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ON SPREADS OF CHARACTERISTIC p ADMITTING NONSOLVABLE GROUPS, WHOSE SYLOW p-SUBGROUPS ARE PLANAR

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Introduction

In this paper, π denotes a translation plane of order n and characteristic p admitting, in its translation complement, a group G such that $\sqrt{n}\,\big|\,|G|$. We use S to denote a Sylow p-subgroup of G and assume that S is planar with fixed plane π_S.

The recently completed classification due to Foulser and Johnson [9, 10], of the finite translation planes of order n that admit $SL(2, \sqrt{n})$ in their translation complement, has the following implication for π.

Theorem A. *Suppose the plane π_S is a Baer subplane of π and let $\mathcal{P}_n$ denote the class of nonisomorphic translation planes of order n that admit $SL(2, \sqrt{n})$ in their translation complement. Then both the following statements are valid.*

(a) *If $S \ntriangleleft G$ then π is isomorphic to a plane in $\mathcal{P}_n$.*

(b) *If G is nonsolvable then either $S \ntriangleleft G$, and so part* (a) *applies, or π_S is a G-invariant Desarguesian subplane such that restriction homomorphism $G \rightarrow G|\pi_S$ has solvable kernel but its image is a* (*meta*) *cyclic extension of* $SL(2, 5)$.

Proof. Suppose $S \ntriangleleft G$. It is now sufficient to verify that $G \supseteq SL(2, \sqrt{n})$. For odd $n(\neq 9)$ this follows from Foulser [7, corollary 4.2 and proposition 5.1], while for n even we get the required conclusion from Dempwolff [2, Satz, p. 1]. Since $n=9$ presents no problems, case (a) applies. If G is nonsolvable and $S \triangleleft G$ then the conclusions of case (b) are easily deduced from Foulser [6]: the details are very similar to the proof of proposition 4.4.

The object of the present paper is to attempt a generalization of theorem A by weakening the assumption imposed on π_S; specifically, instead of assuming that π_S is a Baer subplane of π, we shall assume that π_S has order at least $n^{1/4}$. If n is odd, and this is the case we shall mainly be concerned with, it turns out that the conclusions of theorem A are essentially unchanged; though there may exist a new plane of order 5^4.

Theorem B. *Suppose the plane π_S has order $\geq n^{1/4}$. Then both the following statements hold, provided that n is odd.*

(a) *If $S \lhd G$ then π is a Hall plane except perhaps when π has order 5^4 and contains a G-invariant Hall plane ψ, of order 25, such that $G \supseteq SL(2, 5)$ and the latter group acts faithfully on ψ.*

(b) *Suppose G is nonsolvable. Then either $S \lhd G$, and so part* (a) *applies, or there is a G-invariant Baer chain of subplanes*

$$\pi \supset \pi_{\bar{S}} \supseteq \pi_S$$

where π_S is Desarguesian and the restriction map $G \to G | \pi_S$ has solvable kernel but its image is a (meta) cyclic extension of $SL(2, 5)$.

REMARK. Actually, the Baer subplane $\pi_{\bar{S}}$ is fixed elementwise by $\bar{S}$, the set of all Baer p-elements in G; in particular, $\bar{S}$ is a (normal) subgroup of G (cf. propositions 4.3 and 4.4).

Next let us consider the case when π has even order. Our results now are of a very preliminary nature and are deduced from Dempwolff's theorem [2, Satz, p. 1] and Johnson's work on planes of order 16 [15]; but note that all translation planes of order 16 have recently been determined by Dempwolff and Riefart [4].

Theorem C. *Assume π_S has order $\geq n^{1/4}$ and that $|S| \geq p\sqrt{n}$. Then the following statements are valid.*

(a) *π has even order, $|S| = 2\sqrt{n}$ and π_S has order $n^{1/4}$. Also S contains an elementary abelian subgroup E such that $[S:E]=2$ and π_E is a Desarguesian Baer subplane of π.*

(b) *If G is nonsolvable then π is a plane of order* 16 *isomorphic to the Dempwolff plane* [3], *the Lorimer-Rahilly plane, or its transpose, the Johnson-Walker plane* [16].

