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## NON-STATIONARY AND DISCONTINUOUS QUASICONFORMAL MAPPING CLASS GROUPS

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### Abstract

Every stationary subgroup of the quasiconformal mapping class group of a Riemann surface acts on the Teichmüller space discontinuously if the surface satisfies a certain geometric condition. In this paper, we construct such a Riemann surface that the quasiconformal mapping class group is non-stationary but it still acts on the Teichmüller space discontinuously.

### 1. Introduction and statement of results

For a Riemann surface  $R$  of analytically infinite type whose Teichmüller space  $T(R)$  is infinite dimensional, the action of the quasiconformal mapping class group  $MCG(R)$  on  $T(R)$  is not discontinuous in general. However, we have shown in [9] that certain subgroups of  $MCG(R)$  act on  $T(R)$  discontinuously. For example, under certain geometric conditions on  $R$ , a subgroup  $G_c(R)$  of all quasiconformal mapping classes that preserve the free homotopy class of a simple close geodesic  $c$  acts on  $T(R)$  discontinuously. Also we have shown in [8] that the eventually trivial mapping class group  $E(R)$  acts on  $T(R)$  discontinuously as well as the pure mapping class group  $P(R)$ . These subgroups have a common property: they are stationary.

**DEFINITION 1.1.** A subgroup  $G$  of  $MCG(R)$  is said to be *stationary* if there exists a compact subsurface  $W$  of  $R$  such that  $g(W) \cap W \neq \emptyset$  for every representative  $g$  of every element of  $G$ . An element  $[g] \in MCG(R)$  is said to be stationary if the cyclic group generated by  $[g]$  is stationary.

**REMARK 1.2.** There exists a subgroup  $G \subset MCG(R)$  such that each element of  $G$  is stationary but  $G$  is not stationary. Indeed, there exists an abstract countable infinite group  $\Gamma$  such that every element of  $\Gamma$  is of finite order, and for any countable group  $\Gamma$ , there exists a Riemann surface  $R$  such that the group  $\text{Conf}(R)$  of all conformal automorphisms of  $R$  contains a subgroup  $G$  isomorphic to  $\Gamma$ . Then we may regard  $G$  as a subgroup of  $MCG(R)$ . Every element  $[g] \in G$  is stationary since it is of finite order. On the other hand,  $G$  is not stationary since  $\text{Conf}(R)$  acts on  $R$  properly discontinuously.

Actually, for stationary subgroups in general, we know the following result. The lower and upper bound conditions are defined later in Section 2.

**Proposition 1.3.** [6, Theorem 4.8] *Let  $R$  be a hyperbolic Riemann surface satisfying the lower and upper bound conditions and having no ideal boundary at infinity. Then every stationary subgroup of  $\text{MCG}(R)$  acts on  $T(R)$  discontinuously.*

On the other hand, we did not know any example of a non-stationary subgroup that acts discontinuously, not to say the whole quasiconformal mapping class group  $\text{MCG}(R)$ . In fact, if the genus of  $R$  is positive finite or the number of the punctures of  $R$  is positive finite, then  $\text{MCG}(R)$  must be stationary (see [9, Theorem 2]). Furthermore, a countable quasiconformal mapping class group constructed in [10] acts discontinuously but it is also stationary as is seen in Section 5.

In Section 3, we first give an easy example of a Riemann surface  $R$  such that a non-stationary cyclic subgroup  $G$  of  $\text{MCG}(R)$  acts on  $T(R)$  discontinuously. Actually, this argument tells us certain obstruction for making our desired Riemann surfaces. Then in Section 4, we prove the following, which is the main result of this paper.

**Theorem 1.4.** *There exists a Riemann surface  $R$  such that the whole quasiconformal mapping class group  $\text{MCG}(R)$  is non-stationary but acts on  $T(R)$  discontinuously.*

The existence of non-stationary and discontinuous quasiconformal mapping class groups is crucial for the theory of dynamics on infinite dimensional Teichmüller spaces because it requires further investigations completely different from those in the finite dimensional cases.

## 2. Preliminaries

Throughout this paper, we assume that a Riemann surface  $R$  is hyperbolic. Namely, the universal covering surface of  $R$  is the upper half-plane  $\mathbb{H}$  that admits the hyperbolic metric. We denote the hyperbolic length of an arc  $c$  on  $R$  by  $l(c)$ . We say that  $R$  satisfies the *lower bound condition* if the injectivity radius at every point of  $R$  except cusp neighborhoods is uniformly bounded away from zero, and  $R$  satisfies the *upper bound condition* if there exists a subdomain  $\check{R}$  of  $R$  such that the injectivity radius at every point of  $\check{R}$  is uniformly bounded from above and that the simple closed curves in  $\check{R}$  carry the fundamental group of  $R$ . If the injectivity radius at any point of  $R$  is uniformly bounded from above, then clearly  $R$  satisfies the upper bound condition. The lower and upper bound conditions are invariant under quasiconformal deformations. For a non-trivial and non-cuspidal simple closed curve  $c$  on  $R$ , we denote the simple closed geodesic that is freely homotopic to  $c$  by  $c_*$ .

The *Teichmüller space*  $T(R)$  is the set of all equivalence classes  $[f]$  of quasiconformal homeomorphisms  $f$  on  $R$ . Here we say that two quasiconformal homeomorphisms  $f_1$  and  $f_2$  on  $R$  are *equivalent* if there exists a conformal homeomorphism  $h: f_1(R) \rightarrow f_2(R)$  such that  $f_2^{-1} \circ h \circ f_1$  is homotopic to the identity by a homotopy that keeps every point of the ideal boundary at infinity fixed throughout. The distance between two points  $[f_1]$  and  $[f_2]$  in  $T(R)$  is defined by  $d([f_1], [f_2]) = (1/2) \log K(f)$ , where  $f$  is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation  $K(f)$  is minimal in the homotopy class of  $f_2 \circ f_1^{-1}$ . Then  $d$  is a complete metric on  $T(R)$ , which is called the Teichmüller distance.

