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# SINGULAR VARIATION OF DOMAIN AND SPECTRA OF THE LAPLACIAN WITH SMALL ROBIN CONDITIONAL BOUNDARY I.

Dedicated to Professor M.M. Schiffer on his 80th birthday

SHIN OZAWA

(Received October 23, 1991)

## 1. Introduction

In this paper the author considers the following problem.

Let $\Omega$ be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary $\partial\Omega$. Let $\tilde{w}$ be a fixed point in $\Omega$. Let $B(\varepsilon, \tilde{w})$ be the ball of radius $\varepsilon$ with the center $\tilde{w}$. We put $\Omega_\varepsilon=\Omega\backslash\overline{B(\varepsilon, \tilde{w})}$. Consider the following eigenvalue problem

$$
\begin{aligned}
(1.1)\qquad -\Delta u(x) &= \lambda u(x) && x\in\Omega_\varepsilon \\
u(x) &= 0 && x\in\partial\Omega \\
u(x)+k\,\varepsilon^\sigma \frac{\partial u}{\partial\nu_x}(x) &= 0 && x\in\partial B_\varepsilon\,.
\end{aligned}
$$

Here $k$ denotes the positive constant. And $\sigma$ is a non negative constant. Here $\dfrac{\partial}{\partial\nu_x}$ denotes the derivative along the exterior normal direction with respect to $\Omega_\varepsilon$.

Let $\mu_j(\varepsilon)>0$ be the $j$-th eigenvalue of (1.1). Let $\mu_j$ be the $j$-th eigenvalue of the problem

$$
\begin{aligned}
(1.2)\qquad -\Delta u(x) &= \lambda u(x) && x\in\Omega \\
u(x) &= 0 && x\in\partial\Omega\,.
\end{aligned}
$$

Let $G(x,y)$ be the Green function of the Laplacian in $\Omega$ with the Dirichlet boundary condition on $\partial\Omega$ satisfying $-\Delta G(x,y)=\delta(x-y)$.

Main aim of this paper is to show the following Theorem 1. Let $\varphi_j(x)$ be the $L^2$ normalized eigenfunction associated with $\mu_j$.

**Theorem 1.** *Fix $\sigma\in(0,1)$. Fix $j$. Assume that $\mu_j$ is a simple eigenvalue. Then,*

$$
\begin{aligned}
(1.3)\qquad \mu_j(\varepsilon)-\mu_j &= 2\pi\,k^{-1}\,\varepsilon^{1-\sigma}\,\varphi_j(\tilde{w})^2 \\
&\quad +O(\varepsilon^{2-2\sigma}(\log\varepsilon)^2)\,.
\end{aligned}
$$

*when* $\sigma=0$, *we have*
$(1.3)^{\text{bis}}$ *the remainder of* $(1.3)=0(\varepsilon^{1+\beta})$
*for any* $\beta\in(0, 1)$.

REMARK. The related topics are discussed in Ozawa [9], [10], [11], Besson [2], Courtois [5], Chavel-Feldman [3] and the references in the above papers. It should be noticed that the difference between $\mu_j(\varepsilon)$ and $\mu_j$ is of order $\varepsilon^{1-\sigma}$ (when $\sigma\geq 0$) which is quite different from the case of eigenvalue problem on $\Omega_\varepsilon$ under the Neumann condition on $\partial B_\varepsilon$. In the Neumann case, $\varepsilon^2$ is the order of the difference between $\mu_j(\varepsilon)$ and $\mu_j$.

The other case $\sigma\in\boldsymbol{R}\backslash[0, 1)$ will be treated in part II of the present paper, since we need some change of our method of proof.

Let us notice the related papers on eigenvalues with many small randomly distributed Dirichlet holes. See Ozawa [12], [13], Kac [7], Rauch-Taylor [14], Simon [16], Sznitman [17] and the references of the above papers. It is very interesitng for the author to consider eigenvalue problem of the Laplacian in $\Omega\backslash\overline{\text{many holes}}$ under the Robin condition on the boundaries of holes. Problem of the solution of the Poisson operator with periodically distributed small holes with the Robin condition is discussed in Kaizu [8]. We want to consider statistical problem of eigenvalues of the Laplacian in a domain with randomly distributed Robin holes in the future. In my opinion this paper can be a step for the above problem.

For other related problems on singular variation of domains the readers may be referred to Anné [1], Jimbo [6].

Here the author expresses his hearty thanks to Professor M. M. Schiffer, since my idea of proof of this paper using the Green function was influenced by the fine book Schiffer-Spencer [15]. And the author expresses his sincere thanks to Mr. Roppongi who read this manuscript and gave valuable comments.

