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ON A MIXED PROBLEM OF LINEAR ELASTODYNAMICS WITH A TIME-DEPENDENT DISCONTINUOUS BOUNDARY CONDITION

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1. Introduction

1.1. Problem and Main Result. Let Ω be a bounded domain in $\boldsymbol{R}^n$, $n \geqq 2$, with C^∞-boundary $\Gamma = \partial\Omega$. We consider the following mixed problem of linear elastodynamics:

Problem (D). *Find a vector function* $\boldsymbol{u} = (u^i(t, x))_{1 \leqq i \leqq n}$ *satisfying*

$$\text{(D.1)} \qquad \left(\frac{\partial^2}{\partial t^2} + A(t)\right)\boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \hat{\Omega}_T := (0, T) \times \Omega\,, \quad 0 < T < \infty,$$

$$\text{(D.2)} \qquad \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0\,, \quad \frac{\partial \boldsymbol{u}}{\partial t}(0, \cdot) = \boldsymbol{u}_1 \quad \text{in} \quad \Omega$$

with a time-dependent mixed boundary condition

$$\text{(D.3)} \qquad \begin{cases} \boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \hat{\Gamma}_{D,T} := \bigcup_{t \in [0,T]} \{t\} \times \Gamma_D(t), \\ B(t)\boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \hat{\Gamma}_{N,T} := \bigcup_{t \in [0,T]} \{t\} \times \Gamma_N(t) \end{cases}$$

for given $\boldsymbol{u}_0 = (u_0^i(x))_{1 \leqq i \leqq n}$, $\boldsymbol{u}_1 = (u_1^i(x))_{1 \leqq i \leqq n}$ *and* $\boldsymbol{f} = (f^i(t, x))_{1 \leqq i \leqq n}$.

Here $A(t)$ and $B(t)$ are differential systems operating on $\boldsymbol{v} = (v^i(x))_{1 \leqq i \leqq n}$ for each $t \in [0, T]$ defined by

$$\text{(1.1)} \qquad (A(t)\boldsymbol{v})^i = -\frac{\partial}{\partial x^j}\left(a^{ijkh}(t, x)\frac{\partial v^k}{\partial x^h}\right) \quad \text{in} \quad \Omega\,,$$

$$\text{(1.2)} \qquad (B(t)\boldsymbol{v})^i = \nu_j(\dot{x})a^{ijkh}(t, \dot{x})\frac{\partial v^k}{\partial x^h} \qquad \text{on } \Gamma \quad \text{for } 1 \leqq i \leqq n$$

where $a^{ijkh}(t, x)$ are real-valued C^∞-functions on $\overline{\hat{\Omega}_T}$ with *symmetry relations*

$$\text{(1.3)} \qquad a^{ijkh}(t, x) = a^{khij}(t, x) \qquad \text{for} \quad 1 \leqq i, j, k, h \leqq n\,,$$

and $\boldsymbol{\nu}(\dot{x}) = (\nu_j(\dot{x}))_{1 \leqq j \leqq n}$ denotes the unit outer normal to Γ at $\dot{x} \in \Gamma$. (Super- and subindices i, j, k, h, etc., take their values in the set $\{1, \cdots, n\}$ and the summation convention is adopted concerning the repeated indices.) Moreover, for each $t \in [0, T]$, $\Gamma_D(t)$ and $\Gamma_N(t)$ are nonempty open portions of Γ such that $\Gamma = \Gamma_N(t) \cup \Sigma(t) \cup \Gamma_D(t)$ (disjoint union) with $\Sigma(t)$ an $(n-2)$-dimensional compact C^∞-submanifold of Γ; the interface $\Sigma(t)$ between $\Gamma_D(t)$ and $\Gamma_N(t)$ *changes smoothly with time t*, by which we mean as follows: $\hat{\Gamma}_{D,T}$ and $\hat{\Gamma}_{N,T}$ are relatively open subsets of the lateral boundary $\hat{\Gamma}_T := [0, T] \times \Gamma$ such that their interface $\hat{\Sigma}_T := \cup_{t \in [0,T]} \{t\} \times \Sigma(t)$ is a 1-codimensional C^∞-submanifold (with boundary $\Sigma(0) \cup \Sigma(T)$) of $\hat{\Gamma}_T$ and intersects transversely with $\{t\} \times \Gamma$ for each $t \in [0, T]$.

Problem (D) was posed by Duvaut & Lions: When $\Sigma(t)$ is independent of time, they solved it under hypothesis (H.1) stated below using the Faedo-Galerkin method ([4; Théorème 4.1, Chap. 3]), and proposed that "L'abandon de cette hypothèse ($\Sigma(t)$ ne dépend pas du temps) semble conduire à des problèmes ouvert et fort interéssants" ([4; p. 106]). Subsequently, Inoue [13] studied the same problem as ours for the wave equation case (u: scalar, $A(t) = -\Delta$, $B(t) = \partial/\partial\nu$) to construct a unique weak solution assuming that "the speed of $\Sigma(t)$" is smaller than the propagation speed 1 of the wave governed by $(\partial/\partial t)^2 - \Delta$. See also Čehlov [2], Eskin [5].

The purpose of this paper is to show the existence of a unique *weak solution* $\mathbf{u}$ of (D) under the following two hypotheses:

(H.1) The quadratic form associated with $A(t)$ is *coercive* on $\boldsymbol{V}_D(t) := \boldsymbol{H}_0^1(\Omega \cup \Gamma_N(t))$ for each $t \in [0, T]$ in the sense that there exist positive constants c_1 and c_2 such that

$$a(t; \boldsymbol{u}, \boldsymbol{u}) \geqq c_1 \|\boldsymbol{u}\|_1^2 - c_2 \|\boldsymbol{u}\|^2 \qquad \text{for all} \quad t \in [0, T] \text{ and } \boldsymbol{u} \in \boldsymbol{V}_D(t)$$

where (and in what follows) we use the notation

$$a(t; \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega a^{ijkh}(t, x) \frac{\partial v^k}{\partial x^h} \overline{\frac{\partial w^i}{\partial x^j}} dx \qquad \text{for} \quad \boldsymbol{v} = (v^i), \ \boldsymbol{w} = (w^i) .$$

This hypothesis is equivalent to the following: for each t, the differential system $A(t)$ is *strongly elliptic* on $\bar{\Omega}$ and the boundary-value problem $\{A(t), B(t)\}$ satisfies the *strong complementing condition* on $\overline{\Gamma_N(t)}$ (see Simpson & Spector [21] and Ito [16]).

(H.2) For each $(t_0, \mathring{x}_0) \in \hat{\Sigma}_T$, the trajectory on Γ of the *point* of the intersection of $\Sigma(t)$ with the normal plane to $\Sigma(t_0)$ at $\mathring{x}_0$ moves through $\mathring{x}_0$ at time t_0 *at a speed smaller than* the quantity $c_\Sigma(t_0, \mathring{x}_0)$ defined in Subsection 2.4. (In what follows, we will say simply "the speed of $\Sigma(t)$ at $(t_0, \mathring{x}_0)$" for the speed of that trajectory at $(t_0, \mathring{x}_0)$.) It is remarkable that the $c_\Sigma(t_0, \mathring{x}_0)$ is closely related to the speed of *Rayleigh's wave* which travels over the traction-free boundary of a homogeneous elastic body with the elasticity tensor $(a^{ijkh}(t_0, \mathring{x}_0))$ occupying a half space whose boundary is the tangent hyperplane to Γ at $\mathring{x}_0$. For details, see Appendix.

DEFINITION 1.1. For given data $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\} \in \boldsymbol{V}_D(0) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\hat{\Omega}_T)$, a vector function $\boldsymbol{u}=(u^i(t, x))$ is called a *weak solution* of (D) (for $t \in [0, T)$) if it belongs to $\boldsymbol{H}^1(\hat{\Omega}_T)$ and satisfies $\boldsymbol{u}| =\boldsymbol{0}_{\hat{\Gamma}_{D,T}}$, $\boldsymbol{u}(0, \cdot)=\boldsymbol{u}_0$ in Ω and

$$(1.4) \qquad -\int_0^T \left(\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial \boldsymbol{\eta}}{\partial t}\right) dt + \int_0^T a(t; \boldsymbol{u}, \boldsymbol{\eta}) dt = (\boldsymbol{u}_1, \boldsymbol{\eta}(0, \cdot)) + \int_0^T (\boldsymbol{f}, \boldsymbol{\eta}) dt$$

for all test functions $\boldsymbol{\eta} \in \boldsymbol{H}^1(\hat{\Omega}_T)$ satisfying $\boldsymbol{\eta}|_{\hat{\Gamma}_{D,T}}=\boldsymbol{0}$ and $\boldsymbol{\eta}(T, \cdot)=\boldsymbol{0}$.

For the notation we use, see Subsection 1.3. We note here that (1.4) implies

$$\left(\frac{\partial^2}{\partial t^2} + A(t)\right)\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathscr{D}'(0, T; \boldsymbol{H}^{-1}(\Omega)) \quad \left(\text{and } \frac{\partial \boldsymbol{u}}{\partial t}(0, \cdot) = \boldsymbol{u}_1 \text{ in } \Omega\right)$$

where we regard $A(t)$ as belonging to $\mathcal{L}(\boldsymbol{V}_D(t), \boldsymbol{V}'_D(t)) \subset \mathcal{L}(\boldsymbol{V}_D(t), \boldsymbol{H}^{-1}(\Omega))$ for each t by

$$(1.5) \qquad a(t; \boldsymbol{v}, \boldsymbol{w}) = (A(t)\boldsymbol{v}, \boldsymbol{w}) \qquad \text{for} \quad \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}_D(t)\,.$$

Thus, by Lemma 8.1 of Lions & Magenes [17; Chap. 3], we can show that a weak solution of (D) belongs (after redefining on a set of measure 0 on $[0, T]$) to $C_w^0([0, T]; \boldsymbol{H}^1(\Omega)) \cap C_w^1([0, T]; \boldsymbol{L}^2(\Omega))$ where the subscript w stands for the weak topology. (If $\Sigma(t)$ is independent of t, that is true in the strong sense by Theorem 8.2 of [17; Chap. 3].)

Main Theorem. *Under* (H.1) *and* (H.2), *there exists a unique weak solution* $\boldsymbol{u} \in \boldsymbol{H}^1(\hat{\Omega}_T)$ *of* (D) *for any given data* $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\} \in \boldsymbol{V}_D(0) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\hat{\Omega}_T)$. *Moreover, it satisfies the energy estimate*

$$\|\boldsymbol{u}(t, \cdot)\|_1^2 + \left\|\frac{\partial \boldsymbol{u}}{\partial t}(t, \cdot)\right\|^2 \leqq C(T)(\|\boldsymbol{u}_0\|_1^2 + \|\boldsymbol{u}_1\|^2 + \int_0^t \|\boldsymbol{f}(\tau, \cdot)\|^2 d\tau)$$

for all $t \in [0, T]$ *where* $C(T)>0$ *is a constant independent of given data and time* t.

REMARK 1.2. If $\Sigma(t)$ $(\neq \phi)$ and $A(t)$ are independent of t, we can apply the semi-group theory in the same way as in Hayashida [6] (see also Ibuki [7]). Morevoer, if $\Sigma(t)$ is empty, we may expect that (D) admits a unique *strong solution*; the case $\Gamma \equiv \Gamma_D(t)$ is an exercise of the semi-group theory (see, e.g., Ikawa [8], Tanabe [22; Chap. 4]) and the case $\Gamma \equiv \Gamma_N(t)$ is included in a result of Shibata [20].

Remark 1.3. In the usual linear elasticity theory, an elasticity tensor $(a^{ijkh}(t, x))$ satisfies the *symmetry relations*

$$a^{ijkh}(t, x) = a^{khij}(t, x) = a^{jikh}(t, x)$$

and the *strong convexity condition*

$$a^{ijkh}(t, x)s_{kh}s_{ij} \geq c_0 s_{ij}s_{ij} \qquad \text{for all } (s_{ij}) \text{ with } s_{ij} = s_{ji} \in \boldsymbol{R}$$

where $c_0 > 0$ is constant. However, (1.3) and (H.1), which are weaker conditions than the above, are sufficient for our argument.

Example 1.4. (i) Let $a^{ijkh}(t, x) = \delta^{ik}\delta^{jh}$, that is, $A(t) = -\Delta$ and $B(t) = \partial/\partial\boldsymbol{\nu}$, where (and in what follows) δ^{ij} and also δ^i_j denote the Kronecker delta. Then (H.1) is fulfilled in advance and the $c_\Sigma(t, \dot{x})$ is equal to 1 for any $(t, \dot{x}) \in \hat{\Sigma}_T$ (cf. Inoue [13]).

(ii) In the isotropic elasticity, $(a^{ijkh}(t, x))$ is represented by means of the *Lamé moduli* $\lambda(t, x)$ and $\mu(t, x)$ as

$$a^{ijkh}(t, x) = \lambda(t, x)\delta^{ij}\delta^{kh} + \mu(t, x)(\delta^{ik}\delta^{jh} + \delta^{ih}\delta^{jk}).$$

Then, as seen from Simpson & Spector [21] and Ito [15], hypothesis (H.1) is equivalent to

$$\mu(t, x) > 0,\ \lambda(t, x) + 2\mu(t, x) > 0 \text{ on } \overline{\hat{\Omega}_T},\ \lambda(t, \dot{x}) + \mu(t, \dot{x}) > 0 \text{ on } \overline{\hat{\Gamma}_{N,T}},$$

and the strong convexity condition in Remark 1.3 is given by

$$\mu(t, x) > 0, \quad n\lambda(t, x) + 2\mu(t, x) > 0 \qquad \text{on } \overline{\hat{\Omega}_T}.$$

Moreover, the $c_\Sigma(t, \dot{x})$ in (H.2) is given by $\sqrt{\mu(t, \dot{x})\,\theta(\lambda(t, \dot{x}), \mu(t, \dot{x}))}$ where $\theta(\lambda, \mu)$ is a unique root of the equation

$$\text{(1.6)} \qquad F(\theta) := \theta^3 - 8\theta^2 + 8\Big(3 - \frac{2\mu}{\lambda + 2\mu}\Big)\theta - 16\Big(1 - \frac{\mu}{\lambda + 2\mu}\Big) = 0$$

in the interval (0, 1). This value is nothing but "the speed of Rayleigh's wave" (see Proposition A.5).

1.2. Summary. In principle, our approach to Problem (D) is guided by a plan proposed by Inoue [12], [13]. Let $\{\alpha_\varepsilon(t, \dot{x})\}_{\varepsilon>0}$ be a family of smooth functions on $\hat{\Gamma}_T$ which approximates as $\varepsilon \to 0$ suitably the defining function of $\Gamma_N(t)$ for each t (see Definition 2.1 and (2.1)), and let us call by (D_ε) the mixed problem obtained from (D) by replacing (D.3) with a degenerate boundary condition

$$\text{(1.7)}_\varepsilon \qquad \alpha_\varepsilon(t, \dot{x})B(t)\boldsymbol{u} + (1 - \alpha_\varepsilon(t, \dot{x}))\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \hat{\Gamma}_T$$

and $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\}$ with smooth approximate data $\{\boldsymbol{u}_{0\varepsilon}, \boldsymbol{u}_{1\varepsilon}, \boldsymbol{f}_\varepsilon\}$ satisfying a certain compatibility condition. If (D_ε) admits a unique smooth solution $\boldsymbol{u}_\varepsilon$ for each ε, one hopes that a limit of $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$ will be a weak solution of (D): this was the central idea in [13]. In the present case, however, it seems so difficult to apply directly the method of Inoue [12] (or Ikawa [10]) in order to obtain a smooth solution $\boldsymbol{u}_\varepsilon$ of (D_ε). Hence, we do not solve (D_ε) but modify the discussions in [12], [13] as follows:

We construct a weak solution of (D) by pasting time-local weak solutions, which are obtained through superposition of weak solutions for locally-supported data. We get such a weak solution, which will be also locally-supported, as a limit of smooth solutions $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ to certain approximate problems governing a propagation phenomenon with finite speed (independent of ε). In order to construct a weak solution for given data supported in a "small" neighborhood of a point $(t_0, x_0) \in \hat{\Sigma}_T$, we consider the mixed problem $(\mathrm{D}_\varepsilon)_{(t_0, \dot{x}_0)}$ given by replacing $(1.7)_\varepsilon$ in (D_ε) with

$$(1.8)_\varepsilon \qquad \alpha_\varepsilon(t, \dot{x})(B(t)+\varepsilon X(t))\boldsymbol{u}+(1-\alpha_\varepsilon(t, \dot{x}))\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \hat{\Gamma}_T$$

where $X(t)$ is a differential operator in such a form as

$$(1.9) \qquad X(t)\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t} + \gamma^j(t, x)\frac{\partial \boldsymbol{u}}{\partial x^j} \qquad \text{in} \quad \hat{\Omega}_T$$

(see Definition 2.3 and (4.4)); the initial time will be changed as occasion demands. Our main efforts will be put into this problem with the boundary condition which may change its order near $(t_0, \dot{x}_0)$ not only spatially but also temporally.

In Section 2, we consider a level-preserving local transformation near $(t_0, \dot{x}_0)$ which makes the suitably-defined $\alpha_\varepsilon(t, \dot{x})$ independent of t; thereby we will transform $(\mathrm{D}_\varepsilon)_{(t_0, \dot{x}_0)}$ locally. Sections 3 is devoted to the study of an auxiliary problem to $(\mathrm{D}_\varepsilon)_{(t_0, \dot{x}_0)}$, which is in such a general form that it includes the forms obtained by local transformations given in Section 2 and is invariant locally under Holmgren-like transformations. In treating the degenerate boundary condition there, a result in Ito [16] will be necessary. In Section 4, using the results of Section 3, we prove Main Theorem in such a way as mentioned above. In Appendix, we present some properties of the $c_\Sigma(t, \dot{x})$.

1.3. Notation. We express column vectors in boldface: $\boldsymbol{u} = (u^i) = {}^t(u^1, \cdots, u^n)$, also various $\boldsymbol{C}^n$-valued function spaces. $\boldsymbol{L}^2(\Omega) = L^2(\Omega; \boldsymbol{C}^n)$ (resp. $\boldsymbol{L}^2(\Gamma) = L^2(\Gamma; \boldsymbol{C}^n)$) is a Hilbert space with inner product $(\cdot, \cdot) = (\cdot, \cdot)_\Omega$ (resp. $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\Gamma$) and norm $\|\cdot\| = \|\cdot\|_\Omega$ (resp. $[\cdot] = [\cdot]_\Gamma$) given by

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\, dx = \int_\Omega u^i \overline{v^i}\, dx, \quad \|\boldsymbol{u}\|^2 = (\boldsymbol{u}, \boldsymbol{u})$$
$$\text{for} \quad \boldsymbol{u} = (u^i),\ \boldsymbol{v} = (v^i) \in \boldsymbol{L}^2(\Omega).$$

$$(\text{resp. } \langle \phi, \psi \rangle = \int_\Gamma \phi \cdot \bar{\psi} d\dot{x}\,, \quad [\phi]^2 = \langle \phi, \phi \rangle \qquad \text{for} \quad \phi, \psi \in \boldsymbol{L}^2(\Gamma).)$$

We represent the point of $\bar{\Omega}$ (resp. Γ) by x (resp. $\dot{x}$) and the volume element in Ω (resp. on Γ) by dx (resp. $d\dot{x}$). For an integer $m \geqq 1$, $\boldsymbol{H}^m(\Omega) = H^m(\Omega; \boldsymbol{C}^n)$ (resp. $\boldsymbol{H}^{m-1/2}(\Gamma) = H^{m-1/2}(\Gamma; \boldsymbol{C}^n)$) is the usual $\boldsymbol{C}^n$-valued Sobolev space on Ω of order m (resp. on Γ of order $m-1/2$) with norm denoted by $||\cdot||_m = ||\cdot||_{m,\Omega}$ (resp. $[\cdot]_{m-1/2} = [\cdot])_{m-1/2,\Gamma}$), e.g.,

$$||\boldsymbol{u}||_1^2 = ||\nabla \boldsymbol{u}||^2 + ||\boldsymbol{u}||^2 = \int_\Omega \left(\frac{\partial u^i}{\partial x^k} \overline{\frac{\partial u^i}{\partial x^k}} + u^i \overline{u^i} \right) dx \qquad \text{for} \quad \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)\,.$$

For each t, $\boldsymbol{V}_D(t) = \boldsymbol{H}_0^1(\Omega \cup \Gamma_N(t))$ stands for the closure in $\boldsymbol{H}^1(\Omega)$ of $\boldsymbol{C}_0^\infty(\Omega \cup \Gamma_N(t)) := \{\boldsymbol{u} \in \boldsymbol{C}^\infty(\bar{\Omega});\ \operatorname{supp} \boldsymbol{u} \subset \Omega \cup \Gamma_N(t)\}$, or equivalently $\boldsymbol{V}_D(t) = \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{u}|_{\Gamma_D(t)} = 0\}$. The dual of $\boldsymbol{V}_D(t)$ (resp. $\boldsymbol{H}_0^1(\Omega)$) is denoted by $\boldsymbol{V}'_D(t)$ (resp. $\boldsymbol{H}^{-1}(\Omega)$); we have the inclusion relations

$$\boldsymbol{H}_0^1(\Omega) \subset \boldsymbol{V}_D(t) \subset \boldsymbol{L}^2(\Omega) = (\boldsymbol{L}^2(\Omega))' \subset \boldsymbol{V}'_D(t) \subset \boldsymbol{H}^{-1}(\Omega)\,.$$

Let X and Y be Banach spaces. For an integer $m \geqq 0$, $C^m([0, T]; X)$ represents the space of X-valued functions of $t \in [0, T]$ of class C^m. For simplicity, we often write "$\boldsymbol{u} \in \mathcal{E}_t^m(X)$ (for $t \in [0, T]$)" instead of "$\boldsymbol{u} \in C^m([0, T]; X)$". We denote by $\mathcal{L}(X, Y)$ the Banach space consisting of all bounded linear operators on X into Y.

We can extend $a^{ijkh}(t, \cdot)$, $\Gamma_D(t)$, $\Gamma_N(t)$ and $\Sigma(t)$ in t to $\boldsymbol{R}$ so as to be t-independent on $\boldsymbol{R} \backslash (-1, T+1)$ and to preserve the properties stated before (H.1) (and further hypotheses (H.1), (H.2) if they are satisfied for $t \in [0, T]$). For those extended, we set

$$\hat{\Gamma}_D = \bigcup_{t \in \boldsymbol{R}} \{t\} \times \Gamma_D(t)\,, \quad \hat{\Sigma} = \bigcup_{t \in \boldsymbol{R}} \{t\} \times \Sigma(t); \quad \hat{\Omega},\ \hat{\Gamma},\ \hat{\Gamma}_N \ \text{similarly}\,.$$

2. Local reduction to the case of time-independent $\Sigma(t)$

In this section, we reduce locally the equations (D.1) and $(1.8)_\varepsilon$ in $(\mathrm{D}_\varepsilon)_{(t_0, \dot{x}_0)}$ to the case where $\Sigma(t)$ is independent of t. We do not refer to the initial condition (D.2) for the time being.

