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## ON ARTINIAN RINGS WHOSE INDECOMPOSABLE PROJECTIVES ARE DISTRIBUTIVE

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### 1. Introduction

A module  $L \neq 0$  is called local (or hollow) if  $L = L_1 + L_2$  implies  $L = L_1$  or  $L = L_2$ . Especially a noetherian module is local if and only if it has a unique maximal submodule.

A module  $M$  is called distributive if  $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$  for every submodules  $X, Y, Z$  in  $M$  (cf. [2]). It is clear that any sub- (or factor) module of a distributive module is distributive.

We call a ring  $R$  right locally distributive, right  $LD$  in abbreviation, if it is right artinian and every projective indecomposable right  $R$ -module is distributive. It is evident that every local right module over a right  $LD$ -ring is distributive. The class of right  $LD$ -rings is a generalization of the class of right serial rings.

In this note right  $LD$ -rings are studied, mainly to construct a number of right  $LD$ -algebras.

### 2. Right $LD$ -rings

The following lemma, shown by Fuller, is basic to study distributive modules over a semiperfect ring.

**Lemma 1.** *Let  $R$  be a semiperfect ring. The following conditions on a right  $R$ -module  $M$  are equivalent:*

- (1)  *$M$  is distributive.*
- (2) *For every primitive idempotent  $e$  of  $R$ , the set  $\{xeR \mid x \in M\}$  of all homomorphic images of  $eR$  in  $M$  is linearly ordered.*
- (3) *For every primitive idempotent  $e$  in  $R$ , the right  $eRe$ -module  $Me$  is uniserial.*

Proof. See Fuller [1].

**Theorem 2.** *The following conditions on a right artinian ring  $R$  are equivalent:*

- (1) Every projective indecomposable right  $R$ -module  $eR$  is distributive.  
 (2) For every primitive idempotents  $e$  and  $f$  of  $R$ ,  $eRf$  is a uniserial right  $fRf$ -module.  
 (3) Every submodule in a projective indecomposable right  $R$ -module  $eR$  is characteristic, and the lattice of two-sided ideals in  $R$  is distributive.

Proof. (1)  $\Leftrightarrow$  (2) is a special case of Lemma 1.

(1)  $\Rightarrow$  (3). Every submodule in  $eR$  is a sum of local submodules and every local submodule is characteristic in  $eR$  by Lemma 1. Hence every submodule in  $eR$  is characteristic.

Let  $\{e_i\}_{i=1}^n$  be a complete set of primitive idempotents, and let  $I, J, K$  be two-sided ideals in  $R$ . Then by the distributivity of  $e_iR$ ,

$$\begin{aligned} e_i(I \cap (J+K)) &= e_iI \cap (e_iJ + e_iK) \\ &= (e_iI \cap e_iJ) + (e_iI \cap e_iK) = e_i(I \cap J) + e_i(I \cap K). \end{aligned}$$

Summing up each side of the equations ( $i=1, \dots, n$ ), we have  $I \cap (J+K) = (I \cap J) + (I \cap K)$ .

(3)  $\Rightarrow$  (1). Let  $A$  be a submodule in a projective indecomposable submodule  $eR$ . Since  $A$  is characteristic in  $eR$ ,  $eReA = A$ . We notice that the two-sided ideal  $A' := RA = ReA$  satisfies the equation  $eA' = A$ .

If  $X, Y, Z$  are any submodules in  $eR$ , then

$$\begin{aligned} eX' \cap (eY' + eZ') &= e(X' \cap (Y' + Z')) \\ &= e((X' \cap Y') + (X' \cap Z')) = (eX' \cap eY') + (eX' \cap eZ'). \end{aligned}$$

Hence  $eR$  is distributive.

A right artinian ring is called right  $LD$  if it satisfies the equivalent conditions in Theorem 2.