## 2. Outline of proof of Theorem 1

We introduce the following kernel $p_\varepsilon(x, y)$.

$$(2.1)\qquad p_\varepsilon(x, y) = G(x, y)+g(\varepsilon)\, G(x, \tilde{w})\, G(\tilde{w}, y) \\ +h(\varepsilon)\,\langle\nabla_w G(x, \tilde{w}),\ \nabla_w G(\tilde{w}, y)\rangle\,,$$

where $\langle\nabla_w u(\tilde{w}),\ \nabla_w v(\tilde{w})\rangle=\sum_{j=1}^{2}\frac{\partial u}{\partial w_j}\frac{\partial v}{\partial w_j}\big|_{w=\tilde{w}}$, when $w=(w_1, w_2)$ is an orthonomal frame of $\boldsymbol{R}^2$. Here $g(\varepsilon)$, $h(\varepsilon)$ are determined so that

$$(2.2)\qquad p_\varepsilon(x, y)+k\,\varepsilon^\sigma\frac{\partial}{\partial\nu_x}p_\varepsilon(x, y)\quad x\in\partial\Omega_\varepsilon\backslash\partial\Omega$$

is small in some sense.

If we put

$$g(\varepsilon) = -(\gamma-(2\pi)^{-1}\log\varepsilon+k(2\pi)^{-1}\varepsilon^{\sigma-1})^{-1} \tag{2.3}$$

and

$$h(\varepsilon)\,((2\pi\,\varepsilon)^{-1}+(2\pi)^{-1}k\,\varepsilon^{\sigma-2}) = k\,\varepsilon^{\sigma} \tag{2.4}$$

the above aim for (2.2) to be small is attained. Here

$$\gamma = \lim_{x\to\tilde{w}}(G(x,\tilde{w})+(2\pi)^{-1}\log|x-\tilde{w}|)\,.$$

Let $G_\varepsilon(x,y)$ be the Green function of the Laplacian in $\Omega_\varepsilon$ associated with the boundary condition (1.1).

We put

$$(\boldsymbol{G}f)(x) = \int_\Omega G(x,y)f(y)\,dy$$

$$(\boldsymbol{G}_\varepsilon f)(x) = \int_{\Omega_\varepsilon} G_\varepsilon(x,y)f(y)\,dy$$

and

$$(\boldsymbol{P}_\varepsilon f)(x) = \int_{\Omega_\varepsilon} p_\varepsilon(x,y)f(y)\,dy\,.$$

Let $T$ and $T_\varepsilon$ be operators on $\Omega$ and $\Omega_\varepsilon$, respectively. Then, $||T||_p$, $||T_\varepsilon||_{p,\varepsilon}$ denotes the operator norm on $L^p(\Omega)$, $L^p(\Omega_\varepsilon)$, respectively. Let $f$ and $g_\varepsilon$ be functions on $\Omega$ and $\Omega_\varepsilon$, respectively. Then, $||f||_p$, $||g_\varepsilon||_{p,\varepsilon}$ denotes the norm on $L^p(\Omega)$, $L^p(\Omega_\varepsilon)$, respectively.

A crucial part of our proof of Theorem 1 is the following.

**Theorem 2.** *Fix $\sigma\in(0,1)$, $q>2\sigma^{-1}$. Then, there exists a constant C such that*

$$||\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon||_{q,\varepsilon}\le C\,\varepsilon^{2-\sigma} \tag{2.5}$$

*holds.*

The case $\sigma=0$ is treated in Theorem 7.

By the duality argument we get $||\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon||_{q',\varepsilon}\le C\,\varepsilon^{2-\sigma}$ for $q'$ satisfying $(1/q)+(1/q')=1$. By the Riesz-Thorin interpolation theorem we have the following.

**Theorem $\tilde{2}$.** *Under the same assumption as in Theorem 2, we have*

$$||\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon||_{2,\varepsilon}\le C\,\varepsilon^{2-\sigma}\,.$$

We put

$$\tilde{p}_\varepsilon(x, y) = G(x, y) + g(\varepsilon)\, G(x, \tilde{w})\, G(\tilde{w}, y) \\ + h(\varepsilon)\, \xi_\varepsilon(x) \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle\, \xi_\varepsilon(y)$$

for the characteristic function $\xi_\varepsilon(x)$ of $\Omega_\varepsilon$.

And we put

$$\tilde{\boldsymbol{P}}_\varepsilon f(x) = \int_\Omega \tilde{p}_\varepsilon(x, y) f(y)\, dy\,.$$

It should be noticed that the characteristic function $\xi_\varepsilon$ appears in $\tilde{p}_\varepsilon(x, y)$.

We compare $\boldsymbol{P}_\varepsilon$ with $\tilde{\boldsymbol{P}}_\varepsilon$ and we can get an information of $\boldsymbol{P}_\varepsilon$ from $\tilde{\boldsymbol{P}}_\varepsilon$, because the difference between $\boldsymbol{P}_\varepsilon$ and $\tilde{\boldsymbol{P}}_\varepsilon$ is small in some sense. Since $\boldsymbol{G}_\varepsilon$ is approximated by $\boldsymbol{P}_\varepsilon$, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G} \to \tilde{\boldsymbol{P}}_\varepsilon$. This is our outline of our proof of Theorem 1.

## 3. Preliminary Lemmas

Fix $0 \leq \sigma < 1$. We write $B(w; \varepsilon) = B_\varepsilon$.