2.1. Definition of $\boldsymbol{a}_\varepsilon(t, \dot{x})$. Let $T_{\dot{x}}(\Sigma(t))^\perp$, $(t, \dot{x}) \in \hat{\Sigma}$, denote the normal space at $\dot{x}$ in $T_{\dot{x}}(\Gamma)$ to the submanifold $\Sigma(t)$ of Γ. Then,

$$E(\hat{\Sigma}) := \bigcup_{(t, \dot{x}) \in \hat{\Sigma}} T_{\dot{x}}(\Sigma(t))^\perp (\cong \boldsymbol{R}^1 \times \hat{\Sigma})$$

is regarded as a C^∞-subbundle of the restriction $T(\hat{\Gamma})|_{\hat{\Sigma}}$ to $\hat{\Sigma}$ of the tangent bundle $T(\hat{\Gamma})$. Since $\hat{\Sigma}$ intersects transversely with $\{t\} \times \Gamma$ for each t, there exists a unique C^∞-section Z of $E(\hat{\Sigma})$ such that, when we consider that $Z_{(t, \dot{x})} \in$

$T_{\hat{x}}(\Sigma(t))^{\perp} \subset T_{\hat{x}}(\Gamma)$ for each $(t, \hat{x}) \in \hat{\Sigma}$, $Z_{(t,\hat{x})}$ is a unit vector pointing to the side of $\Gamma_D(t)$. The exponential mapping Exp defined in a neighborhood of the 0-section $\hat{\Gamma}$ in $T(\hat{\Gamma})$ is well-defined also on $E(\hat{\Sigma}) = \{\sigma Z;\ \sigma \in \boldsymbol{R}\}$ by

$$\operatorname{Exp} \sigma Z_{(t,\hat{x})} = \{t\} \times (\exp \sigma Z_{(t,\hat{x})}) \qquad \text{for} \quad \sigma \in \boldsymbol{R},\ (t, \hat{x}) \in \hat{\Sigma}$$

where exp stands for the exponential mapping: $T(\Gamma) \to \Gamma$. For $\mathcal{E}_0 > 0$ small enough, Exp gives a C^∞-diffeomorphism of an open subset $\{\sigma Z;\ |\sigma| < \mathcal{E}_0\}$ $(\cong (-\mathcal{E}_0, \mathcal{E}_0) \times \hat{\Sigma})$ of $E(\hat{\Sigma})$ onto a tubular neighborhood $U_0 := \{(t, \hat{x}) \in \hat{\Gamma};$ $\operatorname{dis}_\Gamma(\hat{x}, \Sigma(t)) < \mathcal{E}_0\}$ of $\hat{\Sigma}$ in $\hat{\Gamma}$ where $\operatorname{dis}_\Gamma(\hat{x}, \Sigma(t))$ denotes the geodesical distance on Γ from $\hat{x}$ to $\Sigma(t)$. Using its inverse, we can show that the Lipschitz function $\sigma(t, \hat{x})$ on $\hat{\Gamma}$ defined by

$$\sigma(t, \hat{x}) = \operatorname{dis}_\Gamma(\hat{x}, \Sigma(t)) \text{ if } \hat{x} \in \overline{\Gamma_D(t)}, \quad = -\operatorname{dis}_\Gamma(\hat{x}, \Sigma(t)) \text{ if } \hat{x} \in \Gamma_N(t)$$

is of class C^∞ in U.

DEFINITION 2.1. For each $\mathcal{E} \in (0, \mathcal{E}_0)$, we define $\alpha_\varepsilon(t, \hat{x})$ by

$$\alpha_\varepsilon(t, \hat{x}) = \int_0^\infty \rho(s - \mathcal{E}^{-1}\sigma(t, \hat{x}))\, ds \qquad \text{for} \quad (t, \hat{x}) \in \hat{\Gamma}$$

where $\rho(s)$ is a C^∞-function on $\boldsymbol{R}$ given by

$$\rho(s) = \left[\int_0^1 e^{1/\tau(\tau-1)} d\tau\right]^{-1} e^{1/s(s-1)} \quad \text{if } 0 < s < 1, \quad = 0 \text{ otherwise}.$$

It is easily seen that for each $\mathcal{E} \in (0, \mathcal{E}_0)$, $\alpha_\varepsilon(t, \hat{x})$ is a C^∞-function on $\hat{\Gamma}$ which depends only on $\operatorname{dis}_\Gamma(\hat{x}, \Sigma(t))$ and satisfies

$$(2.1) \qquad \begin{cases} \alpha_\varepsilon(t, \hat{x}) = 1 & \text{on } \overline{\hat{\Gamma}_N}, \\ \alpha_\varepsilon(t, \hat{x}) = 0 & \text{on } \{(t, \hat{x}) \in \hat{\Gamma}_D;\ \operatorname{dis}_\Gamma(\hat{x}, \Sigma(t)) \geqq \mathcal{E}\}, \\ 0 < \alpha_\varepsilon(t, \hat{x}) < 1 & \text{on } \{(t, \hat{x}) \in \hat{\Gamma}_D;\ 0 < \operatorname{dis}_\Gamma(\hat{x}, \Sigma(t)) < \mathcal{E}\}. \end{cases}$$

2.2. Local transformation. Let $(t_0, \hat{x}_0) \in \hat{\Sigma}$. We can choose a rotation $R = (R^i_j) \in SO(n)$ of the x-coordinates so that, by the transformation: $x \to \tilde{x} = R(x - \hat{x}_0)$, the tangent hyperplane to Γ (resp. hyperline to $\Sigma(t_0)$) at $\hat{x}_0$ in $\boldsymbol{R}^n$ is mapped to $\{\tilde{x} = (\tilde{x}^i);\ \tilde{x}^n = 0\}$ (resp. $\{\tilde{x};\ \tilde{x}^{n-1} = \tilde{x}^n = 0\}$) and the outer unit normal to Γ at $\hat{x}_0$ to the vector $(0, \cdots, 0, -1)$. Under the coordinates $\tilde{x} = (\tilde{x}^i)$, $(a^{ijkh}(t, x))$ and $\boldsymbol{u} = (u^i(t, x))$ are represented, respectively, by

$$\tilde{a}^{ijkh}(t, \tilde{x}) = a^{i'j'k'h'}(t, x) R^i_{i'} R^j_{j'} R^k_{k'} R^h_{h'},\ \tilde{u}^i(t, \tilde{x}) = u^k(t, x) R^i_k.$$

Given data $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\}$ are transformed in the same manner.

Taking the above into consideration, we may assume that $(0, 0)$ is the general point of $\hat{\Sigma}$ and that (i) Ω (resp. Γ) is represented near 0 by $x^n > f(x')$ (resp. $x^n = f(x')$) with f a C^∞-function of $x' = (x^1, \cdots, x^{n-1})$ satisfying $f(0) = 0$ and $\nabla_{x'} f(0) = 0$,

(ii) $\hat{\Gamma}_D$ (resp. $\hat{\Sigma}$) is represented near $(0, 0)$ by $x^{n-1}>g(t, x'')$ (resp. $x^{n-1}=g(t, x'')$) and $x^n=f(x')$ with g a C^∞-function of $(t, x'')=(t, x^1, \cdots, x^{n-2})$ satisfying $g(0, 0)=0$ and $\nabla_{x''}g(0, 0)=0$. Our argument in the rest of this section are under these circumstances.

Proposition 2.2. *There exist a neighborhood U of $(0, 0)\in\hat{\Sigma}$ in $\boldsymbol{R}^{n+1}_{(t,x)}$ and a level-preserving (i.e., $t=s$) diffeomorphism Φ of U onto a neighborhood $\tilde{U}:=\Phi(U)$ of $(0, 0)$ in $\boldsymbol{R}^{n+1}_{(s,y)}$ such that*

(i) $\Phi(\hat{\Omega}\cap U)=\{(s, y)=(s, y^1, \cdots, y^n)\in\tilde{U};\ y^n>0\}$ $[=(\boldsymbol{R}\times\boldsymbol{R}^n_+)\cap\tilde{U}]$,
$\Phi(\hat{\Gamma}\cap U)=\{(s, y)\in\tilde{U};\ y^n=0\}$ $[=(\boldsymbol{R}\times\partial\boldsymbol{R}^n_+)\cap\tilde{U}]$;
(ii) $\Phi(\hat{\Gamma}_D\cap U)=\{(s, y)\in\tilde{U};\ y^{n-1}>y^n=0\}$,
$\Phi(\hat{\Sigma}\cap U)=\{(s, y)\in\tilde{U};\ y^{n-1}=y^n=0\}$;
(iii) $\partial y^n/\partial t=0$ *in U and the Jacobian matrix of Φ at $(0, 0)$ is*

$$\left.\frac{\partial(s, y^1, \cdots, y^n)}{\partial(t, x^1, \cdots, x^n)}\right|_{(t,x)=(0,0)}=\left(\begin{array}{c|cccc} 1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 1 & & & \\ \vdots & & 1 & & 0 \\ 0 & & & \ddots & \\ -\dfrac{\partial g}{\partial t}(0, 0) & & 0 & & \ddots \\ 0 & & & & 1 \end{array}\right);$$

(iv) *For each $\varepsilon\in(0, \varepsilon_0)$, the function $\tilde{\alpha}_\varepsilon(y'):=\alpha_\varepsilon(\Phi^{-1}(s, y))$ is independent of s on $(\boldsymbol{R}\times\partial\boldsymbol{R}^n_+)\cap\tilde{U}$ where $y'=(y^1, \cdots, y^{n-1})$.*

Proof. Let us regard $\{x^1, \cdots, x^{n-1}\}$ as a local coordinate system of Γ near 0. We define a transformation Ψ, which is expected to be the inverse of Φ, of a small neighborhood $\tilde{U}_1$ of $(0, 0)$ in $\boldsymbol{R}^{n+1}_{(s,y)}$ by

$$(2.2)\qquad \begin{cases} t=s\,, \\ x^i=[\exp y^{n-1}Z_{(s,\dot{y})}]^i \qquad \text{for}\quad 1\leqq i\leqq n-1 \\ \qquad\qquad \text{with}\ \dot{y}=(y'', g(s, y''), f(y'', g(s, y'')))\,, \\ x^n=y^n+f(x') \end{cases}$$

where Z is the section of $E(\hat{\Sigma})$ defined at the beginning, and $[\exp y^{n-1}Z_{(s,\dot{y})}]^i$, $1\leqq i\leqq n-1$, stands for the i-th component of $\exp y^{n-1}Z_{(s,\dot{y})}\in\Gamma$ with respect to the above local coordinate system of Γ. Then, Ψ is a level-preserving C^∞-mapping which satisfies

(i)′ $\Psi((\boldsymbol{R}\times\boldsymbol{R}^n_+)\cap\tilde{U}_1)\subset\hat{\Omega}$, $\Psi((\boldsymbol{R}\times\partial\boldsymbol{R}^n_+)\cap\tilde{U}_1)\subset\hat{\Gamma}$;
(ii)′ $\Psi(\{(s, y)\in\tilde{U}_1;\ y^{n-1}>y^n=0\})\subset\hat{\Gamma}_D$,
$\Psi(\{(s, y)\in\tilde{U}_1;\ y^{n-1}=y^n=0\})\subset\hat{\Sigma}$.

Moreover, using the fact that $Z_{(0,0)}=(\partial/\partial x^{n-1})_0$ and fundamental properties of the exponential mapping exp: $T(\Gamma)\to\Gamma$, we calculate the Jacobian matrix of Ψ at (0, 0) to obtain

$$\left.\frac{\partial(t, x^1, \cdots, x^n)}{\partial(s, y^1, \cdots, y^n)}\right|_{(s,y)=(0,0)} = \left(\begin{array}{c|cccc} 1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 1 & & & \\ \vdots & & 1 & & 0 \\ 0 & & & \ddots & \\ \frac{\partial g}{\partial t}(0,0) & & & & \ddots \\ 0 & & 0 & & 1 \end{array}\right);$$

Thus, by the inverse mapping theorem, there exists a neighborhood $\tilde{U}$ of (0, 0) in $\tilde{U}_1$ such that the restriction $\Psi|_{\tilde{U}}$ of Ψ to $\tilde{U}$ is a C^∞-diffeomorphism of $\tilde{U}$ onto $\Psi(\tilde{U})$. We see now from (2.2) that $U:=\Psi(\tilde{U})$ and $\Phi:=(\Psi|_{\tilde{U}})^{-1}$ possess the desired properties (i)—(iv). Q.E.D.

2.3. Change of variables. Now, we study how the equations (D.1) and $(1.8)_\varepsilon$ of $(\mathrm{D}_\varepsilon)_{(t_0,\dot{x}_0)}$ are transformed in U by Φ. When we make the associated change of variables, it is convenient to look at them from a geometric viewpoint (see Inoue & Wakimoto [14; Appendix] and Marsden & Hughes [18; Section 2.4]). Regard U as a manifold with the coordinate system $\{t, x^1, \cdots, x^n\}$ and equip with it the connection ∇ which is the restriction to U of the trivial connection associated with the vector space structure of $\boldsymbol{R}_t\times\boldsymbol{R}^n_x$. Clearly, ∇ restricts to a connection ∇^t on each $U(t):=\{x\in\boldsymbol{R}^n;\ (t, x)\in U\}$, identified with $\{t\}\times U(t)$; ∇^t is the connection of the Euclidean metric $\delta_{ik}dx^i dx^k$ on $U(t)\subset\boldsymbol{R}^n_x$. ($U$ is a subset of the *standard classical spacetime* in the terminology of [18; p. 157].) Taking that into consideration, we regard $\boldsymbol{u}=(u^i(t, x))$ as a vector field $u^i(t, x)\partial/\partial x^i$ tangent to each $U(t)$, $(a^{ijkh}(t, x))$ as a 4th-order contravariant tensor field $a^{ijkh}(t, x)(\partial/\partial x^i)\otimes(\partial/\partial x^j)\otimes(\partial/\partial x^k)\otimes(\partial/\partial x^h)$, the unit outer normal $\boldsymbol{\nu}=(\nu_i(\dot{x}))$ as a 1-form $\nu_i(\dot{x})dx^i$ (on each $U(t)$), etc. Then (D.1) and $(1.8)_\varepsilon$ are rewritten on each $U(t)$ as

$$(2.3)\quad \begin{cases} [\nabla_0\nabla_0 u^i-\nabla^t_j(a^{ijkh}\nabla^t_h u_k)]\dfrac{\partial}{\partial x^i}=f^i\dfrac{\partial}{\partial x^i} & \text{in}\quad \Omega\cap U(t), \\[2ex] [\alpha_\varepsilon(\nu_j a^{ijkh}\nabla^t_h u_k+\varepsilon X u^i)+(1-\alpha_\varepsilon)u^i)]\dfrac{\partial}{\partial x^i}=0 & \text{on}\quad \Gamma\cap U(t) \end{cases}$$

where $\nabla_0:=\nabla_{\partial/\partial t}$ and $\nabla_j:=\nabla_{\partial/\partial x^j}$ are the covariant derivatives, $u_i dx^i$ is the 1-form associated with $\boldsymbol{u}$ on each $U(t)$ (hence $u_i=\delta_{ij}u^j=u^i$ in the present case) and $X=X(t)$ will be clarified below.

Since Φ is level-preserving, $\Phi(s,\cdot)_* v(s,\cdot)=(\Phi_* v)(s,\cdot)$ for any vector field v on U tangnet to each $U(t)$, and the connection $\tilde{\nabla}:=\Phi_*\nabla$ on $\tilde{U}:=\Phi(U)$ also restricts to a connection $\tilde{\nabla}^s:=\Phi(s,\cdot)_*\nabla^s$ on each $\tilde{U}(s):=\Phi(U(s))=\{y\in \boldsymbol{R}^n; (s,y)\in\tilde{U}\}$, identified with $\{s\}\times\tilde{U}(s)$; $\tilde{\nabla}^s$ is the connection of the Riemannian metric $g_{ik}(s,\cdot)dy^i dy^k$ on each $\tilde{U}(s)$ induced by $\Phi(s,\cdot)$ where (and in what follows)

$$g_{ik}(s,y)=\frac{\partial x^j}{\partial y^i}\frac{\partial x^j}{\partial y^k}\quad\left(\text{and } g^{ik}(s,y)=\frac{\partial y^i}{\partial x^j}\frac{\partial y^k}{\partial x^j}\right).$$

We now define X in (2.3) by the following Definition 2.3, so that we have by operating $\Phi(s,\cdot)_*$ to (2.3) with $t=s$

$$(2.4)\qquad \begin{cases} \left[\left(\tilde{\nabla}_0+\dfrac{\partial y^j}{\partial t}\tilde{\nabla}^s_j\right)\left(\tilde{\nabla}_0+\dfrac{\partial y^h}{\partial t}\tilde{\nabla}^s_h\right)\tilde{u}^i-\tilde{\nabla}^s_j(\tilde{a}^{ijkh}\tilde{\nabla}^s_h\tilde{u}_k)\right]\dfrac{\partial}{\partial y^i}=\tilde{f}^i\dfrac{\partial}{\partial y^i} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Phi(\Omega\cap U(s)), \\ [\tilde{\alpha}_\varepsilon(\tilde{\nu}_j\tilde{a}^{ijkh}\tilde{\nabla}^s_h\tilde{u}_k+\varepsilon\tilde{\nabla}_0\tilde{u}^i)+(1-\tilde{\alpha}_\varepsilon)\tilde{u}^i]\dfrac{\partial}{\partial y^i}=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Phi(\Gamma\cap U(s)), \end{cases}$$

where $\tilde{\nabla}_0:=\tilde{\nabla}_{\partial/\partial s}$, $\tilde{\nabla}_j:=\tilde{\nabla}_{\partial/\partial y^j}$ are the covariant derivatives on $\tilde{U}$ and

$$\tilde{a}^{ijkh}(s,y)=a^{i'j'k'h'}(t,x)\frac{\partial y^i}{\partial x^{i'}}\frac{\partial y^j}{\partial x^{j'}}\frac{\partial y^k}{\partial x^{k'}}\frac{\partial y^h}{\partial x^{h'}},$$

$$\tilde{u}^i(s,y)=u^k(t,x)\frac{\partial y^i}{\partial x^k},\qquad \tilde{\nu}_k(s,y')=\nu_i(\dot{x})\frac{\partial x^i}{\partial y^k},$$

$$\tilde{\alpha}_\varepsilon(y')=\alpha_\varepsilon(t,\dot{x})\ \text{(see Proposition 2.2 (iv))};$$

$\tilde{f}^i(s,y)$ and $\tilde{u}_k(s,y)$ are defined similarly. We note that

$$\tilde{u}_k(s,y)=g_{ik}(s,y)\tilde{u}^i(s,y),\quad \tilde{u}^i(s,y)=g^{ik}(s,y)\tilde{u}_k(s,y).$$

Dedinition 2.3. A differential operator X in U is given by

$$X\left(u^i\frac{\partial}{\partial x^i}\right)=(\Phi^{-1})_*\left(\tilde{\nabla}_0\tilde{u}^i\frac{\partial}{\partial y^i}\right)=\left(\nabla_0 u^i+\frac{\partial x^j}{\partial s}\nabla_j u^i\right)\frac{\partial}{\partial x^i}.$$

We set $|\tilde{\nu}|^2=\tilde{\nu}_i\tilde{\nu}_i$, $\tilde{\tilde{\nu}}_i=|\tilde{\nu}|^{-1}\tilde{\nu}_i$ for $(\tilde{\nu}_i)=(\tilde{\nu}_i(s,y'))$ and further

$$(2.5)\qquad \tilde{\tilde{a}}^{ijkh}(s,y)=\tilde{a}^{ijkh}(s,y)-g^{ik}\frac{\partial y^j}{\partial t}\frac{\partial y^h}{\partial t}.$$

Since $(\tilde{\tilde{\nu}}_1,\cdots,\tilde{\tilde{\nu}}_n)=(0,\cdots,0,-1)$ as easily verified, it follows from Proposition 2.2 (iii) that

$$(2.6)\qquad \tilde{\nu}_i\frac{\partial y^i}{\partial t}=-|\tilde{\nu}|\frac{\partial y^n}{\partial t}=0\qquad\text{on}\quad \Phi(\hat{\Gamma}\cap U)\subset\boldsymbol{R}\times\partial\boldsymbol{R}^n_+.$$

Hence the following proposition is derived from (2.4).

Proposition 2.4. *With the above notation, the equations* (D.1) *and* $(1.8)_\varepsilon$ *are transformed in* U *by* Φ *to the following form*:

$$\frac{\partial^2 \tilde{u}^i}{\partial s^2}+2\frac{\partial y^j}{\partial t}\frac{\partial^2 \tilde{u}^i}{\partial s \partial y^j}-\frac{\partial}{\partial y^j}\left(\tilde{\tilde{a}}^{ijkh}\frac{\partial \tilde{u}_k}{\partial y^h}\right)+(\textit{at most 1st order terms of } (\tilde{u}^j))^i=\tilde{f}^i$$

$$\textit{in} \quad \Phi(\hat{\Omega}\cap U)\subset \boldsymbol{R}\times\boldsymbol{R}^n_+ ,$$

$$\tilde{\alpha}_\varepsilon\left[\tilde{\tilde{\nu}}_j\tilde{\tilde{a}}^{ijkh}\frac{\partial \tilde{u}_k}{\partial y^h}+\varepsilon|\tilde{\nu}|^{-1}\frac{\partial \tilde{u}^i}{\partial s}+(\textit{0th order terms of } (\tilde{u}^j) \textit{ depending on } \varepsilon)^i\right]$$

$$+(1-\tilde{\alpha}_\varepsilon)|\tilde{\nu}|^{-1}\tilde{u}^i=0 \qquad \textit{on} \quad \Phi(\hat{\Gamma}\cap U)\subset\boldsymbol{R}\times\partial\boldsymbol{R}^n_+ .$$

Here, $(\tilde{\tilde{a}}^{ijkh}(s, y))$ *satisfies* (1.3), *and* $(\tilde{\tilde{\nu}}_1, \cdots, \tilde{\tilde{\nu}}_n)=(0, \cdots, 0, -1)$ *is the unit outer normal to* $\{y; (s, y)\in\Phi(\hat{\Gamma}\cap U)\}\subset\partial\boldsymbol{R}^n_+$ *for each* s *with respect to the "flat" metric on* $\boldsymbol{R}^n_y$.

2.4. Speed limit of $\Sigma(t)$. In our argument later, it will be essential that $(\tilde{\tilde{a}}^{ijkh}(s, y))$ defined by (2.5) satisfies (H.1) near (0,0), whose condition we present in Proposition 2.6 below. Before stating it, we define an important quantity $c_\Sigma(0, 0)$. (An alternative definition of it will be given in Appendix).

DEFINITION 2.5. We define $c_\Sigma(0, 0)$ by the supremum of $\sqrt{\kappa}$ such that

$$(2.7) \quad \int_{\boldsymbol{R}^n_+} a^{ijkh}(0, 0)\frac{\partial u^k}{\partial x^h}\overline{\frac{\partial u^i}{\partial x^j}}dx-\kappa\left\|\frac{\partial \boldsymbol{u}}{\partial x^{n-1}}\right\|^2_{\boldsymbol{R}^n_+}\geqq 0 \qquad \text{for all} \quad \boldsymbol{u}\in\boldsymbol{H}^1(\boldsymbol{R}^n_+).$$

We note that (H.1) guarantees the existence of $c_3>0$ such that

$$\int_{\boldsymbol{R}^n_+} a^{ijkh}(0, 0)\frac{\partial u^k}{\partial x^h}\overline{\frac{\partial u^i}{\partial x^j}}dx\geqq c_3\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{R}^n_+} \qquad \text{for all} \quad \boldsymbol{u}\in\boldsymbol{H}^1(\boldsymbol{R}^n_+) .$$

Moreover it is easily seen that the value $c_\Sigma(0, 0)$ is independent of the choice of a rotation R at the beginning of Subsection 2.2.

