Title	Remarks on linear Volterra integral equations of parabolic type
Author(s)	Tanabe, Hiroki
Citation	Osaka Journal of Mathematics. 1985, 22(3), p. 519-531
Version Type	VoR
URL	https://doi.org/10.18910/7182
rights	
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Tanabe, H.
Osaka J. Math.
22 (1985), 519–531

REMARKS ON LINEAR VOLTERRA INTEGRAL EQUATIONS OF PARABOLIC TYPE

HIROKI TANABE

(Received December 20, 1983)

The purpose of this paper is to improve the result of the previous paper [6] on the linear Volterra integral equation of parabolic type

$$u(t)+\int_0^t b(t-s)A(s)u(s)ds=f(t)\,,\qquad 0\leqq t\leqq T\,, \tag{0.1}$$

in a Banach space X. Here b is a given complex valued function such that $b(0)=1$, f and u are given and unknown functions with values in X respectively, and $-A(t)$ is a closed linear operator in X which generates an analytic semi-group for each t.

In [6] assuming among others that $\dot{b}$ is absolutely continuous and $\ddot{b}\in L^p(0, T)$ for some $p>1$ (cf. Friedman-Shinbrot [4]), we constructed the fundamental solution $W(t, s)$ of (0.1) which is an operator valued function defined in $0\leqq s\leqq t\leqq T$ satisfying

$$W(t, s)+\lim_{\varepsilon\downarrow 0}\int_{s+\varepsilon}^t b(t-\tau)A(\tau)W(\tau, s)d\tau = I\,, \tag{0.2}$$

$$\left\|\frac{\partial}{\partial t}W(t, s)\right\|\leqq\frac{C}{t-s},\ \ ||A(t)W(t, s)||\leqq\frac{C}{t-s}\,. \tag{0.3}$$

Using the fundamental solution we showed the existence and uniqueness of the solution of (0.1) such that the integral in the left side of the equation exists as an improper integral:

$$\int_0^t b(t-s)A(s)u(s)ds=\lim_{\varepsilon\downarrow 0}\int_\varepsilon^t b(t-s)A(s)u(s)ds\,. \tag{0.4}$$

Recently J. Prüss [5] constructed the fundamental solution for the equation of nonconvolution type

$$\frac{d}{dt}u(t)+A(t)u(t)=\int_0^t K(t, s)A(s)u(s)ds+f(t)\,, \tag{0.5}$$

This work was supported by Grant-in-Aid for Scientific Research 58540079.

which has a bounded operator valued function $K(t, s)$ as its kernel. As for the smoothness of $K(t, s)$ Prüss assumes only Hölder continuity in (t, s) which is less restrictive than that of [6] (compare (0.5) with the equation obtained by formally differentiating both sides of (0.1) in t and note that $\ddot{b}\in L^p(0, T)$, $p>1$, implies $\dot{b}$ is Hölder continuous). In this paper we show that the result of [6] remains valid when b is an absolutely continuous function with derivative $\dot{b}$ of bounded variation.

According to the argument of Clément-MacCamy-Nohel [1] and Clément-Nohel [2] the equation (0.1) is an integral form of the initial value problem for the following integrodifferential equation

$$\begin{aligned}&\frac{d}{dt}\{u(t)+\int_0^t \beta(t-s)u(s)ds\}+A(t)u(t)-\int_0^t \gamma(t-s)A(s)u(s)ds = h(t)\,,\\ &u(0) = u_0 = f(0)\,,\end{aligned} \tag{0.6}$$

where b, f and β, γ, h are linked through the relations (1.6)-(1.10) of section 1. The equation (0.6) is an abstract version of the equation which describes heat flow in material with memory. In this paper it will be shown that the equivalence of (0.1) and (0.6) remains valid when the initial value u_0 is an arbitrary element of X and hence the integral of (0.1) and the latter integral of (0.6) exist only in the improper sense in general.

It is straightforward to extend the results of the present paper to the case where $b(t)$ is an operator valued function.