### 3. Construction of right $LD$ -algebras

We begin with a general remark on modules. For a module  $M$  we denote by  $H(M)$  the inclusion-ordered set of all local submodules in  $M$ . A homomorphism  $f: M \rightarrow N$  of modules induces a correspondence:  $H(M) \rightarrow H(N)$ ,  $X \mapsto f(X)$ . This correspondence is not a mapping in general (the image of some local submodule  $\leq M$  by  $f$  may be 0). If  $M$  is a module of finite length and  $f$  is an epimorphism, then there is a natural surjection

$$(*) \quad \{X \in H(M) \mid X \not\leq \text{Ker}(f)\} \rightarrow H(N).$$

In fact, there exist  $X_1, \dots, X_n \in H(M)$  such that  $f^{-1}(Y) = X_1 + \dots + X_n$  for every  $Y \in H(N)$ , and  $f(X_i) = Y$  for some  $i \in \{1, \dots, n\}$ . Moreover if  $M$  is distributive,  $i$  is unique by Lemma 1, and (\*) is bijective.

In this section a method to construct some right *LD*-algebras is presented. We introduce some terminology.

Suppose  $C$  is a fixed set. A pair  $(P, t)$  of a set  $P$  and a mapping  $t: P \rightarrow C$  is called a  $C$ -set. When  $(P, t)$  and  $(P', t')$  are  $C$ -sets, a mapping  $f: P \rightarrow P'$  is called a  $C$ -set homomorphism if  $t = t'f$ . Moreover, in case that  $P$  and  $P'$  are posets,  $f$  is called a  $C$ -poset homomorphism if  $f$  is both a  $C$ -set homomorphism and a poset homomorphism.

A subset  $U$  of a poset  $P$  is said to be an upper part of  $P$  if  $x \in U, y \in P$  and  $x \leq y$  imply  $y \in U$ . In particular, when  $P$  is finite,  $U$  is an upper part of  $P$  if and only if it is of the form  $\{x \in P \mid x \geq p_1\} \cup \dots \cup \{x \in P \mid x \geq p_n\}$  ( $p_1, \dots, p_n \in P$ ).

DEFINITION. Let  $C$  be a set. A family of  $C$ -posets  $\{(P_1, t_1), \dots, (P_n, t_n)\}$  is called an admissible system (of  $C$ -posets) if it satisfies the following conditions ( $i=1, \dots, n$ ):

- (1) Every poset  $P_i$  has a unique maximal element  $m_i$ .
- (2)  $C = \{t_1(m_1), \dots, t_n(m_n)\}$  ( $t_i(m_i) \neq t_j(m_j)$  if  $i \neq j$ ).
- (3) For every  $c \in C$  the subposet  $\{x \in P_i \mid t_i(x) = c\}$  is linearly ordered.
- (4) For every  $a \in P_i$ , there exist  $j \in \{1, \dots, n\}$  and a  $C$ -poset homomorphism from an upper part of  $P_j$  to  $\{x \in P_i \mid x \leq a\}$ .

REMARK 1. Suppose that the conditions (1), (2), (3) of the above definition are satisfied and that  $f$  is a  $C$ -poset isomorphism from an upper part of  $P_j$  to  $\{x \in P_i \mid x \leq a\}$ . Then  $j$  is determined by  $t_j(m_j) = t_i f(m_j) = t_i(a)$ .

Let  $b_0$  be any element in  $P_j$  and  $b_0 \leq \dots \leq b_r = m_j$  be a chain with  $b_{k-1}$  maximal in  $\{x \in P_j \mid x \leq b_k\}$  ( $k \in \{1, \dots, r\}$ ). Then  $f(b_{k-1})$  is maximal in  $\{x \in P_i \mid x \leq f(b_k)\}$ . Since  $t_j(b_{k-1}) = t_i f(b_{k-1})$ ,  $f(b_{k-1})$  is unique in  $\{x \in P_i \mid x \leq f(b_k)\}$  by (3). Therefore  $f(b_0)$  is determined inductively, and the isomorphism in (4) of the above definition is unique.

REMARK 2. By a similar argument we can replace (4) with (4') If  $a \in P_i$  is maximal in  $P_i \setminus \{m_i\}$ , there exist  $j \in \{1, \dots, n\}$  and a  $C$ -poset isomorphism from an upper part of  $P_j$  to  $\{x \in P_i \mid x \leq a\}$ .