Proposition 2.6. *Let* U, Φ *be as in Proposition* 2.2 *and* $\tilde{\tilde{a}}^{ijkh}$ *as given by* (2.5). *If and only if the moving speed* $|\partial g/\partial t(0, 0)|$ *of* $\Sigma(t)$ *at* (0, 0) *is smaller than* $c_\Sigma(0, 0)$, *there exist an open neighborhood* $\tilde{V}$ *of* 0 *in* $\boldsymbol{R}^n_y$ *and positive constants* δ, c_4, c_5 *such that* $\Phi^{-1}([-\delta, \delta]\times\tilde{V})\subset U$ *and*

$$\int_{\boldsymbol{R}^n_+}\tilde{\tilde{a}}^{ijkh}(s, y)\frac{\partial v^k}{\partial y^h}\overline{\frac{\partial v^i}{\partial y^j}}dy\geqq c_4\|\boldsymbol{v}\|^2_{1,\boldsymbol{R}^n_+}-c_5\|\boldsymbol{v}\|^2_{\boldsymbol{R}^n_+}$$

for all $s\in[-\delta, \delta]$ *and* $\boldsymbol{v}\in\boldsymbol{H}^1(\boldsymbol{R}^n_+)$ *with support in* $\overline{\boldsymbol{R}^n_+}\cap\tilde{V}$.

Proof. We have only to show that, if and only if $|\partial g/\partial t(0, 0)|<c_\Sigma(0, 0)$, there exists a constant $c_6>0$ such that

$$(2.8)\qquad \int_{\boldsymbol{R}_+^n} \tilde{\tilde{a}}^{ijkh}(0,0)\frac{\partial v^k}{\partial y^h}\overline{\frac{\partial v^i}{\partial y^j}}dy \geqq c_6\|\nabla v\|^2 \qquad \text{for all} \quad v\in \boldsymbol{H}^1(\boldsymbol{R}_+^n).$$

Since Proposition 2.2 (iii) indicates

$$\tilde{\tilde{a}}^{ijkh}(0,0) = a^{ijkh}(0,0)-\delta^{ik}\delta^j_{n-1}\delta^h_{n-1}\left|\frac{\partial g}{\partial t}(0,0)\right|^2,$$

we obtain by the definition of $c_\Sigma(0,0)$ that

$$\begin{aligned}
&\text{the left-hand side of (2.8)}\\
&= \int_{\boldsymbol{R}_+^n} a^{ijkh}(0,0)\frac{\partial v^k}{\partial y^h}\overline{\frac{\partial v^i}{\partial y^j}}dy - \left|\frac{\partial g}{\partial t}(0,0)\right|^2 \left\|\frac{\partial v}{\partial y^{n-1}}\right\|^2_{\boldsymbol{R}_+^n}\\
&\geqq \left\{1-\left[\frac{\partial g}{\partial t}(0,0)/c_\Sigma(0,0)\right]^2\right\}\int_{\boldsymbol{R}_+^n} a^{ijkh}(0,0)\frac{\partial v^k}{\partial y^h}\overline{\frac{\partial v^i}{\partial y^j}}dy,
\end{aligned}$$

where the constant in front of the last integral is best possible. From this fact, the desired assertion follows immediately. Q.E.D.

3. Auxiliary problem

This section is self-contained by itself, while it will yield some results essential for our constructing a weak solution of (D).

Let Ω, Γ, $\hat{\Omega}$, etc., be as in Section 1 and let $(g_{ik}(t, x))$ be an $n\times n$ symmetric matrix of C^∞-functions $g_{ik}(t,x)$ on $\overline{\hat{\Omega}}$ such that

$$(3.1)\qquad c_0^{-1}I \leqq (g_{ik}(t,x)) \leqq c_0 I \qquad \text{in } \Omega,\quad c_0\geqq 1:\text{ const}.$$

Using $(g_{ik}(t, x))$, we put at each $t\in\boldsymbol{R}$

$$v_i = g_{ik}(t,\cdot)v^k \qquad \text{for} \quad v=(v^i(x)).$$

Now, we define differential systems $L(t)$ and $B_\alpha(t)$ operating on $\boldsymbol{u}=(u^i(t,x))$ as follows:

$$L(t)\boldsymbol{u} = \left[\frac{\partial^2}{\partial t^2}+a_1(t,x;D)\frac{\partial}{\partial t}+a_2(t,x;D)\right]\boldsymbol{u} \qquad \text{in } \hat{\Omega},$$

$$B_\alpha(t)\boldsymbol{u} = \alpha(\dot{x})\left(b(t,\dot{x};D)+\sigma(t,\dot{x})\frac{\partial}{\partial t}\right)\boldsymbol{u}+(1-\alpha(\dot{x}))\omega(t,\dot{x})\boldsymbol{u} \qquad \text{on } \hat{\Gamma}$$

which we supplement with

$$\begin{aligned}
[a_1(t,x;D)v]^i &= 2d^{ikj}(t,x)\frac{\partial v_k}{\partial x^j}+e^{ik}(t,x)v_k\\
&= 2d^{ikj}(t,x)\frac{\partial}{\partial x^j}(g_{kl}(t,x)v^l)+e^{ik}(t,x)g_{kl}(t,x)v^l,
\end{aligned}$$

$$[a_2(t, x; D)\boldsymbol{v}]^i = -\frac{\partial}{\partial x^j}\left[a^{ijkh}(t, x)\frac{\partial v_k}{\partial x^h}\right]+b^{ikj}(t, x)\frac{\partial v_k}{\partial x^j}+c^{ik}(t, x)v_k ,$$

$$[b(t, \dot{x}; D)\boldsymbol{v}]^i = \nu_j(\dot{x})a^{ijkh}(t, \dot{x})\frac{\partial v_k}{\partial x^h}+\tau^{ik}(t, \dot{x})v_k ,$$

$$[\sigma(t, \dot{x})\boldsymbol{v}]^i = \sigma^{ik}(t, \dot{x})v_k , \qquad [\omega(t, \dot{x})\boldsymbol{v}]^i = \omega^{ik}(t, \dot{x})v_k .$$

Here, all the coefficients (including $g_{ik}(i, x)$, $\alpha(\dot{x})$) are *real-valued* C^∞-functions on $\overline{\hat{\Omega}}$ or $\hat{\Gamma}$ (or Γ), and D stands for the x-derivation.

In this section, we study a mixed problem

$$(3.2) \qquad \begin{cases} L(t)\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \hat{\Omega}_T , & B_\alpha(t)\boldsymbol{u} = \alpha(\dot{x})\boldsymbol{\phi} \quad \text{on } \hat{\Gamma}_T , \\ \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0 , & \dfrac{\partial \boldsymbol{u}}{\partial t}(0, \cdot) = \boldsymbol{u}_1 \quad \text{in } \Omega , \end{cases}$$

under the following conditions:

(a) $0 \leqq \alpha(\dot{x}) \leqq 1$ on Γ, and the boundary of $\Gamma_\alpha := \{\dot{x} \in \Gamma;\ \alpha(\dot{x}) > 0\}$ forms a compact C^∞-submanifold of Γ of codimension 1 or is empty;

(b) $(a^{ijkh}(t, x))$ satisfies (1.3) and (H.1) with $\Gamma_N(t)$ replaced by Γ_α; under (1.3) this condition is, by (3.1), equivalent to

$$\int_\Omega a^{ijkh}(t, x)\frac{\partial v_k}{\partial x^h}\frac{\overline{\partial v_i}}{\partial x^j}dx \geqq c_1\|\boldsymbol{v}\|_1^2 - c_2\|\boldsymbol{v}\|^2 , \quad c_1, c_2 > 0\text{: const} ,$$

$$\text{for all} \quad t \in \boldsymbol{R},\ \boldsymbol{v} = (v^i) \in \boldsymbol{H}_0^1(\Omega \cup \Gamma_\alpha) .$$

(c) $(d^{ikj}(t, x))$ is symmetric with respect to i and k on $\hat{\Omega}$;

(d) $(\omega^{ik}(t, \dot{x}))$ is symmetric and positive definite on $\hat{\Gamma}$;

(e) there exists a constant $c_3 > 0$ such that

$$(d^{ikj}(t, \dot{x})\nu_j(\dot{x})) + (\sigma_s^{ik}(t, \dot{x})) \geqq c_3 I \qquad \text{on} \quad \boldsymbol{R} \times \overline{\Gamma_\alpha}$$

where $\sigma_s^{ik}(t, \dot{x}) = (\sigma^{ik}(t, \dot{x}) + \sigma^{ki}(t, \dot{x}))/2$.

3.1. Function spaces. We treat (3.2) in the following form:

$$\begin{cases} \dfrac{d}{dt}U(t) = \mathcal{A}(t)U(t) + F(t) , \quad U(0) = U_0 , \\ \mathcal{B}_\alpha(t)U(t) = \alpha(\dot{x})\boldsymbol{\phi}(t, \dot{x}) \end{cases}$$

where

$$U(t) = \begin{bmatrix} \boldsymbol{u}(t, x) \\ \dfrac{\partial \boldsymbol{u}(t, x)}{\partial t} \end{bmatrix}, \quad F(i) = \begin{bmatrix} 0 \\ \boldsymbol{f}(t, x) \end{bmatrix}, \quad U_0 = \begin{bmatrix} \boldsymbol{u}_0(x) \\ \boldsymbol{u}_1(x) \end{bmatrix},$$

$$\mathcal{A}(t) = \begin{bmatrix} 0 & I \\ -a_2(t, x; D) & -a_1(t, x; D) \end{bmatrix},$$

$$\mathscr{B}_\alpha(t) = [\alpha(\dot{x})b(t, \dot{x}; D)+(1-\alpha(\dot{x}))\omega(t, \dot{x}) \quad \alpha(\dot{x})\sigma(t, \dot{x}]) .$$

For simplicity, we write as $U_0=\{\boldsymbol{u}_0, \boldsymbol{u}_1\}$ instead of the above.

We introduce several function spaces which we utilize in this section. For an integer $m\geqq 1$, we equip the Banach space $\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega)$ with the norm $|||\cdot|||_m$ given by

$$|||U|||_m^2 = ||\boldsymbol{u}||_m^2+||\boldsymbol{v}||_{m-1}^2 \qquad \text{for} \quad U=\{\boldsymbol{u}, \boldsymbol{v}\}\in\boldsymbol{H}^m(\Omega)\times\boldsymbol{H}^{m-1}(\Omega) .$$

We denote by $\boldsymbol{V}_\alpha(\Omega)$ the completion of $\boldsymbol{C}_0^\infty(\Omega\cup\Gamma_\alpha)$ with respect to the norm $||\cdot||_{V_\alpha(\Omega)}$ given by

$$||\boldsymbol{u}||_{V_\alpha(\Omega)}^2 = ||\boldsymbol{u}||_1^2+\int_{\Gamma_\alpha}\frac{1-\alpha(\dot{x})}{\alpha(\dot{x})}|\boldsymbol{u}(\dot{x})|^2 d\dot{x};$$

thanks to condition (a), $\boldsymbol{V}_\alpha(\Omega)=\{\boldsymbol{u}\in\boldsymbol{H}^1(\Omega);\ ||\boldsymbol{u}||_{V_\alpha(\Omega)}<\infty\}$. For each t, $\mathscr{H}(t)$ denotes the Hilbert space $\boldsymbol{H}^1(\Omega)\times\boldsymbol{L}^2(\Omega)$ equipped with the inner product $(\cdot,\cdot)_{\mathscr{H}(t)}$ and the associated norm $||\cdot||_{\mathscr{H}(t)}$ given by

$$(F, U)_{\mathscr{H}(t)} = a[t; \boldsymbol{f}, \boldsymbol{u}]+c_0c_2(\boldsymbol{f}, \boldsymbol{u})_t+(\boldsymbol{g}, \boldsymbol{v})_t$$
$$\text{for} \quad F=\{\boldsymbol{f}, \boldsymbol{g}\},\ U=\{\boldsymbol{u}, \boldsymbol{v}\}\in\boldsymbol{H}^1(\Omega)\times\boldsymbol{L}^2(\Omega) .$$

Here, for convenience, we use the notation

$$(\boldsymbol{u}, \boldsymbol{v})_t = \int_\Omega u^i\overline{v_i}dx = \int_\Omega u^i\overline{v^k}g_{ik}(t, x)dx ,$$
$$a[t; \boldsymbol{u}, \boldsymbol{v}] = \int_\Omega a^{ijkh}(x, t)\frac{\partial u_k}{\partial x^h}\overline{\frac{\partial v_i}{\partial x^j}}dx \qquad \text{for} \quad \boldsymbol{u}=(u^i),\ \boldsymbol{v}=(v^i) .$$

Remark that (3.1) and (b) guarantee the uniform equivalence in t of the norms $|||\cdot|||_1$ and $||\cdot||_{\mathscr{H}(t)}$ on $\boldsymbol{H}^1(\Omega)\times\boldsymbol{L}^2(\Omega)$. Moreover, $\mathcal{V}_\alpha(t)$ stands for the Hilbert space $\boldsymbol{V}_\alpha(\Omega)\times\boldsymbol{L}^2(\Omega)$ equipped with the inner product $(\cdot,\cdot)_{\mathcal{V}_\alpha(t)}$ and the associated norm $||\cdot||_{\mathcal{V}_\alpha(t)}$ given by

$$(F, U)_{\mathcal{V}_\alpha(t)} = (F, U)_{\mathscr{H}(t)}+\int_{\Gamma_\alpha}\frac{1-\alpha(\dot{x})}{\alpha(\dot{x})}\omega^{ik}(t, \dot{x})f_k\overline{u_i}d\dot{x}$$
$$\text{for} \quad F=\{\boldsymbol{f}, \boldsymbol{g}\},\ U=\{\boldsymbol{u}, \boldsymbol{v}\}\in\boldsymbol{V}_\alpha(\Omega)\times\boldsymbol{L}^2(\Omega);$$

by condition (d), $||\cdot||_{\mathcal{V}_\alpha(t)}$ are equivalent norms of $\boldsymbol{V}_\alpha(\Omega)\times\boldsymbol{L}^2(\Omega)$. Finally, we define $\mathscr{D}_\alpha(t)$ for each t by

$$\mathscr{D}_\alpha(t) = \{U\in\boldsymbol{H}^2(\Omega)\times\boldsymbol{V}_\alpha(\Omega);\ \mathscr{B}_\alpha(t)U=\boldsymbol{0} \text{ on } \Gamma\} .$$

3.2. Energy inequalities. This subsection corresponds to Inoue [12; Section 3].

Lemma 3.1. *There exists a constant $C>0$ such that*

$$\mathrm{Re}\,(\mathcal{A}(t)U,\ U)_{\mathcal{V}_\alpha(t)} \leqq C(\|U\|^2_{\mathcal{H}(t)}+[\boldsymbol{\phi}]^2)$$

for all $U\in \boldsymbol{H}^2(\Omega)\times \boldsymbol{V}_\alpha(\Omega)$ satisfying $\mathcal{B}_\alpha(t)U=\alpha(\dot{x})\boldsymbol{\phi}$ with $\boldsymbol{\phi}\in \boldsymbol{L}^2(\Gamma)$.

Proof. Integrating by parts, we have for $U=\{\boldsymbol{u},\ \boldsymbol{v}\}\in \boldsymbol{H}^2(\Omega)\times \boldsymbol{V}_\alpha(\Omega)$

$$\begin{aligned}
&\mathrm{Re}(\mathcal{A}(t)\ U,\ U)_{\mathcal{H}(t)}\\
&= \mathrm{Re}\ \{a[t;\ \boldsymbol{v},\ \boldsymbol{u}]+c_0c_2(\boldsymbol{v},\ \boldsymbol{u})_t-(a_2(t,\ x;\ D)\boldsymbol{u}+a_1(t,\ x;\ D)\boldsymbol{v},\ \boldsymbol{v})_t\}\\
&\leqq \mathrm{Re}\int_{\Gamma_\alpha}[(b^0(t,\ \dot{x};\ D)\boldsymbol{u})^i-d^{iki}v_jv_k]\overline{v_i}d\dot{x}+C\|U\|^2_{\mathcal{H}(t)}
\end{aligned}$$

where $(b^0(t,\ \dot{x};\ D)\boldsymbol{u})^i=v_j(\dot{x})a^{ijkh}(t,\ \dot{x})\partial u_k/\partial x_h$. On the other hand, since $\mathcal{B}_\alpha(t)U=\alpha(\dot{x})\boldsymbol{\phi}$, it follows that

$$\begin{aligned}
\int_{\Gamma_\alpha}(b^0(t,\ \dot{x};D)\boldsymbol{u})^i\overline{v_i}d\dot{x} &= \int_{\Gamma_\alpha}\{[(b(t,\ \dot{x};\ D)\boldsymbol{u})^i+\sigma^{ik}v_k]-\sigma^{ik}v_k-\tau^{ik}u_k\}\overline{v_i}d\dot{x}\\
&= \int_{\Gamma_\alpha}\left[-\frac{1-\alpha}{\alpha}\omega^{ik}u_k\overline{v_i}+\phi^i\overline{v_i}-\sigma^{ik}v_k\overline{v_i}-\tau^{ik}u_k\overline{v_i}\right]d\dot{x}.
\end{aligned}$$

The combination of the above with the aid of (e) leads us to

$$\begin{aligned}
\mathrm{Re}\,(\mathcal{A}(t)U,\ U)_{\mathcal{V}_\alpha(t)} &\leqq -c_0^{-2}c_3[\boldsymbol{v}]^2+C[\boldsymbol{v}]([\boldsymbol{\phi}]+[\boldsymbol{u}])+C\|U\|^2_{\mathcal{H}(t)}\\
&\leqq C(\|U\|^2_{\mathcal{H}(t)}+[\boldsymbol{\phi}]^2)\,. \qquad \text{Q.E.D.}
\end{aligned}$$

Lemma 3.2. *There exists a number λ_0 such that, for any $\lambda>\lambda_0$, $\lambda I-\mathcal{A}(t)$ is a bijection from $\mathcal{D}_\alpha(t)$ onto $\mathcal{V}_\alpha(t)$ for each t.*

Proof. As easily seen, we have only to show that there exists a number λ_0 such that, for any $\{\boldsymbol{f},\ \boldsymbol{g}\}\in \boldsymbol{V}_\alpha(\Omega)\times \boldsymbol{L}^2(\Omega)$ and any $\lambda>\lambda_0$, the boundary-value problem

$$\begin{cases}
A_\lambda(t)\boldsymbol{u} := (a_2(t,\ x;\ D)+\lambda a_1(t,\ x;\ D)+\lambda^2)\boldsymbol{u} = (a_1(t,\ x;\ D)+\lambda)\boldsymbol{f}+\boldsymbol{g} \qquad \text{in}\quad \Omega,\\
B_{\alpha,\lambda}(t)\boldsymbol{u} := [\alpha(\dot{x})(b(t,\ \dot{x};\ D)+\lambda\sigma(t,\ \dot{x}))+(1-\alpha(\dot{x}))\omega(t,\ \dot{x})]\boldsymbol{u}\\
\qquad = \alpha(\dot{x})\sigma(t,\ \dot{x})\boldsymbol{f} \qquad \text{on}\quad \Gamma
\end{cases}$$

admits a solution $\boldsymbol{u}\in \boldsymbol{H}^2(\Omega)$. Applying Theorem I′ of Ito [16] to the above equations, regarded as, of (u_i), we see that the mapping

$$\{A_\lambda(t),\ B_{\alpha,\lambda}(t)\}:\boldsymbol{H}^2(\Omega)\ni \boldsymbol{u}\rightarrow \{A_\lambda(t)\boldsymbol{u},\ B_{\alpha,\lambda}(t)\boldsymbol{u}\}\in \boldsymbol{L}^2(\Omega)\times \boldsymbol{H}^{1/2}_{(\alpha)}(\Gamma) \tag{3.3}$$

is a Fredholm operator with index 0 for each λ and t where

$$\boldsymbol{H}^{1/2}_{(\alpha)}(\Gamma) := \{\boldsymbol{\phi} = \alpha(\dot{x})\boldsymbol{\phi}_1+(1-\alpha(\dot{x}))\boldsymbol{\phi}_0;\ \boldsymbol{\phi}_1\in \boldsymbol{H}^{1/2}(\Gamma)\,,\ \boldsymbol{\phi}_0\in \boldsymbol{H}^{3/2}(\Gamma)\}$$

is a Banach space equipped with the norm $[\cdot]_{\alpha;1/2}$ defined by

$$[\phi]_{\alpha;1/2}=\inf\{[\phi_1]_{1/2}+[\phi_0]_{3/2};\ \phi=\alpha(\dot{x})\phi_1+(1-\alpha(\dot{x}))\phi_0 \text{ for some } \phi_1\in \boldsymbol{H}^{1/2}(\Gamma),\ \phi_0\in \boldsymbol{H}^{3/2}(\Gamma)\}\,.$$

It is therefore sufficient to show that the kernel of (3.3) is a null space. Let $\boldsymbol{u}\in \boldsymbol{H}^2(\Omega)$ be in the kernel of (3.3), so that

$$\operatorname{Re}(A_\lambda(t)\boldsymbol{u},\boldsymbol{u})_t=0 \quad\text{and}\quad B_{\alpha,\lambda}(t)\boldsymbol{u}=\boldsymbol{0}\,.$$

Integrating by parts and using (b)–(e), we have, when $\lambda>0$,

$$\begin{aligned} a[t;\boldsymbol{u},\boldsymbol{u}]+\lambda^2(\boldsymbol{u},\boldsymbol{u})_t &\leqq -\int_{\Gamma_\alpha}\left\{\lambda(d^{ikj}\nu_j+\sigma_s^{ik})u_k\overline{u_i}+\frac{1-\alpha}{\alpha}\omega^{ik}u_k\overline{u_i}\right\}d\dot{x} \\ &\quad +C([\boldsymbol{u}]^2+\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|+\lambda\|\boldsymbol{u}\|^2) \\ &\leqq \varepsilon\|\boldsymbol{u}\|_1^2+(C(\varepsilon)+C\lambda)\|\boldsymbol{u}\|^2 \end{aligned}$$

for any $\varepsilon>0$ and some $C(\varepsilon)>0$. This estimate with the aid of (b) implies that $\boldsymbol{u}=\boldsymbol{0}$ for sufficiently large λ. Q.E.D.

Lemma 3.3. *Let μ, t be real parameters.*

(i) *If $\boldsymbol{u}\in \boldsymbol{H}^2(\Omega)$ is a solution of the boundary-value problem*

$$\text{(3.4)}\qquad \begin{cases} (a_2(t,x;D)+\mu)\boldsymbol{u}=\boldsymbol{f} & \text{in } \Omega\,,\\ (\alpha(\dot{x})b(t,\dot{x};D)+(1-\alpha(\dot{x}))\omega(t,\dot{x}))\boldsymbol{u}=\alpha(\dot{x})\phi & \text{on } \Gamma \end{cases}$$

with given data $\boldsymbol{f}\in \boldsymbol{L}^2(\Omega)$ and $\phi\in \boldsymbol{H}^{1/2}(\Gamma)$, then we have the estimate

$$\text{(3.5)}\qquad \|\boldsymbol{u}\|_2^2\leqq C(\|\boldsymbol{f}\|^2+[\phi]_{1/2}^2+\|\boldsymbol{u}\|^2)\,,\qquad C>0\text{: const.}$$

And, if μ is sufficiently large, $\boldsymbol{u}$ is a unique solution, and the term $\|\boldsymbol{u}\|^2$ in the right-hand side of (3.5) *can be eliminated.*

(ii) *There exists a constant $C>0$ such that*

$$|||U|||_2^2\leqq C(\|\mathcal{A}(t)U\|_{\mathcal{H}(t)}^2+\|U\|_{\mathcal{H}(t)}^2+[\phi]_{1/2}^2)$$

for all $U\in \boldsymbol{H}^2(\Omega)\times \boldsymbol{V}_\alpha(\Omega)$ satisfying $\mathcal{B}_\alpha(t)U=\alpha(\dot{x})\phi$ with $\phi\in \boldsymbol{H}^{1/2}(\Gamma)$.

