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Title	On isotropy algebras of a Lie albegra of vector fields which satisfies a certain convergence condition
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Citation	Osaka Journal of Mathematics. 1969, 6(1), p. 57-62
Version Type	VoR
URL	https://doi.org/10.18910/7167
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Inoue, T.
Osaka J. Math.
6 (1969), 57-62

ON ISOTROPY ALGEBRAS OF A LIE ALGEBRA OF VECTOR FIELDS WHICH SATISFIES A CERTAIN CONVERGENCE CONDITION

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(Received September 9, 1968)

Let M be a C^∞-manifold and let $L(M)$ be the Lie algebra of all C^∞-vector fields on M. For a Lie subalgebra L of $L(M)$, Matsuda [3] gave a sufficient condition for its integrability. Precisely speaking, for a Lie subalgebra L which satisfies the convergence condition (C) stated below, Matsuda proved that through every point of M there passes a maximal integral manifold of L.

For two elements u and v of $L(M)$, defining $(\mathrm{ad}\, v)^k u$ inductively as $[v, (\mathrm{ad}\, v)^{k-1}u]$, we put

$$g_t(u, v) = \sum_{j=0}^{\infty} (-1)^j \frac{t^j}{j!} (\mathrm{ad}\, v)^j u \,.$$

Let $c(u, v; x)$ be the radius of convergence of $g_t(u, v)$ at x of M.

(C) *For any pair of u and v in L and any compact set K in M, there exists a positive number $c(u, v; K)$ such that*

(i) *we have $c(u, v; x) \geqq c(u, v; K)$ at every $x \in K$, and*

(ii) *$g_t(u, v)$ is continuously differentiable with respect to (t, x) term by term at every (t, x) which satisfies $|t| < c(u, v; K)$ and $x \in K^i$, the interior of K.*

In this paper we say a Lie subalgebra L satisfies the condition (C_k), if '*continously differentiable*' in (ii) of Matsuda's condition (C) can be replaced by '*$(k+1)$-th continuously differentiable*'. Here k is a non-negative integer.

For any Lie subalgebra L of $L(M)$ and a point x of M, we shall define $\mathfrak{g}_k^l(x)$, the *(l, k)-isotropy algebra of L at x*, as follows: Let L_x^k denote the subalgebra of L consisting of vector fields whose coefficients vanish at x with all their derivatives through order k (in one and hence all coordinate systems). Then L_x^k is an ideal in L_x^l for $k \geqq l \geqq 0$. We shall denote the factor algebra L_x^l / L_x^k by $\mathfrak{g}_k^l(x)$ for $k \geqq l \geqq 0$. In particular $\mathfrak{g}_1^0(x)$ is the linear isotropy algebra of L at x.

Theorem 1. *Let M be a connected, paracompact manifold and let L be a Lie subalgebra satisfying the condition (C_k) with $k > 0$. Then the (t, s)-isotropy algebras*

$\mathfrak{g}_s^t(x)$ *and* $\mathfrak{g}_s^t(y)$ *of* L *are isomorphic for* $k \geqq s > t \geqq 0$, *if* x *and* y *lie on the same orbit under* L. *In particular the linear isotropy algebra of* L *is isomorphic each other on every orbit under* L.

Here an orbit under L is the set of all points of M that can be joined each other by finite number of integral curves of L.

Every finite dimensional Lie subalgebra L satisfies the condition (C_k) for all k (see [3]). Also if we are in the real analytic category, then every Lie subalgebra L of $L(M)$ satisfies the condition (C_k) for all k(see [3]).

A subalgebra L is called transitive if $L(x)$ equals the tangent space $T_x(M)$ at every x of M where $L(x)$ is the subspace of $T_x(M)$ defined by $L(x)=\{u(x); u \in L\}$.

Theorem 2. *Under the same assumptions as Theorem* 1, *if moreover* L *is transitive, then there exists a* G*-structure of order* l *with* $1 \leqq l \leqq k$ *such that the Lie algebra of* G *is isomorphic to* $\mathfrak{g}_l^0(x)$ *for every* $x \in M$.

A G-structure of order l is by definition a reduction of the bundle $F^l(M)$ of l frames of M to the group G (see [4]).

The conclusion of Theorem 2 was obtained by Singer and Sternberg ([6], p. 39) under the assumption that L (Lie algebra sheaf in their case) is invariant by a local one parameter transformation group generated by any element of L. Theorem 2 will be proved through the proof of Theorem 1.

The author would like to thank Dr. M. Matsuda and Professor H. Ozeki for their encouragement and kind guidance during the preparation of this paper.