If  $R$  is a right *LD*-ring with the Jacobson radical  $J$ , and  $\{e_i\}_{i=1}^n$  is a basic set of primitive idempotents for  $R$ , then by the first remark of this section, the posets  $H(e_1R), \dots, H(e_nR)$  form an admissible system with the mapping

$$\text{top}(\ ) ( := (\ ) / (\ ) J ): H(e_iR) \rightarrow T(R) \quad (i \in \{1, \dots, n\}),$$

where  $T(R)$  denotes the set of all isomorphism class of simple  $R$ -modules.

**Theorem 3.** For any admissible system  $\{(P_i, t_i)\}_{i=1}^n$  of  $C$ -posets, there exists a right *LD*-ring  $R$  such that  $H(e_iR)$  is isomorphic to  $(P_i, t_i)$  ( $T(R)$  is identified with  $C$  by a bijection  $\beta$ : (the isomorphism class of  $\text{top}(e_iR)) \mapsto t_i(m_i)$ ), where  $\{e_i\}_{i=1}^n$  is a basic set of primitive idempotents for  $R$ .

Proof. Since the  $C$ -poset isomorphism of (4) in Definition is uniquely determined by an element  $a \in P_i$  (Remark 1), we denote the isomorphism by  $\bar{a}$ . Letting any element in  $P_i$  outside the domain of definition of  $\bar{a}$  correspond to no element, the isomorphism  $\bar{a}$  is extended to a correspondence:  $P_j \rightarrow P_i$ , which operates  $P_j$  on the left. This extension is so trivial that it is also denoted by  $\bar{a}$ .

For two correspondences  $\bar{a}_1: P_i \rightarrow P_j$  and  $\bar{a}_2: P_k \rightarrow P_h$  ( $a_1 \in P_j, a_2 \in P_h$ ), we define  $\bar{a}_1 \bar{a}_2 = 0$  if (the composition  $\bar{a}_1 \circ \bar{a}_2$  of the correspondences)  $= \emptyset$  or  $h \neq i$ , and otherwise  $\bar{a}_1 \bar{a}_2 = \bar{a}_1 \circ \bar{a}_2$  the composition of correspondences. Then the disjoint union of  $\{a \mid a \in P_i\}_i$  and  $\{0\}$  forms a semigroup  $S$  with the multiplication defined above. If  $a_1 \leq a_2$  in  $P_i$ , there exists  $x \in S$  satisfying  $\bar{a}_1 = \bar{a}_2 x$  by (4) in Definition.

Let  $R := KS$  be the semigroup algebra of  $S$  over a field  $K$ . Then  $R$  is an artinian algebra over  $K$  with the Jacobson radical  $\{\sum k_a \bar{a} \mid a \neq m_i \text{ for any } i, \text{ and } k_a \in K\}$  and  $\{\bar{m}_i\}_{i=1}^n$  is a basic set of primitive idempotents for  $R$ .

For any element  $x \neq 0$  in  $\bar{m}_j R \bar{m}_i$ ,  $x = k_1 \bar{a}_1 + \dots + k_s \bar{a}_s$  with some distinct  $\bar{a}_1, \dots, \bar{a}_s: P_j \rightarrow P_i$  and  $k_1, \dots, k_s \in K \setminus \{0\}$ . Since  $a_1, \dots, a_s \in P_i$  and  $t_i(a_1) = \dots = t_i(a_s) = t_j(m_j)$ , there exists uniquely the maximal element  $a(x)$  of  $\{a_1, \dots, a_s\}$  by (3) in Definition. If  $a_u = a(x)$  ( $u \in \{1, \dots, s\}$ ),

$$x = \bar{a}_u(k_u + \text{an element of the Jacobson radical})$$

and  $xR = \bar{a}(x)R$ . Therefore  $R$  is right  $LD$  by Theorem 2. It is easily verified that  $\alpha_i: H(\bar{m}_i R) \rightarrow P_i; xR \mapsto a(x)$  is an isomorphism of poset, and that the diagram

$$\begin{array}{ccc} H(\bar{m}_i R) & \xrightarrow{\alpha_i} & P_i \\ \text{top}(\downarrow) & & \downarrow t_i \\ T(R) & \xrightarrow{\beta} & C \end{array}$$

is commutative.