Proof. (i) Similar argument to the proof of the preceding lemma shows the existence of a number μ_0 such that, for any $\mu\geqq\mu_0$, (3.4) has a unique solution $\boldsymbol{u}\in \boldsymbol{H}^2(\Omega)$ with the estimate $\|\boldsymbol{u}\|_2^2\leqq C(\|\boldsymbol{f}\|^2+[\phi]_{1/2}^2)$. It is now easy to get the estimate (3.5) in the case $\mu<\mu_0$.

(ii) An easy application of (i). Q.E.Dl

Lemma 3.4. *For each t, $\mathcal{D}_\alpha(t)$ is dense in $\boldsymbol{V}_\alpha(\Omega)\times \boldsymbol{L}^2(\Omega)$.*

Proof. Since $\boldsymbol{C}_0^\infty(\Omega)$ (resp. $\boldsymbol{C}_0^\infty(\Omega\cup\Gamma_\alpha)$) is dense in $\boldsymbol{L}^2(\Omega)$ (resp. in $\boldsymbol{V}_\alpha(\Omega)$), it is sufficient to show that, for any $\boldsymbol{u}\in \boldsymbol{C}_0^\infty(\Omega\cup\Gamma_\alpha)$, there exists a sequence

$\{u_\mu\}_{\mu=1}^{\infty}$ in $H^2(\Omega)$ such that

$$B_\alpha^1(t)u_\mu := \alpha(\dot{x})b(t, \dot{x}; D)u_\mu + (1-\alpha(\dot{x}))\omega(t, \dot{x})u_\mu = 0 \qquad \text{on} \quad \Gamma,$$
$$u_\mu \to u \qquad \text{in } V_\alpha(\Omega) \quad \text{as } \mu \to \infty .$$

However, for the future use, we will construct one in $C_0^\infty(\Omega \cup \Gamma_\alpha)$.

If we put $v_\mu = u - u_\mu$, a sequence $\{v_\mu\}_{\mu=1}^{\infty}$ in $C_0^\infty(\Omega \cup \Gamma_\alpha)$ satisfying

$$B_\alpha^1(t)v_\mu = B_\alpha^1(t)u \text{ on } \Gamma, \quad v_\mu \to 0 \text{ in } V_\alpha(\Omega) \text{ as } \mu \to \infty$$

is required. Take a vector function $v \in C^\infty(\bar{\Omega}) \cap H_0^1(\Omega)$ such that

$$b^0(t, \dot{x}; D)v = b(t, \dot{x}; D)u + \frac{1-\alpha(\dot{x})}{\alpha(\dot{x})}\omega(t, \dot{x})u \,(\in C^\infty(\Gamma)) \qquad \text{on} \quad \Gamma; \tag{3.6}$$

the existence of such v is verified by using the invertibility of $[a^{ijkh}(t, \dot{x})\nu_j(\dot{x})\nu_h(\dot{x})]_{i,k}$ on Γ for each t (by (H.1)) and the well-known fact that the mapping: $C^\infty(\bar{\Omega}) \ni w \to \{\partial w/\partial \nu, w\} \in C^\infty(\Gamma) \times C^\infty(\Gamma)$ admits a continuous right inverse. And choose a functions $\zeta_\mu \in C^\infty(\bar{\Omega})$ for each μ (resp. $\eta \in C_0^\infty(\Omega \cup \Gamma_\alpha)$) so that

$$0 \leqq \zeta_\mu \leqq 1, \ |\nabla \zeta_\mu| \leqq 2\mu \text{ on } \bar{\Omega};\ \zeta_\mu = 0 \text{ on } \Omega \backslash \Omega_{2/\mu}, \ =1 \text{ on } \overline{\Omega_{1/\mu}} \tag{3.7}$$
$$(\text{resp. } 0 \leqq \eta \leqq 1 \text{ on } \Omega,\ \eta = 1 \text{ in a neighborhood of supp } u)$$

where $\Omega_\delta = \{x \in \Omega; \operatorname{dis}(x, \Gamma) < \delta\}$. If we define $v_\mu = \zeta_\mu \eta v$, the sequence $\{v_\mu\}_{\mu=1}^{\infty}$ in $C_0^\infty(\Omega \cup \Gamma_\alpha)$ is a desired one. In fact, since $\nabla v = 0$ on $\Gamma \backslash \text{supp } u$, we have by (3.6) and (3.7)

$$B_\alpha^1(t)v_\mu = B_\alpha^1(t)v = \alpha(\dot{x})b^0(t, \dot{x}; D)v = B_\alpha^1(t)u, \quad v_\mu = v = 0 \qquad \text{on} \quad \Gamma,$$

and furthermore

$$\|v_\mu\|_{V_\alpha(\Omega)} \leqq C\|\nabla v_\mu\|^2 \leqq C' \int_{\Omega_{2/\mu}} \{(|\nabla \zeta_\mu|^2 + |\nabla \eta|^2)|v|^2 + |\nabla v|^2\} dx$$
$$\leqq C'' \int_{\Omega_{2/\mu}} |\nabla v|^2 dx \to 0 \qquad \text{as} \quad \mu \to \infty$$

where the last inequality is due to a Poincaré-type inequality

$$\int_{\Omega_{2/\mu}} |w|^2 dx \leqq \frac{C}{\mu^2} \int_{\Omega_{2/\mu}} |\nabla w|^2 dx \qquad \text{for all } w \in H_0^1(\Omega), \ \mu = 1, 2, \cdots \qquad \text{Q.E.D.}$$

Using Lemmas 3.1–3.4 and the Hille-Yosida theorem, we have:

Lemma 3.5. *Let $F(t) \in \mathcal{E}_t^1(\mathcal{V}_\alpha(t_0))$ and $U_0 \in \mathcal{D}_\alpha(t_0)$ for $t_0 \in \mathbf{R}$ fixed. Then there exists a unique solution $U(t)$ of the evolution equation*

$$\frac{d}{dt} U(t) = \mathcal{A}(t_0)U(t) + F(t), \quad U(0) = U_0$$

such that $U(t)\in\mathcal{D}_\alpha(t_0)$ *for each* $t\in[0, T]$ *and* $U(t)\in\mathcal{E}_t^1(\mathcal{V}_\alpha(t_0))$.

Now we prove the following two energy inequalities, the latter of which will play and important role in solving (3.2).

Proposition 3.6. *Let* $\boldsymbol{u}\in\mathcal{E}_t^0(\boldsymbol{H}^2(\Omega))\cap\mathcal{E}_t^1(\boldsymbol{V}_\alpha(\Omega))\cap\mathcal{E}_t^2(\boldsymbol{L}^2(\Omega))$ *satisfy*

$$(3.8)\qquad L(t)\boldsymbol{u}=\boldsymbol{f}\quad in\ \hat{\Omega}_T,\quad B_\alpha(t)\boldsymbol{u}=\alpha(\dot{x})\boldsymbol{\phi}\qquad on\ \hat{\Gamma}_T.$$

(i) *If* $\boldsymbol{f}\in\mathcal{E}_t^0(\boldsymbol{L}^2(\Omega))$ *and* $\boldsymbol{\phi}\in\mathcal{E}_t^0(\boldsymbol{L}^2(\Gamma))$, *we have*

$$\|\boldsymbol{u}(t,\cdot)\|_{V_\alpha(\Omega)}^2+\left\|\frac{\partial\boldsymbol{u}}{\partial t}(t,\cdot)\right\|^2\leqq C(T)\Big[\|\boldsymbol{u}(0,\cdot)\|_{V_\alpha(\Omega)}^2+\left\|\frac{\partial\boldsymbol{u}}{\partial t}(0,\cdot)\right\|^2$$
$$+\int_0^t(\|\boldsymbol{f}(\tau,\cdot)\|^2+[\boldsymbol{\phi}(\tau,\cdot)]^2)d\tau\Big].$$

(ii) *If* $\boldsymbol{f}\in\mathcal{E}_t^1(\boldsymbol{L}^2(\Omega))$ *and* $\boldsymbol{\phi}\in\mathcal{E}_t^0(\boldsymbol{H}^{1/2}(\Gamma))\cap\mathcal{E}_t^1(\boldsymbol{L}^2(\Gamma))$, *we have*

$$\|\boldsymbol{u}(t,\cdot)\|_2^2+\|\boldsymbol{u}(t,\cdot)\|_{V_\alpha(\Omega)}^2+\left\|\frac{\partial\boldsymbol{u}}{\partial t}(t,\cdot)\right\|_{V_\alpha(\Omega)}^2+\left\|\frac{\partial^2\boldsymbol{u}}{\partial t^2}(t,\cdot)\right\|^2$$
$$\leqq C(T)\Big[\|\boldsymbol{u}(0,\cdot)\|_2^2+\|\boldsymbol{u}(0,\cdot)\|_{V_\alpha(\Omega)}^2+\left\|\frac{\partial\boldsymbol{u}}{\partial t}(t,\cdot)\right\|_{V_\alpha(\Omega)}^2+\left\|\frac{\partial^2\boldsymbol{u}}{\partial t^2}(t,\cdot)\right\|^2$$
$$+\|\boldsymbol{f}(0,\cdot)\|^2+\sup_{0\leqq\tau\leqq t}[\boldsymbol{\phi}(\tau,\cdot)]_{1/2}^2+\int_0^t\Big(\left\|\frac{\partial\boldsymbol{f}}{\partial t}(\tau,\cdot)\right\|^2+\Big[\frac{\partial\boldsymbol{\phi}}{\partial t}(\tau,\cdot)\Big]^2\Big)d\tau\Big].$$

Proof. (i) By a standard argument (see [10; Proof of Lemma 3.8]).
(ii) Defining $\boldsymbol{u}_\delta(t,x)=(u_\delta^i(t,x))$ for small $\delta>0$ by

$$u_\delta^i(t,x)=\delta^{-1}(u^i(t+\delta,x)-u^i(t,x)),$$

we have from the former of (3.8)

$$(3.9)\qquad L(t)\boldsymbol{u}_\delta(t,x)=\boldsymbol{f}_\delta(t,x)-\Big(a_{1\delta}(t,x;D)\frac{\partial}{\partial t}+a_{2\delta}(t,x;D)\Big)\boldsymbol{u}(t+\delta,x)$$

where $a_{1\delta}(t,x;D)$ is given by

$$(3.10)\qquad (a_{1\delta}(t,x;D)\boldsymbol{v})^i=\Big[2d_\delta^{ikj}(t,x)\frac{\partial}{\partial x^j}+e_\delta^{ik}(t,x)\Big](g_{kl}(t+\delta,x)v^l)$$
$$+\Big[2d^{ikj}(t,x)\frac{\partial}{\partial x^j}+e^{ik}(t,x)\Big](g_{kl,\delta}(t,x)v^l)$$

for $\boldsymbol{v}=(v^i(x))$; $\boldsymbol{f}_\delta(t,x)$ and $a_{2\delta}(t,x;D)$ are defined similarly. The latter of (3.8) are rewritten as

$$\Big\{\alpha(\dot{x})\pi(t,\dot{x})\Big[b(t,\dot{x};D)+\sigma(t,\dot{x})\frac{\partial}{\partial t}\Big]+(1-\alpha(\dot{x}))I\Big\}\boldsymbol{u}(t,\dot{x})$$
$$=\alpha(\dot{x})\pi(t,\dot{x})\boldsymbol{\phi}(t,\dot{x})\qquad\text{on }\hat{\Gamma}_T$$

where $\pi(t, \dot{x})$ denotes the inverse operator of $\omega(t, \dot{x})$, that is,

$$(\pi(t, \dot{x})\boldsymbol{v})^i = g^{ij}(t, \dot{x})\omega_{jk}(t, \dot{x})v^k \qquad \text{for} \quad \boldsymbol{v} = (v^i)$$

with $(\omega_{ik})=(\omega^{ik})^{-1}$ and $(g^{ik})=(g_{ik})^{-1}$. Thus we have

$$\begin{aligned}(3.11)\quad B_\alpha(t)\boldsymbol{u}_\delta(t, \dot{x}) = \alpha(\dot{x})\Big\{&\boldsymbol{\phi}_\delta(t, \dot{x})+\omega(t, \dot{x})\pi_\delta(t, \dot{x})\boldsymbol{\phi}(t+\delta, \dot{x})\\ &-\Big[\omega(t, \dot{x})\pi_\delta(t, \dot{x})\Big(b(t+\delta, \dot{x}; D)+\sigma(t+\delta, \dot{x})\frac{\partial}{\partial t}\Big)\\ &+\Big(b_\delta(t, \dot{x}; D)+\sigma_\delta(t, \dot{x})\frac{\partial}{\partial t}\Big)\Big]\,\boldsymbol{u}(t+\delta, \dot{x})\Big\}\end{aligned}$$

where $b_\delta(t, \dot{x}; D)$, $\sigma_\delta(t, \dot{x})$ and $\pi_\delta(t, \dot{x})$ are defined in a similar manner to (3.10). By applying (i) to (3.9) and (3.11) and then making $\delta\to 0$ there, we obtain for $t\in[0, T-\delta_0]$ with $\delta_0>0$ a small number

$$\begin{aligned}(3.12)\left\|\frac{d}{dt}U(t)\right\|^2_{V_\alpha(t)} \leqq C(T)\Big\{&|||U(0)|||_2^2+\int_{\Gamma_\alpha}\frac{1-\alpha(\dot{x})}{\alpha(\dot{x})}\left|\frac{\partial \boldsymbol{u}}{\partial t}(0, \dot{x})\right|^2 d\dot{x}\\ &+||\boldsymbol{f}(0, \cdot)||^2+\int_0^t\Big(|||U(\tau)|||_2^2+\left\|\frac{\partial \boldsymbol{f}}{\partial t}(\tau, \cdot)\right\|^2+[\boldsymbol{\phi}(\tau, \cdot)]^2+\Big[\frac{\partial\boldsymbol{\phi}}{\partial t}(\tau, \cdot)\Big]^2\Big)d\tau\Big\}\end{aligned}$$

where $C(T)$ is independent of $\delta_0>0$, so that (3.12) holds for all $t\in[0, T]$. Combining (i) above and Lemma 3.3 (ii) with (3.12) and using Gronwall's lemma, we get the desired inequality. Q.E.D.

3.3. Existence and regularity of the solution. This subsection corresponds to Inoue [12; Section 4].

Proposition 3.7. *Let* $\boldsymbol{f}\in\mathcal{E}_t^1(\boldsymbol{L}^2(\Omega))$ *and* $\boldsymbol{\phi}\in\mathcal{E}_t^0(\boldsymbol{H}^{1/2}(\Gamma))\cap\mathcal{E}_t^1(\boldsymbol{L}^2(\Gamma))$. *If* $\boldsymbol{\phi}(0, \dot{x})=\boldsymbol{0}$, *then the mixed problem*

$$(3.13)\qquad \begin{cases} L(t)\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \hat{\Omega}_T, \qquad B_\alpha(t)\boldsymbol{u} = \alpha(\dot{x})\boldsymbol{\phi} \quad \text{on } \hat{\Gamma}_T,\\ \boldsymbol{u}(0, \cdot) = \dfrac{\partial \boldsymbol{u}}{\partial t}(0, \cdot) = \boldsymbol{0} \quad \text{in } \Omega\end{cases}$$

has a unique solution $\boldsymbol{u}\in\mathcal{E}_t^0(\boldsymbol{H}^2(\Omega))\cap\mathcal{E}_t^1(\boldsymbol{V}_\alpha(\Omega))\cap\mathcal{E}_t^2(\boldsymbol{L}^2(\Omega))$.

Proof. Thanks to Proposition 3.6 (ii), it is sufficient to prove when $\boldsymbol{f}\in\mathcal{E}_t^1(\boldsymbol{H}_0^1(\Omega))$ and $\boldsymbol{\phi}\in\mathcal{E}_t^1(\boldsymbol{H}^{1/2}(\Gamma))$.

Let Δ_μ: $0=t_0<t_1<\cdots<t_\mu=T$, $\mu=1, 2, \cdots$, be the subdivision of $[0, T]$ into μ equal parts. For Δ_μ, we construct Cauchy's polygonal line $\boldsymbol{u}_\mu(t, x)$, $t\in[0, T]$, by

$$\boldsymbol{u}_\mu(t, x) = \boldsymbol{u}_{\mu\nu}(t, x) \qquad \text{if} \quad t\in[t_\nu, t_{\nu-1}]\,,\ 0\leqq\nu\leqq\mu-1\,,$$

where $\boldsymbol{u}_{\mu\nu}=\boldsymbol{u}_{\mu\nu}(t, x)$, $t\in[t_\nu, t_{\nu+1}]$, are determined inductively as follows: Let $\boldsymbol{u}_{\mu 0}=\boldsymbol{u}_{\mu 0}(t, x)$ for $t\in[t_0, t_1]$ be the solution of

$$\begin{cases} L(t_0)\boldsymbol{u}_{\mu 0} = \boldsymbol{f} \quad \text{in} \quad (t_0, t_1)\times\Omega\,, \qquad B_\alpha(t_0)\boldsymbol{u}_{\mu 0} = \alpha(\mathring{x})\boldsymbol{\phi} \quad \text{on} \quad [t_0, t_1]\times\Gamma\,, \\ \boldsymbol{u}_{\mu 0}(t_0, \cdot) = \dfrac{\partial \boldsymbol{u}_{\mu 0}}{\partial t}(t_0, \cdot) = \boldsymbol{0} \quad \text{in} \quad \Omega; \end{cases}$$

when $1\leqq\nu\leqq\mu-1$, let $\boldsymbol{u}_{\mu\nu}=\boldsymbol{u}_{\mu\nu}(t, x)$ for $t\in[t_\nu, t_{\nu+1}]$ be the solution of

$$\begin{cases} L(t_\nu)\boldsymbol{u}_{\mu\nu} = \boldsymbol{f} \quad \text{in} \quad (t_\nu, t_{\nu+1})\times\Omega\,, \\ B_\alpha(t_\nu)\boldsymbol{u}_{\mu\nu} = \alpha(\mathring{x})\Big\{\boldsymbol{\phi}(t, \mathring{x}) \\ \qquad + \dfrac{t_{\nu+1}-t}{t_{\nu+1}-t_\nu}[B(t_\nu)-\omega(t_\nu, \mathring{x})\pi(t_{\nu-1}, \mathring{x})B(t_{\nu-1})]\boldsymbol{u}_{\mu,\nu-1}(t, x)\,|_{t=t_\nu} \\ \qquad - \dfrac{t_{\nu+1}-t}{t_{\nu+1}-t_\nu}[I-\omega(t_\nu, \mathring{x})\pi(t_{\nu-1}, \mathring{x})]\boldsymbol{\phi}(t_\nu, \mathring{x})\Big\} \quad \text{on} \quad [t_\nu, t_{\nu+1}]\times\Gamma\,, \\ \boldsymbol{u}_{\mu\nu}(t_\nu, \cdot) = \boldsymbol{u}_{\mu,\nu-1}(t_\nu, \cdot)\,, \qquad \dfrac{\partial \boldsymbol{u}_{\mu\nu}}{\partial t}(t_\nu, \cdot) = \dfrac{\partial \boldsymbol{u}_{\mu,\nu-1}}{\partial t}(t_\nu, \cdot) \quad \text{in} \quad \Omega\,. \end{cases}$$

For each $0\leqq\nu\leqq\mu-1$, since the compatibility condition of order 0 (see Definition 3.8) is satisfied at $t=t_\nu$ inductively, we can show the existence of such $\boldsymbol{u}_{\mu\nu}\in\mathcal{E}_t^0(\boldsymbol{H}^2(\Omega))\cap\mathcal{E}_t^1(\boldsymbol{V}_\alpha(\Omega))\cap\mathcal{E}_t^2(\boldsymbol{L}^2(\Omega))$ for $t\in[t_\nu, t_{\nu+1}]$ (see [12; Proof of Proposition 4.3]). Moreover, as easily seen, $\boldsymbol{u}_\mu$ is in the space $\mathcal{E}_t^0(\boldsymbol{H}^2(\Omega))\cap\mathcal{E}_t^1(\boldsymbol{V}_\alpha(\Omega))\cap\boldsymbol{H}^2(\hat{\Omega}_T)$.

Our remaining task is to show that $\{\boldsymbol{u}_\mu\}_{\mu=1}^{\infty}$ converges in some sense to the desired solution $\boldsymbol{u}$ of (3.13). This process is done in the same manner as in [12; Proof of Lemma 4.5] (see also Ikawa [9; Section 4]) with some modification. We only mention that, in proving what corresponds to Claim 2 of [12; Proof of Lemma 4.5], we need a device used in Proof of Proposition 3.6 (ii). Q.E.D.

With the aid of Proposition 3.6 (ii) and the preceding proposition, we obtain a solution of (3.2). Before stating the result, we introduce the compatibility condition.

DEFINITION 3.8. Let m be an integer$\geqq 0$. For given data $\{\boldsymbol{u}_0, \boldsymbol{u}_1\}$ in Ω, $\boldsymbol{f}$ in $\hat{\Omega}_T$ and $\boldsymbol{\phi}$ on $\hat{\Gamma}_T$ with suitable regularity, we say that they satisfy the compatibility condition to (3.2) of order $m\geqq 0$ at $t=0$ when the following relations hold on Γ:

$$\begin{cases} \displaystyle\sum_{q=0}^{p}\binom{p}{q}\{\alpha(\mathring{x})[b^{(q)}(0, \mathring{x}; D)\boldsymbol{u}_{p-q}+\sigma^{(q)}(0, \mathring{x})\boldsymbol{u}_{p-q+1}]+(1-\alpha(\mathring{x}))\omega^{(q)}(0, \mathring{x})\boldsymbol{u}_{p-q}\} \\ \qquad\qquad = \alpha(\mathring{x})\boldsymbol{\phi}^{(p)}(0, \mathring{x}) \quad \text{for} \quad 0\leqq p\leqq m, \\ \qquad\qquad \boldsymbol{u}_{m+1}\in \boldsymbol{V}_\alpha(\Omega) \end{cases}$$

with $\boldsymbol{u}_p=\boldsymbol{u}_p(x)$, $2\leqq p\leqq m+1$ (when $m\geqq 1$), defined successively by

$$(3.14)\quad \boldsymbol{u}_p = -\sum_{q=0}^{p-2}\binom{p-2}{q}\{a_2^{(q)}(0, x; D)\boldsymbol{u}_{p-q-2}+a_1^{(q)}(0, x; D)\boldsymbol{u}_{p-q-1}\} + \boldsymbol{f}^{(p-2)}(0, x)$$

where we use the notation such as

$$(3.15)\quad \begin{cases} \boldsymbol{\phi}^{(q)}(0, \dot{x}) = \left(\dfrac{\partial}{\partial t}\right)^q \boldsymbol{\phi}(t, \dot{x})\Big|_{t=0}, \\ a_2^{(q)}(0, x; D)\boldsymbol{v} = \left(\dfrac{\partial}{\partial t}\right)^q [a_2(t, x; D)\boldsymbol{v}]\Big|_{t=0} \qquad \text{for} \quad \boldsymbol{v} = (v^i(x)) . \end{cases}$$

Theorem 3.9. *For an integer $m\geqq 0$, let $\{\boldsymbol{u}_0, \boldsymbol{u}_1\}\in \boldsymbol{H}^{m+2}(\Omega)\times \boldsymbol{H}^{m+1}(\Omega)$, $\boldsymbol{f}\in\bigcap_{p=0}^{m-1}\mathcal{E}_t^p(\boldsymbol{H}^{m-p}(\Omega))\cap\mathcal{E}_t^{m+1}(\boldsymbol{L}^2(\Omega))$ and $\boldsymbol{\phi}\in\bigcap_{p=0}^{m}\mathcal{E}_t^p(\boldsymbol{H}^{m-p+1/2}(\Gamma))\cap\mathcal{E}_t^{m+1}(\boldsymbol{L}^2(\Gamma))$. If these data satisfy the compatibility condition to (3.2) of order m at $t=0$, then the mixed problem (3.2) admits a solution $\boldsymbol{u}\in\bigcap_{p=0}^{m+1}\mathcal{E}_t^p(\boldsymbol{H}^{m-p+2}(\Omega)\cap \boldsymbol{V}_\alpha(\Omega))\cap\mathcal{E}_t^{m+2}(\boldsymbol{L}^2(\Omega))$ unique in $\mathcal{E}_t^0(\boldsymbol{H}^2(\Omega))\cap\mathcal{E}_t^1(\boldsymbol{V}_\alpha(\Omega))\cap\mathcal{E}_t^2(\boldsymbol{L}^2(\Omega))$.*

Proof. We show only the case $m=0$; the case $m\geqq 1$ is shown by the same method as in Ikawa [9; Section 5].