1. Assumptions and results

Let X be a real or complex Banach space. For each $t\in[0, T]$ $A(t)$ is a closed, densely defined operator such that $-A(t)$ generates an analytic semigroup $\exp(-\tau A(t))$. We assume that a unique fundamental solution $U(t, s)$ of the evolution equation

$$du(t)/dt+A(t)u(t) = 0 \tag{1.1}$$

exists and has the form

$$U(t, s) = \exp(-(t-s)A(s))+Z(t, s)\,. \tag{1.2}$$

There exists a positive number $\alpha\leqq 1$ such that

$$||A(t)\exp(-(t-s)A(t))-A(s)\exp(-(t-s)A(s))||\leqq C_1(t-s)^{\alpha-1}\,, \tag{1.3}$$

$$||Z(t, s)||\leqq C_2(t-s)^{\alpha},\quad ||(\partial/\partial t)Z(t, s)||\leqq C_3(t-s)^{\alpha-1}\,. \tag{1.4}$$

It follcws from (1.2) and (1.4) that

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\|\leqq\frac{C_4}{t-s}\,,\quad ||A(t)U(t, s)||\leqq\frac{C_5}{t-s}\,. \tag{1.5}$$

As for the precise definition of the fundamental solution see section 2 of [6]. In many cases where the equation (1.1) is solvable, it is known that the assumptions above are satisfied.

In (0.6) β, γ are real valued functions of bounded variation, u_0 is an element of X, and h is a Hölder continuous function with values in X. If we define functions b and f following [1] by

$$b(t) = C(t)+(\rho*C)(t), \tag{1.6}$$

$$f(t) = G(t)+(\rho*G)(t), \tag{1.7}$$

$$C(t) = 1-\int_0^t \gamma(\tau)d\tau, \tag{1.8}$$

$$G(t) = u_0+\int_0^t h(\tau)d\tau, \tag{1.9}$$

$$\rho(t)+(\beta*\rho)(t)+\beta(t) = 0, \tag{1.10}$$

where

$$(\rho*C)(t) = \int_0^t \rho(t-s)C(s)ds,$$

then b is a real valued absolutely continuous function such that $\dot{b}$ is of bounded variation, $b(0)=1$, and f is a function with values in X having a Hölder continuous derivative.

$C([0, T]; X)$ and $C((0, T]; X)$ are the sets of all functions taking values in X which are continuous in $[0, T]$ and $(0, T]$ respectively.

In what follows the notation C denotes constants depending only on the general assumptions stated above unless it stands for the function defined by (1.8).

Theorem 1. *Suppose b is an absolutely continuous complex valued function such that $\dot{b}$ is of bounded variation and $b(0)=1$. Then, there exists a unique fundamental solution $W(t, s)$ of the equation* (0.1) *which is a bounded operator valued function strongly continuous in $0\leqq s\leqq t\leqq T$ and satisfies* (0.2) *and* (0.3).

If f is a function with values in X having a Hölder continuous derivative, then

$$u(t) = W(t, 0)f(0)+\int_0^t W(t, s)\dot{f}(s)ds \tag{1.11}$$

is a unique solution of (0.1) *in the following sense: $u\in C([0, T]; X)$, $u(t)\in D(A(t))$ for $t\in(0, T]$, $Au\in C((0, T]; X)$ and $\int_\varepsilon^t b(t-s)A(s)u(s)ds$ is uniformly bounded in $0<\varepsilon\leqq t\leqq T$, and*

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^t b(t-s)A(s)u(s)ds = f(t)-u(t) \tag{1.12}$$

where the limit in the left side exists uniformly in each compact subset of $(0, T]$.

Theorem 2. *Suppose that* β, γ *are real valued functions of bounded variation in* $[0, T]$, u_0 *is an arbitrary element of* X, *and* h *is a Hölder continuous function with values in* X *in* $[0, T]$. *Let* b *and* f *be functions defined by* (1.6)–(1.10). *Let* u *be a function such that* $u\in C([0, T]; X)$, $u(t)\in D(A(t))$ *for* $t\in(0, T]$, $Au\in C((0, T]; X)$ *and* $u(0)=u_0$. *Then the following two assertions are equivalent.*

(i) u *is the solution of* (0.1) *in the sense of Theorem* 1.

