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ON ARTINIAN RINGS WHOSE INDECOMPOSABLE PROJECTIVES ARE DISTRIBUTIVE

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1. Introduction

A module $L \neq 0$ is called local (or hollow) if $L = L_1 + L_2$ implies $L = L_1$ or $L = L_2$. Especially a noetherian module is local if and only if it has a unique maximal submodule.

A module M is called distributive if $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ for every submodules X, Y, Z in M (cf. [2]). It is clear that any sub- (or factor) module of a distributive module is distributive.

We call a ring R right locally distributive, right *LD* in abbreviation, if it is right artinian and every projective indecomposable right R -module is distributive. It is evident that every local right module over a right *LD*-ring is distributive. The class of right *LD*-rings is a generalization of the class of right serial rings.

In this note right *LD*-rings are studied, mainly to construct a number of right *LD*-algebras.

2. Right *LD*-rings

The following lemma, shown by Fuller, is basic to study distributive modules over a semiperfect ring.

Lemma 1. *Let R be a semiperfect ring. The following conditions on a right R -module M are equivalent:*

- (1) *M is distributive.*
- (2) *For every primitive idempotent e of R , the set $\{xeR \mid x \in M\}$ of all homomorphic images of eR in M is linearly ordered.*
- (3) *For every primitive idempotent e in R , the right eRe -module Me is uniserial.*

Proof. See Fuller [1].

Theorem 2. *The following conditions on a right artinian ring R are equivalent:*

- (1) Every projective indecomposable right R -module eR is distributive.
 (2) For every primitive idempotents e and f of R , eRf is a uniserial right fRf -module.
 (3) Every submodule in a projective indecomposable right R -module eR is characteristic, and the lattice of two-sided ideals in R is distributive.

Proof. (1) \Leftrightarrow (2) is a special case of Lemma 1.

(1) \Rightarrow (3). Every submodule in eR is a sum of local submodules and every local submodule is characteristic in eR by Lemma 1. Hence every submodule in eR is characteristic.

Let $\{e_i\}_{i=1}^n$ be a complete set of primitive idempotents, and let I, J, K be two-sided ideals in R . Then by the distributivity of e_iR ,

$$\begin{aligned} e_i(I \cap (J+K)) &= e_iI \cap (e_iJ + e_iK) \\ &= (e_iI \cap e_iJ) + (e_iI \cap e_iK) = e_i(I \cap J) + e_i(I \cap K). \end{aligned}$$

Summing up each side of the equations ($i=1, \dots, n$), we have $I \cap (J+K) = (I \cap J) + (I \cap K)$.

(3) \Rightarrow (1). Let A be a submodule in a projective indecomposable submodule eR . Since A is characteristic in eR , $eReA = A$. We notice that the two-sided ideal $A' := RA = ReA$ satisfies the equation $eA' = A$.

If X, Y, Z are any submodules in eR , then

$$\begin{aligned} eX' \cap (eY' + eZ') &= e(X' \cap (Y' + Z')) \\ &= e((X' \cap Y') + (X' \cap Z')) = (eX' \cap eY') + (eX' \cap eZ'). \end{aligned}$$

Hence eR is distributive.

A right artinian ring is called right LD if it satisfies the equivalent conditions in Theorem 2.

3. Construction of right LD -algebras

We begin with a general remark on modules. For a module M we denote by $H(M)$ the inclusion-ordered set of all local submodules in M . A homomorphism $f: M \rightarrow N$ of modules induces a correspondence: $H(M) \rightarrow H(N)$, $X \mapsto f(X)$. This correspondence is not a mapping in general (the image of some local submodule $\leq M$ by f may be 0). If M is a module of finite length and f is an epimorphism, then there is a natural surjection

$$(*) \quad \{X \in H(M) \mid X \not\leq \text{Ker}(f)\} \rightarrow H(N).$$

In fact, there exist $X_1, \dots, X_n \in H(M)$ such that $f^{-1}(Y) = X_1 + \dots + X_n$ for every $Y \in H(N)$, and $f(X_i) = Y$ for some $i \in \{1, \dots, n\}$. Moreover if M is distributive, i is unique by Lemma 1, and (*) is bijective.

In this section a method to construct some right *LD*-algebras is presented. We introduce some terminology.

Suppose C is a fixed set. A pair (P, t) of a set P and a mapping $t: P \rightarrow C$ is called a C -set. When (P, t) and (P', t') are C -sets, a mapping $f: P \rightarrow P'$ is called a C -set homomorphism if $t = t'f$. Moreover, in case that P and P' are posets, f is called a C -poset homomorphism if f is both a C -set homomorphism and a poset homomorphism.