**Lemma 3.1.** *Fix $M \in C^\infty(\partial B_\varepsilon)$. Then, the solution of*

$$(3.1) \qquad \begin{aligned} &\Delta u(x) = 0 \quad x \in \Omega \backslash \bar{B}_\varepsilon \\ &u(x) = 0 \quad x \in \partial\Omega \\ &u(x) + k\, \varepsilon^\sigma \frac{\partial u}{\partial \nu_x}(x) = M(\theta) \quad x = (\tilde{w}_1 + \varepsilon \cos\theta,\ \tilde{w}_2 + \varepsilon \sin\theta) \end{aligned}$$

*satisfies*

$$(3.2) \qquad |u(x)| \leq C \operatorname*{Max}_\theta |M(\theta)| (\varepsilon^{1-\sigma} k^{-1} |\log r| + R(\varepsilon, \sigma, r))\,,$$

*where*

$$R(\varepsilon, \sigma, r) = \Big( \sum_{j=1}^\infty k^{-2} j^{-2} \varepsilon^{2j+2-2\sigma} r^{-2j} \Big)^{1/2}.$$

Proof. We put

$$\tilde{u}(x) = a_0 \log r + \sum_{j=1}^\infty (b_j \sin j\theta + c_j \cos j\theta)\, (-j)^{-1} r^{-j}$$

Then, it satisfies $\Delta \tilde{u}(x) = 0$ for $x \in R^2 \backslash \bar{B}_\varepsilon$. We see that

$$\begin{aligned} \tilde{u}(x) + k\, \varepsilon^\sigma \frac{\partial \tilde{u}}{\partial \nu_x}(x) \Big|_{x \in \partial B_\varepsilon} &= M(\theta) \\ &\equiv s_0 + \sum_{j=1}^\infty (s_j \sin j\theta + t_j \cos j\theta) \end{aligned}$$

implies

$$a_0(\log \varepsilon - k\,\varepsilon^{\sigma-1}) = s_0$$
$$b_j\,\varepsilon^{-j}(-(1/j) - k\,\varepsilon^{\sigma-1}) = s_j$$
$$c_j\,\varepsilon^{-j}(-(1/j) - k\,\varepsilon^{\sigma-1}) = t_j$$

for $j \geq 1$. Thus,

$$\begin{aligned}(3.3)\qquad |\tilde{u}(x)| &\leq |s_0 \log r| / |k\,\varepsilon^{\sigma-1} + \log \varepsilon| \\ &+ \Big(\sum_{j=1}^{\infty} j^{-2}\,\varepsilon^{2j}\,r^{-2j}(-(1/j) - k\,\varepsilon^{\sigma-1})^{-2}\Big)^{1/2} \times \Big(\sum_{j=1}^{\infty}(s_j^2 + t_j^2)\Big)^{1/2}.\end{aligned}$$

Since $\sigma < 1$, the right hand side of (3.3) does not exceed

$$C \text{ Max } |M(\theta)|((\varepsilon^{1-\sigma}\,|\log r|/k) + (\varepsilon^{1-\sigma}/k)\,(\sum_{j=1}^{\infty} j^{-2}\,\varepsilon^{2j}\,r^{-2j})^{1/2})$$

from

$$\begin{aligned}s_0^2 + \sum_{j=1}^{\infty}(s_j^2 + t_j^2) &\leq C \int_0^{2\pi} M(\theta)^2\,d\theta \\ &\leq C' \text{ Max } M(\theta)^2.\end{aligned}$$

Thus, $\tilde{u}(x)$ satisfies the first and the third conditions of (3.1). We see that

$$\max_{x \in \partial\Omega} |\tilde{u}(x)| = 0(\varepsilon^{1-\sigma}).$$

We can get the solution $u(x)$ of (3.1) by the same repeating construction of the functions $v_\varepsilon^{(n)}$ in Proposition 1 of Ozawa [10]. That solution satisfies (3.2).

**Lemma 3.2.** *Fix* $q \in (1, \infty)$. *Under the same assumption as in Lemma* 3.1 *we have*

$$||u||_{q,\varepsilon} \leq C_q(\varepsilon^{1-\sigma} \text{ Max} |M(\theta)|).$$

Proof. The second term in the right hand side of (3.2) is a bounded function for $r \geq \varepsilon$. Therefore, we get the desired result.

## 4. Proof of Theorem 2

We recall that $w = (w_1, w_2)$. Assume that $\tilde{w} = (0, 0)$.

We put

$$S(x, y) = G(x, y) + (1/2\pi) \log |x - y|.$$

Then, $S(x, y) \in C^\infty(\Omega \times \Omega)$. We have the following formulas (4.1), (4.2) in p. 263 of Ozawa [10].

$$\begin{aligned}(4.1)\qquad &\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle \\ &= (2\pi\,\varepsilon)^{-1} \frac{\partial}{\partial w_1} G(\tilde{w}, y) + \langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\end{aligned}$$

for $x=(\varepsilon, 0)$, $\tilde{w}=0$,

$$(4.2)\qquad \frac{\partial}{\partial x_1}\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle = \frac{\partial}{\partial x_1}\langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle$$
$$-(2\pi)^{-1}\varepsilon^{-2}\frac{\partial}{\partial w_1} G(\tilde{w}, y)$$

for $x=(\varepsilon, 0)$, $\tilde{w}=0$.

