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Author(s)	Ferrario, Davide L.
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A MÖBIUS INVERSION FORMULA FOR GENERALIZED LEFSCHETZ NUMBERS

DAVIDE L. FERRARIO

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1. Introduction

Let G be a finite group and $f: U \to X$ an equivariant map. A common way of studying the properties of f is looking at the restrictions $f^H: X^H \to Y^H$ to the spaces fixed by the subgroups H of G, as non-equivariant maps. For example, if U and X are G-CW complexes, then $f: U \to X$ is a G-equivariant homotopy equivalence if and only if for every H the map f^H is a homotopy equivalence; a similar result related to a G-retraction is due to Jaworowski: a locally compact, separable metric and finite-dimensional G-space X is a G-ENR if and only if for every H the fixed point set X^H is an ENR [11]. This paper is addressed to studying fixed points (up to compactly fixed G-homotopy) of a G-equivariant self-map $f: U \subset M \to M$, where M is a G-ENR or a smooth G-manifold. If there is a compactly fixed G-homotopy f_t, $t \in I$, such that $f_0 = f$ and f_1 is fixed point free, then for every subgroup H there is a compactly fixed homotopy f_t^H such that $f_0^H = f^H$ and f_1^H is fixed point free, and this means that every restriction f^H can be deformed to a fixed point free map. To investigate under which conditions the converse of this statement is true, it is necessary to exhibit the algebraic obstructions of the existence of the equivariant deformation f_t, and then relate them to the corresponding obstructions of the non-equivariant restrictions f^H. Under some dimensional assumptions, Nielsen theory is exactly what describes these invariants; if M^H is a manifold of dimension different from 2 then the generalized Lefschetz number $\mathcal{L}(f^H)$ or equivalently the Nielsen number $N(f^H)$ vanish if and only if f^H can be deformed to be fixed point free (it is the Converse of the Lefschetz Property). So the problem can be stated algebraically as: under which conditions does the knowledge of the generalized Lefschetz numbers $\mathcal{L}(f^H)$ allow to compute the obstructions to an equivariant deformation? Again, it is necessary to relate the latter obstruction to a set of invariants (as done first in [7]), namely the generalized Lefschetz numbers of some restrictions $\mathcal{L}({f'}^H|U_H)$ of a suitable approximation f' of f. This set of homotopy invariants gives what might be thought as an equivariant generalized Lefschetz number, and, under the same dimension assumptions as above, they vanish if and only if the map has a G-deformation to a fixed point free map (compactly fixed). So the point is to relate the two sets of invariants above.

First results in this direction date back to the papers of Komiya [13],

D. Wilczyński, [20] and Fadell and Wong [4]. The hypotheses needed there were in some sense quite restrictive. It was necessary to assume the codimension of X_s^H in X^H to be at least 2. This assumption was still present in the papers of Wong [18, 19] on equivariant Nielsen theory. In the present paper we use some properties of the generalized Lefschetz numbers that were not known, and the fact that the singular set X_s^H and the singular set of the action of the Weyl group $W_G H$ on X^H are often not the same. It should be clear from the examples given at the end of this paper that the condition needed is actually the weaker assumption (i) of Theorem 9. Another difference in the approach is that by using generalized Lefschetz numbers instead of Nielsen numbers it is now possible to keep track of the indices of the fixed point classes at any isotropy level, so that a formula becomes available, relating the set of the GLN's $\mathcal{L}(f^H)$ to the set of $\mathcal{L}(f'^H|U_H)$. This is the aim of the first half of the paper.

The first step is done applying the addition formula for generalized Lefschetz numbers, first proved in [6], which gives $\mathcal{L}(f^H)$ as a sum of the images of the $\mathcal{L}(f'^H|U_H)$ under some homomorphisms. Inverting such formula in a Möbius way is what can give the solution of the initial problem: if $\mathcal{L}(f^H)$ vanishes for every H, then the obstructions to deform f equivariantly vanish. Furthermore, the inversion formula relates fixed point classes at different isotropy groups: in this way it is also possible to describe some necessary conditions for the weighted sets of fixed point classes and isotropies, extending the result of Dold [3] and Komiya [14].

The main results of the paper are the Möbius inversion formula given in Theorem 3 and the Converse of the Lefschetz Property for isovariant maps given in Theorem 9, which follows from Theorem 3. The structure of the paper follows the steps needed in proving these theorems: in Section 2 the notation and some preliminaries are given on properties of equivariant spaces and the generalized Lefschetz number. Two technical results (Lemma 1 and 2) are proved, which allow to prove the inversion formula in Theorem 3 (Section 3). The first consequence of the formula is the necessary condition that must be satisfied by generalized Lefschetz numbers $\mathcal{L}(f^H)$ of the restrictions of an equivariant map. In Section 4 we start proving results needed in order to apply the inversion formula to the converse of the Lefschetz property: first, with the aid of the key Lemma 5, the existence of a transfer homomorphism is shown, and then its main property is proved in Proposition 6. Finally, in Section 5 the converse of the Lefschetz property (Theorem 9) is proved, after two technical Lemmas (Lemma 7 and 8). An immediate interesting consequence of Theorem 9 is given in Corollary 10: if the group G is 2-split, then for every G-map f the equivariant Nielsen number $N_G(f)$ is zero if and only if f is G-homotopic to a fixed point free G-map. In Section 6 some examples are given, showing the reason for which the hypotheses of Theorem 9 are needed: in Examples 1 and 2 the Hypothesis (i) holds (actually, G is 2-split) but not (ii) (the map is not isovariant, nor G-homotopic to an isovariant map); furthermore, in Example 3 it is shown how if (i) does not hold (of course for group G not 2-split) then the CLP does not hold even for the identity map. At the end, in Section 7, there are some comments which might help in understanding the perspective of the paper and some relevant details.

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2. Preliminaries

Let G be a finite group acting on a space X. The isotropy subgroup $G_x \subset G$ of an element $x \in X$ is the subgroup $G_x = \{g \in G \mid gx = x\}$. The set of isotropy subgroups is a poset, with respect to inclusion, denoted by $\mathrm{iso}(X)$. If $H \subset G$ is a subgroup of G, then X^H denotes the subspace of X fixed by H, i.e. $\{x \in X \mid Hx = x\}$. The singular set in X^H is the set of points with isotropy not equal to H: $X^H_s = \{x \in X^H \mid G_x \not\supset H\}$. If H is an isotropy subgroup then the complement of X^H_s in X^H is denoted by $X_H = \{x \in X^H \mid G_x = H\}$. The Weyl group $W_G H$ of the isotropy subgroup H is the quotient of the normalizer $N_G H = \{g \in G \mid g^{-1}Hg = H\}$ modulo H, that is $W_G H = N_G H/H$. The space X^H is endowed with a natural action of $W_G H$, with trivial isotropy type (whenever H is isotropy): its singular set is denoted by X^H_{s0}. It is contained in X^H_s but it might be properly contained (see examples in [7, 8]). It is not difficult to see that X^H_{s0} is the set of points in $x \in X^H_s$ with the property that $G_x \cap N_G H \neq H$.