#### 4. Right and left $LD$ -rings

If  $R$  is a right  $LD$ -ring with a basic set  $\{e_i\}_{i \in I, R}$  of primitive idempotents, we construct a semigroup  $S_R$  from the admissible system  $\{(H(e_i R), ( ) / ( ) J)\}_{i \in I}$ . Let  $X \in H(e_i R)$  and  $X/XJ \cong e_i R / e_j J$  ( $i, j \in I$ ), then the correspondence  $\bar{X}$  is induced by the left multiplication of some  $x \in e_i R e_j$  (cf. the first paragraph and Remark 1 in the section 3) and  $X = xR$ .

Symmetrically we have a semigroup  ${}_R S$ , the left version of  $S_R$ , from the admissible system  $\{(H(R e_i), ( ) / ( ))\}_{i \in I}$  if  $R$  is a left  $LD$ -ring with a basic set  $\{e_i\}_{i \in I}$  of primitive idempotents, where correspondences operate  $H(R e_i)$  on the right.

The semigroup algebra  $KS_R$  (resp.  $K{}_R S$ ) over a field  $K$  is considered a model of right (resp. left)  $LD$ -ring  $R$  with respect to the submodule-lattice structure of

the projective indecomposable right (resp. left)  $R$ -modules.

However, if  $R$  is a right and left  $LD$ -ring, the “one-sided model”  $KS_R$  or  $K_R S$  is two-sided (see Proposition 5 below).

**Lemma 4.** *Let  $e$  be an idempotent of a ring  $R$ , and suppose that every submodule in  $eR$  is characteristic. Then  $Rx \leq Ry$  implies  $xR \leq yR$  for any  $x, y \leq eR$ .*

Proof. If  $Rx \leq Ry$ , there is  $r \in eRe$  satisfying  $x = ry$ . Since  $yR$  is characteristic in  $eR$ ,  $xR = ryR \leq yR$ .

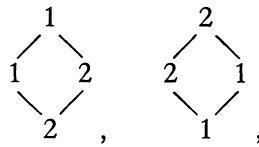
**Proposition 5.** *Let  $R$  be a right and left  $LD$ -ring with a basic set  $\{e_i\}_{i \in I}$  of primitive idempotents. Then  $S_R \cong {}_R S$ ,  $KS_R \cong K_R S$  is a right and left  $LD$ -ring, and one of the admissible systems  $\{H(e_i R)\}_{i \in I}$ ,  $\{H(Re_i)\}_{i \in I}$  is obtained by the other.*

Proof. A bijection  $S_R \rightarrow {}_R S$ ;  $\overline{xR} \mapsto \overline{Rx}$  ( $x \in e_i R e_j$ ) is defined by Lemma 4, where  $\overline{xR}$  (resp.  $\overline{Rx}$ ) is the correspondence  $H(e_j R) \rightarrow H(e_i R)$  (resp.  $H(Re_i) \rightarrow H(Re_j)$ ) associated to  $xR$  (resp.  $Rx$ ) adopting the notation in the proof of Theorem 3. Since  $\overline{xRyR} = \overline{xyR}$  and  $\overline{RxRy} = \overline{Rxy}$  for  $x \in e_i R e_j$  and  $y \in e_k R e_h$  ( $i, j, k, h \in I$ ) (cf. Remark 1 in the section 3), this bijection is an isomorphism. The rest follows immediately.

**5. Examples**

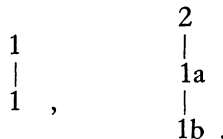
The construction of right  $LD$ -rings in the section 3 is useful especially in case that the Loewy length is small.

(1) From the admissible system of  $C$ -posets ( $C = \{1, 2\}$ ) with the Hasse diagram;



a  $QF$ - $LD$ -ring is given, where the numbers on the vertices are their values in  $C$ .

(2) Let  $R$  be a right  $LD$ -ring with the admissible system of  $\{1, 2\}$ -posets;



Then  $R$  is not left  $LD$ , since there is no element  $x$  in  $S_R$  satisfying  $\overline{b} = x\overline{a}$ .

**References**

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- [2] W. Stephenson: *Modules whose lattice of submodules is distributive*, Proc. London Math. Soc. (3) **28** (1974), 291–310.

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