The *quasiconformal mapping class*  $[g]$  is a homotopy class of quasiconformal automorphisms  $g$  of a Riemann surface  $R$ , and the *quasiconformal mapping class group*  $\text{MCG}(R)$  is the group of all quasiconformal mapping classes on  $R$ . Here we also consider homotopy classes relative to the ideal boundary at infinity. A mapping class  $[g]$  is said to be *eventually trivial* if there exists a compact subsurface  $V_g \subset R$  such that, for each connected component  $W$  of  $R - V_g$  that is not a cusp neighborhood, the restriction  $g|_W: W \rightarrow R$  is homotopic to the inclusion map  $\text{id}|_W: W \rightarrow R$ . The *eventually trivial mapping class group*  $E(R)$  of  $R$  is the group of all eventually trivial mapping classes on  $R$ . Furthermore the *pure mapping class group*  $P(R)$  of  $R$  is the group of mapping classes  $[g]$  such that  $g$  fix all non-cuspidal ends of  $R$ .

Every element  $[g] \in \text{MCG}(R)$  induces a biholomorphic automorphism  $[g]_*$  of  $T(R)$  by  $[f] \mapsto [f \circ g^{-1}]$ , which is also an isometry with respect to the Teichmüller distance. Let  $\text{Aut}(T(R))$  be the group of all biholomorphic automorphisms of  $T(R)$ . Then we have a homomorphism  $\iota: \text{MCG}(R) \rightarrow \text{Aut}(T(R))$  by  $[g] \mapsto [g]_*$  and define the Teichmüller modular group by  $\text{Mod}(R) = \iota(\text{MCG}(R))$ . It is known that the homomorphism  $\iota$  is injective except for a few low dimensional cases. Thus we may identify  $\text{Mod}(R)$  with  $\text{MCG}(R)$ .

We say that a subgroup  $G \subset \text{MCG}(R)$  acts at a point  $p \in T(R)$  *discontinuously* if there exists a neighborhood  $U$  of  $p$  such that the number of elements  $[g] \in G$  satisfying  $[g]_*(U) \cap U \neq \emptyset$  is finite. This is equivalent to that there exist no distinct elements  $[g_n] \in G$  such that  $d([g_n]_*(p), p) \rightarrow 0$  as  $n \rightarrow \infty$  (see [5]). We say that  $G$  acts on  $T(R)$  discontinuously if  $G$  acts at every point in  $T(R)$  discontinuously. If  $R$  has the ideal boundary at infinity, then the action of  $\text{MCG}(R)$  is discontinuous at no points of  $T(R)$ .

### 3. A Riemann surface with length parameters

In this section, we construct a Riemann surface  $R$  from pairs of pants whose quasiconformal mapping class group  $\text{MCG}(R)$  has a cyclic non-stationary subgroup  $G$  that acts on  $T(R)$  discontinuously. Although this property itself is weaker than that of the Riemann surface as in Theorem 1.4, the surface in Proposition 3.1 below has the advantage of flexibility: by changing length parameters, we have quasiconformal mapping classes of various types as is explained in Remark 3.4 below.

Hereafter,  $P(l_1, l_2, l_3)$  denotes a pair of pants whose geodesic boundary components have the hyperbolic lengths  $l_1, l_2$  and  $l_3$ . We allow the case  $l_i = 0$ , which means that the boundary component degenerates into a puncture. A pair of pants  $P$  has three symmetry axes, which are the shortest geodesic arcs connecting two boundary components and which divide  $P$  into two congruent polygons.

First we make a surface  $S$  with indefinite parameters  $\{l_i\}_{i \in \mathbb{Z}}$  as follows. For every  $i \in \mathbb{Z}$ , we take two pairs of pants  $P_i^- = P_i^-(l_i, 1, 1)$  and  $P_i^+ = P_i^+(l_i, 1, 1)$  with geodesic boundary components  $(a_i^-, b_i^-, c_i^-)$  and  $(a_i^+, b_i^+, c_i^+)$  respectively. Let  $\alpha_i^\pm$  be the symmetry axis of  $P_i^\pm$  connecting  $b_i^\pm$  and  $c_i^\pm$ . Similarly,  $\beta_i^\pm$  is the symmetry axis connecting  $c_i^\pm$  and  $a_i^\pm$ , and  $\gamma_i^\pm$  is the one connecting  $a_i^\pm$  and  $b_i^\pm$ . We give an orientation to each boundary component of  $P_i^\pm$  counterclockwise when we view from the inside of  $P_i^\pm$ . Furthermore we parametrize each boundary component of  $P_i^\pm$  by a normalized arc length parameter  $\theta$  ( $0 \leq \theta \leq 1$ ) with respect to the hyperbolic metric (that is, the normalization means the variation of the parameter is one) such that  $a_i^\pm(0) = a_i^\pm(1) \in \gamma_i^\pm$ ,  $b_i^\pm(0) = b_i^\pm(1) \in \alpha_i^\pm$  and  $c_i^\pm(0) = c_i^\pm(1) \in \beta_i^\pm$ .