2.1. The generalized Lefschetz number. Let $f : U \subset X \to X$ be a compactly fixed (continuous) map defined on an open subset of an ENR X. Let I be the unit interval $[0,1]$, and X^I the space of continuous maps $I \to X$ (with the compact-open topology). Let $E(f)$ denote the subspace of X^I consisting of all the maps $\lambda \in X^I$ such that $\lambda(0) \in U$ and $f\lambda(0) = \lambda(1)$ (they are the Brouwer translation paths), and $\mathcal{R}(f)$ the set of (connected) components of $E(f)$. The set $\mathcal{R}(f)$ is called *Reidemeister set* of the map f. Its elements are also called *Reidemeister classes*. The class of a path λ is denoted with $[\lambda]$. It has some interesting functional properties, and if $X = U$ is connected it is isomorphic to the set of twisted Reidemeister conjugacy classes in the fundamental group $\pi_1(X)$. The free Abelian group generated by $\mathcal{R}(f)$ is denoted by $\mathbb{Z}\mathcal{R}(f)$: an element of $\mathbb{Z}\mathcal{R}(f)$ is simply a function $\xi \colon \mathcal{R}(f) \to \mathbb{Z}$, and the sum of two such functions is done component-wise. Any function $\phi \colon \mathcal{R}(f_1) \to \mathcal{R}(f_2)$, induces in a standard way a homomorphism $\phi_* \colon \mathbb{Z}\mathcal{R}(f_1) \to \mathbb{Z}\mathcal{R}(f_2)$.

The fixed point set $\mathrm{Fix}(f) = \{x \in U \mid fx = x\}$ can be embedded in $E(f)$, by sending the fixed point $x \in \mathrm{Fix}(f)$ to the constant path $\lambda(t) \equiv x$, $\lambda \in E(f)$. The composition $\mathrm{cd} \colon \mathrm{Fix}(f) \to E(f) \to \mathcal{R}(f)$ is said the *coordinate function*. The preimage $\mathrm{cd}^{-1}\xi$ of an element $\xi \in \mathcal{R}(f)$ is a set of fixed points called the *Nielsen class* of fixed points corresponding to the class ξ. Every such a class is compact and has an isolating neighborhood in U. Thus there is only a finite number of Nielsen classes of fixed points.

The generalized Lefschetz number of f, denoted by $\mathcal{L}(f)$, is the element $\mathcal{L}(f) \in \mathbb{Z}\mathcal{R}(f)$ defined by $\mathcal{L}(f)(\xi) = \operatorname{ind}(f, \operatorname{cd}^{-1}\xi)$ for all $\xi \in \mathcal{R}(f)$, where $\operatorname{ind}(f, \operatorname{cd}^{-1}\xi)$ denotes the fixed point index of f in any isolating neighborhood of the fixed point class $\operatorname{cd}^{-1}\xi$. With an abuse of notation, by identifying ξ with the dual function $\hat{\xi} \in \mathbb{Z}\mathcal{R}(f)$ which sends ξ to 1 and $\xi' \neq \xi$ to 0, it is possible to write

$$\mathcal{L}(f) = \sum_{\xi \in \mathcal{R}(f)} \operatorname{ind}(f, \operatorname{cd}^{-1}\xi) \cdot \xi.$$

If X is a CW-complex and U a subcomplex, then $\mathcal{L}(f)$ can be written as the Lefschetz number of the homomorphism induced in homology, with respect to the Hattori-Stallings trace with coefficients in $\mathbb{Z}\pi(X)$ (see [5, 9]). The number of elements of the support of $\mathcal{L}(f)$ (i.e. the number of non-zero terms in the sum above) is the *local Nielsen number* $N(f)$ of f. If $U = X$ then it is the classical Nielsen number of f. If G is a finite group and M a G-manifold, then the equivariant Nielsen number is the vector $\left(N(f^H)\right)_H$, where H ranges over the isotropy groups $H \subset G$ for M. It is the analogous of the equivariant Lefschetz number $L_G(f)$. Further details on Nielsen fixed point theory can be found in the books of R.F. Brown [1] and B.J. Jiang [10]; on equivariant Nielsen fixed point theory the papers of P. Wong [18, 19] are a good reference. The common reference for equivariant algebraic topology is the book of tom Dieck [17]. A different approach on generalized Lefschetz numbers (for equivariant maps) was given by E. Laitinen and W. Lück in [15].

2.2. Index properties of the generalized Lefschetz number. We recall here some properties of the generalized Lefschetz number which resemble the properties of the fixed point index (see [2] and [7]) and the Nielsen number.

If X and Y are ENR's, $f_1\colon U \subset X \to X$ and $f_2\colon V \subset Y \to Y$ are compactly fixed maps defined on open subsets U and V, and $i\colon X \to Y$ is a map such that $U = i^{-1}V$ and $if_1 = f_2 i$, then there is a map

$$i_*\colon \mathcal{R}(f_1) \to \mathcal{R}(f_2)$$

defined by $i_*[\lambda] = [i\lambda]$, for every class $[\lambda] \in \mathcal{R}(f)$ (we can use the same symbol i_* for the homomorphism induced on $\mathbb{Z}\mathcal{R}(f_i)$). If i is an inclusion, and there is a retraction r of a neighborhood of X in Y, and if f_2 is defined as $if_1r\colon r^{-1}U \to Y$, then the induced map

$$i_*\colon \mathbb{Z}\mathcal{R}(f_1) \to \mathbb{Z}\mathcal{R}(f_2)$$

sends $\mathcal{L}(f_1)$ to $\mathcal{L}(f_2)$:

$$\mathcal{L}(f_2) = i_*\mathcal{L}(f_1).$$

Another important property is the union property: let $f\colon U \subset X \to X$ be a map; if U (open) is the union of two open subsets U_1 and U_2, and f is compactly fixed in U_1 and U_2, then there exist the generalized Lefschetz numbers $\mathcal{L}(f|U_1)$ (the map f restricted to U_1), $\mathcal{L}(f|U_2)$, $\mathcal{L}(f|U_1 \cap U_2)$ and $\mathcal{L}(f)$. There are functorial maps induced by the inclusions $i_1\colon \mathbb{Z}\mathcal{R}(f|U_1) \to \mathbb{Z}\mathcal{R}(f)$, $i_2\colon \mathbb{Z}\mathcal{R}(f|U_2) \to \mathbb{Z}\mathcal{R}(f)$ and $i_0\colon \mathbb{Z}\mathcal{R}(f|U_1 \cap U_2) \to \mathbb{Z}\mathcal{R}(f)$. The union property is the following:

$$\mathcal{L}(f) = i_1\mathcal{L}(f|U_1) + i_2\mathcal{L}(f|U_2) - i_0\mathcal{L}(f|U_1 \cap U_2).$$

This is a consequence of a more general property of the general Lefschetz number of pushout maps ([6]), but can be proved directly just using the properties of the fixed point index and the Reidemeister sets.

