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# ON ISOTROPY ALGEBRAS OF A LIE ALGEBRA OF VECTOR FIELDS WHICH SATISFIES A CERTAIN CONVERGENCE CONDITION

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Let  $M$  be a  $C^\infty$ -manifold and let  $L(M)$  be the Lie algebra of all  $C^\infty$ -vector fields on  $M$ . For a Lie subalgebra  $L$  of  $L(M)$ , Matsuda [3] gave a sufficient condition for its integrability. Precisely speaking, for a Lie subalgebra  $L$  which satisfies the convergence condition (C) stated below, Matsuda proved that through every point of  $M$  there passes a maximal integral manifold of  $L$ .

For two elements  $u$  and  $v$  of  $L(M)$ , defining  $(\text{ad } v)^k u$  inductively as  $[v, (\text{ad } v)^{k-1} u]$ , we put

$$g_t(u, v) = \sum_{j=0}^{\infty} (-1)^j \frac{t^j}{j!} (\text{ad } v)^j u.$$

Let  $c(u, v; x)$  be the radius of convergence of  $g_t(u, v)$  at  $x$  of  $M$ .

(C) For any pair of  $u$  and  $v$  in  $L$  and any compact set  $K$  in  $M$ , there exists a positive number  $c(u, v; K)$  such that

- (i) we have  $c(u, v; x) \geq c(u, v; K)$  at every  $x \in K$ , and
- (ii)  $g_t(u, v)$  is continuously differentiable with respect to  $(t, x)$  term by term at every  $(t, x)$  which satisfies  $|t| < c(u, v; K)$  and  $x \in K^i$ , the interior of  $K$ .

In this paper we say a Lie subalgebra  $L$  satisfies the condition  $(C_k)$ , if 'continuously differentiable' in (ii) of Matsuda's condition (C) can be replaced by ' $(k+1)$ -th continuously differentiable'. Here  $k$  is a non-negative integer.

For any Lie subalgebra  $L$  of  $L(M)$  and a point  $x$  of  $M$ , we shall define  $g_l^k(x)$ , the  $(l, k)$ -isotropy algebra of  $L$  at  $x$ , as follows: Let  $L_x^k$  denote the subalgebra of  $L$  consisting of vector fields whose coefficients vanish at  $x$  with all their derivatives through order  $k$  (in one and hence all coordinate systems). Then  $L_x^k$  is an ideal in  $L_x^l$  for  $k \geq l \geq 0$ . We shall denote the factor algebra  $L_x^l / L_x^k$  by  $g_l^k(x)$  for  $k \geq l \geq 0$ . In particular  $g_1^0(x)$  is the linear isotropy algebra of  $L$  at  $x$ .

**Theorem 1.** Let  $M$  be a connected, paracompact manifold and let  $L$  be a Lie subalgebra satisfying the condition  $(C_k)$  with  $k > 0$ . Then the  $(t, s)$ -isotropy algebras

$g_s^t(x)$  and  $g_s^t(y)$  of  $L$  are isomorphic for  $k \geq s > t \geq 0$ , if  $x$  and  $y$  lie on the same orbit under  $L$ . In particular the linear isotropy algebra of  $L$  is isomorphic each other on every orbit under  $L$ .

Here an orbit under  $L$  is the set of all points of  $M$  that can be joined each other by finite number of integral curves of  $L$ .

Every finite dimensional Lie subalgebra  $L$  satisfies the condition  $(C_k)$  for all  $k$  (see [3]). Also if we are in the real analytic category, then every Lie subalgebra  $L$  of  $L(M)$  satisfies the condition  $(C_k)$  for all  $k$  (see [3]).

A subalgebra  $L$  is called transitive if  $L(x)$  equals the tangent space  $T_x(M)$  at every  $x$  of  $M$  where  $L(x)$  is the subspace of  $T_x(M)$  defined by  $L(x) = \{u(x); u \in L\}$ .

