

Title	A remark on odd-primary components of special unitary groups
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Citation	Osaka Journal of Mathematics. 21(2) P.457-P.460
Issue Date	1984
Text Version	publisher
URL	<a href="https://doi.org/10.18910/7164">https://doi.org/10.18910/7164</a>
DOI	10.18910/7164
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## A REMARK ON ODD-PRIMARY COMPONENTS OF SPECIAL UNITARY GROUPS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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(Received March 9, 1983)

Let  $G$  be a compact connected Lie group of dimension  $d > 0$ , and let us assume that an orientation of  $G$  is chosen. Let  $\mathfrak{L}$  denote the left invariant framing of the tangent bundle of  $G$ . For the pair  $(G, \mathfrak{L})$  we obtain by the Pontrjagin-Thom construction an element  $[G, \mathfrak{L}]$  in  $\pi_d^S$ . Ossa[6] proved that  $72[G, \mathfrak{L}] = 0$ . Of course this implies that the  $p$ -primary component of  $[G, \mathfrak{L}]$  is zero for any prime  $p > 3$ . As for information on the 3-primary part of general nature we have the following results of Becker-Schultz: For  $G = SO(2n)$ ,  $Spin(2n)$  or  $U(n)$  the 3-primary component of  $[G, \mathfrak{L}]$  is zero [2]. For the exceptional Lie groups Knapp[4] proved that the 3-primary component of  $[F, \mathfrak{L}]$  is zero. In this note we give the following additional information:

\*) For  $n \equiv 0$  or  $3 \pmod{4}$  the 3-primary component of  $[SU(n), \mathfrak{L}]$  vanishes.

Let  $\tau$  be an involutive automorphism of  $G$  and let  $K$  denote the closed subgroup of  $G$  consisting of all elements fixed by  $\tau$ . Then using the equivariant stable homotopy theory for involutions, we have

**Proposition 1.** *If  $K$  is of odd codimension in  $G$ , then*

$$[G, \mathfrak{L}]_{(odd)} = 0$$

where  $a_{(odd)}$  denotes the odd-primary part of  $a$ .

The assertion \*) is an immediate corollary of this proposition. According to the classification theorem of irreducible Riemannian symmetric spaces, examples of Lie groups to which this proposition applies are  $SU(4n)$ ,  $SU(4n+3)$ ,  $Spin(2n)$  and  $SO(2n)$ . For  $SU(n)$  (resp.  $Spin(n+1)$  and  $SO(n+1)$ ) we adopt the involutive automorphism corresponding to the symmetric space of AI-type (resp. of BDII-type), whose fixed point set is of codimension  $(n-1)(n+2)/2$  (resp.  $n$ ).

In the final section we make a remark on the real Adams  $e$ -invariant  $e'_k$  and we show

$$e'_k[SO(16n+6), \mathbb{Z}] = e'_k[Spin(16n+6), \mathbb{Z}] = e'_k[SU(4n), \mathbb{Z}] = 0 \quad [1, 4].$$

1. Proposition 1 is generalized as follows. Let  $R^{p,q}$  denote the euclidean space  $R^{p+q}$  with the linear involution which reverses the first  $p$  coordinates and fixes the last  $q$ . Let  $M$  be a closed smooth manifold with a smooth involution  $\tau$ . If there exist integers  $r, s \geq 0$  and an isomorphism of  $Z_2$ -vector bundles

$$\Phi_\tau: TM \oplus (M \times R^{r,s}) \rightarrow M \times R^{p+r,q+s}$$

for  $r, s$  where  $TM$  is the tangent bundle of  $M$ , then we say that  $M$  is  $(p, q)$ -framed. (This terminology stands for Segal's  $R^{p,q}$ -framed in [7].) Let  $\pi_{p,q}^S$  be the  $(p, q)$ -th equivariant stable homotopy group of sphere with involution of Landweber [5].

When  $M$  is  $(p, q)$ -framed, the pair  $(M, \Phi_\tau)$  defines an element  $[M, \Phi_\tau]^{Z_2}$  in  $\pi_{p,q}^S$  via the equivariant Pontrjagin-Thom construction. Here forget the involution, then  $\Phi_\tau$  becomes the usual stable framing of  $M$  which we denote by  $\Phi$ . We also denote as usual by  $[M, \Phi]$  the element of  $\pi_{p+q}^S$  defined by the pair  $(M, \Phi)$  similarly.

**Proposition 2.** *Suppose that  $M$  is  $(p, q)$ -framed and  $p$  is odd. Then*

$$[M, \Phi]_{(odd)} = 0.$$

Before we prove Proposition 2 we observe that  $G$  is  $(d-\dim K, \dim K)$ -framed. Recall that identifying  $R^d$  with  $T_e G$  in the orientation preserving way, the left invariant framing  $\mathfrak{L}: TG \rightarrow G \times R^d$  of  $G$  is given by  $\mathfrak{L}(v_g) = (g, (L_{g^{-1}})_* v_g)$   $v_g \in T_g G$ , where  $T_g G$  denotes the tangent space at  $g \in G$  and  $L_{g^{-1}}: G \rightarrow G$  the left multiplication by  $g^{-1}$ . If we consider here the action on  $TG$  induced by  $\tau$ , then it is easy to check that  $\mathfrak{L}$  itself is an isomorphism of  $Z_2$ -vector bundles  $TG \rightarrow G \times R^{d-h,h}$  where  $h$  denotes the dimension of  $K$ . Thus we see that Proposition 1 is obtained as a corollary of Proposition 2.