(c) *If G does not contain a normal elementary abelian subgroup of order $\sqrt{n}$, then G is nonsolvable and* (b) *applies.*

REMARKS. (1) For the Dempwolff plane $G = \Gamma L(2, 4)$, while for the other two planes mentioned in part (b) G can be chosen to be $L_3(2)$.

(2) It is important to observe that S in theorem C need not be a "B-group," in the sense of Dempwolff, even when $p=2$. Otherwise we could draw very much stronger conclusions.

1. Preliminaries

For general background on translation planes we refer the reader to standard works [11, 17, 18]. Our notation is fairly standard, but we draw the reader's attention to the following conventions.

(i) If $\mathcal{G}$ is a permutation group on Ω then $\mathcal{G}_\Lambda$ is the elementwise stabilizer of $\Lambda\subseteq\Omega$, and if $\mathcal{G}$ leaves Λ invariant then $\mathcal{G}\to\mathcal{G}|\Lambda$ denotes the restriction homomorphism.

(ii) Let T be a collineation of the affine translation plane π. T is called an *autotopism*, if $\mathrm{Fix}(T)$ includes a triangle in the projective closure of π; if in fact $\mathrm{Fix}(T)$ is a subplane of π then T is said to be *planar*. Similar remarks apply to collineation groups. Also, when T is a planar group or collineation, π_T denotes $\mathrm{Fix}(T)$.

We assume the reader is familiar with the properties of planar groups of translation planes, that fix a Baer subplane elementwise. Such groups have been thoroughly analyzed by Foulser [6] and we summarize below the aspects of his work which will be required in the sequel. Our summary is given in terms of $\mathrm{Aff}(1,p^r)$, the group of bijections of the field $\boldsymbol{F}_{p^r}$ that are of form $x\to ax+b$, with $a\neq 0$.

Result 1. *Let ψ be a translation plane of order N^2 with a Baer subplane Λ. Let p denote the characteristic of ψ and suppose $\mathcal{G}$ is a subgroup of $(\mathrm{Aut}\,\psi)_\Lambda$. Then all the following statements are valid.*

(a) *$\mathcal{G}\subseteq \mathrm{Aff}(1,N)$; in particular $\mathcal{G}$ is solvable with a normal elementary abelian Sylow p-subgroup with order M dividing N.*

(b) *If $\boldsymbol{F}_{p^k}=\mathrm{kern}\,\Lambda$ then $p^k\geq M$.*

(c) *If $\mathcal{G}_1$ is a p'-subgroup of $\mathcal{G}$ then there is a Baer subplane $\Lambda_1\neq\Lambda$ such that $\mathcal{G}_1$ leaves Λ_1 invariant; in fact Λ_1 can be chosen so that Λ and Λ_1 share the same slope set and also have exactly one affine point in common.*

Finally we remind the reader that π, G and S are always as in the first paragraph of the article; also, since our objective is to prove theorems B and C, we shall assume that π_S has order $\geq n^{1/4}$. For convenience we put $n^{1/4}=q$, even though q may not be an integer.

2. When $pq^2\leq|S|$

In this section we prove Theorem C.

Lemma 1. *Suppose $|S|\geq pq^2$. Then $|S|=2q^2$ and S contains an elementary abelian subgroup E such that $|E|=q^2$ and π_E is a Desarguesian Baer subplane of π. So π_S has order q.*

Proof. Coordinatize π with a quasifield Q so that π_S is coordinatized by a subquasifield R, i.e., choose coordinate axes in π_S with $[\infty]$ assigned to the translation axis. Now S may be identified with an isomorphic subgroup $\hat{S}$ of $(\mathrm{Aut}\,Q)_R$. Since $\hat{S}$ is a p-group there is an $\boldsymbol{F}_p\hat{S}$ submodule H such that

(i) $R \subset H \subset Q$; and

(ii) $|H| = p|R|$.

