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$[p]$-TYPICAL FORMAL GROUPS AND THE HOMOMORPHISM $\Omega_*^U \to \Omega_*^{SO}$

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In the present note we construct a $[p]$-typical formal group $F_{[p]}$, p a prime, which is universal for $[p]$-typical formal groups over arbitrary ground rings; and we study structure of the ground ring of $F_{[p]}$ (Corollary 5). Using $F_{[2]}$ we describe the kernel of the forgetful homomorphism S: $\Omega_*^U \to \Omega_*^{SO}$ of complex structures (Corollary 7).

Our basic reference is [1] and we use the notations of [1] freely.

1. Universal $[p]$-typical formal group

Let U be the Lazard ring and F_U the universal one-dimensional formal group defined over U. As usual we identify U with the complex cobordism ring Ω_*^U. Then U is graded by non-negative even degrees (or by non-positive even degrees when we regard Ω_*^U as $U^*(pt)$).

Let p be a prime, R a commutative ring with 1 and F a (commutative one-dimensional) formal group over R. By the terminology of [1] F is $[p]$-typical iff $\boldsymbol{f}_{p,F}\gamma_0=0$, where $\boldsymbol{f}_p$ is the Frobenius operator and γ_0 is the identity curve.

Let F be $[p]$-typical and u: $U \to R$ the unique unitary homomorphism of rings such that $u_* F_U = F$. By the notation of [1] we put

$$(\boldsymbol{f}_{p,U}\gamma_0)(T) = \sum_{n\geq 1}{}^{F_U}(v_{np-1}^{(p)} T^n) .$$

Then $v_{np-1}^{(p)} \in U_{2(np-1)}$. Now

$$u_*(\boldsymbol{f}_{p,U}\gamma_0) = \boldsymbol{f}_{p,F}\gamma_0 = 0 .$$

Hence

$$\sum_{n\geq 1}{}^{F}(u(v_{np-1}^{(p)})T^n) = 0 ,$$

and by [1], Proposition 2.10, we obtain

$$(1.1) \qquad u(v_{np-1}^{(p)}) = 0 \qquad \text{for all } n \geq 1 .$$

Let

$$J_p = (v_{p-1}^{(p)}, v_{2p-1}^{(p)}, \cdots, v_{np-1}^{(p)}, \cdots) ,$$

the ideal of U generated by $v_{np-1}^{(p)}$, $n\geq 1$. By (1.1) u factorizes as the composition of the sequence

$$U \xrightarrow{\pi_p} U/J_p \xrightarrow{u_p} R . \tag{1.2}$$

Define

$$V_{[p]} = U/J_p \quad \text{and} \quad F_{[p]} = \pi_{p*}F_U .$$

Then $u_{p*}F_{[p]}=F$ and the homomorphism u_p: $V_{[p]}\to R$, $u_{p*}F_{[p]}=F$, is unique by the uniqueness of u. And obviously $f_{p,F_{[p]}}\gamma_0=0$. Thus we obtain

Proposition 1. *$F_{[p]}$ is a $[p]$-typical formal group over $V_{[p]}$ and universal for $[p]$-typical formal groups.*

In [1] we observed a $[p]$-typical formal group over a $Z[1/p]$-algebra which is universal for $[p]$-typical formal groups over $Z[1/p]$-algebras. The present $[p]$-typical formal group $F_{[p]}$ differs from that of [1] as it is universal for $[p]$-typical formal groups over arbitrary commutative rings with unities.

2. Structure of $\hat{U}/\hat{J}_p$

Let

$$\log_U T = \sum_{k\geq 0} m_k T^{k+1}, \quad m_0 = 1 ,$$

the logarithm of F_U over $U\otimes Q$ and put

$$\hat{U} = Z[m_1, m_2, \cdots, m_k, \cdots]$$

as in [1]. As usual we can identify $\hat{U}$ with $H_*(MU)$. Then the inclusion map

$$U\subset \hat{U} \tag{2.1}$$

is identified with the Hurewicz homomorphism

$$\pi_*(MU) \to H_*(MU) .$$

Let p be a prime and put

$$\hat{J}_p = (v_{p-1}^{(p)}, v_{2p-1}^{(p)}, \cdots, v_{np-1}^{(p)}, \cdots)_{\hat{U}} ,$$

the ideal of $\hat{U}$ generated by $v_{np-1}^{(p)}$, $n\geq 1$. In this section we observe structure of the quotient ring $\hat{U}/\hat{J}_p$.