The last properties are the most significative: if $f_1 \sim f_2\colon U \subset X \to X$ are compactly fixed and homotopic (via a compactly fixed homotopy) then there is a bijection $b\colon \mathcal{R}(f_1) \cong \mathcal{R}(f_2)$ such that $b_*\mathcal{L}(f_1) = \mathcal{L}(f_2)$. This implies that if the map is fixed point free, then $\mathcal{L}(f) = 0$. This is called the Lefschetz property (in analogy with the Wecken property). Furthermore, if X is a manifold of dimension at least 3, $f\colon U \subset X \to X$ a compactly fixed map with $\mathcal{L}(f) = 0$, then there is a fixed point free compactly fixed deformation of f (i.e. the converse of the Lefschetz property holds for manifolds of dimension at least 3).

2.3. The taut approximation lemma. Let G be a finite group and Y a G-space. If $A \subset Y$ is a G-subset, then a G-map $f\colon Y \to Z$ is *taut* over A if there is a G-retraction $r\colon Y \to A$ such that $f = fr$. If U is a G-neighborhood of A in Y, then f is said to be *taut* over A in U if the restriction $f|U$ is taut over A. A G-map $f\colon U \subset X \to X$ is *taut* if for every isotropy subgroup $H \subset G$ of X the restriction $f^H\colon U^H \to X^H$ is taut over the singular set U_s^H in a suitable neighborhood of U_s^H in U^H. The assertion of the taut approximation lemma is what elsewhere (e.g. [14]) was called the existence of a *normal form* of any G-map $f\colon U \subset X \to X$ defined on enough regular G-spaces. The most general spaces in which such a proposition holds are, to my knowledge, G-ENR's.

Lemma 1 (Taut approximation). *Let G be a finite group, X a G-ENR, U an open G-subspace of X and $f\colon U \to X$ a compactly fixed G-map. Then for every $\epsilon > 0$ there is a compactly fixed ϵ-approximation f' of f, such that f' is taut in a neighborhood of* $\mathrm{Fix}(f')$.

Proof. It is a consequence of the fact that for every isotropy H of X the inclusion $X_s^H \to X^H$ is a W_GH-cofibration (actually, X_s^H is a W_GH-neighborhood retract in X). Then, by induction over orbit types, it is possible to deform f^H relatively to U_s^H and outside a compact W_GH-neighborhood of $\mathrm{Fix}(f^H)$, in order to obtain an ϵ-approximation f'^H of f^H which is taut in a neighborhood of $\mathrm{Fix}(f'^H)$. Alternatively, there is a compact G-neighborhood C of $\mathrm{Fix}(f)$ in U, and a G-deformation of the identity $1_C\colon C \to C$ which can be extended to an equivariant ϵ'-deformation ϕ

of 1_U leaving $U \setminus C$ fixed such that $f \circ \phi$ is the wanted ϵ-deformation, with ϵ' small enough. $\square$

The main point about such taut approximation f' is that the generalized Lefschetz numbers $\mathcal{L}(f'^H|U_H)$ of the maps f'^H restricted to U_H are well-defined, for every isotropy H (because it turns out that if f' is taut in a neighborhood of $\mathrm{Fix}(f')$ then every restriction $f'^H|U_H$ is compactly fixed and $W_G H$-equivariant). Another important property is that if f_1 and f_2 are taut approximations of the same map f (they are taut in a neighborhood of their fixed point sets), then for every isotropy H the generalized Lefschetz numbers $\mathcal{L}(f_1^H|U_H) = \mathcal{L}(f_2^H|U_H)$ coincide (more precisely, there is a compactly fixed homotopy $f_1 \sim f_2$ inducing a bijection $\theta\colon \mathcal{R}(f_1) \cong \mathcal{R}(f_2)$ such that $\theta_* \mathcal{L}(f_1^H|U_H) = \mathcal{L}(f_2^H|U_H)$).

2.4. The Möbius coefficients. Let $\mathrm{iso}(X)$ be the poset of isotropy groups of X in G. The Möbius coefficients $\mu(K, H)$ are the unique integers with the property that if $a, b\colon \mathrm{iso}(X) \to \mathbb{Z}$ are arbitrary integer-valued maps such that

$$b(H) = \sum_{K \supset H} a(K),$$

then

$$a(H) = \sum_{K \supset H} \mu(K, H) b(K).$$

The coefficients $\mu(K, H)$ do not depend upon the functions a and b, but only upon the poset $\mathrm{iso}(X)$. A slight generalization of the Möbius inversion formula is given by the following lemma.

Lemma 2. *Consider* $\mathrm{iso}(X)$ *as a category* (*the inclusions are the morphisms*), *and let* $A\colon \mathrm{iso}(X) \to Ab$ *be a contravariant functor from* $\mathrm{iso}(X)$ *to the category of Abelian groups* ($\mathbb{Z}$*-modules*). *Let* i_K^H *denote the image of the inclusion* $H \to K$ *under* A, *and* $A(H)$ *the Abelian group corresponding to* H. *Then, if* $(\forall H)$ $b(H)$ *is the element of* $A(H)$ *defined by*

$$b(H) = \sum_{K \supset H} i_K^H a(K),$$

the following inversion formula is true

$$a(H) = \sum_{K \supset H} \mu(K, H) i_K^H b(K).$$

Proof. It follows the same lines of the proof of the Möbius inversion formula. Instead of considering a poset, it is possible to consider the image of the poset embedded in Ab, and write the formula as the inverse of a matrix $I + N$, where N is nilpotent and I is the identity. In this case the entries of the matrix N are the homomorphisms i^H_K, and the formal inverse of $I + N$ can be obtained as $\sum_{j=0}^n (-1)^j N^j$ for a suitable integer n. $\square$

3. The Möbius inversion formula

Let G be a finite group, X a G-ENR and $f : U \subset X \to X$ a compactly fixed G-map defined on an open subset U of X and taut in a G-neighborhood of $\mathrm{Fix}(f)$. The generalized Lefschetz number $\mathcal{L}(f^H|U_H)$ is well-defined for every isotropy H (and if f' is another G-map compactly fixed homotopic to f and taut in a G-neighborhood of its fixed point set, $\mathcal{L}(f^H|U_H) = \mathcal{L}(f'^H|U_H)$); let i_H denote the homomorphism $i_H : \mathbb{Z}\mathcal{R}(f^H|U_H) \to \mathbb{Z}\mathcal{R}(f^H)$ induced by the inclusion $U_H \subset U^H$.