Proof of Theorem 1. For simplicity, we shall prove the case $t=0$ and $s=1$. The proof for the general case proceeds similarly, if we replace M by $F^t(M)$ and $F(M)$ *by* $F^{s-t}(F^t(M))$ respectively in the following argument. Let $N(x)$ be the orbit under L through $x \in M$. By the result of Matsuda, $N(x)$ is a maximal connected integral manifold of L through x. We shall denote by $F(M)$ the bundle of linear frames of M with π as projection. Since $N(x)$ is a submanifold of M, $\pi^{-1}(N(x))$ is a submanifold of $F(M)$. Each vector field u on M induces a vector field $F(u)$ on $F(M)$ as follows: Let $\phi_t(u)$ be a local one parameter transformation group of M generated by u and let $F(\phi_t(u))$ be a local one parameter transformation group of $F(M)$ defined by

$$F(\phi_t(u)) \cdot w = \phi_t(u)_{*z} \cdot w \qquad w \in F(M) \quad \text{and} \quad z = \pi(w) .$$

We shall denote its infinitesimal generator by $F(u)$. The mapping F: $u \to F(u)$ of $L(M)$ into $L(F(M))$ is a Lie algebra homomorphism and has following properties:

(i) $\pi_* F(u) = u$

(ii) $R_{a*}F(u)=F(u)$ for any $a \in GL(m, R)$, where R_a is the right translation of $F(M)$ by an element a and m is dim M. We put $F(L)=\{F(u): u \in L\}$, which is a Lie algebra. For $u \in L$ a vector field $F(u)$ on $F(M)$ is, as is clear from the

definition, tangent to the submanifold $\pi^{-1}(N(x))$ at all points and so defines a vector field on $\pi^{-1}(N(x))$. The set of vector fields on $\pi^{-1}(N(x))$ so obtained is a Lie algebra which we shall denote by $\overline{F(L)}$. If L satisfies the condition (C_1) on M, $F(L)$ satisfies the condition (C_0) on $F(M)$ and so $\overline{F(L)}$ satisfies the condition (C_0) on $\pi^{-1}(N(x))$. Let p be a point of $F(M)$ with $\pi(p)=x$. We denote the orbit under $\overline{F(L)}$ through p by $W(p)$. Then again by the result of Matsuda, $W(p)$ is a maximal integral manifold of $\overline{F(L)}$. We have clearly $\pi(W(p))=N(x)$. Put $n=\dim N(x)$ and $r=\dim W(p)-n$.

(1) The intersection $W(p)\cap\pi^{-1}(x)$ is a r-dimensional submanifold of three manifolds $\pi^{-1}(N(x))$, $\pi^{-1}(x)$ and $W(p)$. It has at most countably many connected components.

We shall prove this statement (1). For $w\in\pi^{-1}(N(x))$ the differential π_{*w} at w of π maps $\overline{F(L)}(w)$ onto the tangent space $T_{\pi(w)}(N(x))$ at $\pi(w)$, where $\overline{F(L)}(w)=\{\bar{u}(w)\colon \bar{u}\in\overline{F(L)}\}$, since $\pi_*F(u)=u$ and $L(z)=T_z(N(x))$. Hence we obtain $T_w(\pi^{-1}(N(x)))=T_w(W(p))+T_w(\pi^{-1}(x))$ for any $w\in W(p)\cap\pi^{-1}(x)$, i.e., two submanifolds $W(p)$ and $\pi^{-1}(x)$ of $\pi^{-1}(N(x))$ are transverse at their intersection. From this fact the first statement follows (see [5], p. 30). To prove the second part of (1), we first show that $\overline{F(L)}$ defines an involutive distribution in the sense of Chevalley [1] on $\pi^{-1}(N(x))$. For this, it suffices to show that the dimension of $\overline{F(L)}(w)$ is constant on $\pi^{-1}(N(x))$. The orbit $W(p)$ is an integral manifold of $\overline{F(L)}$ and so dim $\overline{F(L)}(w)$ is constant on $W(p)$. Since $\pi(W(p))=N(x)$, and since each vector field of $F(L)$ is invariant by the action of $GL(m, \mathrm{R})$, we have dim $\overline{F(L)}(w)=$*constant* on $\pi^{-1}(N(x))$. Hence each point $w\in\pi^{-1}(x)$ has a neighbourhood U in $\pi^{-1}(N(x))$ with coordinates $(x_1, \cdots, x_{n+m^2})$ such that the slices given by $x_{n+r+1}=\text{const.}, \cdots, x_{n+m^2}=\text{const.}$ are integral manifolds of $\overline{F(L)}$ ([1], p. 89, Theorem 1). Since two submanifolds $W(p)$ and $\pi^{-1}(x)$ of $\pi^{-1}(N(x))$ are transverse at their intersection, we can change this coordinates such that, in addition to the above condition, the set $V=\{q\in U\colon x_1(q)=\cdots=x_n(q)=0\}$ together with the restriction of $(x_{n+1}, \cdots, x_{n+m^2})$ to V form a local chart on $\pi^{-1}(x)$ containing w (compare the proof in [5], p. 30, Lemma 6). Hence $W(p)\cap V$ is the union of certain sets of planes represented by $x_1=0, \cdots, x_n=0, x_{n+r+1}=\text{const.}, \cdots, x_{n+m^2}=\text{const.}$, each of which is an open set of $W(p)\cap\pi^{-1}(x)$. The intersection $W(p)\cap U$ is the union of certain sets of slices given by $x_{n+r+1}=\text{const.}, \cdots, x_{n+m^2}=\text{const.}$ The paracompactness of M implies that the connected submanifold $N(x)$ of M and hence $\pi^{-1}(N(x))$ satisfies the second axiom of countability and so does the connected submanifold $W(p)$ of $\pi^{-1}(N(x))$. Then it follows that the intersection $W(p)\cap U$ is the union of at most countably many of slices and so the neighbourhood V of w in $\pi^{-1}(x)$ can meet at most countably many connected components of $W(p)\cap\pi^{-1}(x)$. Thus we have shown that each $w\in\pi^{-1}(x)$ has a neighbourhood V in $\pi^{-1}(x)$ which meets at most countably