We first take a sequence $\{\boldsymbol{u}_{1\mu}\}_{\mu=1}^{\infty}$ in $\boldsymbol{H}^2(\Omega)$ such that

$$(3.16)\quad \begin{aligned} &\{\alpha(\dot{x})b(0, \dot{x}; D)+(1-\alpha(\dot{x}))\omega(0, \dot{x})\}\boldsymbol{u}_{1\mu} = \boldsymbol{0} \qquad \text{on} \quad \Gamma, \\ &\boldsymbol{u}_{1\mu}\to\boldsymbol{u}_1 \ \text{in} \ \boldsymbol{V}_\alpha(\Omega) \ \text{as} \ \mu\to\infty \end{aligned}$$

in the same way as in Proof of Lemma 3.4. By virtue of Lemma 3.3 (i), the boundary-value problem

$$(3.17)\quad \begin{cases} (a_2(0, x; D)+\mu_0)\boldsymbol{u}_{0\mu} = (a_2(0, x; D)+\mu_0)\boldsymbol{u}_0 \qquad \text{in} \quad \Omega, \\ [\alpha(\dot{x})b(0, \dot{x}; D)+(1-\alpha(\dot{x}))\boldsymbol{\omega}(0, \dot{x})]\boldsymbol{u}_{0\mu} \\ \qquad = \alpha(\dot{x})[\boldsymbol{\phi}(0, \dot{x})-\sigma(0, \dot{x})\boldsymbol{u}_{1\mu}] \qquad \text{on} \quad \Gamma \end{cases}$$

with $\mu_0>0$ large enough has a unique solution $\boldsymbol{u}_{0\mu}\in\boldsymbol{H}^2(\Omega)$ for each μ, and the sequence $\{\boldsymbol{u}_{0\mu}\}_{\mu=1}^{\infty}$ converges to $\boldsymbol{u}_0$ in $\boldsymbol{H}^2(\Omega)$.

We next consider a mixed problem

$$(3.18)\quad \begin{cases} L(t)\boldsymbol{v}_\mu = \boldsymbol{f}-L(t)\boldsymbol{w}_\mu \quad \text{in} \quad \hat{\Omega}_T, \qquad B_\alpha(t)\boldsymbol{v}_\mu = \alpha(\dot{x})\boldsymbol{\phi}_\mu \quad \text{on} \quad \hat{\Gamma}_T, \\ \boldsymbol{v}_\mu(0, \cdot) = \dfrac{\partial\boldsymbol{v}_\mu}{\partial t}(0, \cdot) = \boldsymbol{0} \qquad \text{in} \quad \Omega \end{cases}$$

where

$$\boldsymbol{w}_\mu(t, x) = \boldsymbol{u}_{0\mu}+t\boldsymbol{u}_{1\mu},$$

$$\boldsymbol{\phi}_\mu(t, \dot{x}) = -[b(t, \dot{x}; D)-\omega(t, \dot{x})\pi(0, \dot{x})b(0, \dot{x}; D)]\boldsymbol{w}_\mu(t, \dot{x})$$
$$-[\sigma(t, \dot{x})-\omega(t, \dot{x})\pi(0, \dot{x})\sigma(0, \dot{x})]\boldsymbol{u}_{1\mu}+\boldsymbol{\phi}(t, \dot{x})-\omega(t, \dot{x})\pi(0, \dot{x})\boldsymbol{\phi}(0, \dot{x}).$$

Applying Proposition 3.7 to (3.18), we obtain its unique solution $\boldsymbol{v}_\mu \in \mathcal{E}_t^0(\boldsymbol{H}^2(\Omega)) \cap \mathcal{E}_t^1(\boldsymbol{V}_\alpha(\Omega)) \cap \mathcal{E}_t^2(\boldsymbol{L}^2(\Omega))$ for each μ. Since (3.16) and (3.17) indicate that $\boldsymbol{B}_\alpha(0)\boldsymbol{w}_\mu = \alpha(\dot{x})\boldsymbol{\phi}(0, \dot{x})$ on $\hat{\Gamma}_T$, we have

$$\begin{aligned}\boldsymbol{B}_\alpha(t)\boldsymbol{w}_\mu &= \alpha(\dot{x})\omega(t, \dot{x})\pi(0, \dot{x})\boldsymbol{\phi}(0, \dot{x})+[\boldsymbol{B}_\alpha(t)-\omega(t, \dot{x})\pi(0, \dot{x})\boldsymbol{B}_\alpha(0)]\boldsymbol{w}_\mu \\ &= \alpha(\dot{x})(\boldsymbol{\phi}-\boldsymbol{\phi}_\mu).\end{aligned}$$

Thus $\boldsymbol{u}_\mu := \boldsymbol{v}_\mu + \boldsymbol{w}_\mu$ is a solution of the mixed problem

$$\begin{cases} L(t)\boldsymbol{u}_\mu = \boldsymbol{f} \quad \text{in} \quad \hat{\Omega}_T, \qquad \boldsymbol{B}_\alpha(t)\boldsymbol{u}_\mu = \alpha(\dot{x})\boldsymbol{\phi} \quad \text{on} \quad \hat{\Gamma}_T, \\ \boldsymbol{u}_\mu(0, \cdot) = \boldsymbol{u}_{0\mu}, \quad \dfrac{\partial \boldsymbol{u}_\mu}{\partial t}(0, \cdot) = \boldsymbol{u}_{1\mu} \quad \text{in} \quad \Omega. \end{cases}$$

Hence Proposition 3.6 (ii) shows that $\{\boldsymbol{u}_\mu\}_{\mu=1}^\infty$ converges to the unique solution $\boldsymbol{u}$ of (3.2) in $\mathcal{E}_t^0(\boldsymbol{H}^2(\Omega)) \cap \mathcal{E}_t^1(\boldsymbol{V}_\alpha(\Omega)) \cap \mathcal{E}_t^2(\boldsymbol{L}^2(\Omega))$. Q.E.D.

3.4. Dependence domain. Denoting by $\lambda_p(t, x; \xi)$, $1 \leqq p \leqq 2n$, the real roots of the characteristic equation of $\boldsymbol{L}(t)$:

$$\det [g^{ik}(t, x)\lambda^2 + 2d^{ikj}(t, x)\xi_j\lambda - a^{ijkh}(t, x)\xi_j\xi_h]_{i,k} = 0$$

for $(t, x) \in \hat{\Omega}$ and $\xi = (\xi_i) \in \boldsymbol{R}^n \setminus \{0\}$, we define

$$\lambda_{\max} = \sup_{\substack{(t,x)\in\hat{\Omega} \\ |\xi|=1}} \max_{1\leqq p\leqq 2n} |\lambda_p(t, x; \xi)|.$$

We begin by studying how the equations

$$(3.19) \qquad L(t)\boldsymbol{u} = \boldsymbol{0} \quad \text{in} \quad \hat{\Omega}, \qquad \boldsymbol{B}_\alpha(t)\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \hat{\Gamma}$$

are transformed by the change of variables

$$(3.20) \qquad s = \phi(t, x) \quad \text{and} \quad y = x$$

where $\phi(t, x)$ is a C^∞-function in a neighborhood of $\overline{\hat{\Omega}}$ such that $\phi(t, x) \equiv t$ for sufficiently large $|t|$ and

$$(3.21) \qquad \frac{\partial \phi}{\partial t} > \lambda_{\max} |\nabla_x \phi| \quad \text{on} \quad \overline{\hat{\Omega}}.$$

Denoting

$$\begin{aligned}&\tilde{u}^i(s, y) = u^i(t, x), \quad \tilde{a}^{ijkh}(s, y) = a^{ijkh}(t, x), \\ &\tilde{u}_k(s, y) = \tilde{g}_{ik}(s, y)\tilde{u}^i(s, y) = u_k(t, x), \\ &\tilde{\alpha}(\dot{y}) = \alpha(\dot{x}), \quad \tilde{\Omega} = \Omega, \quad \text{etc.,}\end{aligned}$$

we obtain from (3.19) for each $1\leqq i\leqq n$

$$\hat{g}^{ik}\frac{\partial^2\tilde{u}_k}{\partial s^2}+\left\{2\tilde{d}^{ikj}\frac{\widetilde{\partial\phi}}{\partial t}-(\tilde{a}^{ihkj}+\tilde{a}^{ijkh})\frac{\widetilde{\partial\phi}}{\partial x^h}\right\}\frac{\partial^2\tilde{u}_k}{\partial s\partial y^j}-\tilde{a}^{ijkh}\frac{\partial^2\tilde{u}_k}{\partial y^j\partial y^h}$$
$$+(\text{at most 1st order terms of } (\tilde{u}^k))^i=0 \qquad \text{in} \quad \boldsymbol{R}\times\tilde{\Omega}\,,$$
$$\tilde{\alpha}\left\{\tilde{\nu}_j\tilde{a}^{ijkh}\frac{\partial\tilde{u}_k}{\partial y^h}+\left[\tilde{\sigma}^{ik}\frac{\widetilde{\partial\phi}}{\partial t}+\tilde{\nu}_j\tilde{a}^{ijkh}\frac{\widetilde{\partial\phi}}{\partial x^h}\right]\frac{\partial\tilde{u}_k}{\partial s}+\tilde{\tau}^{ik}\tilde{u}_k\right\}+(1-\tilde{\alpha})\tilde{\omega}^{ik}\tilde{u}_k=0$$
$$\text{on} \quad \boldsymbol{R}\times\tilde{\Gamma}$$

where $\hat{g}^{ik}\in C^\infty(\boldsymbol{R}\times\overline{\tilde{\Omega}})$ are given by

$$\hat{g}^{ik}=\tilde{g}^{ik}\left[\frac{\widetilde{\partial\phi}}{\partial t}\right]^2+2\tilde{d}^{ikj}\frac{\widetilde{\partial\phi}}{\partial x^j}\frac{\widetilde{\partial\phi}}{\partial t}-\tilde{a}^{ijkh}\frac{\widetilde{\partial\phi}}{\partial x^j}\frac{\widetilde{\partial\phi}}{\partial x^h};$$

$(\hat{g}^{ik})$ is symmetric and positive definite by (3.21). With the notation

$$(\hat{g}_{ik})=(\hat{g}^{ik})^{-1}\,,\quad w_i=\tilde{u}_i\,,\quad w^k=\hat{g}^{ik}w_i=\hat{g}^{ik}\tilde{u}_i\,,$$

the above equations are rewritten as

$$\frac{\partial^2 w_i}{\partial s^2}+\hat{g}_{il}\hat{g}_{km}\left[2\tilde{d}^{lmj}\frac{\widetilde{\partial\phi}}{\partial t}-(\tilde{a}^{lhmj}+\tilde{a}^{ljmh})\frac{\widetilde{\partial\phi}}{\partial x^h}\right]\frac{\partial^2 w^k}{\partial s\partial y^j}-\hat{g}_{il}\hat{g}_{km}\tilde{a}^{ljmh}\frac{\partial^2 w^k}{\partial y^j\partial y^h}$$
$$+(\text{at most 1st order terms of } (w_j))_i=0 \qquad \text{in} \quad \boldsymbol{R}\times\tilde{\Omega}\,,$$
$$\tilde{\alpha}\left\{\tilde{\nu}_j\hat{g}_{il}\hat{g}_{km}\tilde{a}^{ljmh}\frac{\partial w^k}{\partial y^h}+\hat{g}_{il}\hat{g}_{km}\left[\tilde{\sigma}^{lm}\frac{\widetilde{\partial\phi}}{\partial t}+\tilde{\nu}_j\tilde{a}^{ljmh}\frac{\widetilde{\partial\phi}}{\partial x^h}\right]\frac{\partial w^k}{\partial s}\right.$$
$$\left.+(\text{0th order terms of } (w_j))_i\right\}+(1-\tilde{\alpha})\hat{g}_{il}\hat{g}_{km}\tilde{\omega}^{lm}w^k=0 \qquad \text{on} \quad \boldsymbol{R}\times\tilde{\Gamma}\,.$$

We then realize that the transformation (3.20) leaves conditions (a)–(e) invariant by considering the following correspondence:

$$\begin{aligned}
&g_{ik}\leftrightarrow\hat{g}^{ik}\,,\\
&u^i\leftrightarrow w_i:=\tilde{u}_i\,,\\
&a^{ijkh}\leftrightarrow\hat{a}_i{}^j{}_k{}^h:=\hat{g}_{il}\hat{g}_{km}\tilde{a}^{ljmh}\\
&a^{ikj}\leftrightarrow\hat{d}_{ik}{}^j:=\hat{g}_{il}\hat{g}_{km}\left[\tilde{d}^{lmj}\frac{\widetilde{\partial\phi}}{\partial t}-\frac{1}{2}(\tilde{a}^{lhmj}+\tilde{a}^{ljmh})\frac{\widetilde{\partial\phi}}{\partial x^h}\right],\\
&\omega^{ik}\leftrightarrow\hat{\omega}_{ik}:=\hat{g}_{il}\hat{g}_{km}\tilde{\omega}^{lm}\,,\\
&\sigma^{ik}\leftrightarrow\hat{\sigma}_{ik}:=\hat{g}_{il}\hat{g}_{km}\left[\tilde{\sigma}^{lm}\frac{\widetilde{\partial\phi}}{\partial t}+\tilde{\nu}_j\tilde{a}^{ljmh}\frac{\widetilde{\partial\phi}}{\partial x^h}\right].
\end{aligned}$$

Now, using Holmgren's transformation in a neighborhood U of $(0, x_0)\in\boldsymbol{R}^{n+1}$ with $x_0\in\overline{\Omega}$ (that is, $\phi(t, x)=t+|x-x_0|^2$ in U), we have the local uniqueness near $(0, x_0)$ (see, e.g., Inoue [11; Section 5]). Furthermore, the wellknown method of sweeping out shows that the mixed problem (3.2) has a finite propagation speed.

Theorem 3.10. *For* $(t_0, x_0)\in\overline{\hat{\Omega}}_T$, *we denote*

$$\Lambda(t_0, x_0) = \{(t, x);\ |x-x_0|<\lambda_{\max}(t_0-t),\ t\geqq 0\}\ .$$

Suppose that $\boldsymbol{u}\in \boldsymbol{C}^2(\overline{\hat{\Omega}}_T\cap\Lambda(t_0, x_0))$ *satisfies*

$$\begin{cases} L(t)\boldsymbol{u}=0 & \text{in}\quad \hat{\Omega}_T\cap\Lambda(t_0, x_0)\,, \qquad B_\alpha(t)\boldsymbol{u}=0 \quad \text{on}\quad \hat{\Gamma}_T\cap\Lambda(t_0, x_0)\,, \\ \boldsymbol{u}(0,\ \cdot)=\dfrac{\partial \boldsymbol{u}}{\partial t}(0,\ \cdot)=0 & \text{in}\quad \{x\in\Omega;\ (0, x)\in\Lambda(t_0, x_0)\}\,. \end{cases}$$

Then $\boldsymbol{u}$ *is identically zero in* $\hat{\Omega}_T\cap\Lambda(t_0, x_0)$.

4. Proof of Main Theorem

Now we come back to the original problem. All our argument in this section is under hypotheses (H.1) and (H.2).

4.1. Weak solution for locally-supported data. This subsection is devoted to proving the following local version of Main Theorem.

Theorem 4.1. *For any* $(t_0, x_0)\in\overline{\hat{\Omega}}_T$, *there exist a constant* $\delta>0$ *and a neighborhood* $W\subset[t_0-\delta,\ t_0+\delta]\times\boldsymbol{R}^n$ *of* (t_0, x_1) *which satisfy the following: For any* $t_1\in(t_0-\delta,\ t_0+\delta)$ *and any given data* $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\}\in \boldsymbol{V}_D(t_1)\times\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2((t_1,\ t_0+\delta)\times\Omega)$ *satisfying*

$$(4.1)\qquad (\operatorname{supp}\boldsymbol{u}_0)\cup(\operatorname{supp}\boldsymbol{u}_1)\subset\overline{\Omega}\cap W(t_1)\,,\quad \operatorname{supp}\boldsymbol{f}\subset([t_1,\ t_0+\delta]\times\overline{\Omega})\cap W$$

with $W(t_1)=\{x;\ (t_1, x)\in W\}$, *the mixed problem*

$$(4.2)\qquad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2}+A(t)\right)\boldsymbol{u}=\boldsymbol{f} & \text{in}\quad (t_1,\ t_0+\delta)\times\Omega\,, \\ B(t)\boldsymbol{u}=0 & \text{on}\quad \bigcup\limits_{t_1\leqq t\leqq t_0+\delta}\{t\}\times\Gamma_N(t)\,, \\ \boldsymbol{u}=0 & \text{on}\quad \bigcup\limits_{t_1\leqq t\leqq t_0+\delta}\{t\}\times\Gamma_D(t)\,, \\ \boldsymbol{u}(t_1,\ \cdot)=\boldsymbol{u}_0\,,\quad \dfrac{\partial\boldsymbol{u}}{\partial t}(t_1,\ \cdot)=\boldsymbol{u}_1 & \text{in}\quad \Omega \end{cases}$$

admits a weak solution $\boldsymbol{u}\in\boldsymbol{H}^1((t_1,\ t_0+\delta)\times\Omega)$ *such that*

(i) $\operatorname{supp}\boldsymbol{u}\subset([t_1,\ t_0+\delta]\times\overline{\Omega})\cap\overline{W}$;

(ii) $\|\boldsymbol{u}(t,\ \cdot)\|_1^2+\left\|\dfrac{\partial\boldsymbol{u}}{\partial t}(t,\ \cdot)\right\|^2\leqq C(\|\boldsymbol{u}_0\|_1^2+\|\boldsymbol{u}_1\|^2+\displaystyle\int_{t_1}^t\|\boldsymbol{f}(\tau,\ \cdot)\|^2d\tau)$

for all $t\in(t_1,\ t_0+\delta)$ *where* $C>0$ *is independent of* t_1 *and given data.*

The case $(\boldsymbol{t}_0, \boldsymbol{x}_0)\in\hat{\Sigma}_T$. We may assume that $(t_0, x_0)=(0, 0)$ and that Ω, Γ, $\hat{\Gamma}_D$ and $\hat{\Sigma}$ are represented near there as mentioned at the beginning of Subsection 2.2. Let U and Φ (resp. $\tilde{V}$ and δ) be as in Proposition 2.2 (resp. Proposition 2.6). And W is given as follows: choose open balls $\tilde{V}_0$, $\tilde{V}_1$ in $\boldsymbol{R}^n$ with the common center at 0 such that $\tilde{V}_0\Subset\tilde{V}_1\Subset\tilde{V}$, and define $\tilde{\lambda}_{\max}$, after shrinking U if necessary, by

$$\tilde{\lambda}_{\max}=\sup_{\substack{(s,y)\in\Phi(U)\\ |\xi|=1}}\max_{1\leqq p\leqq 2n}|\tilde{\lambda}_p(s, y;\xi)|<+\infty$$

where $\tilde{\lambda}_p(s, y;\xi)$, $1\leqq p\leqq 2n$, are the real roots of the characteristic equation of $\tilde{L}(s)$ (defined in $(4.5)_\varepsilon$):

$$\det\left[g^{ik}(s, y)\lambda^2+2g^{ik}(s, y)\frac{\partial\phi^j}{\partial t}(\Psi(s, y))\xi_j\lambda-\tilde{a}^{ijkh}(s, y)\xi_j\xi_h\right]_{i,k}=0\,.$$

Then, by replacing $\delta>0$ with a smaller value if necessary, we have

$$[-\delta, \delta]\times\tilde{V}_0\supset\tilde{W}:=\{(s, y)\in[-\delta, \delta]\times\boldsymbol{R}^n;\ |y|<\tilde{\lambda}_{\max}(s+\delta)\}\,.$$

We define an neighborhood W of $(0, 0)$ by $W=\Psi(\tilde{W})$ where $\Psi=\Phi^{-1}$.