We glue  $P_i^-$  and  $P_i^+$  by identifying  $a_i^-(\theta)$  with  $a_i^+(1 - \theta)$  and  $b_i^-(\theta)$  with  $b_i^+(1 - \theta)$  for all  $\theta$ . Then we obtain a torus  $A_i$  with two geodesic boundary components  $c_i^-$  and  $c_i^+$  having  $a_i^-(\theta) = a_i^+(1 - \theta)$  and  $b_i^-(\theta) = b_i^+(1 - \theta)$  as simple closed geodesics  $a_i$  and  $b_i$  in it. Note that  $l(b_i) = 1$  for all  $i$ , but  $l(a_i) = l_i$  are indefinite. Furthermore, for each  $i \in \mathbb{Z}$ , we glue  $A_i$  and  $A_{i+1}$  by identifying  $c_i^+(\theta)$  with  $c_{i+1}^-(1 - \theta)$  for all  $\theta$ . Then the resulting surface of infinite genus is denoted by  $S$ , which is our Riemann surface of indefinite parameters  $\{l_i\}_{i \in \mathbb{Z}}$ .

Assume here that all the parameters  $l_i$  are the same. Then this surface admits a conformal automorphism  $g$  determined by a translation such that  $g(A_i) = A_{i+1}$  for all  $i$ . We consider this particular mapping class  $[g]$  of  $S$  hereafter.

After the preparation of those notations, we can state the example of our Riemann surface as follows.

**Proposition 3.1.** *Let  $R$  be a Riemann surface obtained by taking the lengths  $\{l_i\}_{i \in \mathbb{Z}}$  of  $S$  so that  $l_i \rightarrow 0$  as  $i \rightarrow \pm\infty$  and that  $1/e^2 \leq l_i/l_{i+1} \leq e^2$  for every  $i$ . Then the mapping class  $[g]$  of  $R$  belongs to  $\text{MCG}(R)$  and the cyclic non-stationary subgroup  $G$  generated by  $[g]$  acts on  $T(R)$  discontinuously.*

The following two lemmas, which give certain estimates of the maximal dilatations of quasiconformal homeomorphisms, will be used in the proofs of our statements here and later.

**Lemma 3.2** ([2]). *Let  $P = P(l_1, l_2, l_3)$  and  $P' = P'(l'_1, l_2, l_3)$  be pairs of pants (possibly degenerate) with  $\max\{l_1, l'_1, l_2, l_3\} \leq L$ . Suppose that  $\varepsilon := |\log(l_1/l'_1)| \leq 2$ . Then there exists a quasiconformal homeomorphism  $\psi: P \rightarrow P'$  preserving the symmetry axes such that  $K(\psi) \leq 1 + C\varepsilon$  for a constant  $C = C(L)$  depending only on  $L$*

and that it is identical on each boundary component with respect to the normalized arc length parameter.

**Lemma 3.3** ([12], [13]). *Let  $c$  be a simple closed geodesic on a Riemann surface  $R$  with the hyperbolic length  $l(c)$  and  $f: R \rightarrow R'$  a quasiconformal homeomorphism of  $R$  onto another Riemann surface  $R'$ . Then the hyperbolic length  $l(f(c)_*)$  of the geodesic  $f(c)_*$  satisfies*

$$\frac{1}{K(f)}l(c) \leq l(f(c)_*) \leq K(f)l(c).$$

Proof of Proposition 3.1. By applying Lemma 3.2 to each pair of pants, we see that there exists a quasiconformal automorphism  $h$  of  $R$  in the mapping class  $[g]$  such that  $K(h|_{A_i}) \leq 1 + C(L_i)\varepsilon_i$  on  $A_i$ . Here  $\varepsilon_i = |\log(l_i/l_{i+1})| \leq 2$  and  $L_i = \max\{l_i, l_{i+1}, 1\}$  for every  $i$ . Hence the mapping class  $[g]$  belongs to  $\text{MCG}(R)$ .

We will prove that  $G$  acts on  $T(R)$  discontinuously. First we show that  $G$  acts at the base point  $o = [\text{id}] \in T(R)$  discontinuously. Suppose to the contrary that there exists a subsequence  $\{[g^{n_k}]\}$  such that  $d([g^{n_k}]_*(o), o) \rightarrow 0$  as  $k \rightarrow \infty$ . Then there exist representatives  $h_k$  in the mapping classes  $[g^{n_k}]$  such that  $K(h_k) \rightarrow 1$  as  $k \rightarrow \infty$ . However, since  $h_k(a_0)$  is freely homotopic to  $a_{n_k}$ , we have  $K(h_k) \geq l_0/l_{n_k} \rightarrow \infty$  by Lemma 3.3. This is a contradiction.

Next consider an arbitrary point  $p = [f] \in T(R)$ , where  $f$  is a quasiconformal homeomorphism of  $R$ . Then, again by Lemma 3.3, the simple closed geodesics  $f(a_n)_*$  on  $f(R)$  satisfy  $l(f(a_n)_*) \rightarrow 0$  as  $n \rightarrow \pm\infty$ . Then by the same consideration, we see that  $G$  acts at  $p \in T(R)$  discontinuously.  $\square$

**REMARK 3.4.** In Proposition 3.1, we can choose the parameters of  $R$  as  $l_i = L_{-i} = 1/2^i$  for  $i \geq 0$ . Then the quasiconformal mapping class  $[g]$  is not asymptotically conformal. Indeed, since  $|\log(l_i/l_{i+1})| = \log 2$  for every  $i$ , Theorem 3.6 in [7] yields the assertion. For the definition of asymptotically conformal homeomorphisms, see [4]. Also, we can set  $l_0 = 1$  and  $l_i = L_{-i} = 1/i$  for  $i \geq 1$  as well. In this case, the quasiconformal mapping class  $[g]$  is asymptotically conformal. Indeed, by applying Lemma 3.2 as in the proof of Proposition 3.1, there exists a quasiconformal automorphism  $h$  in the mapping class  $[g]$  such that  $K(h|_{A_i}) \leq 1 + C(1)\varepsilon_i$  on  $A_i$ , where  $\varepsilon_i = |\log(l_i/l_{i+1})| = |\log((i+1)/i)| \rightarrow 0$  as  $i \rightarrow \infty$ .