Theorem 3. *If $f : U \to X$ is compactly fixed and taut in a G-neighborhood of* $\mathrm{Fix}(f)$, *then for every isotropy group $H \subset G$,*

$$i_H \mathcal{L}(f^H|U_H) = \sum_{K \supset H} \mu(K, H) i^H_K \mathcal{L}(f^K).$$

Proof. Let H be an isotropy group, and consider the $W_G H$-map $f^H : U^H \to X^H$. Its fixed point set $\mathrm{Fix}(f^H)$ is the disjoint union

$$\mathrm{Fix}(f^H) = \bigcup_{K \supset H} \mathrm{Fix}(f^K|U_K),$$

i.e. of the sets of fixed points with isotropy exactly K, with $K \supset H$. Because f is assumed to be taut in a neighborhood of $\mathrm{Fix}(f)$, every component C of $\mathrm{Fix}(f^H)$ has just one isotropy, K_C. Moreover, if V_C is an isolating neighborhood in U^H of C then, by the retraction property of the fixed point index, $\mathrm{ind}(f^H, V_C) = \mathrm{ind}(f^K, V_C \cap U_K)$. In other words, if we denote with V_K the union of all the V_C, with C component with isotropy exactly K, the fixed point set $\mathrm{Fix}(f^H)$ has a neighborhood V which is the disjoint union

$$\mathrm{Fix}(f^H) \subset V = \bigcup_{K \supset H} V_K.$$

Now we can use the union property (the union is disjoint) and the fact that the indexes are preserved, by the retraction property, i.e.

$$\mathcal{L}(f^H) = \sum_{K \supset H} \mathcal{L}(f^H|V_K) = \sum_{K \supset H} \iota^H_K \mathcal{L}(f^K|U_K),$$

where ι^H_K is the homomorphism induced by the inclusion

$$\iota^H_K : \mathbb{Z}\mathcal{R}(f^K|U_K) \to \mathbb{Z}\mathcal{R}(f^H).$$

It is worthwhile to note that the homomorphism $\iota_H^H = i_H \colon \mathbb{Z}\mathcal{R}(f^H|U_H) \to \mathbb{Z}\mathcal{R}(f^H)$ needs not to be the identity, and that $\mathcal{R}(f^H|U_H)$ and $\mathcal{R}(f^H)$ can differ. Moreover, for every choice of $K \supset H$, the equality $\iota_K^H = i_K^H \circ i_K$ is true. Thus the previous formula can be written as

$$\mathcal{L}(f^H) = \sum_{K \supset H} i_K^H \left[i_K \mathcal{L}(f^K|U_K) \right].$$

The formal properties of the Möbius coefficients $\mu(K, H)$ (Lemma 2) imply that for every isotropy H

$$i_H \mathcal{L}(f^H|U_H) = \sum_{K \supset H} \mu(K, H) i_K^H \mathcal{L}(f^K),$$

i.e. the thesis. □

Corollary 4. *Let G be a finite group, X a G-ENR, U an open subspace of X and $f \colon U \to X$ a compactly fixed G-map. If the element L is defined as*

$$L = \sum_{K \supset H} \mu(K, H) i_K^H \mathcal{L}(f^K) \in \mathbb{Z}\mathcal{R}(f^H),$$

then, for every $\xi \in \mathcal{R}(f^H)$,

$$L(\xi) \equiv 0 \mod |W_\xi|,$$

where W_ξ denotes the isotropy subgroup of the class ξ (relative to the action of the Weyl group $W_G H$ on the Reidemeister set $\mathcal{R}(f^H)$).

Proof. By lemma 1 it is possible to find a taut approximation of f via a compactly fixed G-homotopy (and the generalized Lefschetz numbers involved in the formula remain the same). Without loss of generality we can thus assume that f is taut in a neighborhood of its fixed point set. Because of Theorem 3, for every isotropy group $H \subset G$,

$$L = \sum_{K \supset H} \mu(K, H) i_K^H \mathcal{L}(f^K) = \iota_H \mathcal{L}(f^H|U_H).$$

So the proof is complete once it is shown that, for every $\xi \in \mathcal{R}(f^H)$, $\iota_H \mathcal{L}(f^H|U_H)(\xi) \equiv 0 \mod |W_\xi|$. This comes from the fact that the Weyl group acts freely on U_H: the fixed point set $\mathrm{Fix}(f^H|U_H)$ is the union of the free orbits of its components. Without loss of generality we can assume that $F = \mathrm{Fix}(f^H|U_H)$ is a finite number of points: thus

$$\iota_H \mathcal{L}(f^H|U_H) = \sum_{x \in F} \mathrm{ind}(f, x) \cdot \mathrm{cd}(x),$$

where $\mathrm{cd}(x)$ is the coordinate in $\mathcal{R}(f^H)$ of the fixed point x. By splitting F into orbits we get

$$\iota_H \mathcal{L}(f^H|U_H) = \sum_{[x]\in F/W} \sum_{x\in[x]} \mathrm{ind}(f, x) \cdot \mathrm{cd}(x).$$

But for every $[x] \in F/W$ the elements of $[x]$ have the same index, thus

$$\sum_{x\in[x]} \mathrm{ind}(f, x) \cdot \mathrm{cd}(x) = \mathrm{ind}(f, x) \sum_{x\in[x]} \mathrm{cd}(x) = \mathrm{ind}(f, x) W \cdot \mathrm{cd}(x).$$

This means that every term of the sum is a multiple of an element like $W \cdot \xi$, with $\xi \in \mathcal{R}(f^H)$, and the conclusion follows. □

REMARK 1. Another way of stating the proposition of Corollary 4 is the following. The action of the Weyl group $W_G H$ makes $\mathbb{Z}\mathcal{R}(f^H)$ into a $\mathbb{Z}W_G H$-module. Let $W = \sum_{w\in W_G H} w$ denote the diagonal element in $\mathbb{Z}W_G H$ and $I_W \subset \mathbb{Z}\mathcal{R}(f^H)$ the submodule of the multiples of W in $\mathbb{Z}\mathcal{R}(f^H)$, $I_W = W \cdot \mathbb{Z}\mathcal{R}(f^H)$. The statement of Corollary 4 is simply that

$$\sum_{K\supset H} \mu(K, H)\ i_K^H \mathcal{L}(f^K) \in I_W \subset \mathbb{Z}\mathcal{R}(f^H).$$

4. The transfer homomorphism

Now consider a finite group W acting locally smoothly on a manifold X, and a compactly fixed W-map $f : U \subset X \to X$ defined on an open G-subspace of X. Without loss of generality we can assume that the principal isotropy type of U is the trivial subgroup, so that U is the union of U_1 (the free part) and U_s (the singular part, which is the union of the spaces fixed by the elements of W). The inclusion $U_1 \to U$ induces the map $i_* : \mathcal{R}(f|U_1) \to \mathcal{R}(f)$. A class in $\mathcal{R}(f|U_1)$ is denoted by $[\lambda]$, where λ is a translation path (that is, an element of $E(f|U_1)$, i.e. $\lambda : I \to X$ such that $\lambda(0) \in U_1$ and $f\lambda(0) = \lambda(1)$).

Lemma 5. *If λ_1 and λ_2 are two translation paths such that $\lambda_1 I$, $\lambda_2 I \subset U_1$ and $i_*[\lambda_1] = i_*[\lambda_2] \in \mathcal{R}(f)$, then there is an element $w \in W$ such that $[\lambda_2] = [w\lambda_1] \in \mathcal{R}(f|U_1)$.*

Proof. By definition, because $i_*[\lambda_1] = i_*[\lambda_2]$, λ_1 and λ_2 belong to the same component of $E(f)$, that is, there is a map $\tilde{\gamma} : I \to E(f)$ such that $\gamma(0) = \lambda_1$ and $\tilde{\gamma}(1) = \lambda_2$. The map $\Gamma : I \times I \to X$ defined by $\Gamma(s, t) = \gamma(\lambda)(s)(t)$ is therefore a homotopy such that $\Gamma(0, t) = \lambda_1(t)$ and $\Gamma(1, t) = \lambda_2(t)$, while $f\Gamma(s, 0) = \Gamma(s, 1)$ for every $s \in I$. This is equivalent to the existence of a path $\gamma = \Gamma(s, 0)$ such that $\lambda_1 f(\gamma) \sim \gamma\lambda_2$ (homotopy rel. endpoints). Let us call such a path *a path connecting λ_1 and λ_2*.