We put $p_\varepsilon(x, y)$ as before. Then, we have

$$(4.3)\qquad p_\varepsilon(x, y) - k\,\varepsilon^\sigma \frac{\partial}{\partial x_1} p_\varepsilon(x, y)\,|_{x=(\varepsilon,0)} = \sum_{j=1}^{10} L_j\,,$$

where

$$\begin{aligned}
L_1 &= G(x, y)\\
L_2 &= -(1/2\pi)(\log \varepsilon)\, g(\varepsilon)\, G(\tilde{w}, y)\\
L_3 &= g(\varepsilon)\, S(x, \tilde{w})\, G(\tilde{w}, y)\\
L_4 &= h(\varepsilon)(2\pi\,\varepsilon)^{-1}\frac{\partial}{\partial w_1} G(\tilde{w}, y)\,|_{w=\tilde{w}}\\
L_5 &= h(\varepsilon)\langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\\
L_6 &= -k\,\varepsilon^\sigma \frac{\partial}{\partial x_1} G(x, y)\,|_{x=(\varepsilon,0)}\\
L_7 &= k\,\varepsilon^{\sigma-1} g(\varepsilon)(2\pi)^{-1} G(\tilde{w}, y)\\
L_8 &= -k\,\varepsilon^\sigma \frac{\partial}{\partial x_1} S(x, \tilde{w})\,|_{x=(\varepsilon,0)}\, G(\tilde{w}, y)\, g(\varepsilon)\\
L_9 &= -k\,\varepsilon^\sigma h(\varepsilon)\left(-(2\pi)^{-1}\varepsilon^{-2}\frac{\partial}{\partial w_1} G(\tilde{w}, y)\right)\\
L_{10} &= -k\,\varepsilon^\sigma h(\varepsilon)\frac{\partial}{\partial x_1}\langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle\,.
\end{aligned}$$

Let $S(\tilde{w}, \tilde{w})=\gamma$. Then $S(x, \tilde{w})-S(\tilde{w}, \tilde{w})=0(\varepsilon)$ as $\varepsilon\to 0$.

We put

$$(4.4)\qquad g(\varepsilon)\left(-(2\pi)^{-1}\log \varepsilon+\gamma+0(\varepsilon)+(k/2\pi)\,\varepsilon^{\sigma-1}+0\,(\varepsilon^\sigma|\log \varepsilon|)\right) = -1\,.$$

Then, $L_1+L_2+L_3+L_7+L_8$ is equal to

$$(4.5)\qquad G(x, y)-G(w, y)\,.$$

Here $0(\varepsilon)$, $0(\varepsilon^\sigma|\log \varepsilon|)$ arises from $L_3$, $L_8$, respectively.

We put

$$(4.6)\qquad h(\varepsilon)\left((2\pi\,\varepsilon)^{-1}+k(2\pi)^{-1}\varepsilon^{\sigma-2}\right) = k\,\varepsilon^\sigma$$

Then,

$$(4.7) \qquad L_4+L_9=k\,\varepsilon^{\sigma}\frac{\partial}{\partial w_1}G(w,y)\,|_{w=\tilde{w}}\,.$$

Therefore, (4.3) is equal to

$$G(x,y)-G(\tilde{w},y)+k\,\varepsilon^{\sigma}\Big(\frac{\partial}{\partial w_1}G(w,y)\,|_{w=\tilde{w}}-\frac{\partial}{\partial x_1}G(x,y)\Big)$$
$$+L_5+L_{10}$$

for $x=(\varepsilon,0)$.

Let $\boldsymbol{G}_w$ denote the operator $v(x)\to(\boldsymbol{G}\,v)\,(\tilde{w})$. And $G(\cdot,w)$ denotes the multiplication operator $u(x)\to G(x,w)\,u(x)$.

Using the above facts we get (4.8) for $\tilde{f}$ which is zero on $B_\varepsilon$.

$$(4.8) \qquad \begin{aligned} &\boldsymbol{P}_\varepsilon\tilde{f}(x)-k\,\varepsilon^{\sigma}\frac{\partial}{\partial x_1}(\boldsymbol{P}_\varepsilon\tilde{f})\,(x)\,|_{x=(\varepsilon,0)}\\ &=(\boldsymbol{G}\tilde{f})\,(x)-(\boldsymbol{G}\tilde{f})\,(\tilde{w})\\ &\quad+k\,\varepsilon^{\sigma}\Big(\frac{\partial}{\partial w_1}(\boldsymbol{G}\tilde{f})\,(\tilde{w})-\frac{\partial}{\partial x_1}(\boldsymbol{G}\tilde{f})\,(x)\Big)\\ &\quad+h(\varepsilon)\,\langle\nabla_w\,S(x,\tilde{w}),\,\nabla_w(\boldsymbol{G}_w\tilde{f})\rangle\\ &\quad-k\,\varepsilon^{\sigma}\,h(\varepsilon)\,\frac{\partial}{\partial x_1}\langle\nabla_w\,S(x,\tilde{w}),\,\nabla_w(\boldsymbol{G}_w\tilde{f})\rangle \end{aligned}$$