(ii) u *is differentiable in* $(0, T]$, $\int_\varepsilon^t \gamma(t-s)A(s)u(s)ds$ *is uniformly bounded in* $0<\varepsilon\leqq t\leqq T$,

$$\int_0^t \gamma(t-s)A(s)u(s)ds = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^t \gamma(t-s)A(s)u(s)\,ds$$

exists almost everywhere in $(0, T]$, *and* (0.6) *holds.*

2. Proof of Theorem 1

Let $r(t)$ be the solution of

$$\dot{b}+r+\dot{b}*r = 0 \tag{2.1}$$

As is easily seen $r(t)$ is a function of bounded variation. The equation (2.1) is equivalent to

$$b+b*r = 1\,, \tag{2.2}$$

$$\dot{b}+r(0)b+b*\dot{r} = 0\,, \tag{2.3}$$

where

$$(b*\dot{r})(t) = \int_0^t b(t-s)dr(s)\,.$$

We construct the fundamental solution $W(t, s)$ as the solution of the integral equation

$$W(t, s) = U(t, s)+\int_s^t U(t, \tau)F(\tau, s)d\tau, \tag{2.4}$$

where

$$F(\tau, s) = r(\tau-s)-r(0)W(\tau, s)-\int_s^\tau W(\tau+s-\sigma, s)dr(\sigma-s) \tag{2.5}$$

(cf. Crandall-Nohel [3]). The equation (2.4) is solvable by successive approximation, and $W(t, s)$ is strongly continuous in $0\leqq s\leqq t\leqq T$. From (1.5) it follows that for $0\leqq s<\tau<t\leqq T$

$$\|U(t, s)-U(\tau, s)\|\leqq\frac{C}{\kappa}\left(\frac{t-\tau}{\tau-s}\right)^\kappa, \tag{2.6}$$

where κ is an arbitrary positive number<1. Since $F(\tau, s)$ is uniformly bounded, it follows from (2.4) and (2.6) that

$$||W(t, s)-W(\tau, s)||\leqq\frac{C}{\kappa}\left(\frac{t-\tau}{\tau-s}\right)^{\kappa}+C(t-\tau)+\frac{C}{\kappa(1-\kappa)}(t-\tau)^{\kappa}(\tau-s)^{1-\kappa} \quad (2.7)$$

for $0\leqq s<\tau<t\leqq T$. Applying Fubini's theorem we get

$$W(t, s) = U(t, s)+\int_s^t U(t, \tau)r(\tau-s)d\tau-r(0)\int_s^t U(t, \tau)W(\tau, s)d\tau$$
$$-\int_s^t\int_\sigma^t U(t, \tau)W(\tau+s-\sigma, s)d\tau dr(\sigma-s)\,. \quad (2.8)$$

We first show the differentiability of the last term of (2.8). If we write

$$\frac{\partial}{\partial t}\int_\sigma^t U(t, \tau)W(\tau+s-\sigma, s)d\tau$$
$$=\frac{\partial}{\partial t}\int_0^t \exp\left(-(t-\tau)A(\tau)\right)W(\tau+s-\sigma, s)d\tau$$
$$+\frac{\partial}{\partial t}\int_\sigma^t Z(t, \tau)W(\tau+s-\sigma, s)d\tau = \mathrm{I}+\mathrm{II}\,, \quad (2.9)$$

it follows from (1.4) that

$$||\mathrm{II}||\leqq C\int_\sigma^t (t-\tau)^{\alpha-1}d\tau=\frac{C}{\alpha}(t-\sigma)^{\alpha}\,.$$

For simplicity we set

$$S(t, \tau) = A(t)\exp\left(-(t-\tau)A(t)\right)-A(\tau)\exp\left(-(t-\tau)A(\tau)\right).$$

In view of (1.3)

$$||S(t, \tau)||\leqq C(t-\tau)^{\alpha-1}\,. \quad (2.10)$$

As is easily seen

$$\mathrm{I} = \int_\sigma^t S(t, \tau)W(\tau+s-\sigma, s)d\tau$$
$$-\int_\sigma^t A(t)\exp\left(-(t-\tau)A(t)\right)\left(W(\tau+s-\sigma, s)-W(t+s-\sigma, s)\right)d\tau$$
$$+\exp\left(-(t-\sigma)A(t)\right)W(t+s-\sigma, s)\,. \quad (2.11)$$