Since $\hat{S}$ fixes R elementwise, it is clear that the restriction homomorphism $\hat{S} \to \hat{S}|H$ has kernel K of order $\geq pq$. Now obviously $F = \mathrm{Fix}(K)$ can only be a Baer quasifield of Q, if we note that $|\mathrm{Fix}(\hat{S})| \geq q$. So result 1.1(b), applied to Q and F, shows that $|\mathrm{Kern}\, F| \geq pq$ and hence $F = \boldsymbol{F}_{q^2}$; in particular $(\mathrm{Aut}\, F)_R$ has order 2. Therefore the restriction homomorphism $\hat{S} \to \hat{S}|F$ has kernel $\hat{S}_0$ of order $\geq \frac{1}{2} pq^2$. But $\hat{S}_0$ is clearly semiregular on $Q \backslash F$ and this means $|\hat{S}_0| = q^2$ and also $p = 2$, $|\hat{S}| = 2q^2$. Recalling the connection between $\hat{S}$ and S, and identifying $\hat{S}_0$ with the appropriate subgroup $E \subseteq S$, leads to the lemma.

We now need to use the following special case of Dempwolff's work [2], mentioned in the introduction. Abusing Dempwolff's terminology, we call an elementary abelian planar 2-group E a *B-group* if π_E is a Baer subplane of the (translation) plane on which E acts.

Result 2 (Dempwolff [2, Satz]). *Let ψ be a translation plane of order 2^{4n} admitting a collineation group $\mathcal{G}$ in its translation complement such that the Sylow 2-subgroups of $\mathcal{G}$ are planar. Assume that $E \subseteq \mathcal{G}$ is a B-group of maximal order and that $|E| > 2^n$. Finally, let $\mathcal{G}^*$ denote the group generated by all B-groups in $\mathcal{G}$ that have the same order as E. Then one of the following cases must occur*:

(i) *ψ has order* 16 *and* $\mathcal{G}^* \cong SL(3, 2)$;

(ii) $\mathcal{G}^* \cong SL(2, 2^m)$ *for some m such that* $2^m \geq |E|$;

(iii) *$\mathcal{G}^*$ contains an elementary abelian normal subgroup M of order 2^{2n} such that $\mathcal{G}^*/M \cong D_{2r}$, where r is odd. Also now $\mathcal{G}^*$ is planar and $\pi_{\mathcal{G}^*}$ has order 2^n.*

(iv) *$\mathcal{G}^*$ is an elementary abelian* 2-*group.*

An immediate consequence of lemma 1 and Dempwolff's result is

Corollary 3. *Suppose $pq^2 \leq |S|$ and that G is nonsolvable. Then one of the following cases must occur.*

(i) $G \supseteq SL(3, 2)$ *and π has order* 16;

(ii) $G \supseteq SL(2, q^2)$ *and q is even.*

As all planes of type (i) and (ii) are known, we can refine the corollary above.

Lemma 4. *Suppose $pq^2 \leq |S|$ and that G is nonsolvable. Then π can only be one of the following planes of order* 16:

(i) *the Lorimer-Rahilly plane*;

(ii) *the Johnson-Walker plane*;

(iii) *the Dempwolff plane.*

Proof. Since S is planar, π is certainly not Desarguesian. Thus by corollary 3 and the Foulser-Johnson theorem [9], π is a Hall plane or π has order 16. But since S contains the Klein group, Dempwolff and Riefart [3] implies that when π has order 16 it is either a Hall plane or one of the three planes listed above. So it now remains to check only that a Hall plane π, of even order n, cannot admit a planar 2-group S of order $2\sqrt{n}$ (cf. lemma 1). So assume π to be a Hall plane and recall [18, p. 107] that now $Aut\ \pi$ has a slope orbit Δ of length $\sqrt{n}+1$. Moreover Δ is a derivation set and S must leave invariant one of the $\sqrt{n}+1$ Baer subplanes ψ that contains Δ and shares an affine point 0 with π_S. Since S fixes a line of ψ passing through 0 and $|S|>\sqrt{n}$, a nontrivial element of S fixes a subplane of ψ elementwise. In fact some $\sigma\in S$ induces a nontrivial planar collineation of ψ; for if $\mathcal{O}\in S\setminus\{1\}$ fixes ψ identically then π_S is a Baer subplane of ψ. So π can be coordinatized by a Hall system Q such that $\operatorname{Aut} Q$ contains a nontrivial 2-element which acts faithfully on K, the kern of Q, since ψ, like all subplanes containing Δ, is a kern plane of π. But now transitivity of $\operatorname{Aut} Q$ on $Q\setminus K$ shows that there is a reducible quadratic $x^2-\alpha x-\beta$ in $GF(\sqrt{n})=K$ such that every $t\in Q\setminus K$ satisfies the condition $t^2=t\alpha+\beta$, contradicting Jha [14, proposition 4.2(a)]. So π is not a Hall plane and the lemma is valid.