We know that

$$\begin{aligned} g(\varepsilon)&=-(2\pi/k)\,\varepsilon^{1-\sigma}+0(\varepsilon^{2-2\sigma}|\log\varepsilon|)\\ h(\varepsilon)&=2\pi\,\varepsilon^2+0(\varepsilon^{3-\sigma})\,, \end{aligned}$$

We want to estimate (4.8). It is easy to show that

$$\begin{aligned} |\boldsymbol{G}f(x)-\boldsymbol{G}f(\tilde{w})|_{x=(\varepsilon,0)}&\le C\varepsilon\,||\boldsymbol{G}\tilde{f}||_{C^1(\Omega)}\\ &\le C'\varepsilon\,||\tilde{f}||_{p,\varepsilon} \end{aligned}$$

for $p>2$. We see that the sum of the third and the fourth term in the right hand side of (4.8) does not exceed

$$Ck\,\varepsilon^{\sigma+\tau}\,||\boldsymbol{G}\tilde{f}||_{C^{1+\tau}(\Omega)}\le C'k\varepsilon^{\sigma+\tau}\,||\tilde{f}||_{p,\varepsilon}$$

for $p>2\,(1-\tau)^{-1}$. The fifth term and the sixth term in the right hand side of (4.8) does not exceed $Ch(\varepsilon)\,||\tilde{f}||_{p,\varepsilon}$ $Ck\,\varepsilon^{\sigma}\,h(\varepsilon)\,||\tilde{f}||_{p,\varepsilon}$, respectively, for $p>2$.

We put $(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)f=v$. Then, $v$ satisfies (3.1) and $M(\theta)=(4.8)$, because $\boldsymbol{G}_\varepsilon f$ satisfies the given Robin condition on $\partial B_\varepsilon$. By Lemma 3.2 we have

$$||v||_{q,\varepsilon}\le C\,\varepsilon^{1-\sigma}(\varepsilon+\varepsilon^{\sigma+\tau})\,||\tilde{f}||_{q,\varepsilon}$$

for $q>2\,(1-\tau)^{-1}$, $\tau\in(0,1)$. Therefore,

$$||P_\varepsilon - G_\varepsilon||_{q,\varepsilon} \leq C(\varepsilon^{2-\sigma} + \varepsilon^{1+\tau})$$

We take $0<\tau<1$ so that $1+\tau=2-\sigma$. Then, we get Theorem 2 for $q>2\sigma^{-1}$.

## 5. Estimate of the difference between $P_\varepsilon$ and $\tilde{P}_\varepsilon$

We want to estimate

$$\text{(5.1)} \qquad ||\xi_\varepsilon \tilde{P}_\varepsilon \xi_\varepsilon - \tilde{P}_\varepsilon||_p .$$

It does not exceed $||(1-\xi_\varepsilon)\tilde{P}_\varepsilon \xi_\varepsilon||_p + ||\tilde{P}_\varepsilon(1-\xi_\varepsilon)||_p$. We want to estimate the first term of the above sum.

Fix $f \in L^p(\Omega)$.

$$\text{(5.2)} \qquad ||(1-\xi_\varepsilon)\tilde{P}_\varepsilon \xi_\varepsilon f||_p \leq C(|B_\varepsilon|^{1/p} \max_{x\in\bar{B}_\varepsilon} |G_x(\xi_\varepsilon f)| + g(\varepsilon)\left(\int_{B_\varepsilon} G(x,\tilde{w})^p\, dx\right)^{1/p} |G_w(\xi_\varepsilon f)|)$$

for $p>1$ observing the fact that $(1-\xi_\varepsilon)\xi_\varepsilon=0$ in $h(\varepsilon)$-term. Therefore, we get

$$\begin{aligned}\text{(5.2)} \ &\leq C(\varepsilon^{2/p} + g(\varepsilon)\,\varepsilon^{2/p}(\log \varepsilon))\,||f||_p \\ &\leq C'\,\varepsilon^{2/p}\,||f||_p\end{aligned}$$

for $p>1$. Then, for any $p>1$,

$$\text{(5.3)} \qquad ||(1-\xi_\varepsilon)\tilde{P}_\varepsilon \xi_\varepsilon||_p \leq C'\,\varepsilon^{2/p} .$$

Moreover, $||(1-\xi_\varepsilon)\tilde{P}_\varepsilon||_p$ has the same bound in (5.3).

We have the duality

$$((1-\xi_\varepsilon)\tilde{P}_\varepsilon)^* = \tilde{P}_\varepsilon(1-\xi_\varepsilon) .$$

Therefore,

$$\text{(5.4)} \qquad ||\tilde{P}_\varepsilon(1-\xi_\varepsilon)||_{p'} \leq C'\,\varepsilon^{2/p} .$$

As a corollary of the above facts we get the following.

**Theorem 3.** *There exists a constant $C$ independent of $\varepsilon$ such that*

$$\text{(5.5)} \qquad ||\tilde{P}_\varepsilon - \xi_\varepsilon \tilde{P}_\varepsilon \xi_\varepsilon||_2 \leq C\,\varepsilon .$$

We here want to prove the following.