With the aid of (2.10), (2.7), (2.11) we easily show that I is uniformly bounded, and hence so is the left member of (2.9). Thus, if $t-s$ is a point of continuity of r

$$\frac{\partial}{\partial t}\int_s^t\int_\sigma^t U(t,\tau)W(\tau+s-\sigma,s)d\tau\,dr(\sigma-s)$$
$$=\int_s^t\frac{\partial}{\partial t}\int_\sigma^t U(t,\tau)W(\tau+s-\sigma,s)d\tau\,dr(\sigma-s) \tag{2.12}$$

exists. Next, we examine the second term of the right member of (2.8). If r is continous at t, then

$$\frac{\partial}{\partial t}\int_s^t \exp(-(t-\tau)A(\tau))r(\tau-s)d\tau$$
$$=\int_s^t S(t,\tau)r(\tau-s)d\tau+\exp(-(t-s)A(t))r(0)$$
$$+\int_s^t \exp(-(t-\tau)A(t))dr(\tau-s).$$

Hence it is easily seen that $\frac{\partial}{\partial t}\int_s^t U(t,\tau)r(\tau-s)d\tau$ exists at points of continuity of r and is uniformly bounded. With the aid of (2.7) it is easy to show that the third term of the right member of (2.8) is differentiable with respect to t and the derivative is uniformly bounded. Thus we have shown that $W(t,s)$ is differentiable in t and the first inequality of (0.3) holds. It is not difficult to show

$$(\partial/\partial t)W(t,s)+A(t)W(t,s)=F(t,s). \tag{2.13}$$

Next we prove (0.2). With the aid of (2.13) and integrating by parts

$$\int_{s+\varepsilon}^t b(t-\tau)A(\tau)W(\tau,s)d\tau$$
$$=\int_{\varepsilon+\varepsilon}^t b(t-\tau)\left\{F(\tau,s)-\frac{\partial}{\partial\tau}W(\tau,s)\right\}d\tau$$
$$=\int_{s+\varepsilon}^t b(t-\tau)F(\tau,s)d\tau-W(t,s)+b(t-s-\varepsilon)W(s+\varepsilon,s) \tag{2.14}$$
$$-\int_{s+\varepsilon}^t \dot{b}(t-\tau)W(\tau,s)d\tau.$$

This shows that $\int_{s+\varepsilon}^t b(t-\tau)A(\tau)W(\tau,s)d\tau$ is uniformly bounded and

$$\lim_{\varepsilon\downarrow 0}\int_{s+\varepsilon}^t b(t-\tau)A(\tau)W(\tau,s)d\tau$$
$$=\int_s^t b(t-\tau)F(\tau,s)d\tau-W(t,s)+b(t-s)-\int_s^t \dot{b}(t-\tau)W(\tau,s)d\tau. \tag{2.15}$$

With the aid of (2.2) one obtains

$$\int_s^t b(t-\tau)F(\tau, s)d\tau = 1-b(t-s)-r(0)\int_s^t b(t-\tau)W(\tau, s)d\tau$$
$$-\int_s^t b(t-\tau)\int_s^\tau W(\tau+s-\sigma, s)dr(\sigma-s)d\tau . \tag{2.16}$$

By integıation by parts one gets

$$\int_{s+\varepsilon}^t b(t-\tau)\int_s^{\tau-\varepsilon} W(\tau+s-\sigma, s)dr(\sigma-s)d\tau$$
$$= \int_{s+\varepsilon}^t b(t-\tau)\{W(s+\varepsilon, s)r(\tau-\varepsilon-s)-W(\tau, s)r(0)$$
$$-\int_s^{\tau-\varepsilon} \frac{\partial}{\partial\sigma} W(\tau+s-\sigma, s)r(\sigma-s)d\sigma\}d\tau \tag{2.17}$$