To complete the proof of theorem C, cf. lemmas 1, 4 and result 2(iii), we need only check that the three planes of lemma 4 admit nonsolvable autotopism groups containing 2-groups of order 8, since now the 2-groups will automatically be planar. But it is known that $\Gamma L(2, 4)$ is an autotopism group of the Dempwolff plane [3] and $L_3(2)$ is an autotopism group of the two other planes mentioned in lemma 4; in fact, $L_3(2)$ fixes three slopes precisely [16]. Hence theorem C is valid.

3. The Sylow p-subgroups of G

The object of this section is to list some properties of S that we shall need. We begin with a slight modification of lemma 2.1.

Lemma 1. *The plane π_S has order q or q^2.*

Proof. If the lemma is false then π_S has order q^r where $1<r<2$. Now as in lemma 2.1, coordinatize π with a quasifield Q such that π_S is coordinatized by a subquasifield R. As before we have $\hat{S}\subseteq(\operatorname{Aut} Q)_R$ such that $\hat{S}\cong S$ and $\hat{S}$ leaves invariant H such that $Q\supset H\supset R$ and $|H|=p|R|=pq^r$. The restriction map $\hat{S}\to\hat{S}|H$ clearly has kernel $\hat{S}_0$ such that $|\hat{S}_0|\geq q^2/|R|=q^{2-r}$. Since $r<2$, $\hat{S}_0$ is nontrivial and so we have a chain of quasifields $Q\supset Q_0\supset R$. But this means $q^4=|Q|\geq|R|^4=q^{4r}$. We now contradict our assymption that $r>1$ and so the lemma is valid.

Everything else we need to know about S is a corollary to the following special case of Foulser [8, Corollary 3.5(1) and Theorem 4.7].

Result 2. *Let Q be a finite quasifield with characteristic p that admits an automorphism p-group Σ. Also write $Q_0 = Fix(\Sigma)$ and $[Q:Q_0] = dim\, Q/dim\, Q_0$, where dimensions are over the prime field in Q. Then the following are all valid.*

(a) *$[Q:Q_0]$ is an integer.*

(b) *Assume $p \not| [Q:Q_0]$. Now*

(i) *Σ is elementary abelian;*

(ii) *if $\sigma_1, \sigma_2 \in \Sigma$ then $Fix(\sigma_1) \subseteq Fix(\sigma_2)$ or $Fix(\sigma_2) \subseteq Fix(\sigma_1)$; and*

(iii) *$[Q:Q_0] < p$.*

The following notation will be used throughout the article.

NOTATION 3. $\bar{S}$ is the subset of the Sylow p-subgroup of G consisting of all Baer p-elements in S together with the identity.

We now summarize everything we need to know about S and $\bar{S}$ in the following corollary to Foulser's theorem (result 2).

Proposition 4. *Assume π has odd order q^4. Then all the following statements are valid.*

(a) *The characteristic $p > 3$, provided $S \neq \bar{S}$.*

(b) *π_S is a Desarguesian plane of order q or q^2, and S is an elementary abelian group of order q^2.*

(c) *$\bar{S}$ is a planar group and $\pi_{\bar{S}}$ is a Baer subplane of π; thus either $S = \bar{S}$ or $\pi_{\bar{S}}$ is a Baer extension of π_S.*

(d) *The restriction homomorphism $S \to S \mid \pi_{\bar{S}}$ has image and kernel of order precisely q, provided $S \neq \bar{S}$.*

Proof. Part (c) follows from result 2(b) (ii) and lemma 1. Now part (d) is immediate if we recall that $|S|$ is precisely q^2 when q is odd (lemma 2.1) and note that the kernel and image of $S \to S \mid \pi_{\bar{S}}$ are both semiregular on their nonfixed points. Part (a) follows from result 2(b) (iii). If π_S has order q^2 then part (b) is a consequence of Foulser, result 1.1; if, however, π_S has order q then result 2(b) (i) shows that S is elementary abelian; also by part (d), $\mathrm{Aut}\, \pi_{\bar{S}}$ contains a group of order q fixing π_S identically, and so by result 1.1 (b) π_S must be Desarguesian. This completes the proof of part (b) and the proposition.