**REMARK 3.5.** Let  $R_1$  be a Riemann surface such that the parameters  $l_i$  on  $S$  are bounded from above and away from zero. Then  $G = \langle [g] \rangle \subset \text{MCG}(R_1)$  does not act on  $T(R_1)$  discontinuously. Indeed, let  $R_0$  be a Riemann surface with  $l_i = 1$  for all  $i \in \mathbb{Z}$ . Then  $R_1$  is a quasiconformal deformation of  $R_0$  and hence  $T(R_1) = T(R_0)$ . On the Riemann surface  $R_0$ , the mapping class  $[g]$  has a conformal representative. Then  $G$  does not act discontinuously at  $o = [\text{id}] \in T(R_0)$ .

#### 4. Proof of main theorem

In this section, we will prove Theorem 1.4. If a Riemann surface  $R$  has a sequence of simple closed geodesics whose hyperbolic lengths tend to 0, namely, if  $R$  does not satisfy the lower bound condition, then the action of  $\text{MCG}(R)$  on  $T(R)$  is not discontinuous (see [5, Theorem 1]). In particular, the Riemann surface as in Proposition 3.1 is not appropriate for Theorem 1.4. The Riemann surface as in Remark 3.5 is not appropriate either by the reason explained there.

**Proof of Theorem 1.4.** First we define a sequence  $\{l_n\}_{n \in \mathbb{N}}$  of positive numbers as follows. Fix a constant  $K > 1$  once and for all. Set  $l_1 = 1$  and take a degenerate pair of pants  $P_1 = P(l_1, l_1, 0)$ . Let  $\hat{\eta}$  be the supremum of  $\eta \leq l_1 + 1$  such that there exist  $K$ -quasiconformal homeomorphisms  $\varphi: P_1 \rightarrow P(l_1, \eta, 0)$  and  $\varphi': P_1 \rightarrow P(\eta, \eta, 0)$  that preserve the symmetry axes and that are identical on each boundary component with respect to the normalized arc length parameter. Then we set  $l_2 = \hat{\eta}$ .

Here the above supremum is actually attained. Indeed, we take a sequence  $\{\eta_j\}$  converging to  $\hat{\eta}$  such that there exist  $K$ -quasiconformal homeomorphisms  $\varphi_j: P_1 \rightarrow P(l_1, \eta_j, 0)$  and  $\varphi'_j: P_1 \rightarrow P(\eta_j, \eta_j, 0)$ . It is enough to consider  $\varphi_j$  and  $\varphi'_j$  on the symmetric half  $D_1$  of  $P_1$  and we may assume that their images  $\varphi_j(D_1) = D(l_1, \eta_j, 0)$  and  $\varphi'_j(D_1) = D(\eta_j, \eta_j, 0)$  are embedded in the hyperbolic plane  $\mathbb{H}$  in such a way that  $D(l_1, \eta_j, 0)$  and  $D(\eta_j, \eta_j, 0)$  converge to pentagons  $D(l_1, \hat{\eta}, 0)$  and  $D(\hat{\eta}, \hat{\eta}, 0)$  respectively in the sense of Carathéodory. Then  $\varphi_j$  and  $\varphi'_j$  converge to  $K$ -quasiconformal homeomorphisms  $\varphi_\infty$  and  $\varphi'_\infty$  respectively (see [11, Theorem 5.2]). Moreover, by an application of the Carathéodory convergence theorem (see [3, Theorem 3.1]), their images  $\varphi_\infty(D_1)$  and  $\varphi'_\infty(D_1)$  are coincident with  $D(l_1, \hat{\eta}, 0)$  and  $D(\hat{\eta}, \hat{\eta}, 0)$  respectively and they are affine on the two sides of  $D_1$  with respect to the hyperbolic metric. This implies that  $\varphi_\infty$  and  $\varphi'_\infty$  attain the supremum.

Assuming that  $l_n$  has been determined, we define  $l_{n+1}$  as follows. For a degenerate pair of pants  $P_n = P(l_n, l_n, 0)$ , let  $l_{n+1}$  be the supremum of  $\eta \leq l_n + 1$  (which is actually the maximum by the same reason as above) such that there exist  $K$ -quasiconformal homeomorphisms  $\varphi: P_n \rightarrow P(l_n, \eta, 0)$  and  $\varphi': P_n \rightarrow P(\eta, \eta, 0)$  that preserve the symmetry axes and that are identical on each boundary component with respect to the normalized arc length parameter. In this way, we have  $l_n$  for  $n \geq 1$  inductively.

Next we prove that  $l_n \rightarrow \infty$ . Suppose to the contrary that  $\sup l_n =: \hat{l} < \infty$ . Let  $C(\cdot)$  be the constant as in Lemma 3.2. Since  $\sup l_n = \hat{l}$ , we can take an integer  $n$  such that

$$l_n \geq \max \left[ \hat{l} \cdot \exp \left( -\frac{K^{1/4} - 1}{C(\hat{l})} \right), \hat{l} - \frac{1}{2} \right].$$

Then by Lemma 3.2, there exist  $K^{1/4}$ -quasiconformal homeomorphisms between pairs of pants  $P(l_n, l_n, 0)$  and  $P(l_n, \hat{l}, 0)$ , and between pairs of pants  $P(l_n, \hat{l}, 0)$  and  $P(\hat{l}, \hat{l}, 0)$ .