A $\sigma\colon I \to U$ connecting λ_1 and λ_2 can be deformed in a way such that $\sigma I \cap U_s$ is a finite number of singular points with minimal isotropy (being the action locally smooth, if $\sigma(t) \in U^{w_1} \cap U^{w_2}$, σ can be deformed in a way that this single intersection point gives rise to at least two points with isotropy $\langle w_1 \rangle$ and $\langle w_2 \rangle$). Moreover, without loss of generality we can assume that σ is smooth and crosses each U^w transversally. If σ has these properties, we say that σ is a *normal connecting path*. Every connecting path is homotopic to a normal connecting path rel. endpoints.

For each normal connecting path σ, let $|\sigma|$ be defined as follows: $|\sigma| = \#\{t \in \sigma^{-1}U_s\}$. Thus $|\sigma| = 0$ if and only if σ does not touch the singular set (and thus it is a path in U_1). Now consider λ_1 and λ_2. Using this terminology, the assertion of the lemma says that there is a $w \in W$ such that λ_1 and $w \cdot \lambda_2$ can be connected by a normal connecting path σ with $|\sigma| = 0$. Assume it is false: thus, up to replacing λ_2 with $w \cdot \lambda_2$ for a suitable $w \in W$, λ_1 and λ_2 can be connected by a normal connecting path σ with $|\sigma| > 0$ and such that $|\sigma|$ is the minimum among all the possible choices of w and normal connecting paths. Because for all $w \in W$ the fixed subspaces X^w have transverse regular intersection (the action of W is locally smooth), if σ attains the minimum of $|\sigma|$, then also the sets $\sigma^{-1}U^w$ have minimum cardinality among all the normal connection paths homotopic to σ. Let $s_0 = \min \sigma^{-1}U_s$ and W_0 the isotropy of $\sigma(s_0)$.

Consider the map $\Gamma\colon I \times I \to X$ defined as before by the connecting path σ. Because f is supposed to be isovariant, $|f\sigma| = |\sigma|$ and $\sigma^{-1}U^w = (f\sigma)^{-1}U^w$ for every w. Moreover, because W acts locally smoothly on X, and U^{W_0} must have codimension 1 in U, W_0 must be a cyclic group of order 2 generated by an element $w_0 \in W$ (locally it must be a reflection along a hyperplane). Furthermore, let A be the component in $I \times I \setminus \Gamma^{-1}U^{w_0}$ containing $\{0\} \times I$. By minimality of $|\sigma|$, and consequently by minimality of $\sigma^{-1}U^{w_0}$, A cannot contain points in $I \times \partial I$ different from $[0, s_0) \times \{0, 1\}$ (otherwise the minimality of $|\sigma|$ would be contradicted, because there would exist a normal connecting path σ' with less intersection points with U_s than σ). Therefore the complement $A' = I \times I \setminus A$ of A in $I \times I$ contains $\{1\} \times I$, and $I \times I = \overline{A} \cup A'$. Because $\overline{A} \cap A' \subset \Gamma^{-1}U^{w_0}$, the following homotopy is well-defined:

$$\Gamma'(s,t) = \begin{cases} w_0 \cdot \Gamma(s,t) & \text{if } (s,t) \in \overline{A} \\ \Gamma(s,t) & \text{if } (s,t) \in A'. \end{cases}$$

But if $\sigma' = \Gamma'(-, 0)$, then σ' is a path connecting $w_0 \cdot \lambda_1$ and λ_2 and it can be easily deformed only in a neighborhood of s_0 so that $\sigma'(s_0) \notin U_s$, thus $w_0 \cdot \lambda_1$ and λ_2 can be connected by a normal connecting path with less than $|\sigma|$ transversal simple intersection points with U_s, which contradicts the hypothesis of minimality of $|\sigma|$. This means that the minimal $|\sigma|$ cannot be different from 0, that is the thesis. $\square$

Let $f\colon U \subset X \to X$ be an isovariant W-map, with trivial principal isotropy type in U, and U_1 the complement of the singular part U_s of U. The purpose of the previous lemma is to define a *transfer homomorphism* $\mathrm{Tr}\colon \mathbb{Z}\mathcal{R}(f) \to \mathbb{Z}\mathcal{R}(f|U_1)$. We need only to consider the value of Tr on the free generators of $\mathbb{Z}\mathcal{R}(f)$, i.e. the elements of $\mathcal{R}(f)$. If $\lambda_0 \in E(f)$ is such that $\lambda_0^{-1}U_s = \emptyset$, then let $\mathrm{Tr}([\lambda_0]) = \sum_{w\in W}[w \cdot \lambda_0]$. It is an element of $\mathbb{Z}\mathcal{R}(f|U_1)$, because $\lambda_0(0) \notin U_s$ by assumption. Otherwise, if $\lambda \in E(f)$ is an element such that $\lambda^{-1}U_s \neq \emptyset$, there are two cases: either every translation path in the same component $[\lambda] \in \mathcal{R}(f)$ has a nontrivial intersection with U_s, or there is at least $\lambda_0 \in [\lambda]$ such that $\lambda_0^{-1}U_s = \emptyset$. In the first case, define $\mathrm{Tr}([\lambda]) = 0$. In the second case, define simply $\mathrm{Tr}([\lambda]) = \mathrm{Tr}([\lambda_0])$. What now we need is to show that the definition does not depend upon the choice of such a λ_0 or upon the representative in a class $[\lambda] \in \mathcal{R}(f)$.

Proposition 6 (Transfer). *If $f: U \subset X \to X$ is an isovariant W-map, then the transfer homomorphism* Tr *defined as above is well-defined*; *moreover, if f is compactly fixed in U_1 then the generalized Lefschetz number verifies the identity*

$$\mathrm{Tr} \circ i_* \mathcal{L}(f|U_1) = |W| \mathcal{L}(f|U_1)$$

in $\mathbb{Z}\mathcal{R}(f|U_1)$, where $i_*\colon \mathbb{Z}\mathcal{R}(f|U_1) \to \mathbb{Z}\mathcal{R}(f)$ is the homomorphism induced by the inclusion $U_1 \subset U$.

Proof. It is only needed to show that if λ_1 and λ_2 are two translation paths in the same component of $E(f)$, that do not touch U_s, then $\mathrm{Tr}([\lambda_1]) = \mathrm{Tr}([\lambda_2])$. By Lemma 5, because $i_*[\lambda_1] = i_*[\lambda_2] \in \mathcal{R}(f)$, there is an element $w_1 \in W$ such that $[\lambda_2] = [w_1\lambda_1] \in \mathcal{R}(f|U_1)$. But this implies that

$$\mathrm{Tr}([\lambda_1]) = \sum_{w\in W}[w \cdot \lambda_1] = \sum_{w\in W}[ww_1^{-1}\lambda_2] = \mathrm{Tr}([\lambda_2]),$$

that is the thesis.