**Theorem 2.** *Under the same assumptions as Theorem 1, if moreover  $L$  is transitive, then there exists a  $G$ -structure of order  $l$  with  $1 \leq l \leq k$  such that the Lie algebra of  $G$  is isomorphic to  $g_l^0(x)$  for every  $x \in M$ .*

A  $G$ -structure of order  $l$  is by definition a reduction of the bundle  $F'(M)$  of  $l$  frames of  $M$  to the group  $G$  (see [4]).

The conclusion of Theorem 2 was obtained by Singer and Sternberg ([6], p. 39) under the assumption that  $L$  (Lie algebra sheaf in their case) is invariant by a local one parameter transformation group generated by any element of  $L$ . Theorem 2 will be proved through the proof of Theorem 1.

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**Proof of Theorem 1.** For simplicity, we shall prove the case  $t=0$  and  $s=1$ . The proof for the general case proceeds similarly, if we replace  $M$  by  $F'(M)$  and  $F(M)$  by  $F^{s-t}(F'(M))$  respectively in the following argument. Let  $N(x)$  be the orbit under  $L$  through  $x \in M$ . By the result of Matsuda,  $N(x)$  is a maximal connected integral manifold of  $L$  through  $x$ . We shall denote by  $F(M)$  the bundle of linear frames of  $M$  with  $\pi$  as projection. Since  $N(x)$  is a submanifold of  $M$ ,  $\pi^{-1}(N(x))$  is a submanifold of  $F(M)$ . Each vector field  $u$  on  $M$  induces a vector field  $F(u)$  on  $F(M)$  as follows: Let  $\phi_t(u)$  be a local one parameter transformation group of  $M$  generated by  $u$  and let  $F(\phi_t(u))$  be a local one parameter transformation group of  $F(M)$  defined by

$$F(\phi_t(u)) \cdot w = \phi_t(u)_{*z} \cdot w \quad w \in F(M) \quad \text{and} \quad z = \pi(w).$$

We shall denote its infinitesimal generator by  $F(u)$ . The mapping  $F: u \rightarrow F(u)$  of  $L(M)$  into  $L(F(M))$  is a Lie algebra homomorphism and has following properties:

$$(i) \quad \pi_* F(u) = u$$

(ii)  $R_a * F(u) = F(u)$  for any  $a \in GL(m, R)$ , where  $R_a$  is the right translation of  $F(M)$  by an element  $a$  and  $m$  is  $\dim M$ . We put  $F(L) = \{F(u); u \in L\}$ , which is a Lie algebra. For  $u \in L$  a vector field  $F(u)$  on  $F(M)$  is, as is clear from the

definition, tangent to the submanifold  $\pi^{-1}(N(x))$  at all points and so defines a vector field on  $\pi^{-1}(N(x))$ . The set of vector fields on  $\pi^{-1}(N(x))$  so obtained is a Lie algebra which we shall denote by  $\overline{F(L)}$ . If  $L$  satisfies the condition  $(C_1)$  on  $M$ ,  $F(L)$  satisfies the condition  $(C_0)$  on  $F(M)$  and so  $\overline{F(L)}$  satisfies the condition  $(C_0)$  on  $\pi^{-1}(N(x))$ . Let  $p$  be a point of  $F(M)$  with  $\pi(p)=x$ . We denote the orbit under  $\overline{F(L)}$  through  $p$  by  $W(p)$ . Then again by the result of Matsuda,  $W(p)$  is a maximal integral manifold of  $\overline{F(L)}$ . We have clearly  $\pi(W(p))=N(x)$ . Put  $n=\dim N(x)$  and  $r=\dim W(p)-n$ .

(1) The intersection  $W(p) \cap \pi^{-1}(x)$  is a  $r$ -dimensional submanifold of three manifolds  $\pi^{-1}(N(x))$ ,  $\pi^{-1}(x)$  and  $W(p)$ . It has at most countably many connected components.