By a suitable change of variables, Fubini's theorem, and integration by parts

$$\int_{s+\varepsilon}^t b(t-\tau)\int_s^{\tau-\varepsilon} \frac{\partial}{\partial\sigma} W(\tau+s-\sigma, s)r(\sigma-s)d\sigma\, d\tau$$
$$= -\int_{s+\varepsilon}^t b(t-\tau)\int_{s+\varepsilon}^\tau \frac{\partial}{\partial\sigma} W(\sigma, s)\, r(\tau-\sigma)d\sigma\, d\tau$$
$$= -\int_{s+\varepsilon}^t (b*r)(t-\sigma)\frac{\partial}{\partial\sigma} W(\sigma, s)d\sigma$$
$$= (b*r)(t-s-\varepsilon)W(s+\varepsilon, s)+\int_{s+\varepsilon}^t \frac{\partial}{\partial\sigma}(b*r)(t-\sigma)W(\sigma, s)d\sigma$$
$$= (b*r)(t-s-\varepsilon)W(s+\varepsilon, s)+\int_{s+\varepsilon}^t \dot{b}(t-\sigma)W(\sigma, s)d\sigma .$$

Combining this with (2.17) and letting $\varepsilon \to 0$ we obtain

$$\int_s^t b(t-\tau)\int_s^\tau W(\tau+s-\sigma, s)dr(\sigma-s)d\tau$$
$$= -\int_s^t b(t-\tau)W(\tau, s)\,d\tau r(0)-\int_s^t \dot{b}(t-\sigma)W(\sigma, s)d\sigma . \tag{2.18}$$

Substituting (2.18) in (2.16) gives

$$\int_s^t b(t-\tau)F(\tau, s)d\tau = 1-b(t-s)+\int_s^t \dot{b}(t-\sigma)W(\sigma, s)d\sigma . \tag{2.19}$$

Combining (2.15) with (2.19) we obtain (0.2), and the proof of the first assertion of Theorem 1 is complete.

Let f be a function in the statement of Theorem 1. With the aid of Fubini's theorem we easily get

$$\int_0^t W(t, s)\dot{f}(s)ds$$
$$= \int_0^t U(t, \tau)\{\dot{f}(\tau)+\int_0^\tau r(\tau-s)\dot{f}(s)ds-r(0)\int_0^\tau W(\tau, s)\dot{f}(s)ds\}d\tau$$
$$-\int_0^t\int_s^t\int_\sigma^t U(t, \tau)W(\tau+s-\sigma, s)d\tau dr(\sigma-s)\dot{f}(s)ds\,. \tag{2.20}$$

Using (2.12), (2.7) it follows that

$$\frac{\partial}{\partial t}\int_0^t\int_s^t\int_\sigma^t U(t, \tau)W(\tau+s-\sigma, s)d\tau\, dr(\sigma-s)\dot{f}(s)ds$$
$$= \int_0^t\int_s^t \frac{\partial}{\partial t}\int_\sigma^t U(t, \tau)W(\tau+s-\sigma, s)d\tau dr(\sigma-s)\dot{f}(s)\,ds$$
$$= \int_0^t\int_s^t \{W(t+s-\sigma, s)$$
$$-A(t)\int_\sigma^t U(t, \tau)W(\tau+s-\sigma, s)d\tau\}dr(\sigma-s)\dot{f}(s)ds$$
$$= \int_0^t\int_s^t W(t+s-\sigma, s)dr(\sigma-s)\dot{f}(s)ds$$
$$-A(t)\int_0^t\int_s^t\int_\sigma^t U(t, \tau)W(\tau+s-\sigma, s)d\tau\, dr(\sigma-s)\dot{f}(s)ds\,. \tag{2.21}$$

The inside of the bracket { } of the right side of (2.20) is Holder continuous. Hence, from (2.20) and (2.21) it follows that

$$\frac{\partial}{\partial t}\int_0^t W(t, s)\dot{f}(s)ds = \dot{f}(t)+\int_0^t r(t-s)\dot{f}(s)ds$$
$$-r(0)\int_0^t W(t, s)\dot{f}(s)ds-\int_0^t\int_s^t W(t+s-\sigma, s)dr(\sigma-s)\dot{f}(s)ds$$
$$-A(t)\int_0^t W(t, s)\dot{f}(s)ds\,. \tag{2.22}$$