Recall the relation (6.2) of [1]:

$$pm_{np-1} = v_{np-1}^{(p)} + \sum_{\substack{ij=n \\ 1\leq i<n}} m_{j-1}(v_{ip-1}^{(p)})^j . \tag{2.2}$$

This is the basic relation we use here. This shows that

(2.3) $$pm_{np-1}\in\hat{J}_p$$

on one hand, and by an induction on n,

(2.4) $$(1/p)v^{(p)}_{np-1}\in\hat{U}$$

on the other hand.

Let p and q be different primes. For each integer $k\geq 1$ we have

(2.5) $$pv^{(q)}_{pq^k-1}\in\hat{J}_p$$

Proof by induction on k. By (2.2) we have

$$qm_{pq-1}=v^{(q)}_{pq-1}+m_{p-1}(v^{(q)}_{q-1})^p\,.$$

Hence

$$pv^{(q)}_{pq-1}=qpm_{pq-1}-pm_{p-1}(v^{(q)}_{q-1})^p\in\hat{J}_p$$

by (2.3). Thus (2.5) is true for $k=1$. Now assume that $pv^{(q)}_{pq^j-1}\in\hat{J}_p$ for $j<k$. Then by (2.2) we have

$$pv^{(q)}_{pq^k-1}=pqm_{pq^k-1}-\sum_{\substack{ij=pk\\1\leq i<pk}}pm_{j-1}(v^{(q)}_{q^i-1})^j\,.$$

$pqm_{pq^k-1}\in\hat{J}_p$ by (2.3). For each term under the summation, if $p|j$ then $pm_{j-1}\in\hat{J}_p$, and if $p\not| j$ then $p|i$ and $pv^{(q)}_{q^i-1}\in\hat{J}_p$ by induction hypothesis. Thus

$$pv^{(q)}_{pq^k-1}\in\hat{J}_p\,, \qquad \text{Q.E.D.}$$

Here we recall Milnor basis of U. Let s_n denote the Chern number corresponding to $\sum t_i^n$. As is well-known a series of elements $u_n\in U_{2n}$, $n\geq 1$, forms a polynomial basis of U if it satisfies

$$\begin{aligned}s_n(u_n)&=q\ \text{ when }\ n=q^s-1\ \text{ for some prime }\ q\,,\\&=1\ \text{ otherwise.}\end{aligned}$$

Such a basis is called Milnor basis. We shall choose a Milnor basis in a specific form.

By (2.2) we see that

(2.6) $$s_{nq-1}(v^{(q)}_{nq-1})=q$$

for any prime q and $n\geq 1$. First we choose

(2.7) $$u_n=v^{(q)}_{q^s-1}\ \text{ when }\ n=q^s-1,\ q\text{ a prime.}$$

Now let p be the specified prime. When $p|n+1$ and $n+1$ is not a power of p, choosing the smallest prime q dividing $n+1$ and differing from p, we can express n as $n=pqk-1$, k a pitive integer. In such a case we put

(2.8) $$u_{qpk-1}=sv^{(q)}_{pq^k-1}+tv^{(p)}_{pq^k-1}\,,$$

where s and t are integers such that $sq+tp=1$. Then

$$s_{pqk-1}(u_{pqk-1}) = 1$$

by (2.6).

For remaining n, i.e., $p \not| n+1$ and $n+1$ is not a prime power, we choose u_n arbitrarily so that $s_n(u_n)=1$.

Hereafter we use only the above special choice of Milnor basis. First of all we have

$$pu_{pqk-1} \in \hat{J}_p \tag{2.9}$$

for elements of type (2.8), which follows from (2.5).

Put

$$\begin{aligned} m'_n &= (1/q)u_n \quad \text{when } n+1=q^s,\ q \text{ a prime,} \\ &= u_n \qquad\quad \text{when } n+1 \text{ is not a prime power.} \end{aligned} \tag{2.10}$$

These are well defined elements of $\hat{U}$ by (2.4) and

$$\hat{U} = Z[m'_1, m'_2, \cdots, m'_k, \cdots] \tag{2.11}$$

since $s_k(m'_k)=1$.

For degrees of type (2.8) we observe the elements $pm'_{pqk-1} - v^{(p)}_{pqk-1}$. These belong to $\hat{J}_p$ by (2.9) and are decomposable in $\hat{U}$ since s_n-numbers are zero. Thus by induction on qk we can replace the ideal basis elements $v^{(p)}_{pqk-1}$ of $\hat{J}_p$ by pm'_{pqk-1} for such degrees and we obtain

Proposition 2. $\hat{J}_p = (pm'_{pn-1},\ n \geq 1)$.

Corollary 3. *$\hat{U}/\hat{J}_p$ is a direct sum of copies of Z and Z/pZ of which each direct summand is generated by a monomial of m'_k's. A monomial is of order p when it contains an element m'_k with $p \mid k+1$ as a factor, and otherwise of infinite order.*

3. Structure of $V_{[p]}$

Under our special choice of Milnor basis of U we could choose a polynomial basis of $\hat{U}$ so that its each element is a constant multiple of the corresponding element of the Milnor basis (cf., (2.10)—(2.11)).