Furthermore, take a constant  $\mu > \hat{l}$  such that

$$\mu \leq \min \left[ \hat{l} \cdot \exp \left( \frac{K^{1/4} - 1}{C(\hat{l} + 1/2)} \right), \hat{l} + \frac{1}{2} \right].$$

Then by Lemma 3.2, there exist  $K^{1/4}$ -quasiconformal homeomorphisms between  $P(\hat{l}, \hat{l}, 0)$  and  $P(\hat{l}, \mu, 0)$ , and between  $P(\hat{l}, \mu, 0)$  and  $P(\mu, \mu, 0)$ . By composing the four  $K^{1/4}$ -quasiconformal homeomorphisms, we obtain a  $K$ -quasiconformal homeomorphism between  $P(l_n, l_n, 0)$  and  $P(\mu, \mu, 0)$ . Also, since there is a  $K^{1/4}$ -quasiconformal homeomorphism between  $P(l_n, \hat{l}, 0)$  and  $P(l_n, \mu, 0)$ , we have a  $K^{1/2}$ -quasiconformal homeomorphism between  $P(l_n, l_n, 0)$  and  $P(l_n, \mu, 0)$ . Remark that  $\mu - l_n \leq 1$ . Since  $\mu > \hat{l}$ , they contradict the definition of  $l_{n+1}$ .

Now we construct the desired Riemann surface  $R$ . For each  $i \in \mathbb{Z} - \{0\}$ , we take degenerate pairs of pants  $A_i = P(l_{|i|}, l_{|i|}, 0)$  with geodesic boundary components  $(a_i^-, a_i^+, x_i)$  and  $B_i = P(l_{|i|}, l_{|i|+1}, 0)$  with geodesic boundary components  $(b_i^-, b_i^+, y_i)$ . Here  $x_i$  and  $y_i$  are punctures. For  $i = 0$ , we set  $B_0 = P(1, 1, 0)$  (namely  $l_0 = 1$ ) with geodesic boundary components  $(b_0^+, b_0^+, y_0)$  (the two components have the same name).

Let  $s_i^\pm \subset A_i$  be the symmetry axis connecting  $a_i^\pm$  with  $x_i$ , and let  $t_i^\pm \subset B_i$  be the symmetry axis connecting  $b_i^\pm$  with  $y_i$ . We parametrize the boundary components of  $A_i$  and  $B_i$  counterclockwise by a normalized arc length parameter  $\theta$  ( $0 \leq \theta \leq 1$ ) with respect to the hyperbolic metric such that  $a_i^\pm(0) = a_i^\pm(1) \in s_i^\pm$  and  $b_i^\pm(0) = b_i^\pm(1) \in t_i^\pm$ .

We glue  $B_0$  and  $A_1$  by identifying one  $b_0^+(\theta)$  with  $a_1^-(1 - \theta)$ , and glue  $B_0$  and  $A_{-1}$  by identifying the other  $b_0^+(\theta)$  with  $a_{-1}^-(1 - \theta)$ . For each  $i \geq 1$ , we glue  $A_i$  and  $B_i$  by identifying  $a_i^+(\theta)$  with  $b_i^-(1 - \theta)$ , and glue  $B_i$  and  $A_{i+1}$  by identifying  $b_i^+(\theta)$  with  $a_{i+1}^-(1 - \theta)$ . Also for each  $i \leq -1$ , we glue  $A_i$  and  $B_i$  by identifying  $a_i^+(\theta)$  with  $b_i^-(1 - \theta)$ , and glue  $B_i$  and  $A_{i-1}$  by identifying  $b_i^+(\theta)$  with  $a_{i-1}^-(1 - \theta)$ . In this manner, we obtain a planar Riemann surface  $R$ .

Let  $v$  be the geodesic line consisting of all the symmetry axes of  $A_i$  and  $B_i$  other than  $s_i^\pm$  and  $t_i^\pm$ . If the hyperbolic length of  $v$  is infinite, then  $R$  has no ideal boundary at infinity. Otherwise, we reconstruct  $R$  as follows. For each  $i$ , we prepare more than  $1/l(v \cap A_i)$  copies of  $A_i$  and glue them in the same way as above to obtain  $\tilde{A}_i$  whose boundary components other than punctures are more than one apart in the hyperbolic distance. Then, replacing  $A_i$  with  $\tilde{A}_i$ , we make  $R$ . In this sense, we may assume that the Riemann surface  $R$  constructed above has no ideal boundary at infinity.

The union of the symmetry axes  $s_i^+ \cup t_i^-$  ( $i \neq 0$ ) makes a geodesic line connecting the punctures  $x_i$  with  $y_i$ . Similarly,  $t_{i-1}^+ \cup s_i^-$  ( $i \geq 1$ ) or  $t_i^+ \cup s_{i-1}^-$  ( $i \leq 0$ ) makes a geodesic line connecting  $y_{i-1}$  with  $x_i$  ( $i \geq 1$ ) or  $y_i$  with  $x_{i-1}$  ( $i \leq 0$ ). All these geodesic lines together with  $v$  divide the Riemann surface  $R$  into the symmetric halves  $R^\circ$  and  $R^\bullet$ , which are simply connected. Also they divide the pair of pants  $A_i$  into the symmetric halves  $A_i^\circ = A_i \cap R^\circ$  and  $A_i^\bullet = A_i \cap R^\bullet$ , and divide the pair of pants  $B_i$  as well.



The quasiconformal mapping class group  $\text{MCG}(R)$  is non-stationary. Indeed, by the definition of the sequence  $\{l_n\}$ , there exist  $K^2$ -quasiconformal homeomorphisms between  $A_i$  and  $B_i$  ( $i \neq 0$ ), between  $B_{i-1}$  and  $A_i$  ( $i \geq 1$ ) and between  $B_i$  and  $A_{i-1}$  ( $i \leq 0$ ). Hence there exists a  $K^2$ -quasiconformal automorphism  $g$  of  $R$  that maps  $\{A_i\}$  to  $\{B_i\}$ . Clearly this mapping class  $[g] \in \text{MCG}(R)$  is non-stationary.