We want to construct a weak solution of (4.2) by approximating with a solution of the following mixed problem with $\varepsilon\in(0, \varepsilon_0)$:

$$(4.3)\quad\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}+A(t)\right)\boldsymbol{u}=\boldsymbol{f}_\varepsilon & \text{in}\quad(t_1, \delta)\times\Omega\,,\\ \alpha_\varepsilon(t, \dot{x})(B(t)+\varepsilon X(t))\boldsymbol{u}+(1-\alpha_\varepsilon(t, \dot{x}))\boldsymbol{u}=\boldsymbol{0} & \text{on}\quad(t_1, \delta)\times\Gamma\,,\\ \boldsymbol{u}(t_1, \cdot)=\boldsymbol{u}_{0\varepsilon}\,,\quad\dfrac{\partial\boldsymbol{u}}{\partial t}(t_1, \cdot)=\boldsymbol{u}_{1\varepsilon} & \text{in}\quad\Omega\end{cases}$$

where ε_0 and $\alpha_\varepsilon(t, \dot{x})$ are as in Subsection 2.1, $\{\boldsymbol{u}_{0\varepsilon}, \boldsymbol{u}_{1\varepsilon}, \boldsymbol{f}_\varepsilon\}$ converge to $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\}$ as $\varepsilon\to 0$ (see Lemma 4.2) and $X(t)$ is a C^∞-extension outside U of the $X(t)$ given in Definition 2.3 in the form that

$$(4.4)\qquad X(t)\boldsymbol{u}=\frac{\partial\boldsymbol{u}}{\partial t}+\gamma^j(t, x)\frac{\partial\boldsymbol{u}}{\partial x^j}\qquad\text{for}\quad(t, x)\in[-\delta, \delta]\times\bar{\Omega}\,.$$

According to Proposition 2.4, $(4.3)_\varepsilon$ is transformed in U by Φ as

$$(4.5)_\varepsilon\quad\begin{cases}\tilde{L}(s)\boldsymbol{v}:=\left(\dfrac{\partial^2}{\partial s^2}+a_1(s, y;D)\dfrac{\partial}{\partial s}+a_2(s, y;D)\right)\boldsymbol{v}=\boldsymbol{g}_\varepsilon\\ \qquad\qquad\text{in}\quad\Phi(((t_1, \delta)\times\Omega)\cap U)\subset(t_1, \delta)\times\boldsymbol{R}^n_+\,,\\ \tilde{\alpha}_\varepsilon(y')\left(b_\varepsilon(s, y';D)+\sigma_\varepsilon(s, y')\dfrac{\partial}{\partial s}\right)\boldsymbol{v}+(1-\tilde{\alpha}_\varepsilon(y'))\omega(s, y')\boldsymbol{v}=\boldsymbol{0}\\ \qquad\qquad\text{on}\quad\Phi(([t_1, \delta]\times\Gamma)\cap U)\subset[t_1, \delta]\times\partial\boldsymbol{R}^n_+\,,\\ \boldsymbol{v}(t_1, \cdot)=\boldsymbol{v}_{\varepsilon 0}\,,\quad\dfrac{\partial\boldsymbol{v}}{\partial s}(t_1, \cdot)=\boldsymbol{v}_{1\varepsilon}\qquad\text{in}\quad\Phi((\{t_1\}\times\Omega)\cap U)\subset\{t_1\}\times\boldsymbol{R}^n_+\end{cases}$$

where

$$
(4.6)\quad
\begin{cases}
\boldsymbol{v} = (v^i) \quad \text{with} \quad v^i = \tilde{u}^i = \dfrac{\partial \phi^i}{\partial x^k}(\Psi(s, y))u^k(\Psi(s, y)), \\
\boldsymbol{g}_\varepsilon = (g^i_\varepsilon) \quad \text{with} \quad g^i_\varepsilon = \tilde{f}^i_\varepsilon, \\
\boldsymbol{v}_{0\varepsilon} = (v^i_{0\varepsilon}) \quad \text{with} \quad v^i_{0\varepsilon} = \tilde{u}^i_{0\varepsilon} = \dfrac{\partial \phi^i}{\partial x^k}(\Psi(t_1, y))u^k(\Psi(t_1, y)), \\
\boldsymbol{v}_{1\varepsilon} = (v^i_{1\varepsilon}) \quad \text{with} \quad v^i_{1\varepsilon} = \tilde{u}^i_{1\varepsilon} - \dfrac{\partial \phi^j}{\partial t}(\Psi(t_1, y))\dfrac{\partial \tilde{u}^i_{0\varepsilon}}{\partial y^j} \\
\qquad\qquad\qquad\qquad + \dfrac{\partial^2 \phi^i}{\partial t \partial x^j}(\Psi(t_1, y))\dfrac{\partial \psi^j}{\partial y^k}(t_1, y)\tilde{u}^k_{0\varepsilon}
\end{cases}
$$

with the notation as in Subsection 2.3 and

$$y^j = \phi^j(t, x) \quad \text{for } (s, y) = \Phi(t, x), \quad x^i = \psi^i(s, y) \quad \text{for } (t, x) = \Psi(s, y);$$

the last of (4.6) is due to the formula

$$\Phi_*\Big(\nabla_0 u^i \frac{\partial}{\partial x^i}\Big) = \Big(\tilde{\nabla}_0 \tilde{u}^i + \frac{\partial y^j}{\partial t}\tilde{\nabla}_j \tilde{u}^i\Big)\frac{\partial}{\partial y^i} = \Big(\frac{\partial \tilde{u}^i}{\partial s} + \frac{\partial y^j}{\partial t}\frac{\partial \tilde{u}^i}{\partial y^j} - \frac{\partial^2 y^i}{\partial t \partial x^j}\frac{\partial x^j}{\partial y^k}\tilde{u}^k\Big)\frac{\partial}{\partial y^i}.$$

Moreover, with the notation at each s

$$w_i = g_{ik}(s, \cdot)w^k \qquad \text{for} \quad \boldsymbol{w} = (w^i(y)),$$

the operators a_1, a_2, b_ε, σ_ε and ω are in the following forms:

$$(a^1(s, y; D)\boldsymbol{w})^i = 2g^{ik}\frac{\partial \phi^j}{\partial t}\frac{\partial w_k}{\partial y^j} + (\text{0th order terms of } (w^j))^i,$$

$$(a_2(s, y; D)\boldsymbol{w})^i = -\frac{\partial}{\partial y^j}\Big(\tilde{\tilde{a}}^{ijkh}\frac{\partial w_k}{\partial y^h}\Big) + (\text{at most 1st order terms of } (w^j))^i,$$

$$(b_\varepsilon(s, y'; D)\boldsymbol{w})^i = \tilde{\nu}_j \tilde{\tilde{a}}^{ijkh}\frac{\partial w_k}{\partial y^h} + (\text{0th order terms of } (w^j) \text{ depending on } \varepsilon)^i,$$

$$(\sigma_\varepsilon(s, y')\boldsymbol{w})^i = \varepsilon|\tilde{\boldsymbol{\nu}}|^{-1}g^{ik}w_k, \quad (\omega(s, y')\boldsymbol{w})^i = |\tilde{\boldsymbol{\nu}}|^{-1}g^{ik}w_k.$$

In order to apply the results in the preceding section, we consider the following mixed problem modified from $(4.5)_\varepsilon$, $0<\varepsilon<\varepsilon_0$:

$$
(4.7)_\varepsilon\quad
\begin{cases}
\tilde{L}(s)\boldsymbol{v} = \Big(\dfrac{\partial^2}{\partial s^2} + a_1(s, y; D)\dfrac{\partial}{\partial s} + a_2(s, y; D)\Big)\boldsymbol{v} = \boldsymbol{g}_\varepsilon \qquad \text{in} \quad (t_1, \delta)\times\omega, \\
\tilde{B}_\varepsilon(s)\boldsymbol{v} := \tilde{\beta}_\varepsilon(\mathring{y})\Big(b_\varepsilon(s, \mathring{y}; D) + \sigma_\varepsilon(s, \mathring{x})\dfrac{\partial}{\partial s}\Big)\boldsymbol{v} \\
\qquad\qquad + (1-\tilde{\beta}_\varepsilon(\mathring{y}))\omega(s, \mathring{y})\boldsymbol{v} = \boldsymbol{0} \qquad \text{on} \quad [t_1, \delta]\times\gamma, \\
\boldsymbol{v}(t_1, \cdot) = \boldsymbol{v}_{0\varepsilon}, \quad \dfrac{\partial \boldsymbol{v}}{\partial s}(t_1, \cdot) = \boldsymbol{v}_{1\varepsilon} \qquad \text{in} \quad \omega.
\end{cases}
$$

Here ω is a bounded subdomain of $\tilde{V}$ with C^∞-boundary $\gamma=\partial\omega$ chosen so that $\boldsymbol{R}^n_+ \cap \tilde{V}_1 \subset \omega \subset \boldsymbol{R}^n_+ \cap \tilde{V}$ and γ is represented by $y^n=0$ near $\tilde{V}_1$. Choose a C^∞-submanifold σ, diffeomorphic to an $(n-2)$-dimensional sphere, of γ of codimension 1 so that $\sigma \subset \{y \in \tilde{V}_1; y^{n-1} \leqq y^n=0\}$ and is represented by $y^{n-1}=y^n=0$ near $\tilde{V}_0$. Then γ is divided by σ into two open subsets; the one including $\{y \in \tilde{V}_0; y^{n-1}<y^n=0\}$ is referred to as γ_N and the other as γ_D: $\gamma=\gamma_N \cup \sigma \cup \gamma_D$ (disjoint union). Making $\varepsilon_0>0$ smaller if necessary, we define $\tilde{\beta}_\varepsilon(\dot{y}) \in C^\infty(\gamma)$, $0<\varepsilon<\varepsilon_0$, in the same way as $\alpha_\varepsilon(t, \dot{x})$ with Γ and $\Sigma(t)$ replaced by γ and σ, so that $\tilde{\beta}_\varepsilon(\dot{y})=\tilde{\alpha}_\varepsilon(y')$ on $\gamma \cap \tilde{V}_0$. Further, an appropriate extension of $|\tilde{\boldsymbol{\nu}}|^{-1}$ outside $[-\delta, \delta] \times (\gamma \cap \tilde{V}_1)$ (resp. $(\tilde{\tilde{\nu}}_j)$ outside $\gamma \cap \tilde{V}_1$) makes σ_ε and ω_ε (resp. b_ε) to be forms to which we can apply the results of the preceding section. Finally, the data $\{\boldsymbol{v}_{0\varepsilon}, \boldsymbol{v}_{1\varepsilon}, \boldsymbol{g}_\varepsilon\}$ are given in the following lemma. The meaning of "the compatibility condition to $(4.5)_\varepsilon$" appearing below will be understood from Definition 3.8.

Lemma 4.2. *Let* $t_1 \in (-\delta, \delta)$ *and let* $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\} \in \boldsymbol{V}_D(t_1) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2((t_1, \delta) \times \Omega)$ *satisfy* (4.1). *For any integer* $m \geqq 0$, *there exists a family* $\{\boldsymbol{v}_{0\varepsilon}, \boldsymbol{v}_{1\varepsilon}, \boldsymbol{g}_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ *in* $\boldsymbol{C}^\infty(\bar{\omega}) \times \boldsymbol{C}^\infty(\bar{\omega}) \times \boldsymbol{C}^\infty(([t_1, \delta] \times \bar{\omega})$ *such that*

(i) $(\operatorname{supp} \boldsymbol{v}_{\varepsilon 0}) \cup (\operatorname{supp} \boldsymbol{v}_{1\varepsilon}) \cup (\operatorname{supp} \boldsymbol{g}_\varepsilon(t_1, \cdot)) \subset (\omega \cup \gamma_N) \cap \tilde{W}(t_1)$,
$\operatorname{supp} \boldsymbol{g}_\varepsilon \subset ([t_1, \delta] \times \bar{\omega}) \cap \tilde{W}$

where $\tilde{W}(t_1)=\{y \in \boldsymbol{R}^n; (t_1, y) \in \tilde{W}\}$;

(ii) $\{\boldsymbol{v}_{0\varepsilon}, \boldsymbol{v}_{1\varepsilon}, \boldsymbol{g}_\varepsilon\} \to \{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{g}\}$ *as* $\varepsilon \to 0$ *in* $\boldsymbol{H}^1(\omega) \times \boldsymbol{L}^2(\omega) \times \boldsymbol{L}^2((t_1, \delta) \times \omega)$ *where* $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{g}\}$ *are obtained by* (4.6) *from the given data* $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\}$;

(iii) $\{\boldsymbol{v}_{0\varepsilon}, \boldsymbol{v}_{1\varepsilon}, \boldsymbol{g}_\varepsilon\}$ *satisfy the compatibility condition to* $(4.5)_\varepsilon$ *of order* m *at* $s=t_1$.

Proof. We construct a desired family by induction with respect to m. The case $m=0$ is easy (see Proof of Lemma 3.4). We show the validity for the case $m \geqq 1$ assuming that for $m-1$. Define $\boldsymbol{v}_{p\varepsilon} \in \boldsymbol{C}^\infty(\bar{\omega})$, $2 \leqq p \leqq m+1$, from $\{\boldsymbol{v}_{0\varepsilon}, \boldsymbol{v}_{1\varepsilon}, \boldsymbol{g}_\varepsilon\}$ by (3.14) and choose $\boldsymbol{w}_\varepsilon \in \boldsymbol{H}^1_0(\omega) \cap \boldsymbol{C}^\infty(\bar{\omega})$ so that $\|\boldsymbol{w}_\varepsilon\|_1 \leqq \varepsilon$, $\operatorname{supp} \boldsymbol{w}_\varepsilon \subset (\omega \cup \gamma_N) \cap \tilde{W}(t_1)$ and

$$b_\varepsilon(t_1, y'; D)\boldsymbol{w}_\varepsilon = \sum_{p=0}^{m} \binom{m}{p} (b_\varepsilon^{(p)}(t_1, y'; D)\boldsymbol{v}_{m-p,\varepsilon} + \sigma_\varepsilon^{(p)}(t_1, y')\boldsymbol{v}_{m-p+1,\varepsilon})$$

(see Proof of Lemma 3.4) where $b_\varepsilon^{(p)}$ and $\sigma_\varepsilon^{(p)}$ are defined as in (3.15). Then the family $\{\boldsymbol{v}_{0\varepsilon}, \boldsymbol{v}'_{1\varepsilon}, \boldsymbol{g}'_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ defined by

$$\boldsymbol{v}'_{1\varepsilon}=\boldsymbol{v}_{1\varepsilon}-\boldsymbol{w}_\varepsilon, \quad \boldsymbol{g}'_\varepsilon = \boldsymbol{g}_\varepsilon - a_1(t_1, y; D)\boldsymbol{w}_\varepsilon \qquad \text{if} \quad m=1,$$

$$\boldsymbol{v}'_{1\varepsilon} = \boldsymbol{v}_{1\varepsilon}, \quad \boldsymbol{g}'_\varepsilon = \boldsymbol{g}_\varepsilon - \frac{(t-t_1)^{m-2}}{(m-2)!}\left[I + \frac{t-t_1}{m-1} a_1(t, y; D)\right]\boldsymbol{w}_\varepsilon \qquad \text{if} \quad m \geqq 2$$

satisfies (i)-(iii) by inductive assumption. Q.E.D.

Proposition 4.3. *When* $m \geqq \left[\frac{n}{2}\right]+1$, *for the* $\{v_{0\varepsilon}, v_{1\varepsilon}, g_\varepsilon\}$ *given in Lemma 4.2 with small* $\varepsilon>0$, *say* $0<\varepsilon\leqq\varepsilon_1$, $(4.7)_\varepsilon$ *admits a unique solution* $v_\varepsilon \in C^2([t_1, \delta]\times\varpi)$, *which is supported in* $([t_1, \delta]\times\varpi)\cap\tilde{W}$.

Proof. For each $\varepsilon\in(0, \varepsilon_1]$ with $\varepsilon_1>0$ small enough, $(4.7)_\varepsilon$ fulfills conditions (a)—(e) given at the beginning of Section 3 under the appropriate correspondence. By virtue of Lemma 4.2 and Theorem 3.9, $(4.7)_\varepsilon$ has a unique solution $v_\varepsilon\in C^2([t_1, \delta]\times\varpi)$ for each $\varepsilon\in(0, \varepsilon_1]$. Applying Theorem 3.10 to this solution and using the property (i) in Lemma 4.2 of $\{v_{0\varepsilon}, v_{1\varepsilon}, g_\varepsilon\}$, we have

$$v_\varepsilon = 0 \quad \text{in} \quad ([t_1, \delta]\times\varpi)\cap\overline{\tilde{\Lambda}(\delta_0, y_0)} \quad \text{for any } y_0\in\varpi\backslash\tilde{W}(\delta)$$

where $\tilde{\Lambda}(\delta, y_0)=\{(s, y);\ |y-y_0|<\tilde{\lambda}_{\max}(\delta-s),\ s\geqq t_1\}$. Consequently, since the definition of $\tilde{W}$ implies

$$([t_1, \delta]\times\varpi)\backslash\overline{\bigcup_{y_0\in\bar{\varpi}\backslash\tilde{W}(\delta)}\tilde{\Lambda}(\delta, y_0)} = ([t_1, \delta]\times\varpi)\cap\tilde{W},$$

the support of v_ε is included in $([t_1, \delta]\times\varpi)\cap\tilde{W}$, as desired. Q.E.D.

Let $\{u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon\}$ be the data of $(4.3)_\varepsilon$ which are obtained, using (4.6) and 0-extension, from the $\{v_{0\varepsilon}, v_{1\varepsilon}, g_\varepsilon\}$ given in Lemma 4.2 with $m\geqq\left[\frac{n}{2}\right]+1$, $\varepsilon\in(0, \varepsilon_1]$. Then, a C^2-solution u_ε of $(4.3)_\varepsilon$ is obtained similarly from the above v_ε.

Corollary 4.4. *Under the above circumstances, the mixed problem* $(4.3)_\varepsilon$ *adimits a solution* $u_\varepsilon\in C^2([t_1, \delta]\times\bar{\Omega})$ *with support in* $([t_1, \delta]\times\bar{\Omega})\cap\tilde{W}\subset([t_1, \delta]\times\bar{\Omega})\cap U$.

Before constructing a weak solution as a limit of the sequence $\{u_\varepsilon\}_{0<\varepsilon\leqq\varepsilon_1}$ obtained above, we examine some properties of $X(t)$.

Lemma 4.5. *The operator* $X(t)$ of (4.4) *satisfies the following*:

(i) $\nu_j(\dot{x})\gamma^j(t, \dot{x})=0$ *on* $\hat{\Gamma}\cap U$;

(ii) *If* $\tilde{V}$ *and* δ *are sufficiently small, there exist positive constants* δ_0, c, C *such that*

$$a(t;\, v, v)-(1+\delta_0)\left\|\gamma^j(t, \cdot)\frac{\partial v}{\partial x^j}\right\|^2 \geqq c\|v\|_1^2-C\|v\|^2 \tag{4.8}$$

for all $t\in[-\delta, \delta]$ *and* $v\in H^1(\Omega)$ *with support in* $\{x\in\bar{\Omega};\ (\tau, x)\in\Psi(\tilde{V})$ *for some* $\tau\in[-\delta, \delta]\}$.

Proof. (i) Since $\gamma^j(t, x)=\dfrac{\partial\psi^j}{\partial s}(\Phi(t, x))$ in U, we have by (2.6)

$$\nu_j(\mathring{x})\gamma^j(t, \mathring{x}) = -\tilde{\nu}_i(\Phi(t, \mathring{x}))\frac{\partial \phi^i}{\partial t}(t, \mathring{x}) = 0 \qquad \text{on} \quad \hat{\Gamma} \cap U\,.$$

(ii) By the definition of $c_\Sigma(0, 0)$ and the fact that

$$\gamma^j(0, 0)\frac{\partial v}{\partial x^j} = -\frac{\partial g}{\partial t}(0, 0)\frac{\partial v}{\partial x^{n-1}} \qquad \text{for} \quad v \in \boldsymbol{H}^1(\boldsymbol{R}^n_+)\,.$$

we deduce the desired result (see Proof of Proposition 2.6). Q.E.D.

We may assume that $\tilde{V}$ and δ are, in advance, chosen so that (4.8) is satisfied. Now we present an energy inequality for $\boldsymbol{u}_\varepsilon$.

Proposition 4.6. *The C^2-solution $\boldsymbol{u}_\varepsilon$ of* $(4.3)_\varepsilon$, $0 < \varepsilon \leqq \varepsilon_1$, *obtained in Corollary* 4.4 *satisfies the energy estimate*

$$(4.9) \qquad \|\boldsymbol{u}_\varepsilon(t, \cdot)\|_1^2 + \left\|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(t, \cdot)\right\|^2 \leqq C\left(\|\boldsymbol{u}_{0\varepsilon}\|_1^2 + \|\boldsymbol{u}_{1\varepsilon}\|^2 + \int_{t_1}^t \|\boldsymbol{f}_\varepsilon(\tau, \cdot)\|^2 d\tau\right)$$

for all $t \in [t_1, \delta]$ where $C > 0$ is independent of t_1 and ε.

Proof. The proof is similar to Inoue [13; Proof of Theorem 3.5]. By integrating by parts after taking the scalar product of $((\partial/\partial t)^2 + A(t))\boldsymbol{u}_\varepsilon = \boldsymbol{f}_\varepsilon$ with $X(t)\boldsymbol{u}_\varepsilon$, and by using Lemma 4.5 (i), the fact that supp $\boldsymbol{u}_\varepsilon \subset ([t_1, \delta] \times \bar{\Omega}) \cap U$ and the inequality for $\boldsymbol{v} \in \boldsymbol{C}^1([t_1, \delta] \times \bar{\Omega})$

$$\frac{d}{dt}\|\boldsymbol{v}\|^2 = 2\,\mathrm{Re}\left(\frac{\partial \boldsymbol{v}}{\partial t}, \boldsymbol{v}\right) \leqq \left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2 + \|\boldsymbol{v}\|^2 \qquad \text{at each} \quad t \in [t_1, \delta]\,,$$

we obtain

$$(4.10) \qquad \frac{1}{2}\frac{d}{dt}\left[\|\boldsymbol{u}_\varepsilon\|_{E(t)}^2 + 2\,\mathrm{Re}\left(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}, \gamma^j \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x^j}\right)\right] - \mathrm{Re}\,\langle B(t)\boldsymbol{u}_\varepsilon, X(t)\boldsymbol{u}_\varepsilon\rangle$$
$$\leqq C(K)(\|\boldsymbol{u}_\varepsilon\|_{E(t)}^2 + \|\boldsymbol{f}_\varepsilon\|^2)$$

where $\|\cdot\|_{E(t)}$ is given at each $t \in [t_1, \delta]$ by

$$\|\boldsymbol{v}\|_{E(t)}^2 = \left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2 + a(t; \boldsymbol{v}, \boldsymbol{v}) + K\|\boldsymbol{v}\|^2 \qquad \text{for} \quad \boldsymbol{v} \in \boldsymbol{C}^1([t_1, \delta] \times \bar{\Omega})$$

with $K > 0$ so large that there exists a constant $c > 0$ satisfying

$$a(t; \boldsymbol{v}, \boldsymbol{v}) \geqq c\|\boldsymbol{v}\|_1^2 - K\|\boldsymbol{v}\|^2 \qquad \text{for all} \quad t \in [-\delta, \delta],\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega \cup \Gamma_{\varepsilon_1}(t))\,.$$

The integration of (4.10) over (t_1, t) gives

$$(4.11) \qquad \frac{1}{2}\|\boldsymbol{u}_\varepsilon(t, \cdot)\|_{E(t)}^2 - \mathrm{Re}\int_{t_1}^t \langle B(\tau)\boldsymbol{u}_\varepsilon, X(\tau)\boldsymbol{u}_\varepsilon\rangle d\tau$$

$$+\mathrm{Re}\Big(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(t, \cdot), \gamma^j(t, \cdot)\frac{\partial \boldsymbol{u}_\varepsilon}{\partial x^j}(t, \cdot)\Big)$$
$$\leqq C\Big[\|\boldsymbol{u}_\varepsilon(t_1, \cdot)\|_{E(t_1)}+\int_{t_1}^t (\|\boldsymbol{u}_\varepsilon\|_{E(\tau)}^2+\|\boldsymbol{f}_\varepsilon\|^2)d\tau\Big].$$

Using the facts that supp $\boldsymbol{u}_\varepsilon\subset([t_1, \delta]\times\bar{\Omega})\cap U$ and that supp $(\boldsymbol{u}_{0\varepsilon}|_\Gamma)\subset\Gamma_N(t_1)$, we have by the change of variables by $\Phi(\tau, \cdot)$ for each τ

$$-\mathrm{Re}\int_{t_1}^t \langle B(\tau)\boldsymbol{u}_\varepsilon, X(\tau)\boldsymbol{u}_\varepsilon\rangle d\tau \geqq \mathrm{Re}\int_{t_0}^t \Big\langle\frac{1-\alpha_\varepsilon}{\alpha_\varepsilon}\boldsymbol{u}_\varepsilon, X(\tau)\boldsymbol{u}_\varepsilon\Big\rangle_{\Gamma_\varepsilon(\tau)}d\tau$$
$$=\mathrm{Re}\int_{t_1}^t ds\int_{\tilde{\Gamma}_\varepsilon}\frac{1-\tilde{\alpha}_\varepsilon}{\tilde{\alpha}_\varepsilon}g_{ik}\tilde{u}_\varepsilon^i\Big[\frac{\partial \overline{\tilde{u}_\varepsilon^k}}{\partial s}+\frac{\partial y^k}{\partial x^h}\frac{\partial^2 x^h}{\partial s\partial y^j}\overline{\tilde{u}_k^j}\Big]\sqrt{|g|}dy'$$
$$=\frac{1}{2}\int_{t_1}^t ds\int_{\tilde{\Gamma}_\varepsilon}\Big\{\frac{d}{ds}\Big[\frac{1-\tilde{\alpha}_\varepsilon}{\tilde{\alpha}_\varepsilon}\tilde{u}_\varepsilon^i\overline{\tilde{u}_\varepsilon^k}g_{ik}\sqrt{|g|}\Big]-\frac{1-\tilde{\alpha}_\varepsilon}{\tilde{\alpha}_\varepsilon}\tilde{u}_\varepsilon^i\overline{\tilde{u}_\varepsilon^k}\frac{\partial}{\partial s}(g_{ik}\sqrt{|g|})\Big\}dy'$$
$$+\mathrm{Re}\int_{t_1}^t ds\int_{\tilde{\Gamma}_\varepsilon}\frac{1-\tilde{\alpha}_\varepsilon}{\tilde{\alpha}_\varepsilon}\tilde{u}_\varepsilon^i\overline{\tilde{u}_\varepsilon^j}g_{ik}\frac{\partial y^k}{\partial x^h}\frac{\partial^2 x^h}{\partial s\partial y^j}\sqrt{|g|}dy'$$
$$\geqq\frac{1}{2}\Big\langle\frac{1-\alpha_\varepsilon}{\alpha_\varepsilon}\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon\Big\rangle_{\Gamma_\varepsilon(t)}-C\int_{t_1}^t\Big\langle\frac{1-\alpha_\varepsilon}{\alpha_\varepsilon}\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon\Big\rangle_{\Gamma_\varepsilon(\tau)}d\tau.$$

where $\Gamma_\varepsilon(t)=\{\mathring{x}\in\Gamma;\ \alpha_\varepsilon(t, \mathring{x})>0\}$, $\hat{\Gamma}_\varepsilon=\{y'\in\partial\boldsymbol{R}_+^n;\ y^{n-1}<\varepsilon\}$ and $|g|=\det(g_{ij})$. Therefore, (4.11) yields

$$\|\boldsymbol{u}_\varepsilon(t, \cdot)\|_{E(t)}^2+\Big\langle\frac{1-\alpha_\varepsilon(t, \cdot)}{\alpha_\varepsilon(t, \cdot)}\boldsymbol{u}_\varepsilon(t, \cdot), \boldsymbol{u}_\varepsilon(t, \cdot)\Big\rangle_{\Gamma_\varepsilon(t)}$$
$$\leqq C\Big\{\|\boldsymbol{u}_\varepsilon(t_1, \cdot)\|_{E(t_1)}^2+\int_{t_1}^t\Big(\|\boldsymbol{f}_\varepsilon\|^2+\Big\langle\frac{1-\alpha_\varepsilon}{\alpha_\varepsilon}\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon\Big\rangle_{\Gamma_\varepsilon(\tau)}\Big)d\tau\Big\}$$
$$+\Big|\Big(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(t, \cdot), \gamma^j(t, \cdot)\frac{\partial \boldsymbol{u}_\varepsilon}{\partial x^j}(t, \cdot)\Big)\Big|.$$

Using (4.8) and Gronwall's lemma and taking a larger value of K if necessary, we arrive at the desired inequality (4.9). Q.E.D.