Because the action of W in U_1 is free, one can assume without loss of generality that $\mathrm{Fix}(f) \cap U_1$ is the disjoint union of a finite number of W-orbits of fixed points. To prove the second part of the proposition, consider an isolated fixed point $x \in \mathrm{Fix}(f) \cap U_1$ and its coordinate $\mathrm{cd}(x) \in \mathcal{R}(f|U_1)$. The orbit $W \cdot x$ contributes in the expression of $\mathcal{L}(f|U_1)$ with the term

$$\mathrm{ind}(f, x) \sum_{w\in W} w \cdot \mathrm{cd}(x).$$

Its image in $\mathcal{R}(f)$ is simply the sum

$$\mathrm{ind}(f, x) \sum_{w\in W} w \cdot i_* \mathrm{cd}(x).$$

The image under the transfer Tr is given by

$$\begin{aligned}\operatorname{ind}(f,x)\sum_{w\in W}\operatorname{Tr}(w\cdot i_*\operatorname{cd}(x)) &= \operatorname{ind}(f,x)\sum_{w\in W}\sum_{v\in W} v\cdot w\cdot\operatorname{cd}(x)\\ &= |W|\operatorname{ind}(f,x)\sum_{w\in W} w\cdot\operatorname{cd}(x).\end{aligned}$$

By adding these terms in $\mathcal{L}(f|U_1)$ the proof is complete. □

5. Consequences: the converse of the Lefschetz property

Lemma 7. *Let G be a finite group, M a smooth G-manifold and $U \subset M$ an open G-subspace. Assume that for every isotropy H the space U_H has no component of dimension* 2. *Then a compactly fixed G-map $f: U \to M$ is G-homotopic to a fixed point free G-map* (*via a compactly fixed G-homotopy*) *if and only if there exists a G-map $f': U \to M$, compactly fixed homotopic to f and taut in a neighborhood of* $\operatorname{Fix}(f')$, *such that the generalized Lefschetz number $\mathcal{L}({f'}^H|U_H)$ vanishes for every isotropy H.*

Proof. A fixed point free G-map f_0 compactly fixed G-homotopic to f is also taut in a neighborhood of $\operatorname{Fix}(f_0)$, and clearly the generalized Lefschetz number $\mathcal{L}(f_0^H|U_H)$ vanishes for every isotropy H. On the other hand, if there exists a G-map $f': U \to M$ compactly fixed homotopic to f and taut in a neighborhood of $\operatorname{Fix}(f')$, then by induction over orbit types it is possible to deform it to be fixed point free, once one can prove that for every isotropy H the restriction ${f'}^H: U_H \to M^H$ can be deformed to be fixed point free via a compactly fixed W_GH-homotopy. This can be done by using the Wecken-Jiang modification on simple paths connecting fixed points in the same Nielsen class, in the standard way, because we are assuming that every component of U_H has dimension different from 2. □

REMARK 2. If there is a G-map $f': U \to M$, compactly fixed homotopic to f and taut in a neighborhood of $\operatorname{Fix}(f')$, such that the generalized Lefschetz number $\mathcal{L}({f'}^H|U_H)$ vanishes for every isotropy H, then every other G-map $f'': U \to M$, compactly fixed homotopic to f and taut in a neighborhood of $\operatorname{Fix}(f'')$ has $\mathcal{L}({f''}^H|U_H) = 0$ for every isotropy H. A simple way of proving this is to consider a compact G-submanifold of M and a W_GH-deformation retraction of an equivariant neighborhood of $M^H \times \partial I \cup M_s^H \times I$ in M^H (due to the fact that $M_s^H \to M^H$ is a W_GH-cofibration).

Lemma 8. *Let G be a finite group, M be a smooth G-manifold and $U \subset M$ an open G-subspace. If $f: U \to M$ is an isovariant compactly fixed G-map then there is a compactly fixed deformation $f_t: U \to M$, $t \in [0,2]$, such that*

(i) $f_0 = f$;

(ii) f_2 *is taut in a neighborhood of* $\mathrm{Fix}(f_2)$;
(iii) f_t *is isovariant for every* $t \in [0, 1]$;
(iv) *for every isotropy* H *the restriction* f_1^H *is compactly fixed in* U_H *and* $\mathcal{L}(f_1^H|U_H) = \mathcal{L}(f_2^H|U_H)$.

Proof. It is possible to define f_t by induction over orbit types as follows. For every isotropy H the singular set M_s^H has a W_GH-neighborhood N which is W_GH-homeomorphic to the mapping cylinder of a suitable map $j: N_1 \to M_s^H \subset M^H$ from a free smooth compact W_GH-manifold N_1 to M_s^H.

$$\begin{tikzcd}
N_1 \arrow[r, "j"] \arrow[d] & M_s^H \arrow[d] \arrow[ddr, bend left] & \\
N_1 \times I \arrow[r, "\overline{j}"] \arrow[drr, bend right] & N \arrow[dr] & \\
 & & M^H
\end{tikzcd}$$

Thus $N \supset M_s^H$ has coordinates (n, t), with $n \in N_1$ and $t \in I$, with $(n, 0) \equiv (n', 0)$ whenever $j(n) = j(n')$. Assume now that f_t^H is defined on U_s^H: we want to extend it to a map $f_t^H: U^H \to M^H$. First, consider a compact W_GH-subspace (with boundary) C of U_s^H containing the fixed point set $\mathrm{Fix}(f_0^H) \cap U_s^H$. There exists $\epsilon > 0$ such that $f\overline{j}(n, t) \in N$ for all $(n, t) \in C \times [0, \epsilon]$. Now, it is easy to define a deformation through a W_GH-isotopy $h_t: C \to C$, $t \in I$, with the property that $h_0 = 1_C$ and $pr_2 f h_1 \overline{j}(n, t) \leq (1/2)t$, where pr_2 means the projection onto the second factor. Such a deformation can be extended to U^H, identically outside a compact neighborhood of $\overline{j}C \times [0, \epsilon]$ (small enough so that no other fixed point is involved outside the fixed points in N), and thus we obtain a deformation $f_t = f \circ h_t$, $t \in [0, 1]$, with the property that f_t is isovariant for every $t \in [0, 1]$ and that for $(n, t) \in C \times [0, \epsilon]$,

$$pr_2 f_1 \overline{j}(n, t) = pr_2 f h_1 \leq \frac{1}{2} pr_2 \overline{j}(n, t).$$

Furthermore, C contains the fixed points of f_1 in the singular part U_s^H (actually $f_1 = f$ in the singular part), and by construction $\mathrm{Fix}(f_1^H) \cap U_H$ is compact. Then we proceed by induction over orbit types and we have defined a G-map f_1 with the desired properties. Now we need to show that a taut G-approximation f_2 of f_1 has the same generalized Lefschetz numbers $\mathcal{L}(f_1^H|U_H)$. To prove it, simply consider that a taut approximation can be defined by considering, for every H, a function $\phi: [0, \epsilon] \to I$, constant in a neighborhood $[0, \epsilon']$ of 0, with $0 < \epsilon' < \epsilon$ (notation as before), and equal to the identity in ϵ: if $(n, t) \in C \times [0, \epsilon]$, and then defining

$$f_2 \overline{j}(n, t) = \left(pr_1 f_1 \overline{j}(n, t), \phi(t) \cdot pr_2 f_1 \overline{j}(n, t) \right).$$

Such a function has a suitable $W_G H$-extension to U^H, and it is clear that it gives rise to a $W_G H$-homotopy, relative to U_s^H and compactly fixed in U_H. This completes the proof. $\square$

Theorem 9 (CLP). *Let G be a finite group, M a smooth G-manifold and $f : U \to M$ a compactly fixed G-map. Assume that the following conditions hold*:

(i) *For every isotropy $H \subset G$ the codimension of $M_s^H \setminus M_{s0}^H$ in M^H is at least* 2;

(ii) *The map f is isovariant.*

(iii) *For every H no component of U_H has dimension* 2.

