_\tau]^{Z_2} = \phi[M, (1 \times I)\Phi_\tau]^{Z_2}$. So it follows from (3) that

$$(4) \quad [M, \Phi_\tau]^{Z_2}_{\langle odd \rangle} = [M, (1 \times I)\Phi_\tau]^{Z_2}_{\langle odd \rangle}.$$

By definition we also have $\psi[M, (1 \times I)\Phi_\tau]^{Z_2} = -[M, \Phi_\tau]^{Z_2}$ where ψ is the forgetful homomorphism. By this and (4) we therefore have $[M, \Phi_\tau]_{\langle odd \rangle} = 0$, which completes the proof of Proposition 2.

3. We prove here the following

Proposition 3. *Suppose that $4n-1 > p \geq 0$ and $p \equiv 1, 5 \pmod 8$, n even; $p \equiv 3, 7 \pmod 8$. Then the composite*

$$e'_R \psi: \pi_{p,4n-p-1}^S \rightarrow Q/Z$$

is zero.

Proof. By $\Sigma^{p,q}$ we denote the one-point compactification of $R^{p,q}$ with ∞ as base point. Let $f: \Sigma^{8r+p,8r+4n-p-1} \rightarrow \Sigma^{8r,8r}$ be a base point preserving Z_2 -map for large r and \bar{f} be the map obtained from f by forgetting the action. Applying \widetilde{KO} to the cofibre sequence

$$S^{16r+4n-1} \xrightarrow{\bar{f}} S^{16r} \xrightarrow{i} C_{\bar{f}} \rightarrow S^{16r+4n} \rightarrow S^{16r+1}$$

where $C_{\bar{f}}$ is the mapping cone of \bar{f} and i is the injection map, we have an exact sequence

$$(1) \quad 0 \leftarrow \widetilde{KO}(S^{16r}) \xleftarrow{i^*} \widetilde{KO}(C_{\bar{f}}) \leftarrow \widetilde{KO}(S^{16r+4n}) \leftarrow 0. \\ \cong Z \qquad \qquad \qquad \cong Z$$

Using the periodicity theorem and Lemma 4.1 in [5] we get

$$(2) \quad \widetilde{KO}_{Z_2}(\Sigma^{8r+p,8r+4n-p-\varepsilon}) \cong \begin{cases} 0 \text{ or } Z_2 & \text{if } \varepsilon = 0 \\ 0 \text{ or } Z & \text{if } \varepsilon = 1. \end{cases}$$

Furthermore it follows from the argument in §2 that $\lambda_{p,4n-p-1}^S \otimes Q = 0$ and hence $\pi_{p,4n-p-1}^S$ is a finite group. So applying \widetilde{KO}_{Z_2} to the cofibre sequence

induced by f , we see that

$$i^*: \widetilde{KO}_{Z_2}(C_f) \rightarrow \widetilde{KO}_{Z_2}(\Sigma^{8r,8r}) \text{ is onto}$$

where i is the injection map $\Sigma^{8r,8r} \subset C_f$. Therefore we can choose a generator ξ of $\widetilde{KO}_{Z_2}(C_f)$ such that $i^*\psi(\xi)$ generates $\widetilde{KO}(S^{16r})$ where ψ denotes the forgetful homomorphism in K -theory. Let ψ^k be the k -th Adams operation in KO - or KO_{Z_2} -theory. Since $\psi^k\psi^k = \psi^{2k}$ we have by [5], Lemma 3.3 and (2)

$$\psi^k\psi^k(\xi) = k^{8r}\psi(\xi),$$

which implies that (1) splits as an exact sequence of groups admitting operations ψ^k . Hence by definition

$$e_k[\tilde{f}] = 0$$

where $[\tilde{f}]$ is the stable homotopy class of \tilde{f} . This completes the proof.

Again considering the classification of involutive automorphisms of Lie groups, we have the examples above to which this proposition applies.

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