We shall prove this statement (1). For  $w \in \pi^{-1}(N(x))$  the differential  $\pi_{*w}$  at  $w$  of  $\pi$  maps  $\overline{F(L)}(w)$  onto the tangent space  $T_{\pi(w)}(N(x))$  at  $\pi(w)$ , where  $\overline{F(L)}(w) = \{u(w) : u \in \overline{F(L)}\}$ , since  $\pi_* F(u) = u$  and  $L(z) = T_z(N(x))$ . Hence we obtain  $T_w(\pi^{-1}(N(x))) = T_w(W(p)) + T_w(\pi^{-1}(x))$  for any  $w \in W(p) \cap \pi^{-1}(x)$ , i.e., two submanifolds  $W(p)$  and  $\pi^{-1}(x)$  of  $\pi^{-1}(N(x))$  are transverse at their intersection. From this fact the first statement follows (see [5], p. 30). To prove the second part of (1), we first show that  $\overline{F(L)}$  defines an involutive distribution in the sense of Chevalley [1] on  $\pi^{-1}(N(x))$ . For this, it suffices to show that the dimension of  $\overline{F(L)}(w)$  is constant on  $\pi^{-1}(N(x))$ . The orbit  $W(p)$  is an integral manifold of  $\overline{F(L)}$  and so  $\dim \overline{F(L)}(w)$  is constant on  $W(p)$ . Since  $\pi(W(p)) = N(x)$ , and since each vector field of  $F(L)$  is invariant by the action of  $GL(m, \mathbb{R})$ , we have  $\dim \overline{F(L)}(w) = \text{constant}$  on  $\pi^{-1}(N(x))$ . Hence each point  $w \in \pi^{-1}(x)$  has a neighbourhood  $U$  in  $\pi^{-1}(N(x))$  with coordinates  $(x_1, \dots, x_{n+m^2})$  such that the slices given by  $x_{n+r+1} = \text{const.}, \dots, x_{n+m^2} = \text{const.}$  are integral manifolds of  $\overline{F(L)}$  ([1], p. 89, Theorem 1). Since two submanifolds  $W(p)$  and  $\pi^{-1}(x)$  of  $\pi^{-1}(N(x))$  are transverse at their intersection, we can change this coordinates such that, in addition to the above condition, the set  $V = \{q \in U : x_1(q) = \dots = x_n(q) = 0\}$  together with the restriction of  $(x_{n+1}, \dots, x_{n+m^2})$  to  $V$  form a local chart on  $\pi^{-1}(x)$  containing  $w$  (compare the proof in [5], p. 30, Lemma 6). Hence  $W(p) \cap V$  is the union of certain sets of planes represented by  $x_1 = 0, \dots, x_n = 0, x_{n+r+1} = \text{const.}, \dots, x_{n+m^2} = \text{const.}$ , each of which is an open set of  $W(p) \cap \pi^{-1}(x)$ . The intersection  $W(p) \cap U$  is the union of certain sets of slices given by  $x_{n+r+1} = \text{const.}, \dots, x_{n+m^2} = \text{const.}$ . The paracompactness of  $M$  implies that the connected submanifold  $N(x)$  of  $M$  and hence  $\pi^{-1}(N(x))$  satisfies the second axiom of countability and so does the connected submanifold  $W(p)$  of  $\pi^{-1}(N(x))$ . Then it follows that the intersection  $W(p) \cap U$  is the union of at most countably many of slices and so the neighbourhood  $V$  of  $w$  in  $\pi^{-1}(x)$  can meet at most countably many connected components of  $W(p) \cap \pi^{-1}(x)$ . Thus we have shown that each  $w \in \pi^{-1}(x)$  has a neighbourhood  $V$  in  $\pi^{-1}(x)$  which meets at most countably

many connected components of  $W(p) \cap \pi^{-1}(x)$ . Since  $\pi^{-1}(x)$  can be covered with countably many of these open sets, we have the second part of (1).