Then there exists a fixed point free G-map f' compactly fixed G-homotopic to f if and only if for every isotropy $H \subset G$ the Nielsen number $N(f^H) = 0$, i.e. if and only if the equivariant Nielsen number $N_G(f)$ vanishes.

Proof. By definition $N_G(f) = 0$ if and only if $N(f^H) = 0$ for every isotropy $H \subset G$, and this happens if and only if $\mathcal{L}(f^H) = 0$ for every isotropy $H \subset G$. Of course, if f' is compactly fixed G-homotopic to f and fixed point free, then $(\forall H)\ 0 = \mathcal{L}({f'}^H) = \mathcal{L}(f^H)$. Thus it remains to prove that if $(\forall H)\ \mathcal{L}(f^H) = 0$ then there exists such a fixed point free deformation of f. Let f_t be the G-deformation of Lemma 8. Because f_2 is taut in a neighborhood of $\mathrm{Fix}(f_2)$ and $\forall K \mathcal{L}(f_2^K) = \mathcal{L}(f^K) = 0$, by applying Theorem 3, one gets that for every isotropy $H \subset G$,

$$i_H \mathcal{L}(f_2^H | U_H) = \sum_{K \supset H} \mu(K, H) i_K^H \mathcal{L}(f_2^K) = 0.$$

Thus $i_H \mathcal{L}(f_2^H | U_H) = 0$. But by property (iv) of Lemma 8 the generalized Lefschetz numbers coincide $\mathcal{L}(f_2^H | U_H) = \mathcal{L}(f_1^H | U_H)$, with f_1^H compactly fixed in U_H. This means that $\forall H$, $i_H \mathcal{L}(f_1^H | U_H) = 0$. Now consider $f_1^H : U^H \to M^H$ as a $W = W_G H$-equivariant map. Because we are assuming that for every isotropy $H \subset G$ the codimension of $M_s^H \setminus M_{s0}^H$ in M^H is at least 2 (i), actually the homomorphism induced by the inclusion $U_H = U^H \setminus U_s^H \subset U^H \setminus U_{s0}^H$ is an isomorphism

$$\mathbb{Z}\mathcal{R}(f_1^H | U_H) \cong \mathbb{Z}\mathcal{R}\left(f_1^H | \left(U^H \setminus U_{s0}^H\right)\right).$$

But $U^H \setminus U_{s0}^H$ is exactly the free part of the action of W on U: thus, when considering only the action of W on M^H, we obtain that the free part is $U_1^H = U^H \setminus U_{s0}^H$ and so

$$i_* \mathcal{L}(f_1^H | U_1) = 0 \in \mathbb{Z}\mathcal{R}(f_1^H),$$

where i_* is the homomorphism induced by the inclusion $U_1 \subset U$.

Now apply the transfer Tr to both sides of the equality (f_1^H is an isovariant W-map, and compactly fixed in U_1): by Proposition 6,

$$0 = \mathrm{Tr}\left(i_* \mathcal{L}(f_1^H | U_1)\right) = |W| \mathcal{L}(f_1^H | U_1) \in \mathbb{Z}\mathcal{R}(f^H | U_1).$$

Hence $\mathcal{L}(f_1^H|U_1) = 0$, and thus $\mathcal{L}(f_1^H|U_H) = \mathcal{L}(f_1^H|U_1) = 0$. This holds for every H, and thus f_1 is an isovariant G-map, such that for every H it is compactly fixed in U_H and $\mathcal{L}(f_1^H|U_H) = 0$. Again by property (iv) of Lemma 8 this implies that for every H

$$\mathcal{L}(f_2^H|U_H) = \mathcal{L}(f_1^H|U_H) = 0.$$

Now apply Lemma 7: we have found a G-map $f_2\colon U \to M$, compactly fixed homotopic to f and taut in a neighborhood of $\mathrm{Fix}(f_2)$, such that the generalized Lefschetz number $\mathcal{L}(f_2^H|U_H)$ vanishes for every H. The proof is therefore complete. □

A finite group is 2-split if it is the direct product of its 2-Sylow subgroup and the odd-order complement.

Corollary 10. *Let G be a 2-split finite group, M a smooth G-manifold and $f\colon U \to M$ an isovariant compactly fixed G-map. If for every H no component of U_H has dimension 2, then f is compactly fixed G-homotopic to a fixed point free G-map if and only if $N_G(f) = 0$.*

Proof. If G is 2-split, then condition (i) of Theorem 9 is automatically satisfied. Under this assumption the Corollary follows directly from the Theorem. □

6. Examples

EXAMPLE 1. Let G be the cyclic group of order 2, and M the genus 3 surface (torus with three holes). Let x_0 be a base-point in M and $a(t)$, $b(t)$ and $c(t)\colon I \to M$ the three simple closed paths in M based in x_0 such that the closure of the interior of M (as a 3-manifold with boundary) is homotopy equivalent to the wedge (in x_0) of the loops a, b, and c. We will denote by a the image of a in M as a set (and the same for b and c) or, equivalently, as a homotopy class of paths. It will be clear from the context which meaning is correct. Embed smoothly M in $\mathbb{R}^3$ so that, if g denotes the reflection $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $g(x_1, x_2, x_3) = (-x_1, x_2, x_3)$, the following properties hold

(i) $gM = M$;
(ii) $\forall t \in I\colon ga(t) = b(t),\ gb(t) = a(t),\ gc(t) = c(t)$.
(iii) $M^g = a \sqcup S^1$.

That is, M is endowed with a smooth action of G (see [7], example 5.2 for a figure). Let $f\colon a \to M$ be the map defined by $f(a) = a^{-1}b^{-1}ab$. We can extend it to the wedge of a, b and c by setting $f(b) = f(ga) = gf(a)$ and $f(c) = c$. Now, M can be retracted on $a \vee b \vee c$, and the composition of the retraction with f gives rise to a G-equivariant self-map $f\colon M \to M$. This map is not isovariant, and actually is not G-homotopic to any G-isovariant map: it is not difficult to see that $N(f) = N(f^G) =$

0, but for any taut approximation h of f $\mathcal{L}(h|M \setminus M^G) \neq 0$. Now, it is possible to consider the manifold $X = M \times S^2$ (trivial action of G on S^1), with the map $F = f \times 1$, where $1\colon S^2 \to S^2$ is the identity map. By the multiplication property of the generalized Lefschetz number, the equivariant Nielsen number $N_G(F)$ vanishes, while $\mathcal{L}(F^G|X \setminus X^G) \neq 0$. The assumptions of Corollary 10 and thus of Theorem 9 are not fulfilled: G is 2-split, there are no components in X^G or $X \setminus X^G$ of dimension 2, but f is not isovariant. Moreover, the conclusion of the Corollary does not hold.