4. $\bar{S}$ is normal in G

For the rest of this work we shall assume that π has odd order. In this section we shall show that now π is a Hall plane or $\bar{S} \lhd G$. We end the section by exploring the connection between the solvability of G and the normality of

S in G; it turns out that the two conditions are usually equivalent.

Lemma 1. *If π is not a Hall plane then π_S has order q^2 only if $S \lhd G$.*

Proof. If the lemma is false then G contains a subgroup T of order q^2 such that $T \neq S$ and π_T, π_S are both Baer subplanes of π. Hence by Foulser, result 1.1 (a), we find $\pi_T \neq \pi_S$. Now by Foulser [7], $G \supseteq SL(2, q^2)$ such that the latter group fixes a set of q^2+1 slopes: specifically, appeal to [7, corollary 4.2(2)] for the case $p>3$ and to [7, propositions 5.1, 5.4] when $p=3$. Now the Foulser-Johnson theorem [10] shows that π must be a Hall plane. Hence the lemma is valid.

Lemma 2. *Assume $S \ntriangleleft G$ and let Σ be the slope set of the plane $\pi_{\bar{S}}$. Then either π is a Hall plane or $\pi_{\bar{S}}$ is the only (proper) subplane of π that contains $\Sigma \cup \{0\}$, where 0 is any affine point fixed by G.*

Proof. Assume π is not a Hall plane and that $\psi (\neq \pi_{\bar{S}})$ is also a Baer subplane of π that contains $\Sigma \cup \{0\}$. Now consider S_1, the global stabilizer of ψ in S. So if $\sigma \in S_1 \cap \bar{S}$ then $\mathrm{Fix}(\sigma)$ includes, in addition to $\pi_{\bar{S}}$, a Baer subplane of ψ; thus $\sigma=1$. On the other hand if $\sigma \in S_1 \backslash \bar{S}$ then clearly σ fixes a Baer subplane π_S of $\pi_{\bar{S}}$ and also a Baer subplane of ψ. Thus σ is a Baer p-element of S and so lies in $\bar{S}$. As this is a contradiction, S_1 must be trivial and hence the S-orbit of ψ includes q^2 Baer subplanes containing $\Sigma \cup \{0\}$. So Σ must be a derivation set and hence, by the Prohaska-Cofman theorem [17, theorem 51.1], $\pi_{\bar{S}}$ is Desarguesian. Now proposition 3.4(c) shows that $\pi_{\bar{S}}=\pi_S$ and so by lemma 1, $S \lhd G$, contradicting our hypothesis. The lemma follows.

Proposition 3. *Suppose π is not a Hall plane. Then $\bar{S} \lhd G$.*

Proof. If $S \ntriangleleft G$ then by proposition 3.4(a), $p>3$ and so Foulser [7] shows that Σ, the slope set of $\pi_{\bar{S}}$, is invariant under $\mathrm{Aut}\,\pi$. Now lemma 2 implies that G leaves $\pi_{\bar{S}}$ invariant and so by Foulser (result 1.1) we have $\bar{S} \lhd G$. It remains to consider the case when $S \lhd G$ but $\bar{S} \ntriangleleft G$. Since we still have $p>3$, Σ continues to be $\mathrm{Aut}\,\pi$ invariant. But now G leaves π_S and Σ invariant and hence G also leaves $\pi_{\bar{S}}$ invariant. So a double application of result 1.1 implies that $\bar{S} \lhd G$ and the proposition follows.

We now note that the solvability of G is equivalent to the normality of S in G, except when G has a unique nonsolvable composition factor and this factor is A_5.