**Theorem 4.** *There exists a constant $C$ such that*

$$\begin{aligned}||\tilde{P}_\varepsilon - G||_2 &\leq C(|g(\varepsilon)| + |\log \varepsilon|\,|h(\varepsilon)|) \\ &\leq C\,|g(\varepsilon)|\end{aligned}$$

*holds.*

Proof of Theorem 4.

We put

$$A_1 f(x) = G(x, \tilde{w})\, \boldsymbol{G}_w f$$
$$A_2 f(x) = \xi_\varepsilon(x) \langle \nabla_w G(x, \tilde{w}), \nabla_w \boldsymbol{G}_w(\xi_\varepsilon f)\rangle .$$

Then, $\tilde{\boldsymbol{P}}_\varepsilon = \boldsymbol{G}_\varepsilon + g(\varepsilon) A_1 + h(\varepsilon) A_2$.

We have the following.

(5.6) $$\|A_1\|_p \leq C$$

for any $p \in (1, \infty)$. And

(5.7) $$\|A_2\|_2 \leq C\, |\log \varepsilon| .$$

This is observed by

$$\|A_2 f\|_2 \leq \Big(\int_{\Omega_\varepsilon} |\nabla_w G(x, \tilde{w})|^2\, dx\Big)^{1/2} |\nabla_w \boldsymbol{G}_w(\xi_\varepsilon f)| .$$

Now we get the desired result.

## 6. Convergence of eigenvalues

Notice that the $j$-th eigenvalue of $\boldsymbol{P}_\varepsilon$ is equal to the $j$-th eigenvalue of $\chi_\varepsilon \tilde{\boldsymbol{P}}_\varepsilon \chi_\varepsilon$. By virtue of Theorems $\tilde{2}$, 3, 4 we see that there exists a constnat $C$ independent of $j$ such that

(6.1)
$$\begin{aligned} &|\mu_j(\varepsilon)^{-1} - \mu_j^{-1}| \\ &\leq C(\varepsilon^{2-\sigma} + \varepsilon + |g(\varepsilon)| + |\log \varepsilon|\, |h(\varepsilon)|) \\ &\leq C\, \varepsilon^{1-\sigma} \end{aligned}$$

holds.

We need more precise estimate for the left hand side of (6.1) to get Theorem 1. By (6.1) we know that the multiplicity of $\mu_j(\varepsilon)$ is one for small $\varepsilon$ when the multiplicity of $\mu_j$ is one.

## 7. Perturbation theory for $\tilde{\boldsymbol{P}}_\varepsilon$

In this section we consider the behaviour of eigenvalues of $\tilde{\boldsymbol{P}}_\varepsilon$ as $\varepsilon$ tends to 0. We set $A_0 = \boldsymbol{G}$ and $A_1$, $A_2$ as mentioned before.

For the present we discuss a formal treatment of perturbation theory for eigenvalues. We put

$$A(\varepsilon) = A_0 + g(\varepsilon) A_1 + h(\varepsilon) A_2$$

$$\lambda(\varepsilon) = \lambda_0 + g(\varepsilon)\lambda_1 + h(\varepsilon)\lambda_2$$
$$\psi(\varepsilon) = \psi_0 + g(\varepsilon)\psi_1 + h(\varepsilon)\psi_2$$

so that $\lambda(\varepsilon)$ and $\psi(\varepsilon)$ is an approximate eigenvalue of $A(\varepsilon)$ and an approximate eigenfunction of $A(\varepsilon)$, respectively. We consider the following equations:

$$(A(\varepsilon)-\lambda(\varepsilon))\psi(\varepsilon) = o \text{ (small term)}, \tag{7.1}$$

where the meaning of $o$(small term) is not specified here. We set

$$\|\psi_0\|_2 = 1, \quad (\psi_0, \psi_j) = 0 \quad (j = 1, 2),$$

where ( , ) denotes the inner porduct on $L^2(\Omega)$. Here $\varepsilon \to \lambda(\varepsilon)$ is thought as a perturbation family. To get (7.1) we examine the folloiwng equations:

$$(A_0-\lambda_0)\psi_0 = 0 \tag{7.2}$$

$$(A-\lambda_0)\psi_1 = (\lambda_1 - A_1)\psi_0 \tag{7.3}$$

$$(A_0-\lambda_0)\psi_2 = (\lambda_2-A_2)\psi_0 + (\lambda_1-A_1)\psi_1 . \tag{7.4}$$

By the Fredholm alternative theory we see that

$$\lambda_1(\varepsilon) = (A_1\psi_0, \psi_0)$$
$$\lambda_2(\varepsilon) = (A_2\psi_0, \psi_0) + (A_1\psi_1, \psi_0)$$

is the conditions to solve (7.2), (7.3), (7.4) when $\lambda_0$ has multiplicity one.