Theorem 4. $J_p = (u_{p^k-1},\ k \geq 1,\ pu_{np-1},\ n \neq p^s)$.

Proof. Inductively on n we replace generators $v^{(p)}_{np-1}$ of J_p by the elements stated in Theorem. Since $u_{p-1} = v^{(p)}_{p-1}$ the replacement is already done for $n=1$. Assume the replacement is done for $k<n$. When $n=p^s$ it is done already. Suppose n is not a power of p Since $pu_{pn-1} - v^{(p)}_{pn-1}$ is decomposable we can

express it as a polynomial of u_k's such that $1 \leq k < pn-1$, say, P. The polynomial expression P' of P in $\hat{U}$ can be obtained by replacing each monomial in P by the corresponding monomial of m'_k multiplied with a non-zero integer. Now $pu_{pn-1} - v^{(p)}_{pn-1} \in \hat{J}_p$ by (2.9). Then by Proposition 2 each summand of P' belongs to $\hat{J}_p$. This implies that each monomial in P with non-zero coefficient contains a u_m with $m=ps-1$ as a factor and, when it contains no u_m with $m=p^j-1$ as a factor, then p divides its coefficient. Hence each summand of P belongs to J_p and $pu_{pn-1} - v^{(p)}_{pn-1} \in J_p$. Now we can replace $v^{(p)}_{pn-1}$ by pu_{pn-1} in the system of generators of J_p, Q.E.D.

Corollary 5. *$V_{[p]}$ is a direct sum of copies of Z and Z/pZ of which each direct summand is generated by a monomial of u_n's such that $n \neq p^s-1$. A monomial is of order p when it contains an element u_k with $p \mid k+1$ as a factor, and otherwise of infinite order.*

4. The forgetful homomorphism $\Omega^U_* \to \Omega^{SO}_*$

Let

$$S\colon \Omega^U_* \to \Omega^{SO}_* \quad \text{and} \quad \Psi\colon \Omega^{SO}_* \to \mathfrak{N}_*$$

be the forgetful homomorphisms of complex structures and orientations respectively. Milnor [2] observed that

$$(\Psi \circ S)(\Omega^U_*) = (\mathfrak{N}_*)^2,$$

where $(\mathfrak{N}_*)^2$ is the subalgebra of $\mathfrak{N}_*$ consisting of bordism classes of manifold squares $N \times N$. Let M be a weakly complex $2n$-manifold and $\Psi \circ S(M) = [N \times N]$. Then the Milnor's result shows that

$$s_n(M) \equiv s'_n(N) \mod 2,$$

where s'_n denotes the Whitney number corresponding to $\sum t_i{}^n$. Thus we have a polynomial basis $\{x_n, n \neq 2^h-1\}$ of $\mathfrak{N}_*$ such that

$$(\Psi \circ S)(u_n) = x_n{}^2, \quad n \neq 2^h-1.$$

and $\Psi \circ S$ induces an isomorphism

$$\Omega^U_*/(u_{2^k-1}, k \geq 1) \otimes Z/2Z \cong (\mathfrak{N}_*)^2. \tag{4.1}$$

As we remarked in [1], §5, the oriented cobordism $\Omega^*(\ )$ is complex-oriented, [2]-typical and $S_* F_U = F_{SO}$. Thus S factorizes as the composition of the sequence

$$\Omega^U_* = U \xrightarrow{\pi_2} V_{[2]} \xrightarrow{\Phi} \Omega^{SO}_*,$$

By Corollary 5 we have

(4.2) $$V_{[2]}\otimes Z/2Z=Z/2Z[u_n,\ n\neq 2^h-1]\,.$$

By (4.1) and (4.2) we see that $\Psi\circ\Phi$ induces the isomorphism

(4.3) $$V_{[2]}\otimes Z/2Z\cong(\mathfrak{N}_*)^2\,.$$

By Corollary 5 we have

$$V_{[2]}/\mathrm{Tors}=Z[u_{2n},\ n\geq 1]\,.$$

Then by [3], p. 180, we conclude that

(4.4) $$\Phi/\mathrm{Tors}\colon\ V_{[2]}/\mathrm{Tors}\cong\Omega_*^{SO}/\mathrm{Tors}.$$

Finally by (4.3) and (4.4) we obtain

Theorem 6. *Φ: $V_{[2]}\to\Omega_*^{SO}$ is an injection.*

Corollary 7. Ker $S=J_2$, Im $S\cong V_{[2]}$.

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