Proposition 4. *Suppose π is not a Hall plane but, as usual, has odd order. Then the following are valid.*

(a) *G is solvable only if $S \lhd G$ and so there is a G-invariant (strict) Baer* *chain* $\pi \supset \pi_{\bar{S}} \supseteq \pi_S$.

(b) *Suppose* $S \lhd G$. *Then the Baer chain* $\pi \supset \pi_{\bar{S}} \supseteq \pi_S$ *continues to be G-invariant and G is nonsolvable only when the restriction map* $G \to G | \pi_S$ *has solvable kernel but its image is a metacyclic extension of* $SL(2, 5)$.

Proof. Suppose, if possible, that G is solvable but that $S \ntriangleleft G$. So by proposition 3 we have $S \neq \bar{S}$ and now proposition 3.4(a) implies $p > 3$. The usual argument involving Foulser [7] now shows that G induces a nonsolvable group on $\pi_{\bar{S}}$, while moving π_S around. Hence (a) follows. Now assume $S \lhd G$. Using result 1.1 we readily find that the kernel of the restriction homomorphism $G \to G | \pi_S$ is solvable and hence the restriction $G | \pi_S$ is nonsolvable. But π_S is an affine Desarguesian plane (proposition 3.4(b)) and so its only nonsolvable collineation p'-groups, which fix an affine point, are metacyclic extensions of $SL(2, 5)$, e.g., use [12, Hauptsatz 8.27]. Hence the proposition is proved.

Corollary 5. *Assume G is an autotopism group. Then G is solvable if and only if* $S \lhd G$.

Proof. Affine Desarguesian planes do not admit nonsolvable p'-autotopism groups. So the proposition applies, since Hall planes do not present any problems.

5. Proof of theorem B

Lemma 1. *Suppose* π *is not a Hall plane. Then*

(i) $G = \bar{S}H$ *where H is a complement of* $\bar{S}$: *and*

(ii) *the restriction homomorphism* $G \to G | \pi_{\bar{S}}$ *has* $\bar{S}$ *as its kernel and so H acts faithfully on* $\pi_{\bar{S}}$.

Proof. Since the Sylow p-subgroups of G are elementary abelian, every Sylow p-subgroup of G splits over $\bar{S}$. So by Gaschütz's theorem [20, theorem 9.3.7, p. 225] G also splits over $\bar{S}$. So (i) is valid. Now if part (ii) is false then result 1.1(c) contradicts lemma 4.2. Hence the lemma is valid.

We now extend the lemma above and at the same time summarize the results that we require from earlier sections.

Proposition 2. *Assume* π *is not a Hall plane and that* $S \ntriangleleft G$. *Then all the following hold.*

(i) *The characteristic of* $\pi > 3$;

(ii) *G contains* $\Sigma \cong SL(2, q)$ *such that*

$$\langle \bar{S}, \Sigma \rangle = \bar{S} \oplus \Sigma$$

and Σ *fixes elementwise the* $q+1$ *slopes of* π_S;

(iii) *Σ leaves invariant the Baer subplane $\pi_{\bar{S}}$;*
(iv) *the central involution in Σ is the kern involution of π.*