Each element of  $F(L_x^0) = \{F(u) : u \in L_x^0\}$  is tangent to submanifold  $\pi^{-1}(x)$  at all points and so defines a vector field on  $\pi^{-1}(x)$ . The vector fields on  $\pi^{-1}(x)$  so obtained is a Lie algebra which we shall denote by  $\overline{F(L_x^0)}$ . Then the orbit  $W_0(w)$  under  $\overline{F(L_x^0)}$  through  $w$  is the connected component of  $W(p) \cap \pi^{-1}(x)$  containing  $w$  and so a maximal integral manifold of  $\overline{F(L_x^0)}$  through  $w$ . Since  $\overline{F(L_x^0)}$  is invariant by the action of  $GL(m, R)$ , the translations  $R_a$  ( $a \in GL(m, R)$ ) permute among themselves the maximal integral manifolds of  $\overline{F(L_x^0)}$ . Thus for  $w = p \cdot a$  with  $a \in GL(m, R)$  we have  $W_0(w) = W_0(p) \cdot a$  and so  $W(p) \cap \pi^{-1}(x)$  can be written as the disjoint union

$$(2) \quad W(p) \cap \pi^{-1}(x) = \bigcup_{a \in A} W_0(p) \cdot a.$$

By (1), the index set  $A$  is at most countable. Next consider the diffeomorphism  $\gamma: p \cdot a \rightarrow a$  of  $\pi^{-1}(x)$  onto  $GL(m, R)$ . We put  $\gamma(W(p) \cap \pi^{-1}(x)) = G(x, p)$ . By the definitions of  $\gamma$  and  $G(x, p)$ , we have  $G(x, p) = \{a \in GL(m, R) : p \cdot a \in W(p)\}$ . Since  $W(p)$  is the set of all points that can be joined to  $p$  by finite number of integral curves of  $\overline{F(L)}$  and since each vector field of  $\overline{F(L)}$  is invariant by the action of  $GL(m, R)$ , it follows that  $G(x, p)$  is a group. If we introduce the differential structure on  $G(x, p)$  by  $\gamma$ ,  $G(x, p)$  is also a submanifold of  $GL(m, R)$ .

(3)  $G(x, p)$  is a Lie subgroup of  $GL(m, R)$  and the Lie algebra of  $G(x, p)$  is isomorphic to  $\mathfrak{gl}_1(x)$ .

To prove this statement, we put  $\mathfrak{g}(x, p) = \{\gamma_* \bar{u} : \bar{u} \in \overline{F(L_x^0)}\}$ . If we regard  $\mathfrak{gl}(m, R)$  as the Lie algebra of right invariant vector fields on  $GL(m, R)$ ,  $\mathfrak{g}(x, p)$  is a subalgebra of  $\mathfrak{gl}(m, R)$ . Let  $G_0(x, p)$  be the connected Lie subgroup of  $GL(m, R)$  whose Lie algebra is  $\mathfrak{g}(x, p)$ . Since  $G_0(x, p)$  is the maximal integral manifold of  $\mathfrak{g}(x, p)$  containing the identity element of  $GL(m, R)$ , we have  $G_0(x, p) = \gamma(W_0(p))$ . Corresponding to (2), the submanifold  $G(x, p)$  of  $GL(m, R)$  is the disjoint union of integral manifolds  $G_0(x, p) \cdot a$  ( $a \in A$ ) of  $\mathfrak{g}(x, p)$

$$G(x, p) = \bigcup_{a \in A} G_0(x, p) \cdot a.$$

Since  $A$  is countable,  $G(x, p)$  satisfies the second axiom of countability and so the mapping  $(a, b) \rightarrow ab^{-1}$  of  $G(x, p) \times G(x, p) \rightarrow G(x, p)$  is differentiable (see [1], p. 95, Proposition 1. To prove this fact the connectedness of  $G(x, p)$  is not needed. See also [2], p. 10, Proposition 1.3). Hence  $G(x, p)$  is a Lie subgroup of  $GL(m, R)$ . To prove the second part of (3), let  $\bar{F}$  denote the mapping of  $L_x^0 \rightarrow \overline{F(L_x^0)}$  induced by  $F: L(M) \rightarrow L(F(M))$ . The mapping  $\bar{F}$  is a Lie algebra homomorphism of  $L_x^0$  onto  $\overline{F(L_x^0)}$  and the kernel of  $\bar{F}$  is precisely