EXAMPLE 2. Similar to the previous example, the following shows better the crossing of the fixed point classes on non-isovariant maps. Let X be the union of the two circles C_1 and C_2 in the complex plane $\mathbb{C}$ with centers in i and $-i$, and radius 2. Let G be the cyclic group of order 2, generated by the conjugation $g\colon \mathbb{C} \to \mathbb{C}$, $gz = \overline{z}$. Consider the unique G-map $f\colon X \to X$ with the property that for every $t \in \mathbb{R}$

$$f(i - 2ie^{it}) = \begin{cases} i - 2ie^{5it} & \text{if} \quad t \in [-\pi/3, \pi/3] \\ i - 2ie^{-it} & \text{otherwise.} \end{cases}$$

It is well defined, because $i - 2ie^{5/3\pi i} = i - 2ie^{-1/3\pi} = -\sqrt{3}$ and $i - 2ie^{-5/3\pi i} = i - 2ie^{1/3\pi} = \sqrt{3}$. Moreover, the fixed points are i, $-i$, $3i$, $-3i$, with indices -1, -1, $+1$, $+1$. It is easy to see that $N(f^G) = 0$. Moreover, the fixed point classes have mixed indices (that is, they contain a fixed point of index $+1$ and a fixed point of index -1), so that they are inessential, and thus $N(f) = 0$. Now take a neighborhood of X in $\mathbb{C}$ and consider this as X. The Nielsen numbers are preserved. The local Lefschetz numbers of the map f restricted to $X \setminus X^G = X_1$ (which is equal to X minus the intersection of X with the real axis) are non-zero, because for example i and $3i$ do not belong to the same Reidemeister class. This again shows how even if the group G is 2-split, without a further assumption (i.e. the map is isovariant) the transfer is not well-defined, and therefore no analogous of Corollary 10 holds. Of course, it might be that there is a weaker hypothesis than being f isovariant.

EXAMPLE 3. Let G be the dihedral group D_{2n} of order $2n$, with $n \geq 3$ odd, and $\mathbb{C}$ the complex plane with the canonical action of G. Then G acts on the 4-dimensional unit sphere $M = S^4 \subset \mathbb{C} \oplus \mathbb{R}^3$, where the action of G on $\mathbb{R}^3$ is trivial. Representatives of the isotropy classes in G are G, H and 1, where H is any subgroup of order 2 representing a reflection in $\mathbb{C}$. The fixed subspaces are $M^G \approx S^2$, $M^H \approx S^3$, $M^1 = M$. The Weyl group of H in G is trivial, thus $M^H_{s0} = \emptyset$, while $M^H_s = M^G \neq \emptyset$, so that $M^H_s \setminus M^H_{s0} = M^H_s$, and the codimension of M^H_s in M^H is 1. Thus hypothesis (i) of Theorem 9 is not satisfied. In fact, G is not 2-split. As it can be seen in [8], if a group G is not 2-split, then there are examples of smooth G-manifolds with the property that the identity map (which is a G-isovariant self-map) cannot be deformed to be fixed point free. That result was proved with an *ad hoc* proof: it is easy to prove it in a different way, by using Lemma 7 and Lemma 8, together with the basic properties of the generalized Lefschetz numbers of taut approximations (see also [7] for further properties of an equivariant fixed point index).

7. Remarks

REMARK 3. We have seen in examples 1, 2, 3 how the hypotheses in Theorem 9 are not redundant. On the other hand, it is possible to consider other assumptions on f or M: for example, if M^H is simply connected for every H, then instead of the generalized Lefschetz numbers it is possible to use the classical fixed point index, and the Möbius formula reduces to the one of Komiya [14]. In this case it is only needed the codimension assumption (i) of Theorem 9. To overcome this problem, it is possible to suppose directly G of odd order: in this case M^H_s has always codimension at least 2 in M^H, and thus we only need to suppose that no component of M^H has dimension 2. Of course, this latter assumption can not be canceled, because the converse of the Lefschetz property does not hold on surfaces (it holds only in the case f is a self-homeomorphism, because of the Nielsen-Thurston classification of surface homeomorphisms; for maps locally defined, in general it does not hold).

REMARK 4. The techniques used in this paper can be easily extended to the case of a compact Lie group G. The fact is that if $H \subset G$ is an isotropy group of a smooth G-manifold M, then by equivariant obstruction theory it is easy to reduce the problem to the $W_G H$ manifolds M^H with $W_G H$ finite. So that most of the tools used in this paper can be literally translated to the case G compact Lie. The only difficulty is that the Möbius inversion formula cannot be used as it is, because there would be an infinite sum. But considering the Euler characteristic of suitable homogeneous spaces and some counting theorems of conjugacy classes of closed subgroups of G, it is possible to develop a theorem similar to the one for finite groups.

REMARK 5. If $f: M \to M$ is an equivariant self-homeomorphism of a smooth G-manifold M, then it is isovariant. It might be of some interest to know whether under the same assumptions of Theorem 9 the map f can be deformed isotopically to a fixed point free map. This, due to a modified version of a result of M. Kelly [12], is true if $(\forall H)$ the dimension of the components of M^H is at least 5. As far as I know, it is still unknown for lower dimensions. There is an equivariant version of the Nielsen-Thurston classification theorem, but it is needed a stronger stratified version, because M^H is not only $W_G H$-equivariant, but stratified by isotropy strata, which might not coincide with the singular sets of the W-action.

REMARK 6. As it was shown by Dold [3], Komiya [14] and others (consider e.g. the Lefschetz or Nielsen dynamical zeta-functions), equivariant fixed point theory can be used to analyze properties of periodic points of maps, using the natural action of the cyclic group $\mathbb{Z}_n$ on the sets of n-periodic points $\mathrm{Fix}(f^n)$. It might be of some interest to give a characterization of fixed point classes and indices that can occur as generalized Lefschetz numbers of iterated maps f^n, and of the homomorphisms (the so-called *boosting functions*) $\mathbb{Z}\mathcal{R}(f^m) \to \mathbb{Z}\mathcal{R}(f^n)$, when m divides n. Theorem 3 can be seen as a first step in this direction, giving a necessary condition.

REMARK 7. The generalized Lefschetz number used in this paper is the obstruction in deforming the map to be fixed point free. What we have proved is also that, under some assumptions, the set of generalized Lefschetz numbers $(\mathcal{L}(f^H))$ is a kind of *universal invariant* for G-homotopy classes of maps, with respect to the fixed point problem. All the properties of $\mathcal{L}(f^H)$ and a different formulation as a fixed point index in a suitable ring can be found in [7]. The idea of universal invariant is a slight modification of the one exposed in [16].

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Dipartimento di Matematica
del Politecnico di Milano
Piazza Leonardo da Vinci 32
20133 Milano, Italy
e-mail: ferrario@mate.polimi.it