Proof. Parts (i) and (iii) follow from propositions 3.4(a) and 4.4(a). Now let H be as in lemma 1 and consider the possibility of H leaving π_S invariant. This clearly means that all the Sylow p-subgroups of G fix π_S elementwise and leave $\pi_{\bar{S}}$ invariant. It now follows easily that, in addition to S, there is another Sylow p-subgroup T such that $T|\pi_{\bar{S}}=S|\pi_{\bar{S}}$. Hence a p'-element $h\neq 1$ in G fixes $\pi_{\bar{S}}$ elementwise. Now lemma 4.2 and result 1.1(c) contradict each other. Thus H cannot fix π_S. But now again Foulser [7, corollary 4.2(2)] shows that H fixes (globally) the $q+1$ slopes of π_S and the restriction $H|\pi_{\bar{S}}$ contains $SL(2, q)$. But lemma 1 now shows that $H\supseteq\Sigma\cong SL(2, q)$ acts faithfully on $\pi_{\bar{S}}$; also clearly Σ fixes the $q+1$ slopes of π_S. Now because $\bar{S}\lhd G$, we see that $\bar{S}P$ is a Sylow p-subgroup of G whenever P is a Sylow p-subgroup of Σ and hence, since $P\bar{S}$ is abelian (proposition 3.4(b)), and Σ is generated by its Sylow p-subgroups, we conclude that the commutator $[\bar{S}, \Sigma]=1$. But Σ and $\bar{S}$ are disjoint because Σ is faithful on $\pi_{\bar{S}}$ and so $\langle\Sigma, \bar{S}\rangle=\Sigma\oplus\bar{S}$. It only remains to check (iv). If false, then we have a Baer involution $\alpha\in\Sigma$ such that Σ leaves invariant the Baer subplane π_α, a plane of order q^2. But since $\alpha\in\Sigma$ we find that α fixes the $q+1$ slopes of π_S, fixed by all members of Σ. Thus π_α and $\pi_{\bar{S}}$ either meet in a Baer subplane of $\pi_{\bar{S}}$ or π_α and $\pi_{\bar{S}}$ have identical slope sets. The latter possibility contradicts lemma 4.2 and so Σ leaves invariant a Baer subplane π_0 of $\pi_{\bar{S}}$, while fixing its $q+1$ slopes. This also yields a contradiction, e.g., all the Sylow p-subgroups in Σ fix π_0 elementwise. Hence α must be a kern involution and the proposition is proved.

We now derive some module theoretic consequences of proposition 2 which eventually yield a contradiction, unless π is a Hall plane. So let Γ be a spread associated with π and assume $S\ntriangleleft G$ and that π is not a Hall plane. Now Γ has a component V which is invariant under $\bar{S}\oplus\Sigma$. We shall regard V as an $\boldsymbol{F}_pSL(2, q)$ module, defined by the action of Σ on V, and let W be the submodule of order q^2 associated with $\mathrm{Fix}(\bar{S})\cap V$.

Lemma 3. *If π is not a Hall plane and $S\ntriangleleft G$ then there is an $\boldsymbol{F}_pSL(2, q)$ module V to which the following apply:*
(i) *$|V|=q^4$ and p, the characteristic of V, is at least 5;*
(ii) *V contains a submodule W of order q^2;*
(iii) *the central involution in $SL(2, q)$ induces -1 on V; and*
(iv) *W has no complement in V.*

Proof. Parts (i), (ii), (iii) are immediate consequences of proposition 2 and the remarks following it. To see (iv), assume $V=W\oplus W_1$ where W_1 is also an $SL(2, q)$ submodule. Part (iii) and the simplicity of $L_2(q)$ shows that $\Sigma\cong SL(2, q)$

acts faithfully on both W and W_1. But it is generally known that there is, up to isomorphism, only one $\boldsymbol{F}_p SL(2, q)$ module of order q^2 on which $SL(2, q)$ acts faithfully, provided $p>2$ (cf. result 2(ii) of appendix). But this means that if σ is a nontrivial p-element in $SL(2, q)$ then $\mathrm{Fix}(\sigma)$ meets the isomorphic modules W and W_1 in precisely q points. But $W \cap W_1 = 0$ now implies that the p-elements of $\Sigma \cong SL(2, q)$ are all Baer p-elements of π. But this is a contradiction because the Baer p-elements of G lie in $\bar{S}$ and $|\bar{S} \cap \Sigma| = 1$. So part (iv) and the lemma follows.

But modules satisfying conditions (i) through (iv) cannot exist unless $p=q=5$; this fact is verified in the following appendix (proposition 3). Thus proposition 2 and lemma 3 lead to theorem B, part (a). Part (b) now follows from proposition 4.4(b).

Appendix. $SL(2, p^n)$ modules of order p^{4n}

Let W be an irreducible $K\mathcal{G}$ module, where K is a finite field and $\mathcal{G}$ is a finite group. Also let W^L be the natural $L\mathcal{G}$ module on $W \otimes L$, where $L=\tilde{K}$ denotes the algebraic closure of K. Then by [13, lemma 1.15, pp. 17–18] we have

$$W^L \cong \oplus \sum_{i=0}^{r} U_i$$

where the U_i are absolutely irreducible $L\mathcal{G}$ modules that we call the components of W.