$L_x^1$ . Thus we have

$$\mathfrak{g}_1^0(x) \cong \overline{F(L_x^0)} \cong \mathfrak{g}(x, p).$$

Now we shall conclude the proof of Theorem 1. For any  $y \in N(x)$ , take  $q \in W(p)$  with  $\pi(q) = y$ . Then  $W(p)$  is also a maximal integral manifold of  $\overline{F(L)}$  through  $q$ . If we put  $G(y, q) = \{a \in GL(m, R) : q \cdot a \in W(p)\}$ , then by the same argument as (3),  $G(y, q)$  is a Lie subgroup of  $GL(m, R)$  having at most countably many connected components and whose Lie algebra is isomorphic to  $\mathfrak{g}_1^0(y)$ . Since a differential system  $\overline{F(L)}$  on  $\pi^{-1}(N(x))$  is invariant by the action of  $GL(m, R)$ , the translations  $R_a$  ( $a \in GL(m, R)$ ) permute among themselves the maximal integral manifolds of  $\overline{F(L)}$ . It follows that two Lie subgroups  $G(x, p)$  and  $G(y, q)$  of  $GL(m, R)$  coincide as set. Since  $G(x, p)$  and  $G(y, q)$  satisfy the second axiom of countability, they coincide as Lie subgroups (see [2], p. 40) and so do their Lie algebras. Hence we have  $\mathfrak{g}_1^0(x) \cong \mathfrak{g}_1^0(y)$ . This completes the proof.

REMARKS 1. If  $L$  is transitive, the submanifold  $W(p)$  of  $F(M)$  and the group  $G(x, p)$  in the preceding proof form the required  $G$ -structure of order 1 in Theorem 2. The higher order case can be proved, if we put  $t=0$  and  $s=l$  in the above proof of Theorem 1.

2. If we restrict vector fields of  $L$  to an orbit  $N(x)$ , we get a transitive Lie algebra  $\bar{L}$  of vector fields on  $N(x)$ . In general the  $(t, s)$ -isotropy algebras  $\mathfrak{g}_s^t(y)$  of  $L$  are different from the  $(t, s)$ -isotropy algebras  $\bar{\mathfrak{g}}_s^t(y)$  of  $\bar{L}$  for  $y \in N(x)$ . For example let  $L$  be the Lie algebra generated by a vector field  $-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$  on  $\mathbf{R}^2$ . Then the origin  $o$  is an orbit under  $L$  and the linear isotropy algebra  $\mathfrak{g}_1^0(o)$  of  $L$  at  $o$  is not zero, while  $\bar{\mathfrak{g}}_1^0(o)$  is zero. Thus in order to prove Theorem 1 it is not sufficient to prove only the transitive case.

3. Let us give an example  $L$  with is not invariant under the automorphism  $\phi_t(u)_*$  of  $L(M)$  generated by any non-zero element  $u$  of  $L$ . Let  $M$  be  $\mathbf{R}^1$  with a coordinate  $(x)$  and let  $L$  be the Lie algebra consisting of vector fields of the form  $x^2 f(x) \frac{\partial}{\partial x}$ , where  $f(x)$  is a polynomial function on  $\mathbf{R}^1$ . Then this  $L$  has the desired property. In fact if we put  $\phi_t(u)_* x^2 f(x) \frac{\partial}{\partial x} = g(t, x) \frac{\partial}{\partial x}$ , then  $g(t, x)$  is a meromorphic function of  $x$  and  $t$ .

4. There exists a Lie algebra  $L$  to which our theorems cannot be applied. Let  $M$  be  $\mathbf{R}^1$  with a coordinate  $(x)$ . Take a  $C^\infty$ -function  $f(x)$  on  $\mathbf{R}^1$  with a compact support which is not identically zero. Let  $L$  be the Lie algebra generated by  $\frac{\partial}{\partial x}$  and  $f(x) \frac{\partial}{\partial x}$ . Then  $L$  is transitive and so  $L$  has only one orbit  $M$ . We have  $\mathfrak{g}_k^0(x) = \{0\}$  for  $x \notin \text{supp } f$  and all  $k$ , while  $\mathfrak{g}_k^0(x) \neq \{0\}$  for  $x$  with  $\frac{d}{dx} f(x) \neq 0$  and all  $k$ .

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