We finish the proof of Theorem 4.1 for the present case. Since $\{\boldsymbol{u}_\varepsilon\}_{0<\varepsilon\leqq\varepsilon_1}$ is bounded in $\boldsymbol{H}^1((t_1, \delta)\times\Omega)$ by (4.9), we can select a subsequence, denoted by $\{\boldsymbol{u}_{\varepsilon_\mu}\}_{\mu=1}^\infty$ with $\varepsilon_\mu\downarrow 0$, having a weak limit $\boldsymbol{u}$ in $\boldsymbol{H}^1((t_1, \delta)\times\Omega)$. On the other hand, taking the scalar product of $((\partial/\partial t)^2+A(t))\boldsymbol{g}_\varepsilon=\boldsymbol{f}_\varepsilon$ with any $\boldsymbol{\eta}\in\boldsymbol{C}_0^\infty([t_1, \delta)\times\bar{\Omega})$ such that $\boldsymbol{\eta}(t, x)=\boldsymbol{0}$ near $\Gamma_D(t)$ for each t and integrating by parts, we have

$$(4.12)\qquad -\int_{t_1}^\delta\Big(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}, \frac{\partial \boldsymbol{\eta}}{\partial t}\Big)dt+\int_{t_1}^\delta a(t;\ \boldsymbol{u}_\varepsilon, \boldsymbol{\eta})dt$$
$$=(\boldsymbol{u}_{1\varepsilon}, \boldsymbol{\eta}(t_1, \cdot))+\int_{t_1}^\delta(\boldsymbol{f}_\varepsilon, \boldsymbol{\eta})dt+\varepsilon\int_{t_1}^\delta\langle X(t)\boldsymbol{u}_\varepsilon, \boldsymbol{\eta}\rangle dt.$$

In order to pass to the limit, we need to estimate the last integral

$$\int_{t_1}^{\delta}\langle X(t)\boldsymbol{u}_\varepsilon, \boldsymbol{\eta}\rangle dt = -\langle \boldsymbol{u}_{0\varepsilon}, \boldsymbol{\eta}(t_1, \cdot)\rangle + \int_{t_1}^{\delta}\left[\left\langle \gamma^j \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x^j}, \boldsymbol{\eta}\right\rangle - \left\langle \boldsymbol{u}_\varepsilon, \frac{\partial \boldsymbol{\eta}}{\partial t}\right\rangle\right] dt\,.$$

Since $\boldsymbol{\tau}_j := \boldsymbol{e}_j - \nu_j \boldsymbol{\nu}$ is tangential to Γ for each j where $\boldsymbol{e}_j = (\delta_j^i)_i$, we have by Lemma 4.5 (i)

$$\left|\left\langle \gamma^j \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x^j}, \boldsymbol{\eta}\right\rangle\right| = |\langle \gamma^j \partial_{\tau_j} \boldsymbol{u}_\varepsilon, \boldsymbol{\eta}\rangle| \leqq C(\|\boldsymbol{u}_\varepsilon(t, \cdot)\|_1^2 + \|\boldsymbol{\eta}(t, \cdot)\|_2^2)\,,$$

from which it follows that

$$\left|\int_{t_1}^{\delta}\langle X(t)\boldsymbol{u}_\varepsilon, \boldsymbol{\eta}\rangle dt\right| \leqq C\left\{\|\boldsymbol{u}_{0\varepsilon}\|_1^2 + \sup_{t\in(t_1,\delta)}\left(\|\boldsymbol{u}_\varepsilon\|_1^2 + \|\boldsymbol{\eta}\|_2^2 + \left\|\frac{\partial \boldsymbol{\eta}}{\partial t}\right\|_1^2\right)\right\}.$$

Therefore, by letting $\mu\to\infty$ in (4.12) with $\varepsilon=\varepsilon_\mu$, we obtain

$$-\int_{t_1}^{\delta}\left(\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial \boldsymbol{\eta}}{\partial t}\right) dt + \int_{t_1}^{\delta} a(t; \boldsymbol{u}, \boldsymbol{\eta}) dt = (\boldsymbol{u}_1, \boldsymbol{\eta}(t_1, \cdot)) + \int_{t_1}^{\delta}(\boldsymbol{f}, \boldsymbol{\eta}) dt\,.$$

By the density argument, we have (1.4) for this $\boldsymbol{u}$.

Since $\boldsymbol{u}_{\varepsilon_\mu}|_{(t_1,\delta)\times\Gamma} \to \boldsymbol{u}|_{(t_1,\delta)\times\Gamma}$ in $\boldsymbol{L}^2((t_1, \delta)\times\Gamma)$, we have $\boldsymbol{u}|_{(t_1,\delta)\times\Gamma}=\boldsymbol{0}$ on $\bigcup_{t_1\leqq t\leqq\delta}\{t\}\times\Gamma_D(t)$. Moreover, $\boldsymbol{u}(t_1, \cdot)=\boldsymbol{u}_0$ since $\boldsymbol{u}_{\varepsilon_\mu}(t_1, \cdot)\to\boldsymbol{u}(t_1, \cdot)$ in $\boldsymbol{L}^2(\Omega)$. Hence $\boldsymbol{u}$ is a weak solution of $(4.3)_\varepsilon$.

That the $\boldsymbol{u}$ satisfies (i) and (ii) of Theorem 4.1 is due to the fact that, for each $\varepsilon\in(0, \varepsilon_1]$, $\boldsymbol{u}_\varepsilon$ does satisfy them (see Corollary 4.4 and (4.9)).

The other cases. In the case $(t_0, x_0)\in\hat{\Gamma}_{N,T}$, we consider the approximate problem modified from $(4.3)_\varepsilon$, for small $\varepsilon>0$, by replacing $\Sigma(t)$ (resp. $\Gamma_N(t)$) with $\{\dot{x}\in\Gamma;\ \mathrm{dis}_\Gamma(x_0, \dot{x})=\varepsilon^0$ (resp. $\mathrm{dis}_\Gamma(x_0, \dot{x})<\varepsilon^0)\}$, $\alpha_\varepsilon(t, \dot{x})$ accordingly and $X(t)$ with $\partial/\partial t$ where $\varepsilon^0>0$ is sufficiently small. In the case $(t_0, x_0)\in\hat{\Omega}_T\backslash\overline{\hat{\Gamma}_{N,T}}$, we have only to take $\Gamma_D(t)=\Gamma$, so $\alpha_\varepsilon(t, \dot{x})\equiv 0$, in $(4.3)_\varepsilon$. In both cases, we can apply the results of Section 3 directly to the approximate problems thus obtained, and the remainder processes of the proof are similar to but much easier than the preceding case.

4.2. Proof of Main Theorem. Before proceeding with the proof of Main Theorem, we prepare the following simple lemma, whose proof we leave to the reader.

Lemma 4.7. *Let* $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\}\in \boldsymbol{V}_D(0)\times\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\hat{\Omega}_T)$. *When* $0<t_1<t_2\leqq T$, *we assume the following*:

(i) (D) *admits a weak solution* $\boldsymbol{v}=\boldsymbol{v}(t, x)$ *for* $t\in[0, t_1)$;

(ii) *With the initial data* $\left\{\boldsymbol{v}(t_1, \cdot), \dfrac{\partial \boldsymbol{v}}{\partial t}(t_1, \cdot)\right\}\in\boldsymbol{V}_D(t_1)\times\boldsymbol{L}^2(\Omega)$ *at* $t=t_1$, (D) *has a weak solution* $\boldsymbol{w}=\boldsymbol{w}(t, x)$ *for* $t\in[t_1, t_2)$. *Then the vector function* $\boldsymbol{u}=\boldsymbol{u}(t, x)$ *defined by*

$$\boldsymbol{u}(t, x)=\boldsymbol{v}(t, x) \quad \textit{for} \quad t\in[0, t_1), \quad =\boldsymbol{w}(t, x) \quad \textit{for} \quad t\in[t_1, t_2)$$

is a weak solution of (D) *for* $t\in[0, t_2)$.

Proof of Main Theorem. There exist a finite number of points $(t_l, x_l)\in\overline{\hat{\Omega}}_T$, $l\in L$, such that $\{\mathring{W}_l\}_{l\in L}$ is an open covering of $\overline{\hat{\Omega}}_T$ where $\mathring{W}_l$ denotes the interior of W given in Theorem 4.1 with respect to (t_l, x_l). Moreover, as well-known, there exists a number $r>0$ such that the r-neighborhood $B_r(t, x)$ of any $(t, x)\in\overline{\hat{\Omega}}_T$ is included in $\mathring{W}_{l_0}$ for some $l_0\in L$. We now choose a finite number of points x_m, $m\in M$, so that $\Omega\times[0, r_1]\subset\bigcup_{m\in M}B_r(0, x_m)$ with r_1 fixed in $(0, r)$, and further a partition of unity $\{\phi_m\}_{m\in M}\subset C_0^\infty(\boldsymbol{R}^{n+1})$ subordinate to an open covering $\{B_r(0, x_m)\}_{m\in M}$ of $[0, r_1]\times\overline{\Omega}$, that is,

$$\operatorname{supp}\phi_m\subset B_r(0, x_m), \quad 0\leqq\phi_m\leqq 1 \quad \text{for all} \quad m\in M; \quad \sum_{m\in M}\phi_m=1 \text{ on } [0, r_1]\times\overline{\Omega}.$$

Then, by virtue of Theorem 4.1, the mixed problem

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2}+A(t)\right)\boldsymbol{u}=\phi_m\boldsymbol{f} & \text{in} \quad \Omega\times(0, r_1), \\ B(t)\boldsymbol{u}=\boldsymbol{0} \quad \text{on} \quad \bigcup\limits_{0\leqq t\leqq r_1}\{t\}\times\Gamma_N(t), \quad \boldsymbol{u}=\boldsymbol{0} \quad \text{on} \quad \bigcup\limits_{0\leqq t\leqq r_1}\{t\}\times\Gamma_D(t), \\ \boldsymbol{u}(0, \cdot)=\phi_m(0, \cdot)\boldsymbol{u}_0, \quad \dfrac{\partial\boldsymbol{u}}{\partial t}(0, \cdot)=\phi_m(0, \cdot)\boldsymbol{u}_1 & \text{in} \quad \Omega \end{cases}$$

admits a weak solution $\boldsymbol{u}_m$ for each $m\in M$. Obviously, $\boldsymbol{v}:=\sum_{m\in M}\boldsymbol{u}_m$ is a weak solution of (D) for $t\in[0, r_1)$.

By the same method as above, we can construct a weak solution $\boldsymbol{w}$ of (D) for $t\in[r_1, 2r_1)$ with the initial data $\left\{\boldsymbol{v}(r_1, \cdot), \dfrac{\partial\boldsymbol{v}}{\partial t}(r_1, \cdot)\right\}\in\boldsymbol{V}_D(r_1)\times\boldsymbol{L}^2(\Omega)$ at $t=r_1$. Thus, by Lemma 4.7, a weak solution of (D) for $t\in[0, 2r_1)$ is obtained. Repeating the same argument $([T/r_1]+1)$ times, we arrive at a weak solution $\in\boldsymbol{H}^1(\hat{\Omega}_T)$ of (D) for $t\in[0, T)$.

For uniqueness, see Duvaut & Lions [4; p. 130] and Inoue [13; Section 5]. The energy inequality is easily obtained from Theorem 4.1 and the construction of the weak solution. Q.E.D.

Let $\lambda_p(t, x; \xi)$, $1\leqq p\leqq n$, be the positive roots of the characteristic equation $\det[\delta^{ik}\lambda^2-a^{ijkh}(t, x)\xi_j\xi_h]_{i,k}=0$ of $(\partial/\partial t)^2+A(t)$. Putting $\lambda_{\max}=\sup\limits_{\substack{(t,x)\in\hat{\Omega}_T\\|\xi|=1}}\max\limits_{1\leqq p\leqq n}\lambda_k(t, x; \xi)$, we have the following.

Corollary to Main Theorem. *For* $(t^0, x^0)\in\boldsymbol{R}_+\times\boldsymbol{R}^n$, *we set*

$$\Lambda(t^0, x^0) = \{(t, x);\ |x-x^0| < \lambda_{\max}(t^0-t),\ t \geqq 0\}$$

If given data $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\} \in \boldsymbol{V}_D(0) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\hat{\Omega}_T)$ *satisfy*

$$\boldsymbol{u}_0 = \boldsymbol{u}_1 = \boldsymbol{0} \quad \text{in } \{x \in \Omega;\ (0, x) \in \Lambda(t^0, x^0)\}\,, \qquad \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \hat{\Omega}_T \cap \Lambda(t^0, x^0)\,,$$

then the weak solution $\boldsymbol{u} \in \boldsymbol{H}^1(\hat{\Omega}_T)$ *of* (D) *vanishes in* $\hat{\Omega}_T \cap \Lambda(t^0, x^0)$.

REMARK 4.8. Let Ω be an interior or exterior domain of a compact C^∞-hypersurface Γ in $\boldsymbol{R}^n$. Consider the following mixed problem of linear elastodynamics in a more general form than (D):

$$\text{(4.13)} \qquad \begin{cases} \left(\rho(t, x)\dfrac{\partial^2}{\partial t^2} + A(t)\right)\boldsymbol{u} = \boldsymbol{f} & \text{in } \hat{\Omega}_T\,, \\ B(t)\boldsymbol{u} = \boldsymbol{\phi} \quad \text{on } \hat{\Gamma}_{N,T}\,, \quad \boldsymbol{u} = \boldsymbol{\psi} & \text{on } \hat{\Gamma}_{D,T}\,, \\ \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0\,, \quad \dfrac{\partial \boldsymbol{u}}{\partial t}(0, \cdot) = \boldsymbol{u}_1 & \text{in } \Omega \end{cases}$$

where $\rho(t, x)$ is a positive C^∞-function on $\overline{\hat{\Omega}}_T$. For simplicity, we assume that $a^{ijkh}(t, x)$ and $\rho(t, x)$ are constant in x outside some bounded set in Ω for each t. If we redefine $c_\Sigma(t, \dot{x})$, $(t, \dot{x}) \in \hat{\Sigma}_T$, by replacing $a^{ijkh}(t, \dot{x})$ with $a^{ijkh}(t, \dot{x})/\rho(t, \dot{x})$ in Definition 2.5, we can show the following under hypotheses (H.1) and (H.2):

Let $\{\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{f}\} \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\hat{\Omega}_T)$, and let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are vector functions on $\hat{\Gamma}_T$ such that $\boldsymbol{u}_0(\dot{x}) = \boldsymbol{\psi}(0, x)$ on $\Gamma_D(0)$ and $B(t)\boldsymbol{v} = \boldsymbol{\phi}$, $\boldsymbol{v} = \boldsymbol{\psi}$ on $\hat{\Gamma}_T$ for some $\boldsymbol{v} \in \boldsymbol{H}^2(\hat{\Omega}_T)$. Then (4.13) admits uniquely a weak solution $\boldsymbol{u} \in \boldsymbol{H}^1(\hat{\Omega}_T)$, i.e., $\boldsymbol{u} = \boldsymbol{\psi}$ on $\hat{\Gamma}_{D,T}$, $\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0$ in Ω and

$$-\int_0^T \left(\frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial(\rho\boldsymbol{\eta})}{\partial t}\right) dt + \int_0^T a(t; \boldsymbol{u}, \boldsymbol{\eta})\, dt$$
$$= (\boldsymbol{u}_1, \rho(0, \cdot)\boldsymbol{\eta}(0, \cdot)) + \int_0^T \{(\boldsymbol{f}, \boldsymbol{\eta}) + \langle \boldsymbol{\phi}, \boldsymbol{\eta} \rangle\}\, dt$$

for all test function $\boldsymbol{\eta}$ as in Definition 1.1 (cf. Duvaut & Lions [4; Théorème 4.1, Chap. 3]).

Appendix. Some properties of $c_\Sigma(t, \dot{x})$

Our argment given below is under (H.1) and the same circumstances as in the latter half of Section 2. Let us write $a^{ijkh} = a^{ijkh}(0, 0)$ and $c_\Sigma = c_\Sigma(0, 0)$ for short, and let A and B be those of (1.1) and (1.2) associated with these a^{ijkh} and $\Omega = \boldsymbol{R}^n_+$, $\Gamma = \partial\boldsymbol{R}^n_+$.

A.1. An alternative definition of c_Σ. We present another algebraic definition of c_Σ than Definition 2.5. The idea here is much the same as in Ito [15; Section 4].

First of all, we note that (2.7) implies the coercivity on $\boldsymbol{H}^1(\boldsymbol{R}^n_+)$ of the quadratic form assoicated with $A_\kappa:=A+\kappa(\partial/\partial x^{n-1})^2$, whose symbol is denoted by $a_\kappa(\xi)=a(\xi)-\kappa\xi_{n-1}^2 I$ with $a(\xi)=(a^{ijkh}\xi_j\xi_h)_{i,k}$ the symbol of A. Since A is strongly elliptic, its symbol $a(\xi)$, which is real symmetric matrix-valued and homogeneous in $\xi\in\boldsymbol{R}^n$ of degree 2, is positive definite for all $\xi\in\boldsymbol{R}^n\setminus\{0\}$. Defining $c_A>0$ by

$$\begin{aligned} c_A^2 &= \sup\{\kappa;\ a_\kappa(\xi)\geqq 0 \quad \text{for all } \xi\neq 0\} \\ &= \min\{\kappa;\ \det a_\kappa(\xi)=0 \quad \text{for some } \xi\neq 0\}\ , \end{aligned}$$

we see that A_κ is strongly elliptic if and only if $\kappa<c_A^2$, and that (2.7) holds for all $\boldsymbol{u}\in\boldsymbol{H}^1_0(\boldsymbol{R}^n_+)$ if and only if $\kappa\leqq c_A^2$.

Let $\kappa<c_A^2$. Since A_κ is strongly elliptic, the Dirichlet problem

$$\text{(A.1)} \qquad A_\kappa\boldsymbol{u}=\boldsymbol{0} \quad \text{in} \quad \boldsymbol{R}^n_+\,, \quad \boldsymbol{u}|_{\partial\boldsymbol{R}^n_+}=\boldsymbol{\phi}\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^{n-1})$$

admits a unique bounded solution $\boldsymbol{u}_\kappa\in\boldsymbol{C}^\infty(\overline{\boldsymbol{R}^n_+})$, where we define a mapping P_κ: $\boldsymbol{C}_0^\infty(\boldsymbol{R}^{n-1})\to\boldsymbol{C}^\infty(\overline{\boldsymbol{R}^n_+})$ by $\boldsymbol{u}_\kappa=P_\kappa\boldsymbol{\phi}$. This is carried out as follows: By the Fourier transformation, (A.1) is reduced to a system of ordinary differential equations in $x^n\geqq 0$ with a parameter $\eta=(\eta_1,\cdots,\eta_{n-1})\in\boldsymbol{R}^{n-1}\setminus\{0\}$

$$\text{(A.2)} \qquad a_\kappa(\eta, D_n)\hat{\boldsymbol{u}}(\eta, x^n)=\boldsymbol{0} \quad \text{for} \quad x^n>0\,, \quad \hat{\boldsymbol{u}}(\eta, 0)=\hat{\boldsymbol{\phi}}(\eta)$$

where $D_n=-\sqrt{-1}\partial/\partial x^n$ and ^ denotes the Fourier transform with respect to $x'=(x^1,\cdots,x^{n-1})$: for example,

$$\hat{\boldsymbol{u}}(\eta, x^n)=(2\pi)^{-(n-1)/2}\int_{\boldsymbol{R}^{n-1}} e^{-\sqrt{-1}x'\cdot\eta}\boldsymbol{u}(x', x^n)dx'\ .$$

It is well-known that (A.2) has a unique solution $\hat{\boldsymbol{u}}_\kappa(\eta, x^n)$ which dies down exponentially as $x^n\to+\infty$; its inverse Fourier transform $\boldsymbol{u}_\kappa(x)$ is the desired solution of (A.1) and satisfies the estimates

$$\text{(A.3)} \qquad \sum_{|\alpha|=m}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha \boldsymbol{u}_\kappa\right\|^2_{\boldsymbol{R}^n_+}\leqq C(\kappa, m)\int_{\boldsymbol{R}^{n-1}}|\eta|^{2m-1}|\hat{\boldsymbol{\phi}}(\eta)|^2 d\eta\,, \qquad m=0, 1, 2, \cdots.$$

We next define a mapping T_κ: $\boldsymbol{C}_0^\infty(\boldsymbol{R}^{n-1})\to\boldsymbol{C}^\infty(\boldsymbol{R}^{n-1})$ by $T_\kappa\boldsymbol{\phi}=B\boldsymbol{u}_\kappa=BP_\kappa\boldsymbol{\phi}$; T_κ is a formally self-adjoint classical pseudo-differential operator on $\boldsymbol{R}^{n-1}\cong\partial\boldsymbol{R}^n_+$ of order 1. Its symbol $t_\kappa(\eta)$, which is Hermitian matrix-valued and homogeneous in $\eta\in\boldsymbol{R}^{n-1}\setminus\{0\}$ of degree 1, is calculated from the formula

$$\text{(A.4)} \qquad t_\kappa(\eta)\hat{\boldsymbol{\phi}}(\eta)=b(\eta, D_n)\hat{\boldsymbol{u}}(\eta, x^n)|_{x^n=0}$$

where $b(\xi)$ is the symbol of B. We note that the strong complementing condition of $\{A, B\}$ is equivalent to the positive definiteness of $t_{\kappa=0}(\eta)$ for all $\eta\in\boldsymbol{R}^{n-1}\setminus\{0\}$.