Proposition 1. *Let the $K\mathcal{G}$ module V be an extension of an irreducible submodule A by another irreducilbe $K\mathcal{G}$ module B. Suppose $(A_i)_1^n$ and $(B_j)_1^m$ are the components of A and B. Then either*

(i) $V \cong A \oplus B$; *or*

(ii) $Ext^1_{LG}(B_J, A_I) \neq 0$ *for some I, J where $L=\tilde{K}$.*

Proof. Consider first the case when $\mathrm{Ext}^1_{LG}(B^L, A^L) = 0$. Since $\frac{V}{A} \cong B$ we also have $\frac{V^L}{A^L} \cong B^L$ [13, lemma 1.2(e), pp. 5–6]. But the nullity assumption on $\mathrm{Ext}^1_{LG}(B^L, A^L)$ means that [19, theorem 7.8, p. 145]

$$V^L \cong A^L \oplus B^L$$

and so clearly $V^L \cong (A \oplus B)^L$. Thus by cancellation [13, p. 26] we have $V \cong A \oplus B$, giving part (i). It remains to consider the case when $\mathrm{Ext}^1_{LG}(B^L, A^L) \neq 0$. As *Ext* is additive on finite direct sums [19, theorems 7.10, 7.11] we have

$$\sum_{i,j} \mathrm{Ext}^1_{LG}(B_j, A_i) \neq 0 .$$

Hence part (ii) and the proposition follow.

We now apply proposition 1 to certain $\boldsymbol{F}_pG(n)$ modules, where $G(n) = SL(2, p^n)$. For this purpose we need to recall the $\boldsymbol{F}_pG(n)$ modules of order $\leq p^{2n}$ on which $G(n)$ acts faithfully, when p is odd. The result we require is implied by Fong and Seitz [5, 4.7, p. 20] but it may be helpful to note [9, (ii), p. 6].

Result 2. *Let W be a nontrivial $\boldsymbol{F}_pG(n)$ module on which $G(n)$ acts faithfully. Also write $L = \tilde{\boldsymbol{F}}_p$. Then*

(i) $|W| \geq p^{2n}$; *and*

(ii) *if* $|W| = p^{2n}$ *and* $p > 2$ *then*

$$W^L \cong \oplus \sum_{i=0}^{n-1} \lambda(p^i)_n$$

where $(\lambda(p^i)_n)_{i=0}^{n-1}$ *denotes the family of the distinct conjugates of* $\lambda(1)_n$, *the* $LG(n)$ *module of* 1*-forms in* $L[x, y]$.

Corollary. *If* $|W| = p^{2n}$ *and* $p > 2$ *then* W *is irreducible and unique, up to isomorphism.*

We can now prove the result we need.

Proposition 3. *Let V be an $\boldsymbol{F}_pSL(2, p^n)$ module of order p^{4n}, where $p \geq 5$ and $p^n \neq 5$. Assume the central involution in $SL(2, p^n)$ acts like -1 on V. Then if X is a submodule of order p^{2n} we have*

$$V = X \oplus Y$$

where Y is an irreducible submodule on which $SL(2, p^n)$ acts faithfully. (*N.B. For $p < 5$ the conclusions of the proposition become more complicated.*)

Proof. The action of the central involution on V implies that $SL(2, p^n)$ acts faithfully on both X and $\frac{V}{X}$. Hence result 2(i) shows that both X and $Y \cong \frac{V}{X}$ are irreducible $\boldsymbol{F}_pSL(2, p^n)$ modules, with $SL(2, p^n)$ acting faithfully on both modules.

To get a contradiction assume that $V \ncong X \oplus Y$. Now proposition 1 and result 2(ii) together imply that

$$\mathrm{Ext}^1_{LG(n)}(\lambda(p^i)_n, \lambda(p^j)_n) \neq 0$$

for some i, j. But the formulae of Andersen et al. [1, p. 4.6] show precisely when the above Ext does not vanish; for $p \geq 5$ it turns out that the relation above holds only when $p^n = 5$. Hence the proposition is valid.

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