many connected components of $W(p)\cap\pi^{-1}(x)$. Since $\pi^{-1}(x)$ can be covered with countably many of these open sets, we have the second part of (1).

Each element of $F(L_x^0)=\{F(u)\colon u\in L_x^0\}$ is tangent to submanifold $\pi^{-1}(x)$ at all points and so defines a vector field on $\pi^{-1}(x)$. The vector fields on $\pi^{-1}(x)$ so obtained is a Lie algebra which we shall denote by $\overline{F(L_x^0)}$. Then the orbit $W_0(w)$ under $\overline{F(L_x^0)}$ through w is the connected component of $W(p)\cap\pi^{-1}(x)$ containing w and so a maximal integral manifold of $\overline{F(L_x^0)}$ through w. Since $\overline{F(L_x^0)}$ is invariant by the action of $GL(m, \mathrm{R})$, the translations R_a ($a\in GL(m, \mathrm{R})$) permute among themselves the maximal integral manifolds of $\overline{F(L_x^0)}$. Thus for $w=p\cdot a$ with $a\in GL(m, \mathrm{R})$ we have $W_0(w)=W_0(p)\cdot a$ and so $W(p)\cap\pi^{-1}(x)$ can be written as the disjoint union

$$W(p)\cap\pi^{-1}(x)=\bigcup_{a\in A}W_0(p)\cdot a\,. \tag{2}$$

By (1), the index set A is at most countable. Next consider the diffeomorphism $\gamma\colon p\cdot a\to a$ of $\pi^{-1}(x)$ onto $GL(m, \mathrm{R})$. We put $\gamma(W(p)\cap\pi^{-1}(x))=G(x, p)$. By the definitions of γ and $G(x, p)$, we have $G(x, p)=\{a\in GL(m, \mathrm{R})\colon p\cdot a\in W(p)\}$. Since $W(p)$ is the set of all points that can be joined to p by finite number of integral curves of $\overline{F(L)}$ and since each vector field of $\overline{F(L)}$ is invariant by the action of $GL(m, \mathrm{R})$, it follows that $G(x, p)$ is a group. If we introduce the differential structure on $G(x, p)$ by γ, $G(x, p)$ is also a submanifold of $GL(m, \mathrm{R})$.

(3) $G(x, p)$ is a Lie subgroup of $GL(m, \mathrm{R})$ and the Lie algebra of $G(x, p)$ is isomorphic to $\mathfrak{g}_1^0(x)$.

To prove this statement, we put $\mathfrak{g}(x, p)=\{\gamma_*\bar{u}\colon \bar{u}\in\overline{F(L_x^0)}\}$. If we regard $\mathfrak{gl}(m, \mathrm{R})$ as the Lie algebra of right invariant vector fields on $GL(m, \mathrm{R})$, $\mathfrak{g}(x, p)$ is a subalgebra of $\mathfrak{gl}(m, \mathrm{R})$. Let $G_0(x, p)$ be the connected Lie subgroup of $GL(m, \mathrm{R})$ whose Lie algebra is $\mathfrak{g}(x, p)$. Since $G_0(x, p)$ is the maximal integral manifold of $\mathfrak{g}(x, p)$ containing the identity element of $GL(m, \mathrm{R})$, we have $G_0(x, p)=\gamma(W_0(p))$. Corresponding to (2), the submanifold $G(x, p)$ of $GL(m, \mathrm{R})$ is the disjoint union of integral manifolds $G_0(x, p)\cdot a$ ($a\in A$) of $\mathfrak{g}(x, p)$