2. Proof of Proposition 2. According to [5] there are different equivariant stable homotopy groups of spheres with involutions  $\lambda_{p,q}^S$  and an exact sequence involving them

$$(1) \quad \rightarrow \lambda_{p,q}^S \rightarrow \pi_{p,q}^S \xrightarrow{\phi} \pi_q^S \rightarrow \lambda_{p,q-1}^S \rightarrow$$

where  $\phi$  is the fixed-point homomorphism. Furthermore we have by [5], Proposition 6.1 and Corollary 6.3 an isomorphism

$$\lambda_{p,q}^S \cong \pi_{2^a+q}^S(P^\infty/P^{2^a-p-1})$$

for a suitably large  $a$  (depending on  $p$  and  $q$ ) where  $P^h$  is the  $h$ -dimensional real projective space. Using the Atiyah-Hirzebruch spectral sequence we therefore obtain

$$(2) \quad \lambda_{p,q}^S_{\langle odd \rangle} = 0 \quad \text{for } p \text{ odd.}$$

(For a finite abelian group  $A$ ,  $A_{\langle odd \rangle}$  denotes the odd components of  $A$ .) From (1) and (2) we have an isomorphism

$$(3) \quad \phi: \pi_{p,q}^S_{\langle odd \rangle} \cong \pi_{q}^S_{\langle odd \rangle} \quad \text{for } p \text{ odd [3].}$$

We now prove Proposition 2. Let  $I$  be an involution of  $R^{p+r,q+s}$  defined by  $I(x_1, x_2, \dots, x_{p+q+r+s}) = (-x_1, x_2, \dots, x_{p+q+r+s})$ . Assuming that  $M$  has the equivariant stable framing  $\Phi_\tau$  as above, we see by definition that  $\phi[M, \Phi_\tau]^{Z_2} = \phi[M, (1 \times I)\Phi_\tau]^{Z_2}$ . So it follows from (3) that

$$(4) \quad [M, \Phi_\tau]^{Z_2_{\langle odd \rangle}} = [M, (1 \times I)\Phi_\tau]^{Z_2_{\langle odd \rangle}}.$$

By definition we also have  $\psi[M, (1 \times I)\Phi_\tau]^{Z_2} = -[M, \Phi_\tau]^{Z_2}$  where  $\psi$  is the forgetful homomorphism. By this and (4) we therefore have  $[M, \Phi_\tau]_{\langle odd \rangle} = 0$ , which completes the proof of Proposition 2.

3. We prove here the following

**Proposition 3.** *Suppose that  $4n-1 > p \geq 0$  and  $p \equiv 1, 5 \pmod{8}$ ,  $n$  even;  $p \equiv 3, 7 \pmod{8}$ . Then the composite*

$$e'_R \psi: \pi_{p,4n-p-1}^S \rightarrow Q/Z$$

is zero.

Proof. By  $\Sigma^{p,q}$  we denote the one-point compactification of  $R^{p,q}$  with  $\infty$  as base point. Let  $f: \Sigma^{8r+p,8r+4n-p-1} \rightarrow \Sigma^{8r,8r}$  be a base point preserving  $Z_2$ -map for large  $r$  and  $\bar{f}$  be the map obtained from  $f$  by forgetting the action. Applying  $\widetilde{KO}$  to the cofibre sequence

$$S^{16r+4n-1} \xrightarrow{\bar{f}} S^{16r} \xrightarrow{i} C_{\bar{f}} \rightarrow S^{16r+4n} \rightarrow S^{16r+1}$$

where  $C_{\bar{f}}$  is the mapping cone of  $\bar{f}$  and  $i$  is the injection map, we have an exact sequence

$$(1) \quad 0 \leftarrow \widetilde{KO}(S^{16r}) \xleftarrow{i^*} \widetilde{KO}(C_{\bar{f}}) \leftarrow \widetilde{KO}(S^{16r+4n}) \leftarrow 0. \\ \cong Z \qquad \qquad \qquad \cong Z$$

Using the periodicity theorem and Lemma 4.1 in [5] we get

$$(2) \quad \widetilde{KO}_{Z_2}(\Sigma^{8r+p,8r+4n-p-\varepsilon}) \cong \begin{cases} 0 \text{ or } Z_2 & \text{if } \varepsilon = 0 \\ 0 \text{ or } Z & \text{if } \varepsilon = 1. \end{cases}$$

Furthermore it follows from the argument in §2 that  $\lambda_{p,4n-p-1}^S \otimes Q = 0$  and hence  $\pi_{p,4n-p-1}^S$  is a finite group. So applying  $\widetilde{KO}_{Z_2}$  to the cofibre sequence

induced by  $f$ , we see that

$$i^*: \widetilde{KO}_{Z_2}(C_f) \rightarrow \widetilde{KO}_{Z_2}(\Sigma^{8r,8r}) \text{ is onto}$$

where  $i$  is the injection map  $\Sigma^{8r,8r} \subset C_f$ . Therefore we can choose a generator  $\xi$  of  $\widetilde{KO}_{Z_2}(C_f)$  such that  $i^*\psi(\xi)$  generates  $\widetilde{KO}(S^{16r})$  where  $\psi$  denotes the forgetful homomorphism in  $K$ -theory. Let  $\psi^k$  be the  $k$ -th Adams operation in  $KO$ - or  $KO_{Z_2}$ -theory. Since  $\psi^k\psi^k = \psi^{2k}$  we have by [5], Lemma 3.3 and (2)

$$\psi^k\psi^k(\xi) = k^{8r}\psi^k(\xi),$$

which implies that (1) splits as an exact sequence of groups admitting operations  $\psi^k$ . Hence by definition

$$e_k[\tilde{f}] = 0$$

where  $[\tilde{f}]$  is the stable homotopy class of  $\tilde{f}$ . This completes the proof.

Again considering the classification of involutive automorphisms of Lie groups, we have the examples above to which this proposition applies.

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