With the aid of (2.22) and integration by parts

$$\int_0^t b(t-\tau)A(\tau)\int_0^\tau W(\tau, s)\dot{f}(s)ds\,d\tau$$
$$= \int_0^t b(t-\tau)\{\dot{f}(\tau)+\int_0^\tau r(\tau-s)\dot{f}(s)ds-r(0)\int_0^\tau W(\tau, s)\dot{f}(s)ds$$
$$-\int_0^\tau\int_s^\tau W(\tau+s-\sigma, s)dr(\sigma-s)\dot{f}(s)ds\}\,d\tau$$
$$-\int_0^t W(t, s)\dot{f}(s)ds-\int_0^t \dot{b}(t-\tau)\int_0^\tau W(\tau, s)\dot{f}(s)dsd\tau$$
$$= (b*\dot{f})(t)+(b*r*\dot{f})(t)$$

$$-r(0)\int_0^t b(t-\tau)\int_0^\tau W(\tau, s)\dot{f}(s)ds\,d\tau$$
$$-\int_0^t b(t-\tau)\int_0^\tau\int_s^\tau W(\tau+s-\sigma, s)dr(\sigma-s)\dot{f}(s)ds\,d\tau$$
$$-\int_0^t W(t, s)\dot{f}(s)ds-\int_0^\tau \dot{b}(t-\tau)\int_0^\tau W(\tau, s)\dot{f}(s)ds\,d\tau\,. \tag{2.23}$$

By the change of variables and (2.3)

$$\int_0^t b(t-\tau)\int_0^\tau\int_s^\tau W(\tau+s-\sigma, s)dr(\sigma-s)\dot{f}(s)ds\,d\tau$$
$$=\int_0^t\int_s^t\int_\sigma^t b(t-\tau)dr(\tau-\sigma)W(\sigma, s)d\sigma\dot{f}(s)ds$$
$$=-\int_0^t\int_s^t(\dot{b}(t-\sigma)+r(0)b(t-\sigma))W(\sigma, s)d\sigma\dot{f}(s)\,ds\,. \tag{2.24}$$

With the aid of (2.2), (2.23), (2.24) we conclude

$$\int_0^t b(t-\tau)A(\tau)\int_0^\tau W(\tau, s)\dot{f}(s)ds\,d\tau$$
$$=f(t)-f(0)-\int_0^t W(t, s)\dot{f}(s)ds\,. \tag{2.25}$$

The remaining part of the proof of Theorem 1 is easy and omitted.

3. Proof of Theorem 2

For simplicity we write $w(t)=A(t)u(t)$. Suppose first that (i) holds. Set

$$F_\varepsilon(t)=\int_\varepsilon^t b(t-s)w(s)ds\,.$$

By (1.6)

$$F_\varepsilon(t)=\int_\varepsilon^t\{C(t-s)+(\rho*C)(t-s)\}w(s)\,ds$$
$$=\int_\varepsilon^t C(t-s)w(s)ds+\int_\varepsilon^t\rho(t-\sigma)\int_\varepsilon^\sigma C(\sigma-s)w(s)dsd\sigma\,. \tag{3.1}$$

Hence

$$\int_\varepsilon^t b(t-\tau)F_\varepsilon(\tau)d\tau=\int_\varepsilon^t\beta(t-\tau)\int_\varepsilon^\tau C(\tau-s)w(s)ds$$
$$+\int_\varepsilon^t\int_\sigma^t\beta(t-\tau)\,\rho(\tau-\sigma)d\tau\int_\varepsilon^\sigma C(\sigma-s)w(s)dsd\sigma$$
$$=\int_\varepsilon^t\{\beta(t-\tau)+(\beta*\rho)(t-\tau)\}\int_\varepsilon^\tau C(\tau-s)w(s)ds\,d\tau$$
$$=-\int_\varepsilon^t\rho(t-\tau)\int_\varepsilon^\tau C(\tau-s)w(s)ds\,d\tau$$
$$=-\int_\varepsilon^t(\rho*C)(t-s)w(s)ds\,. \tag{3.2}$$

The first member of (3.2) is uniformly bounded and converges to $(\beta*(f-u))(\tau)$ uniformly in each compact subset of $(0, T]$. Hence the same holds for the last member of (3.2) and

$$-\int_0^t(\rho*C)(t-s)w(s)ds=(\beta*(f-u))(t)\,. \tag{3.3}$$

Combining this with (3.1) we see that $\int_\varepsilon^t C(t-s)w(s)ds$ is uniformly bounded and tends to

$$\int_0^t C(t-s)w(s)ds=f(t)-u(t)+(\beta*(f-u))(t) \tag{3.4}$$

uniformly in each compact subset of $(0, T]$.