We see that

$$\begin{aligned}(A(\varepsilon)-\lambda(\varepsilon))\psi(\varepsilon) &= (g(\varepsilon)^2-h(\varepsilon))(A_1-\lambda_1)\psi_1 + h(\varepsilon)^2(A_2-\lambda_2)\psi_2 \\ &\quad + g(\varepsilon)h(\varepsilon)((A_1-\lambda_1)\psi_2 + (A_2-\lambda_2)\psi_1) .\end{aligned} \tag{7.5}$$

From now on we give a rigorous treatment of perturbation theory for eigenvalue of $\tilde{\boldsymbol{P}}_\varepsilon$. Let $\mu_j$ and $\varphi_j$ be as in Theorem 1. Thus, $\mu_j$ is a simple eigenvalue. We see that

$$\lambda_1(\varepsilon) = |\boldsymbol{G}_w\psi_0|^2 = \mu_j^{-2}\varphi_j(\tilde{w})^2 \tag{7.6}$$

and

$$\lambda_2(\varepsilon) = \langle \nabla_w \boldsymbol{G}_w(\xi_\varepsilon\psi_0), \nabla_w \boldsymbol{G}_w(\xi_\varepsilon\psi_0)\rangle|_{w=\tilde{w}} + \boldsymbol{G}_w\psi_1 \cdot \boldsymbol{G}_w\psi_0 . \tag{7.7}$$

Then,

$$|\lambda_2(\varepsilon)| \le C|\log\varepsilon| . \tag{7.8}$$

By the Fredholm theory we see that

$$\|\psi_1\|_2 \le C\|(\lambda_1-A_1)\|_2\|\psi_0\|_2 \le C' \tag{7.9}$$

and

$$\begin{aligned}\|\psi_2\|_2 &\leq C\,\|(\lambda_2-A_2)\|_2\,\|\psi_0\|_2+C\,\|(\lambda_1-A_1)\|_2\,\|\psi_1\|_2 \\ &\leq C(1+|\log\varepsilon|)\,.\end{aligned} \tag{7.10}$$

Summing up (7.5), (7.8), (7.9) and (7.10) we have the following inequality.

$$|(7.5)|\leqq C'(g(\varepsilon)^2+h(\varepsilon)^2)\,(\log\varepsilon)^2\equiv S(\varepsilon)\,. \tag{7.11}$$

Therefore, we have the following.

**Theorem 5.** *There exists a constant $C$ independent of $\varepsilon$ such that*

$$\|(\tilde{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\,\psi(\varepsilon)\|_2\leq S(\varepsilon) \tag{7.12}$$

*holds.*

We put $(\Phi(\varepsilon))\,(x)=\xi_\varepsilon(x)\,(\psi(\xi))\,(x)$. Then, $(\boldsymbol{P}_\varepsilon-\lambda(\varepsilon))\,\Phi(\varepsilon)=(\tilde{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))$ $(\xi_\varepsilon\,\psi(\varepsilon))$ on $\Omega_\varepsilon$. Fix $\sigma\in[0, 1)$. Then, there exists a constant $C$ independent of $\varepsilon$ such that

$$\|(\boldsymbol{P}_\varepsilon-\lambda(\varepsilon))\,\Phi(\varepsilon)\|_{2,\varepsilon}\leq S(\varepsilon)+\|T(\varepsilon)\|_{2,\varepsilon}\,, \tag{7.13}$$

where

$$T(\varepsilon)=(\tilde{\boldsymbol{P}}_\varepsilon-\lambda(\varepsilon))\,(1-\xi_\varepsilon)\,\psi(\xi)\,.$$

## 8. On $\boldsymbol{T}(\varepsilon)$

We want to get an upper bound for $T(\varepsilon)$. We have

$$T(\varepsilon)=\sum_{h=1}^{7}T_h\,,$$

where

$$\begin{aligned}
T_1&=\boldsymbol{G}(1-\xi_\varepsilon)\,\varphi_j(x)\\
T_2&=g(\varepsilon)\,\boldsymbol{G}(1-\xi_\varepsilon)\,\psi_1\\
T_3&=h(\varepsilon)\,\boldsymbol{G}(1-\xi_\varepsilon)\,\psi_2\\
T_4&=g(\varepsilon)\,A_1(1-\xi_\varepsilon)\,\varphi_j\\
T_5&=g(\xi)^2\,A_1(1-\xi_\varepsilon)\,\psi_1\\
T_6&=g(\varepsilon)\,h(\varepsilon)\,A_1(1-\xi_\varepsilon)\,\psi_2\\
T_7&=h(\varepsilon)\,A_2(1-\xi_\varepsilon)\,\psi(\varepsilon)
\end{aligned}$$

on $\Omega_\varepsilon$, since $\lambda(\varepsilon)\,(1-\xi_\varepsilon)\,\psi(\varepsilon)=0$ on $\Omega_\varepsilon$.