Next we will prove that  $\text{MCG}(R)$  acts on  $T(R)$  discontinuously. To see this, we use the following.

**Proposition 4.1.** *The Riemann surface  $R$  satisfies the lower and upper bound conditions.*

*Proof.* The hyperbolic distances between geodesic arcs in  $A_i$  and  $B_i$  ( $i \neq 0$ ) satisfy

$$\begin{aligned} \cosh d(s_i^+, a_i^-) &= \frac{2 \cosh(l_{|i|}/2)}{\sinh(l_{|i|}/2)}; \\ \cosh d(s_i^-, a_i^+) &= \frac{2 \cosh(l_{|i|}/2)}{\sinh(l_{|i|}/2)}; \\ \cosh d(t_i^+, b_i^-) &= \frac{\cosh(l_{|i|}/2) + \cosh(l_{|i|+1}/2)}{\sinh(l_{|i|}/2)}; \\ \cosh d(t_i^-, b_i^+) &= \frac{\cosh(l_{|i|}/2) + \cosh(l_{|i|+1}/2)}{\sinh(l_{|i|+1}/2)}. \end{aligned}$$

These are obtained by the combination of formulae for hyperbolic pentagons (see [1, Theorem 7.18.1]). Since  $l_{|i|+1} \leq l_{|i|} + 1$ , the above four distances are uniformly bounded from above and away from zero. In fact, we have

$$\begin{aligned} \limsup_{i \rightarrow \pm\infty} \cosh d(t_i^+, b_i^-) &\leq 1 + e^{1/2}; \\ \liminf_{i \rightarrow \pm\infty} \cosh d(t_i^-, b_i^+) &\geq 1 + e^{-1/2}. \end{aligned}$$

First we prove that  $R$  satisfies the lower bound condition. We will show that the hyperbolic lengths  $l(c)$  of all simple closed geodesics  $c$  on  $R$  are uniformly bounded away from zero. Take  $c$  arbitrarily other than  $a_i^\pm$  or  $b_i^\pm$ . (Remark that  $l(a_i^\pm) \geq 1$  and  $l(b_i^\pm) \geq 1$ .) Let  $i$  ( $\neq 0$ ) be an integer of the largest absolute value satisfying either  $c \cap A_i \neq \emptyset$  or  $c \cap B_i \neq \emptyset$ . In the case where  $c \cap A_i \neq \emptyset$  and  $c \cap B_i = \emptyset$ , we consider the connected components of  $c \cap A_i^\circ$  and  $c \cap A_i^\bullet$ , which are simple geodesic arcs. Then at least one of them, say  $c'$ , connects either  $s_i^+$  with  $a_i^-$  or  $s_i^+$  with  $v \cap A_i$ . Indeed, otherwise both  $c \cap A_i^\circ$  and  $c \cap A_i^\bullet$  connect  $s_i^+$  and  $s_i^-$ , which means that  $c$  surrounds only one puncture  $x_i$ . If  $c'$  connects  $s_i^+$  with  $a_i^-$ , then  $l(c') \geq d(s_i^+, a_i^-) \geq \text{arccosh } 2$  by the above formula. If  $c'$  connects  $s_i^+$  with  $v \cap A_i$ , then  $l(c') \geq l(a_i^+)/2 \geq 1/2$ . In both cases, we have  $l(c) \geq 1/2$ . Also in the case where  $c \cap B_i \neq \emptyset$ , we can apply the same

argument since  $d(t_i^+, b_i^-) \geq \operatorname{arccosh} 2$  and  $l(b_i^+)/2 \geq 1/2$ . Hence in all cases, we have  $l(c) \geq 1/2$  and conclude that  $R$  satisfies the lower bound condition.

Next we prove that  $R$  satisfies the upper bound condition. We consider a dividing simple closed geodesic  $\zeta_{2i}$  ( $i \neq 0$ ) that bounds a doubly-connected domain together with  $s_i^+ \cup t_i^-$ , which surrounds  $x_i$  and  $y_i$ . Also we take a simple closed geodesic  $\zeta_{2i-1}$  surrounding either  $y_{i-1}$  and  $x_i$  ( $i \geq 1$ ) or  $y_i$  and  $x_{i-1}$  ( $i \leq 0$ ) in the same manner as above. For each integer  $m \neq 0$ , let  $Z_m$  be one of the connected components of  $R - \zeta_m$  that contains the two punctures, and set

$$\check{R} = B_0 \cup \bigcup_{m \neq 0} Z_m.$$

The homomorphism  $\pi_1(\check{R}) \rightarrow \pi_1(R)$  induced by the inclusion map  $\check{R} \hookrightarrow R$  is surjective because the connected components of the complement  $R - \check{R}$  are simply connected. Hence we have only to show that the injectivity radii of all points in  $\check{R}$  are uniformly bounded from above.

We will show that the hyperbolic lengths of  $\zeta_m$  are uniformly bounded from above. For disjoint geodesic arcs  $s$  and  $a$  in the simply-connected domain  $R^\circ$ , we denote by  $e\langle s \rightarrow a \rangle \in a$  the end point of the shortest geodesic arc connecting  $s$  with  $a$ . Then we see that

$$l(\zeta_{2i}) \leq 2\{d(s_i^-, a_i^+) + d(e\langle s_i^- \rightarrow a_i^+ \rangle, e\langle t_i^+ \rightarrow b_i^- \rangle) + d(t_i^+, b_i^-)\}$$

for example. Hence we have only to estimate the distances between these end points.