A subset U of a poset P is said to be an upper part of P if $x \in U, y \in P$ and $x \leq y$ imply $y \in U$. In particular, when P is finite, U is an upper part of P if and only if it is of the form $\{x \in P \mid x \geq p_1\} \cup \dots \cup \{x \in P \mid x \geq p_n\}$ ($p_1, \dots, p_n \in P$).

DEFINITION. Let C be a set. A family of C -posets $\{(P_1, t_1), \dots, (P_n, t_n)\}$ is called an admissible system (of C -posets) if it satisfies the following conditions ($i=1, \dots, n$):

- (1) Every poset P_i has a unique maximal element m_i .
- (2) $C = \{t_1(m_1), \dots, t_n(m_n)\}$ ($t_i(m_i) \neq t_j(m_j)$ if $i \neq j$).
- (3) For every $c \in C$ the subposet $\{x \in P_i \mid t_i(x) = c\}$ is linearly ordered.
- (4) For every $a \in P_i$, there exist $j \in \{1, \dots, n\}$ and a C -poset homomorphism from an upper part of P_j to $\{x \in P_i \mid x \leq a\}$.

REMARK 1. Suppose that the conditions (1), (2), (3) of the above definition are satisfied and that f is a C -poset isomorphism from an upper part of P_j to $\{x \in P_i \mid x \leq a\}$. Then j is determined by $t_j(m_j) = t_i f(m_j) = t_i(a)$.

Let b_0 be any element in P_j and $b_0 \leq \dots \leq b_r = m_j$ be a chain with b_{k-1} maximal in $\{x \in P_j \mid x \leq b_k\}$ ($k \in \{1, \dots, r\}$). Then $f(b_{k-1})$ is maximal in $\{x \in P_i \mid x \leq f(b_k)\}$. Since $t_j(b_{k-1}) = t_i f(b_{k-1})$, $f(b_{k-1})$ is unique in $\{x \in P_i \mid x \leq f(b_k)\}$ by (3). Therefore $f(b_0)$ is determined inductively, and the isomorphism in (4) of the above definition is unique.

REMARK 2. By a similar argument we can replace (4) with (4') If $a \in P_i$ is maximal in $P_i \setminus \{m_i\}$, there exist $j \in \{1, \dots, n\}$ and a C -poset isomorphism from an upper part of P_j to $\{x \in P_i \mid x \leq a\}$.

If R is a right *LD*-ring with the Jacobson radical J , and $\{e_i\}_{i=1}^n$ is a basic set of primitive idempotents for R , then by the first remark of this section, the posets $H(e_1R), \dots, H(e_nR)$ form an admissible system with the mapping

$$\text{top}(\) (:= (\) / (\) J): H(e_iR) \rightarrow T(R) \quad (i \in \{1, \dots, n\}),$$

where $T(R)$ denotes the set of all isomorphism class of simple R -modules.

Theorem 3. For any admissible system $\{(P_i, t_i)\}_{i=1}^n$ of C -posets, there exists a right *LD*-ring R such that $H(e_iR)$ is isomorphic to (P_i, t_i) ($T(R)$ is identified with C by a bijection β : (the isomorphism class of $\text{top}(e_iR)) \mapsto t_i(m_i)$), where $\{e_i\}_{i=1}^n$ is a basic set of primitive idempotents for R .

Proof. Since the C -poset isomorphism of (4) in Definition is uniquely determined by an element $a \in P_i$ (Remark 1), we denote the isomorphism by \bar{a} . Letting any element in P_i outside the domain of definition of \bar{a} correspond to no element, the isomorphism \bar{a} is extended to a correspondence: $P_j \rightarrow P_i$, which operates P_j on the left. This extension is so trivial that it is also denoted by \bar{a} .

For two correspondences $\bar{a}_1: P_i \rightarrow P_j$ and $\bar{a}_2: P_k \rightarrow P_h$ ($a_1 \in P_j, a_2 \in P_h$), we define $\bar{a}_1 \bar{a}_2 = 0$ if (the composition $\bar{a}_1 \circ \bar{a}_2$ of the correspondences) $= \emptyset$ or $h \neq i$, and otherwise $\bar{a}_1 \bar{a}_2 = \bar{a}_1 \circ \bar{a}_2$ the composition of correspondences. Then the disjoint union of $\{a \mid a \in P_i\}_i$ and $\{0\}$ forms a semigroup S with the multiplication defined above. If $a_1 \leq a_2$ in P_i , there exists $x \in S$ satisfying $\bar{a}_1 = \bar{a}_2 x$ by (4) in Definition.

Let $R := KS$ be the semigroup algebra of S over a field K . Then R is an artinian algebra over K with the Jacobson radical $\{\sum k_a \bar{a} \mid a \neq m_i \text{ for any } i, \text{ and } k_a \in K\}$ and $\{\bar{m}_i\}_{i=1}^n$ is a basic set of primitive idempotents for R .