Noting that

$$(d/dt)(\beta*u)(t)=\beta(0)u(t)+(u*\dot{\beta})(t)\,,$$

where $(u*\dot{\beta})(t)=\int_0^t u(t-\tau)d\beta(\tau)$, and

$$\dot{f}(t)=h(t)+\rho(t)u_0+(h*\rho)(t)\,, \tag{3.5}$$

we get from (3.3) and (1.10)

$$\begin{aligned}
&-\frac{d}{dt}\int_0^t(\rho*C)(t-s)w(s)ds\\
&=\beta(t)f(0)+(\beta*\dot{f})(t)-\frac{d}{dt}(\beta*u)(t)\\
&=\beta(t)u_0+(\beta*h)(t)+(\beta*\rho)(t)u_0\\
&\quad+(\beta*h*\beta)(t)-\frac{d}{dt}(\beta*u)(t)\\
&=\beta(t)u_0-(\rho*h)(t)+(\beta*\rho)(t)u_0-\frac{d}{dt}(\beta*u)(t)\,.
\end{aligned} \tag{3.6}$$

Let ξ be the solution of

$$-\gamma+\xi-\gamma*\xi=0\,. \tag{3.7}$$

This is equivalent to

$$C+C*\xi=1\,. \tag{3.8}$$

Clearly,

$$\xi(0)=\gamma(0)\,. \tag{3.9}$$

In view of (3.7) and (3.9)

$$\int_\varepsilon^t \int_\varepsilon^\tau C(\tau-s)w(s)ds\,d_\tau\xi(t-\tau)$$
$$= \int_\varepsilon^t \int_s^t C(\tau-s)d_\tau\xi(t-\tau)w(s)ds$$
$$= \int_\varepsilon^t \{\xi(0)C(t-s)-\xi(t-s)+(\xi*\gamma)(t-s)\}w(s)ds$$
$$= \gamma(0)\int_\varepsilon^t C(t-s)\,w(s)ds - \int_\varepsilon^t \gamma(t-s)w(s)ds\,.$$

Hence $\int_\varepsilon^t \gamma(t-s)w(s)ds$ is uniformly bounded and converges almost everywhere in $(0, T]$ as $\varepsilon \to 0$. Since

$$\frac{d}{dt}\int_\varepsilon^t C(t-s)w(s)ds = u(t) - \int_\varepsilon^t \gamma(t-s)w(s)ds\,,$$

we get for $0<\varepsilon<t<t'$

$$\int_\varepsilon^{t'} C(t'-s)\,w(s)ds - \int_\varepsilon^t C(t-s)w(s)\,ds$$
$$= \int_t^{t'} w(\tau)d\tau - \int_t^{t'}\int_\varepsilon^\tau \gamma(\tau-s)w(s)ds\,d\tau\,.$$

Letting $\varepsilon \to 0$ we get

$$\int_0^{t'} C(t'-s)\,w(s)\,ds - \int_0^t C(t-s)w(s)\,ds$$
$$= \int_t^{t'} w(\tau)d\tau - \int_t^{t'}\int_0^\tau \gamma(\tau-s)w(s)ds\,d\tau\,.$$

Hence $\int_0^t C(t-s)w(s)ds$ is differentiable almost everywhere, and

$$\frac{d}{dt}\int_0^t C(t-s)w(s)ds = w(t) - \int_0^t \gamma(t-s)w(s)ds\,. \tag{3.10}$$

From (1.11) we get

$$u(t) + \int_0^t C(t-s)w(s)ds + \int_0^t (\rho*C)(t-s)w(s)\,ds = f(t)\,.$$

Consequently $u(t)$ is differentiable in $(0, T]$ and with the aid of (3.6), (3.10) we obtain

$$\frac{d}{dt}u(t) + w(t) - \int_0^t \gamma(t-s)w(s)ds - \beta(t)u_0$$
$$+(\rho*h)(t) - (\beta*\rho)(t)u_0 + \frac{d}{dt}(\beta*u)(t)$$
$$= h(t) + \rho(t)u_0 + (h*\rho)(t)\,. \tag{3.11}$$

From (1.10) and (3.11) it follows that

$$\frac{d}{dt}\{u(t)+(\beta*u)(t)\}+w(t)-\int_0^t\gamma(t-s)w(s)ds$$
$$=h(t)+\beta(t)u_0+(\beta*\rho)(t)u_0+\rho(t)u_0=h(t)\,.$$

Thus the proof that (i) implies (ii) is complete.