By a formula for the Lambert quadrilaterals (see [1, Theorem 7.17.1 (i)]), we have

$$\begin{aligned} & d(e\langle s_i^- \rightarrow a_i^+ \rangle, e\langle t_i^+ \rightarrow b_i^- \rangle) \\ &= \operatorname{arcsinh} \left\{ \frac{1}{\sinh d(s_i^-, a_i^+)} \right\} - \operatorname{arcsinh} \left\{ \frac{1}{\sinh d(t_i^+, b_i^-)} \right\} \quad (i \neq 0); \\ & d(e\langle t_i^- \rightarrow b_i^+ \rangle, e\langle s_{i+1}^+ \rightarrow a_{i+1}^- \rangle) \\ &= \operatorname{arcsinh} \left\{ \frac{1}{\sinh d(t_i^-, b_i^+)} \right\} - \operatorname{arcsinh} \left\{ \frac{1}{\sinh d(s_{i+1}^+, a_{i+1}^-)} \right\} \quad (i \geq 1); \\ & d(e\langle t_i^- \rightarrow b_i^+ \rangle, e\langle s_{i-1}^+ \rightarrow a_{i-1}^- \rangle) \\ &= \operatorname{arcsinh} \left\{ \frac{1}{\sinh d(t_i^-, b_i^+)} \right\} - \operatorname{arcsinh} \left\{ \frac{1}{\sinh d(s_{i-1}^+, a_{i-1}^-)} \right\} \quad (i \leq 0), \end{aligned}$$

which are uniformly bounded from above. Hence we conclude that  $l(\zeta_m) \leq \delta$  for some constant  $\delta > 0$ .

Since the hyperbolic area of  $Z_m$  is  $2\pi$ , there exists a constant  $r > 0$  independent of  $m$  such that the radius of any embedded disk in any  $Z_m$  is not greater than

$r$ . This means that, for every  $z \in Z_m$ , there exists either a non-trivial closed curve passing through  $z$  whose length is not greater than  $2r$ , or an arc connecting  $z$  with  $\zeta_m = \partial Z_m$  whose length is not greater than  $r$ . Hence, for every  $z \in Z_m$ , there is a non-trivial closed curve passing through  $z$  whose length is not greater than  $2r + \delta$ . Thus we conclude that the injectivity radii of all points of  $\check{R}$  are uniformly bounded from above.  $\square$

Proof of Theorem 1.4 continued. We prove that  $\text{MCG}(R)$  acts on  $T(R)$  discontinuously. First we show that  $\text{MCG}(R)$  acts at the base point  $o = [\text{id}] \in T(R)$  discontinuously. Suppose to the contrary that there is a sequence of distinct elements  $[g_n] \in \text{MCG}(R)$  such that  $d([g_n]_*(o), o) \rightarrow 0$  as  $n \rightarrow \infty$ . If the sequence  $\{[g_n]\}$  is stationary, namely, if there exists a compact subsurface  $W$  of  $R$  such that  $g_n(W) \cap W \neq \emptyset$  for every representative  $g_n$  for every  $n$ , then we have a contradiction by Proposition 1.3 (applied to the sequence instead of a subgroup) and Proposition 4.1. Thus we may assume that the sequence  $\{[g_n]\}$  is non-stationary.

Let  $X_i$  and  $Y_i$  be horocyclic cusp neighborhoods of  $x_i$  and  $y_i$  respectively whose hyperbolic areas are 1. For  $k \geq 1$ , set

$$W_k = (B_0 - Y_0) \cup \bigcup_{1 \leq |i| \leq k} \{(A_i - X_i) \cup (B_i - Y_i)\},$$

which is a compact subsurface of  $R$ . Then there exist  $n_k \in \mathbb{N}$  and a representative  $g_{n_k} \in [g_{n_k}]$  such that  $g_{n_k}(W_k) \cap W_k = \emptyset$ . In particular,  $g_{n_k}(c_0)_* \cap W_k = \emptyset$ , where  $c_0 := b_0^+$  is a geodesic boundary component of  $B_0$  and  $g_{n_k}(c_0)_*$  is the simple closed geodesic that is freely homotopic to  $g_{n_k}(c_0)$ . Without loss of generality, we may assume that  $g_{n_k}(c_0)_*$  belongs to  $\bigcup_{i=i_k}^\infty \{(A_i - X_i) \cup (B_i - Y_i)\}$ , where  $i_k \geq k + 1$  is the minimum integer satisfying this property. We may also assume that  $g_{n_k}(c_0)_*$  is neither  $a_i^\pm$  nor  $b_i^\pm$ , for if  $g_{n_k}(c_0)_*$  is either  $a_i^\pm$  or  $b_i^\pm$  then the estimate below is obvious.

First we consider the case where  $g_{n_k}(c_0)_* \cap A_{i_k} \neq \emptyset$ . The geodesic  $g_{n_k}(c_0)_*$  has intersection with  $s_{i_k}^-$ . Indeed, otherwise, the homotopy class of  $g_{n_k}(c_0)$  has a closed curve that is shorter than  $g_{n_k}(c_0)_*$ . We consider the connected components of  $g_{n_k}(c_0)_* \cap R^\circ$  and  $g_{n_k}(c_0)_* \cap R^\bullet$ , which are simple geodesic arcs. Then one of these arcs, which is denoted by  $c'_k$ , connects  $s_{i_k}^-$  with  $v$ . Indeed, suppose that  $g_{n_k}(c_0)_*$  has no intersection with  $v$ . Then one connected component of  $R - g_{n_k}(c_0)_*$  has only finitely many punctures. However, since  $c_0$  divides  $R$  into two connected components both of which have infinitely many punctures and since  $g_{n_k}$  is homeomorphic,  $g_{n_k}(c_0)_*$  has the same property as  $c_0$ . This is a contradiction. Also in the case where  $g_{n_k}(c_0)_* \cap B_{i_k} \neq \emptyset$  but  $g_{n_k}(c_0)_* \cap A_{i_k} = \emptyset$ , by applying the same argument as above, we conclude that one of the simple geodesic arcs  $g_{n_k}(c_0)_* \cap R^\circ$  and  $g_{n_k}(c_0)_* \cap R^\bullet$  connects  $t_{i_k}^-$  with  $v$ .