For any element $x \neq 0$ in $\bar{m}_j R \bar{m}_i$, $x = k_1 \bar{a}_1 + \dots + k_s \bar{a}_s$ with some distinct $\bar{a}_1, \dots, \bar{a}_s: P_j \rightarrow P_i$ and $k_1, \dots, k_s \in K \setminus \{0\}$. Since $a_1, \dots, a_s \in P_i$ and $t_i(a_1) = \dots = t_i(a_s) = t_j(m_j)$, there exists uniquely the maximal element $a(x)$ of $\{a_1, \dots, a_s\}$ by (3) in Definition. If $a_u = a(x)$ ($u \in \{1, \dots, s\}$),

$$x = \bar{a}_u(k_u + \text{an element of the Jacobson radical})$$

and $xR = \bar{a}(x)R$. Therefore R is right LD by Theorem 2. It is easily verified that $\alpha_i: H(\bar{m}_i R) \rightarrow P_i; xR \mapsto a(x)$ is an isomorphism of poset, and that the diagram

$$\begin{array}{ccc} H(\bar{m}_i R) & \xrightarrow{\alpha_i} & P_i \\ \text{top}(\downarrow) & & \downarrow t_i \\ T(R) & \xrightarrow{\beta} & C \end{array}$$

is commutative.

4. Right and left LD -rings

If R is a right LD -ring with a basic set $\{e_i\}_{i \in I, R}$ of primitive idempotents, we construct a semigroup S_R from the admissible system $\{(H(e_i R), () / () J)\}_{i \in I}$. Let $X \in H(e_i R)$ and $X/XJ \cong e_i R / e_j J$ ($i, j \in I$), then the correspondence \bar{X} is induced by the left multiplication of some $x \in e_i R e_j$ (cf. the first paragraph and Remark 1 in the section 3) and $X = xR$.

Symmetrically we have a semigroup ${}_R S$, the left version of S_R , from the admissible system $\{(H(R e_i), () / ())\}_{i \in I}$ if R is a left LD -ring with a basic set $\{e_i\}_{i \in I}$ of primitive idempotents, where correspondences operate $H(R e_i)$ on the right.

The semigroup algebra KS_R (resp. $K{}_R S$) over a field K is considered a model of right (resp. left) LD -ring R with respect to the submodule-lattice structure of

the projective indecomposable right (resp. left) R -modules.

However, if R is a right and left LD -ring, the “one-sided model” KS_R or $K_R S$ is two-sided (see Proposition 5 below).

Lemma 4. *Let e be an idempotent of a ring R , and suppose that every submodule in eR is characteristic. Then $Rx \leq Ry$ implies $xR \leq yR$ for any $x, y \leq eR$.*

Proof. If $Rx \leq Ry$, there is $r \in eRe$ satisfying $x = ry$. Since yR is characteristic in eR , $xR = ryR \leq yR$.

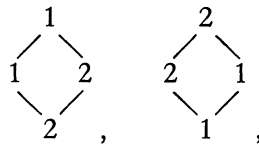
Proposition 5. *Let R be a right and left LD -ring with a basic set $\{e_i\}_{i \in I}$ of primitive idempotents. Then $S_R \cong {}_R S$, $KS_R \cong K_R S$ is a right and left LD -ring, and one of the admissible systems $\{H(e_i R)\}_{i \in I}$, $\{H(Re_i)\}_{i \in I}$ is obtained by the other.*

Proof. A bijection $S_R \rightarrow {}_R S$; $\overline{xR} \mapsto \overline{Rx}$ ($x \in e_i R e_j$) is defined by Lemma 4, where \overline{xR} (resp. \overline{Rx}) is the correspondence $H(e_j R) \rightarrow H(e_i R)$ (resp. $H(Re_i) \rightarrow H(Re_j)$) associated to xR (resp. Rx) adopting the notation in the proof of Theorem 3. Since $\overline{xRyR} = \overline{xyR}$ and $\overline{RxRy} = \overline{Rxy}$ for $x \in e_i R e_j$ and $y \in e_k R e_h$ ($i, j, k, h \in I$) (cf. Remark 1 in the section 3), this bijection is an isomorphism. The rest follows immediately.

5. Examples

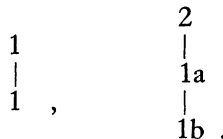
The construction of right LD -rings in the section 3 is useful especially in case that the Loewy length is small.

(1) From the admissible system of C -posets ($C = \{1, 2\}$) with the Hasse diagram;



a QF - LD -ring is given, where the numbers on the vertices are their values in C .

(2) Let R be a right LD -ring with the admissible system of $\{1, 2\}$ -posets;



Then R is not left LD , since there is no element x in S_R satisfying $\overline{b} = x\overline{a}$.

References

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