$$G(x, p)=\bigcup_{a\in A}G_0(x, p)\cdot a\,.$$

Since A is countable, $G(x, p)$ satisfies the second axiom of countability and so the mapping $(a, b)\to ab^{-1}$ of $G(x, p)\times G(x, p)\to G(x, p)$ is differentiable (see [1], p. 95, Proposition 1. To prove this fact the connectedness of $G(x, p)$ is not needed. See also [2], p. 10, Proposition 1.3). Hence $G(x, p)$ is a Lie subgroup of $GL(m, R)$. To prove the second part of (3), let $\bar{F}$ denote the mapping of $L_x^0\to\overline{F(L_x^0)}$ induced by $F\colon L(M)\to L(F(M))$. The mapping $\bar{F}$ is a Lie algebra homomorphism of L_x^0 onto $\overline{F(L_x^0)}$ and the kernel of $\bar{F}$ is precisely

L_x^1. Thus we have

$$\mathfrak{g}_1^0(x) \cong \overline{F(L_x^0)} \cong \mathfrak{g}(x, p) .$$

Now we shall conclude the proof of Theorem 1. For any $y \in N(x)$, take $q \in W(p)$ with $\pi(q)=y$. Then $W(p)$ is also a maximal integral manifold of $\overline{F(L)}$ through q. If we put $G(y, q)=\{a \in GL(m, R) : q \cdot a \in W(p)\}$, then by the same argument as (3), $G(y, q)$ is a Lie subgroup of $GL(m, R)$ having at most countably many connected components and whose Lie algebra is isomorphic to $\mathfrak{g}_1^0(y)$. Since a differential system $\overline{F(L)}$ on $\pi^{-1}(N(x))$ is invariant by the action of $GL(m, R)$, the translations R_a $(a \in GL(m, R))$ permute among themselves the maximal integral manifolds of $\overline{F(L)}$. It follows that two Lie subgroups $G(x, p)$ and $G(y, q)$ of $GL(m, R)$ coincide as set. Since $G(x, p)$ and $G(y, q)$ satisfy the second axiom of countability, they coincide as Lie subgroups (see [2], p. 40) and so do their Lie algebras. Hence we have $\mathfrak{g}_1^0(x) \cong \mathfrak{g}_1^0(y)$. This completes the proof.

REMARKS 1. If L is transitive, the submanifold $W(p)$ of $F(M)$ and the group $G(x, p)$ in the preceding proof form the required G-structure of order 1 in Theorem 2. The higher order case can be proved, if we put $t=0$ and $s=l$ in the above proof of Theorem 1.

2. If we restrict vector fields of L to an orbit $N(x)$, we get a transitive Lie algebra $\bar{L}$ of vector fields on $N(x)$. In general the (t, s)-isotropy algebras $\mathfrak{g}_s^t(y)$ of L are different from the (t, s)-isotropy algebras $\bar{\mathfrak{g}}_s^t(y)$ of $\bar{L}$ for $y \in N(x)$. For example let L be the Lie algebra generated by a vector field $-y\dfrac{\partial}{\partial x}+x\dfrac{\partial}{\partial y}$ on $\boldsymbol{R}^2$. Then the origin o is an orbit under L and the linear isotropy algebra $\mathfrak{g}_1^0(o)$ of L at o is not zero, while $\bar{\mathfrak{g}}_1^0(o)$ is zero. Thus in order to prove Theorem 1 it is not sufficient to prove only the transitive case.

3. Let us give an example L with is not invariant under the automorphism $\phi_t(u)_*$ of $L(M)$ generated by any non-zero element u of L. Let M be $\boldsymbol{R}^1$ with a coordinate (x) and let L be the Lie algebra consisting of vector fields of the form $x^2 f(x)\dfrac{\partial}{\partial x}$, where $f(x)$ is a polynomial function on $\boldsymbol{R}^1$. Then this L has the desired property. In fact if we put $\phi_t(u)_* x^2 f(x)\dfrac{\partial}{\partial x}=g(t, x)\dfrac{\partial}{\partial x}$, then $g(t, x)$ is a meromorphic function of x and t.

4. There exists a Lie algebra L to which our theorems cannot be applied. Let M be $\boldsymbol{R}^1$ with a coordinate (x). Take a C^∞-function $f(x)$ on $\boldsymbol{R}^1$ with a compact support which is not identically zero. Let L be the Lie algebra generated by $\dfrac{\partial}{\partial x}$ and $f(x)\dfrac{\partial}{\partial x}$. Then L is transitive and so L has only one orbit M. We have $\mathfrak{g}_k^0(x)=\{0\}$ for $x \notin \operatorname{supp} f$ and all k, while $\mathfrak{g}_k^0(x) \neq \{0\}$ for x with $\dfrac{d}{dx}f(x) \neq 0$ and all k.

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