Here we see that  $l(c'_k) \geq (1/2)l(a_{i_k}^\pm)$  since  $a_{i_k}^\pm$  restricted to  $R^\circ$  or  $R^\bullet$  are shortest geodesic arcs connecting  $s_{i_k}^-$  with  $v$  and  $t_{i_k}^-$  with  $v$ . Then we have  $l(g_{n_k}(c_0)_*) \geq (1/2)l_{i_k} \rightarrow \infty$  as  $k \rightarrow \infty$ . On the other hand, we can choose representatives  $g'_{n_k} \in [g_{n_k}]$

such that  $K(g'_{n_k}) \rightarrow 1$  as  $k \rightarrow \infty$ . However by Lemma 3.3, we have

$$K(g'_{n_k}) \geq \frac{l(g'_{n_k}(c_0)_*)}{l(c_0)} = l(g_{n_k}(c_0)_*),$$

which is a contradiction. Hence we conclude that  $\text{MCG}(R)$  acts at the base point  $o = [\text{id}] \in T(R)$  discontinuously.

For an arbitrary point  $p = [f] \in T(R)$ , the Riemann surface  $f(R)$  satisfies the lower and upper bound conditions and  $l(f(a_i^\pm)_*) = \infty$  as  $i \rightarrow \pm\infty$  because these properties are quasiconformally invariant. Thus we can apply the same argument as above and conclude that  $\text{MCG}(R)$  acts at  $p$  discontinuously.  $\square$

### 5. A stationary countable mapping class group

In this section, we will prove that  $\text{MCG}(R)$  is stationary for the Riemann surface  $R$  that was constructed in [10]. This surface  $R$  has a property that  $\text{MCG}(R)$  consists only of a countable number of elements, and as a consequence,  $\text{MCG}(R)$  acts on  $T(R)$  discontinuously (see [10, Theorem 1]).

The Riemann surface  $R$  was constructed as follows. Set  $P_0 = P(1, 1, 1)$  and  $P_n = P(n!, (n+1)!, (n+1)!)$  for every integer  $n \geq 1$ . We denote the geodesic boundary component of length  $n!$  in each pair of pants by  $c_n$ . We prepare  $2^{n+1}$  copies of  $P_n$  for each  $n \geq 0$  and glue the geodesic boundary components as follows: We glue the geodesic boundary components  $c_0$  of the 2 copies of  $P_0$  together. The resulting hyperbolic surface with 4 geodesic boundary components  $c_1$  is denoted by  $R_1$ . Next we glue the geodesic boundary component  $c_1$  of each copy of  $P_1$  to the 4 boundary components of  $R_1$ . The resulting hyperbolic surface with 8 geodesic boundary components  $c_2$  is denoted by  $R_2$ . Continuing this process, for every integer  $n \geq 1$ , we obtain a hyperbolic surface  $R_n$  with  $2^{n+1}$  geodesic boundary components  $c_n$  which is made of  $R_{n-1}$  and  $2^n$  copies of  $P_{n-1}$ . Then take the exhaustion of these compact subsurfaces  $R_n$ , which is  $R = \bigcup_{n=0}^\infty R_n$ . Each connected component of  $R - R_n$  is denoted by  $E_n$ . At each step of gluing, we give an appropriate amount of twist along the geodesic boundaries so that  $R$  is a complete hyperbolic surface without ideal boundary at infinity. Then  $R$  has the following property.

**Lemma 5.1** ([10, Theorem 3]). *Let  $g : R \rightarrow R$  be a  $K$ -quasiconformal automorphism of the Riemann surface  $R$ . Then, on each component  $E_n$  of  $R - R_n$  for  $n \geq \max\{K, 5\}$ , the  $g$  restricted to  $E_n$  is homotopic to a conformal homeomorphism of  $E_n$  onto another component of  $R - R_n$ .*

We will prove the following.

**Proposition 5.2.** *Let  $R$  be the Riemann surface constructed above. Then  $\text{MCG}(R)$  is stationary.*

Proof. Let  $R_1$  be the compact subsurface defined as above. We will prove that  $g(R_1) \cap R_1 \neq \emptyset$  for every representatives  $g$  of every element  $[g] \in \text{MCG}(R)$ . Suppose to the contrary that there exists some  $[g]$  such that  $g(R_1) \cap R_1 = \emptyset$ . Let  $K$  be the maximal dilatation of  $g$  and take an integer  $n$  with  $n \geq \max\{K, 5\}$ . The number of the components  $E_n$  of  $R - R_n$  is  $2^{n+1}$  and precisely  $2^{n+1}/4$  of them belong to each of the four components  $E_1$  of  $R - R_1$ .

By Lemma 5.1,  $[g]$  gives a permutation of the  $2^{n+1}$  components  $E_n$ . Since  $g$  is homeomorphic, there are  $2^{n+1}/4$  components  $E_n$  in each of the four components of  $R - g(R_1)$ . By the assumption that  $g(R_1) \cap R_1 = \emptyset$ , the image  $g(R_1)$  belongs to some  $E_1$ . Then we see that there should be at least  $3 \cdot 2^{n+1}/4$  components  $E_n$  belonging to this  $E_1$ . This is a contradiction. Hence we conclude that  $g(R_1) \cap R_1 \neq \emptyset$  for every representatives  $g$  of every  $[g] \in \text{MCG}(R)$ , which means that  $\text{MCG}(R)$  is stationary.  $\square$

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