We have

$$\|T_3+T_5+T_6+T_7\|_{2,\varepsilon} = 0(|h(\varepsilon)\log\varepsilon|+g(\varepsilon)^2+|g(\varepsilon)h(\varepsilon)|\,|\log\varepsilon|)\,.$$

We get

$$\|T_1\|_{2,\varepsilon}\leq|\Omega|^{1/2}\|T_1\|_{\infty,\varepsilon}\leq C\,\|(1-\xi_\varepsilon)\,\varphi_j\|_{p,\varepsilon} \leq C\,\varepsilon^{2/p}$$

for any $p>1$.
Also,

$$\|T_2\|_{2,\varepsilon}\leq Cg(\varepsilon)\,\|\psi_1\|_{L^p(B_\varepsilon)}$$

for $p>1$. Notice that

$$\psi_1=(-\lambda_0)^{-1}((\lambda_1-A_1)\,\psi_0-A_0\,\psi_1)\,.$$

Then,

$$\begin{aligned}\|\psi_1\|_{L^p(B_\varepsilon)}&\leq C(\|\psi_0\|_{L^p(B_\varepsilon)}+\|A_1\,\psi_0\|_{L^p(B_\varepsilon)}+\|A_0\,\psi_1\|_{L^p(B_\varepsilon)})\\ &\leq C(\varepsilon^{2/p}+(\int_{B_\varepsilon}G(x,w)^p\,dx)^{1/p})\\ &\leq C(\varepsilon^{2/p}\,|\log\varepsilon|)\,.\end{aligned}$$

Therefore, $\|T_2\|_{2,\varepsilon}=0(g(\varepsilon)\,\varepsilon^{2/p}\,|\log\varepsilon|)$ for any $p>1$. We have

$$\begin{aligned}\|T_4\|_{2,\varepsilon}&\leq\|G(\cdot,w)\|_{2,\varepsilon}\,\|G(1-\xi_\varepsilon)\,\varphi_j\|_\infty\,g(\varepsilon)\\ &\leq C\,\varepsilon^{2/p}\,g(\varepsilon)\,\varepsilon|\log\varepsilon|,\end{aligned}$$

for any $p>1$.

We take $p<1$ as close as 1 to get Theorem 6.

Summing up these facts we get the following.

**Theorem 6.** The estimate

$$\|T(\varepsilon)\|_{2,\varepsilon}=0(\varepsilon^{2-2\sigma})$$

*holds.*

## 9. Proof of Theorem 1 for $\sigma>0$

We recall the fact (6.1). This is given by Theorems $\tilde{2}$, 3, 4. To prove Theorem $\tilde{2}$ we used the fact that $\sigma\in(0,1)$. Now, we know by (7.13), Theorem 6 that there exists at least one eigenvalue $H(\varepsilon)$ of $\boldsymbol{P}_\varepsilon$ satisfying

$$|H(\varepsilon)-\lambda(\varepsilon)|\leq S(\varepsilon)+C\,\varepsilon^{2-2\sigma}\,.$$

Here we used the fact that $\|\Phi(\varepsilon)\|_{2,\varepsilon}\in(1/2,2)$ for small $\varepsilon$. Since $H(\varepsilon)$ tends to $\mu_j^{-1}$ as $\varepsilon \to 0$, it must be the $j$-th eigenvalue of $\boldsymbol{P}_\varepsilon$. Combine with Theorem 2 and the above fact. Then we get

$$|\mu_j(\varepsilon)^{-1}-(\mu_j^{-1}+g(\varepsilon)\,\mu_j^{-2}\,\varphi_j(\tilde{w})^2)| \\ \leq C(S(\varepsilon)+\varepsilon^{2-2\sigma}).$$

Therefore, we have

$$|\mu_j(\varepsilon)^{-1}-(\mu_j^{-1}-(2\pi/k)\,\mu_j^{-2}\,\varepsilon^{1-\sigma}\,\varphi_j(\tilde{w})^2)| \\ \leq C\,\varepsilon^{2-2\sigma}\,|\log \varepsilon|^2.$$

using an explicit representation of $\lambda(\varepsilon)$.

## 10. Proof of Theorem 1 for $\sigma=0$

Under the same assumption as in Lemma 3.1 we have, for $p\in(1, \infty)$

$$\|(\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon)\,\tilde{f}\|_{p,\varepsilon} \leq C\,\varepsilon \max_\theta\,|M(\theta)| \leq C\,\varepsilon \max_{x\in\partial B_\varepsilon}\,|(4.8)|.$$

The right hand side of the above formula does not exceed

$$C\,\varepsilon(\varepsilon\,\|\boldsymbol{G}\tilde{f}\|_{C^1(\Omega)}+\varepsilon^{\sigma+\beta}\,\|\boldsymbol{G}\tilde{f}\|_{C^{1+\beta}(\Omega)} \\ +|h(\varepsilon)|\,\varepsilon^{1-(2/p)}\,\|\tilde{f}\|_{p,\varepsilon})$$

for any finite $p>2(1-\beta)^{-1}$, $\beta\in(0, 1)$. Then, we can get the following using duality and interpolation argment.

**Theorem 7.** *Assume that* $\sigma=0$. *Fix* $\beta\in(0, 1)$. *Then, there exists a constant C independent of* $\varepsilon$ *such that*

$$\|\boldsymbol{P}_\varepsilon-\boldsymbol{G}_\varepsilon\|_{2,\varepsilon} \leq C\,\varepsilon^{1+\beta}$$

*holds.*

Summing up the above facts we get the desired Theorem 1 for $\sigma=0$.

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