Conversely, suppose that (ii) holds. Integrating (0.7) from ε to t

$$u(t)+(\beta*u)(t)-u(\varepsilon)-(\beta*u)(\varepsilon)+\int_\varepsilon^t w(\tau)\,d\tau$$
$$-\int_\varepsilon^t\int_0^\tau\gamma(\tau-s)\,w(s)ds\,d\tau=\int_\varepsilon^t h(\tau)d\tau\,. \tag{3.12}$$

Hence $\int_\varepsilon^t w(\tau)d\tau$ is uniformly bounded and converges as $\varepsilon\to 0$. Since

$$\frac{d}{dt}\int_\varepsilon^t C(t-s)w(s)ds=w(t)-\int_\varepsilon^t\gamma(t-s)w(s)ds\,,$$

we obtain

$$\int_\varepsilon^t C(t-s)w(s)ds=\int_\varepsilon^t\{w(\tau)-\int_\varepsilon^\tau\gamma(\tau-s)w(s)ds\}\,d\tau\,. \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$u(t)+(\beta*u)(t)-u(\varepsilon)-(\beta*u)(\varepsilon)+\int_\varepsilon^t C(t-s)w(s)ds$$
$$-\int_\varepsilon^t\int_0^\varepsilon\gamma(\tau-s)w(s)ds\,d\tau=\int_\varepsilon^t h(\tau)d\tau\,. \tag{3.14}$$

Since

$$\int_0^\varepsilon\gamma(\tau-s)w(s)\,ds=\int_0^\tau\gamma(\tau-s)w(s)ds-\int_\varepsilon^\tau\gamma(\tau-s)w(s)ds$$

is uniformly bounded and tends to 0 as $\varepsilon\to 0$, we see from (3.14) that $\int_\varepsilon^t C(t-s)\,w(s)ds$ is also uniformly bounded and converges as $\varepsilon\to 0$, and hence the same holds for $\int_\varepsilon^t(\rho*C)(t-s)w(s)ds$. From (3.14) we get

$$\int_\varepsilon^t\rho(t-\tau)u(\tau)d\tau+\int_\varepsilon^t\rho(t-\tau)(\beta*u)(\tau)d\tau-\int_\varepsilon^t\rho(t-\tau)d\tau\{u(\varepsilon)+(\rho*u)(\varepsilon)\}$$
$$+\int_\varepsilon^t\rho(t-\tau)\int_\varepsilon^\tau C(\tau-s)w(s)ds\,d\tau-\int_\varepsilon^t\rho(t-\tau)\int_\varepsilon^\tau\int_0^\varepsilon\gamma(\sigma-s)w(s)ds\,d\sigma\,d\tau$$
$$=\int_\varepsilon^t\rho(t-\tau)\int_\varepsilon^\tau h(\sigma)d\sigma\,d\tau\,.$$

Letting $\varepsilon\to 0$ in the above

$$(\rho*u)(t)+(\rho*\beta*u)(t)-\int_0^t \rho(t-\tau)d\tau u_0$$
$$+\int_0^t (\rho*C)(t-s)w(s)ds = \int_0^t \rho(t-\tau)\int_0^\tau h(\sigma)d\sigma\, d\tau\,. \tag{3.15}$$

It follows from (1.10),

$$-(\beta*u)(t)+\int_0^t (\rho*C)(t-s)w(s)ds = (\rho*G)(t)\,. \tag{3.16}$$

Letting $\varepsilon \to 0$ in (3.14) we obtain

$$u(t)+(\beta*u)(t)-u_0+\int_0^t C(t-s)w(s)\,ds = \int_0^t h(\tau)d\tau\,.$$

Combining this with (3.16) we see that (1.11) holds.

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