Proposition A.1. *The* $c_\Sigma>0$ *in Definition* 2.5 *is given also by*

$$\text{(A.5)}\qquad \begin{aligned} c_\Sigma^2 &= \sup\{\kappa<c_A^2;\ t_\kappa(\eta)\geqq 0\quad \textit{for all } \eta\neq 0\} \\ &= \min[\{\kappa<c_A^2;\ \det t_\kappa(\eta)=0\quad \textit{for some } \eta\neq 0\}\cup\{c_A^2\}]\,. \end{aligned}$$

Proof. Denoting the sesquilinear form associated with A_κ by

$$a_\kappa(\boldsymbol{u}, \boldsymbol{v}) = \int_{\boldsymbol{R}_+^n} a^{ijkh}\frac{\partial u^k}{\partial x^h}\overline{\frac{\partial v^i}{\partial x^j}}dx - \kappa\left(\frac{\partial \boldsymbol{u}}{\partial x^{n-1}},\ \frac{\partial \boldsymbol{v}}{\partial x^{n-1}}\right),$$

we have Green's formula for A_κ:

$$\text{(A.6)}\qquad (A_\kappa \boldsymbol{u}, \boldsymbol{v}) = a_\kappa(\boldsymbol{u}, \boldsymbol{v}) - \langle B\boldsymbol{u}, \boldsymbol{v}\rangle \qquad \text{for}\quad \boldsymbol{u}\in \boldsymbol{H}^2(\boldsymbol{R}_+^n),\ \boldsymbol{v}\in \boldsymbol{H}^1(\boldsymbol{R}_+^n)\,.$$

What we have to do is to show that the supremum of κ such that (2.7) holds is given by the right-hand sides of (A.5).

The first equality. Let $n\geqq 3$. Given any $\boldsymbol{u}\in \boldsymbol{C}_0^\infty(\overline{\boldsymbol{R}_+^n})$, we set $\boldsymbol{\phi}=\boldsymbol{u}|_{\partial\boldsymbol{R}_+^n}\in \boldsymbol{C}_0^\infty(\boldsymbol{R}^{n-1})$, $\boldsymbol{v}=P_\kappa\boldsymbol{\phi}\in \boldsymbol{C}^\infty(\overline{\boldsymbol{R}_+^n})$ and $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{v}\in \boldsymbol{C}^\infty(\overline{\boldsymbol{R}_+^n})$; by (A.3), $\boldsymbol{v}\in \boldsymbol{H}^2(\boldsymbol{R}_+^n)$ and $\boldsymbol{w}\in \boldsymbol{H}_0^1(\boldsymbol{R}_+^n)\cap \boldsymbol{H}^2(\boldsymbol{R}_+^n)$. Using (A.6), we have

$$\begin{aligned} a_\kappa(\boldsymbol{u}) &= a_\kappa(\boldsymbol{v}+\boldsymbol{w}) = a_\kappa(\boldsymbol{v})+a_\kappa(\boldsymbol{w}) = \langle T_\kappa\boldsymbol{\phi}, \boldsymbol{\phi}\rangle + (A_\kappa\boldsymbol{w}, \boldsymbol{w}) \\ &= (t_\kappa(\eta)\hat{\boldsymbol{\phi}}(\eta), \hat{\boldsymbol{\phi}}(\eta))_{\boldsymbol{R}_\eta^{n-1}} + (a_\kappa(\xi)\hat{\boldsymbol{w}}(\xi), \hat{\boldsymbol{w}}(\xi))_{\boldsymbol{R}_\xi^n} \end{aligned}$$

where $a_\kappa(\boldsymbol{u})=a_\kappa(\boldsymbol{u}, \boldsymbol{u})$, $\hat{\boldsymbol{w}}(\xi)$ denotes the Fourier transform with respect to x of the 0-extension of $\boldsymbol{w}$ outside $\boldsymbol{R}_+^n$. Since $a_\kappa(\xi)$ is positive definite for any $\xi\neq 0$ and $\kappa<c_A^2$, the assertion follows immediately. In the case $n=2$, the above discussion is valid if we replace $\boldsymbol{C}_0^\infty(\overline{\boldsymbol{R}_+^2})$ with the following dense subspace of $\boldsymbol{H}^1(\boldsymbol{R}_+^2)$:

$$\{\boldsymbol{u}\in \boldsymbol{C}^\infty(\overline{\boldsymbol{R}_+^2})\cap \boldsymbol{H}^1(\boldsymbol{R}_+^2);\ \boldsymbol{\phi}:=\boldsymbol{u}|_{\partial\boldsymbol{R}_+^2}\in \mathcal{S}(\boldsymbol{R}^1;\ \boldsymbol{C}^2) \text{ and } \int|\eta|^{-1}|\hat{\boldsymbol{\phi}}(\eta)|^2 d\eta<\infty\}\,;$$

see Ito [15; Proof of Theorem 4.6].

The second equality. We have only to show that, if the minimum eigenvalue of $t_\kappa(\eta_0)$ is zero for some $\kappa_0\in(0, c_A^2)$ and $\eta_0\in \boldsymbol{R}^{n-1}\setminus\{0\}$, then $t_\kappa(\eta_0)$ has a negative eigenvalue for any $\kappa\in(\kappa_0, c_A^2)$. Using $\rho(\eta)\in C_0^\infty(\boldsymbol{R}^{n-1})$ such that $\int\rho(\eta)^2 d\eta=1$, define $\boldsymbol{\phi}_\varepsilon\in \mathcal{S}(\boldsymbol{R}^{n-1};\ \boldsymbol{C}^n)$ for $\varepsilon>0$ by $\hat{\boldsymbol{\phi}}_\varepsilon(\eta)=\varepsilon^{(n-1)/2}\rho((\eta-\eta_0)/\varepsilon)\boldsymbol{p}$ with an eigenvector $\boldsymbol{p}\neq \boldsymbol{0}$ associated with the eigenvalue 0 of $t_{\kappa_0}(\eta_0)$. Then, $\boldsymbol{v}_\varepsilon := P_{\kappa_0}\boldsymbol{\phi}_\varepsilon$ satisfies

$$\text{(A.7)}\quad \begin{aligned} a_{\kappa_0}(\boldsymbol{v}_\varepsilon) &= a_{\kappa=0}(\boldsymbol{v}_\varepsilon) - \kappa_0\left\|\frac{\partial \boldsymbol{v}_\varepsilon}{\partial x^{n-1}}\right\|_{\boldsymbol{R}_+^n}^2 = (t_{\kappa_0}(\eta)\hat{\boldsymbol{\phi}}_\varepsilon(\eta), \hat{\boldsymbol{\phi}}_\varepsilon(\eta))_{\boldsymbol{R}_\eta^{n-1}} \\ &= \int_{\boldsymbol{R}^{n-1}}\rho(\eta)^2 t_{\kappa_0}(\eta_0+\varepsilon\eta)\boldsymbol{p}\cdot\overline{\boldsymbol{p}}\,d\eta \to t_{\kappa_0}(\eta_0)\boldsymbol{p}\cdot\overline{\boldsymbol{p}} = 0 \qquad (\varepsilon\to 0)\,. \end{aligned}$$

Moreover, if $\varepsilon>0$ is sufficiently small, by (A.7) and the fact that

$$a_{\kappa=0}(\boldsymbol{v}_\varepsilon) \geqq (t_{\kappa=0}(\eta)\hat{\boldsymbol{\phi}}_\varepsilon(\eta), \hat{\boldsymbol{\phi}}_\varepsilon(\eta))_{\boldsymbol{R}_\eta^{n-1}} \to t_{\kappa=0}(\eta_0)\boldsymbol{p}\cdot\overline{\boldsymbol{p}}>0 \qquad (\varepsilon\to 0)\,,$$

there exists a constant $C_1>0$, independent of small ε, such that $\|\partial \boldsymbol{v}_\varepsilon/\partial x^{n-1}\|^2_{\boldsymbol{R}^n_+}\geqq C_1$. We therefore obtain for any $\kappa\in(\kappa_0,\ c_A^2)$

$$\begin{aligned} a_{\kappa_0}(\boldsymbol{v}_\varepsilon)-C_1(\kappa-\kappa_0) &\geqq a_\kappa(\boldsymbol{v}_\varepsilon)\\ &\geqq (t_\kappa(\eta)\hat{\boldsymbol{\phi}}_\varepsilon(\eta),\ \hat{\boldsymbol{\phi}}_\varepsilon(\eta))_{\boldsymbol{R}^{n-1}_\eta}\to t_\kappa(\eta_0)\boldsymbol{p}\cdot\overline{\boldsymbol{p}} \qquad \text{as} \quad \varepsilon\to 0\,, \end{aligned}$$

which yields $t_\kappa(\eta_0)\boldsymbol{p}\cdot\overline{\boldsymbol{p}}<0$. Q.E.D.

Remark A.2. To tell the truth, $t_\kappa(\eta)$ is well-defined for $\kappa<c_L^2$ with $c_L>0$ the so-called *limiting speed* (see [1], [3]) defined by

$$\begin{aligned} c_L^2 &= \sup\{\kappa;\ a_\kappa(\eta,\ 0)\geqq 0 \ \text{ for all }\ \eta\in\boldsymbol{R}^{n-1}\setminus\{0\}\}\\ &= \min\{\kappa;\ \det a_\kappa(\eta,\ 0)=0\ \text{ for some }\ \eta\in\boldsymbol{R}^{n-1}\setminus\{0\}\}\,. \end{aligned}$$

Moreover, by putting $\kappa=(\tau/\eta_{n-1})^2$ in $t_\kappa(\eta)$, we get the *Lopatinski matrix* $L(\tau,\ \eta)$ in an *elliptic region* $\{(\tau,\ \eta)\in\boldsymbol{R}\times\boldsymbol{R}^{n-1};\ 0<|\tau|<c_L|\eta_{n-1}|\}$ for the mixed problem $\{(\partial/\partial t)^2+A,\ B\}$ with τ the dual variable of t.

A.2. Relation between c_R and c_A. Regard $\overline{\boldsymbol{R}^n_+}$ as an homogeneous elastic body with the elasticity tensor (a^{ijkh}) (with unit mass density). We consider subsonic waves (i.e., with propagation speed $<c_L$) which propagate in the direction x^{n-1} along the traction-free boundary $\partial\boldsymbol{R}^n_+$ of $\overline{\boldsymbol{R}^n_+}$ with body force absent, do not vary with $x''=(x^1,\ \cdots.\ x^{n-2})$ and decay exponentially as $x^n\to+\infty$; such a wave classically called a *Rayleigh wave.* (For the Rayleigh wave as a propagation of singularity phenomenon, see Taylor [23], Yamamoto [24], Nakamura [19].) Let us examine one with propagation speed $c>0$ (independent of the form of motion) in the following form:

$$\text{(A.8)} \qquad \boldsymbol{u}(t,\ x)=e^{\sqrt{-1}K(x^{n-1}-ct)}\boldsymbol{\phi}(x^n)$$

where $\boldsymbol{u}$ is a solution of the equations

$$\text{(A.9)} \qquad ((\partial/\partial t)^2+A)\boldsymbol{u}=\boldsymbol{0}\quad \text{in }\ \boldsymbol{R}\times\boldsymbol{R}^n_+\,,\quad B\boldsymbol{u}=\boldsymbol{0}\quad\text{on }\ \boldsymbol{R}\times\partial\boldsymbol{R}^n_+\,,$$

$K>0$ is a wave number and $\boldsymbol{\phi}(x^n)\in \boldsymbol{C}^\infty(\overline{\boldsymbol{R}_+})$ decays exponentially as $x^n\to+\infty$. When $n=3$ (or 2) and (a^{ijkh}) has the properties in Remark 1.3, Barnett & Lothe gave a necessary and sufficient condition on (a^{ijkh}) for the existence of a Rayleigh wave and showed that its speed, called a *Rayleigh speed,* is at most unique (see Chadwick & Smith [3], Barnett & Lothe [1], Nakamura [19; Appendix]). Since there may be more than one Rayleigh speed in the other cases, we define $c_R>0$ by the slowest if there exist.

Proposition A.3. *Assume that there exists a Rayleigh wave propagating along $\partial\boldsymbol{R}^n_+$ in the direction x^{n-1}, that is,* (A.9) *has a solution $\boldsymbol{u}$ in the form* (A.8). *If $c_R<c_A$, the $c_R>0$ is given by*

(A.10) $$c_R^2 = \min\{\kappa < c_A^2\,;\ \det t_\kappa(e_{n-1}) = 0\}$$

where $e_{n-1}=(0, \cdots, 0, 1)\in \boldsymbol{R}^{n-1}$. *Therefore,* $0<c_\Sigma \leqq c_R$ *in general, and* $c_\Sigma = c_R$ *if* $n=2$ *and* $c_R \leqq c_A$.

Proof. If we set $v(x)=e^{\sqrt{-1}Kx^{n-1}}\phi(x^n)$, then (A.9) with (A.8) can be rewritten, by the change of variable: $x^{n-1}-ct \rightarrow x^{n-1}$, as

$$A_\kappa v = 0 \quad \text{in } \boldsymbol{R}_+^n\,, \qquad Bv = 0 \quad \text{on } \partial\boldsymbol{R}_+^n$$

where we assume $\kappa := c^2 < c_A^2$; remark that $\boldsymbol{u}$ of (A.8) depends only on x^n and $x^{n-1}-ct$. Since $v(x', 0)=e^{\sqrt{-1}Kx^{n-1}}\phi_0$ with $\phi_0=\phi(0)$, the first equation of (A.9) gives $v=P_\kappa(e^{\sqrt{-1}Kx^{n-1}}\phi_0)$, from which it follows

$$Bv = T_\kappa(e^{\sqrt{-1}^{-1}Kx^{n-1}}\phi_0) = \mathcal{F}^*[t_\kappa(\eta)\mathcal{F}[e^{\sqrt{-1}Kx^{n-1}}\phi_0](\eta)](x')$$

where $\mathcal{F}$ (resp. $\mathcal{F}^*$) denotes the Fourier (resp. the inverse Fourier) transformation. Denoting by $\delta(\cdot)$ the Dirac delta, we have

$$\mathcal{F}[e^{\sqrt{-1}Kx^{n-1}}\phi_0](\eta) = (2\pi)^{-(n-1)/2}\delta(\eta_1)\cdots\delta(\eta_{n-2})\delta(\eta_{n-1}-K)\phi_0\,,$$

so that

$$Bv = (2\pi)^{1-n}Ke^{\sqrt{-1}Kx^{n-1}}t_\kappa(e_{n-1})\phi_0\,.$$

Hence we obtain $\det t_\kappa(e_{n-1})=0$. Conversely, if this equation in κ admits a root $\kappa_1\in(0, c_A^2)$, we can construct a Rayleigh wave in the form (A.8) with speed $\sqrt{\kappa_1}$ by taking an eigenvector $\phi_0 \neq 0$ corresponding to the eigenvalue 0 of $t_{\kappa_1}(e_{n-1})$. Thus we have (A.10). The last claims follow immediately from Proposition A.1. (We finally remark that, as a matter of fact, the c_R is smaller than c_L and is given by (A.10) with c_A replaced by c_L; see Remark A.2.) Q.E.D.

In the isotropic case, the elasticity tensor (a^{ijkh}) are, as stated in Example 1.4 (ii), expressed by the Lamé moduli λ, $\mu\in\boldsymbol{R}$ as

$$a^{ijkh} = \lambda\delta^{ij}\delta^{kh} + \mu(\delta^{ik}\delta^{jh} + \delta^{ih}\delta^{jk});$$

remark that, in this case, (a^{ijkh}) is invariant under translation and rotation of the x-coordinates. Hypothesis (H.1) implies that λ and μ satisfy $\mu>0$ and $\lambda+\mu>0$ (see Example 1.4 (ii)).

Proposition A.4. *In the isotropic case, we have* $c_R=c_\Sigma=\sqrt{\theta_0\mu}$ *where* θ_0 *is a unique root of equation* (1.6) *in the interval* (0, 1).

Proof. The eigenvalues of the symbol $a_\kappa(\xi)$ of A_κ:

$$a_\kappa(\xi)=\begin{pmatrix} \mu|\xi|^2+(\lambda+\mu)\xi_1^2-\kappa\xi_{n-1}^2 & (\lambda+\mu)\xi_1\xi_2 & \cdots\cdots & (\lambda+\mu)\xi_1\xi_n \\ (\lambda+\mu)\xi_1\xi_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & (\lambda+\mu)\xi_{n-1}\xi_n \\ (\lambda+\mu)\xi_1\xi_n & \cdots\cdots & (\lambda+\mu)\xi_{n-1}\xi_n & \mu|\xi|^2+(\lambda+\mu)\xi_n^2-\kappa\xi_{n-1}^2 \end{pmatrix}$$

are given by

$$\underbrace{\mu|\xi|^2-\kappa\xi_{n-1}^2, \cdots, \mu|\xi|^2-\kappa\xi_{n-1}^2}_{n-1}, \quad (\lambda+2\mu)|\xi|^2-\kappa\xi_{n-1}^2,$$

from which we obtain $c_A(=c_L)=\sqrt{\mu}$.

If $\kappa<\mu$, the decaying solution $\hat{\boldsymbol{u}}(\eta, x^n)$ of (A.2) for $\boldsymbol{\phi}\in\mathcal{S}(\boldsymbol{R}^{n-1}; \boldsymbol{C}^n)$ is calculated as

$$\hat{\boldsymbol{u}}(\eta, x^n)=\left[\hat{\boldsymbol{\phi}}(\eta)-\frac{\tilde{\boldsymbol{\eta}}_p\cdot\boldsymbol{\phi}(\eta)}{|\eta|^2-pq}\tilde{\boldsymbol{\eta}}_q\right]e^{-px^n}+\frac{\tilde{\boldsymbol{\eta}}_p\cdot\boldsymbol{\phi}(\eta)}{|\eta|^2-pq}\tilde{\boldsymbol{\eta}}_q e^{-qx^n}$$

where p, q, $\tilde{\boldsymbol{\eta}}_p$, $\tilde{\boldsymbol{\eta}}_q$ are given by

$$p=\sqrt{|\eta|^2-\frac{\kappa}{\mu}\eta_{n-1}^2}, \quad q=\sqrt{|\eta|^2-\frac{\kappa}{\lambda+2\mu}\eta_{n-1}^2},$$
$$\tilde{\boldsymbol{\eta}}_p={}^t(\eta, \sqrt{-1}p), \quad \tilde{\boldsymbol{\eta}}_q={}^t(\eta, \sqrt{-1}q).$$

Using (A.4), we obtain from the above

$$t_\kappa(\eta)=\mu\begin{pmatrix} p+r\eta_1^2 & r\eta_1\eta_2 & \cdots\cdots & r\eta_1\eta_n & -\sqrt{-1}(1-rp)\eta_1 \\ r\eta_1\eta_1 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & r\eta_{n-2}\eta_{n-1} & \vdots \\ r\eta_1\eta_{n-1} \cdots & r\eta_{n-2\,n-1} & & p+r\eta_{n-1}^2 & -\sqrt{-1}(1-rp)\eta_{n-1} \\ \sqrt{-1}(1-rp)\eta_1 & \cdots\cdots & & \sqrt{-1}(1-rp)\eta_{n-1} & p(1+rp) \end{pmatrix}$$

where $r=(q-p)/(|\eta|^2-pq)$, its eigenvalues are calculated as

$$\underbrace{\mu p, \cdots, \mu p}_{n-2}, \ \mu p+\frac{1}{2}\mu\{r(|\eta|^2+p^2)\pm\sqrt{r^2(|\eta|^2+p^2)^2+4(1-2rp)|\eta|^2}\}.$$

These calculations are similar to those in Ito [15; Section 4]. We note here that, even when $\kappa=0$ or $\eta_{n-1}=0$, the above expressions are valid in the limiting sense. Thus $\det t_\kappa(\eta)=0$ reduces to

$$2p+r(|\eta|^2+p^2)=\sqrt{r^2(|\eta|^2+p^2)^2+4(1-2rp)|\eta|^2},$$

which is equivalent to $(|\eta|^2+p^2)^2=4pq|\eta|^2$, or $F\left(\frac{\kappa}{\mu}(\eta_{n-1}/|\eta|)^2\right)=0$. We note that, since $F(0)<0$, $F(1)>0$ and $F''(\theta)<0$ on $[0, 1]$, $F(\theta)$ has exactly one zero θ_0 in $(0,1)$. Hence we have by Proposition A.1

$$c_\Sigma^2=\min_{|\eta|=1}\theta_0\mu(|\eta|/\eta_{n-1})^2=\theta_0\mu \qquad (<\mu)\,,$$

where the minimum in the middle is attained by $\eta=e_{n-1}$. Therefore, by the definition of c_Σ, we arrive at the desired result. Q.E.D.

A.3. Charaterization of c_A and c_Σ by wave speeds. We only state the results, which will be verified by paying attention to the discussion of the preceding subsections in Appendix.

Regard $\boldsymbol{R}^n$ and $\overline{\boldsymbol{R}^n_+}$ as elastic bodies with constant elasticity tensor (a^{ijkh}) (with unit mass density) as before. We denote by $c_I(\xi'', \xi_n; \phi)$ the speed of the slowest body wave propagating in $\boldsymbol{R}^n$ in the direction $(\xi''\sin\phi, \cos\phi, \xi_n\sin\phi)$ where $\xi''=(\xi_1, \cdots, \xi_{n-2})$, $|\xi''|^2+|\xi_n|^2=1$ and $0\leqq\phi<\pi/2$, i.e. $c_I(\xi'', \xi_n; \phi)^2$ is the minimum eigenvalue of $a(\xi''\sin\phi, \cos\phi, \xi_n\sin\phi)$. Then $c_A>0$ is characterized as

$$c_A=\inf_{\substack{|\xi''|^2+|\xi_n|^2=1\\ 0\leqq\phi<\pi/2}} c_I(\xi'', \xi_n; \phi)\sec\phi\,. \quad [\text{cf. } c_L=\inf_{\substack{|\xi''|=1\\ 0\leqq\phi<\pi/2}} c_I(\xi'', 0; \phi)\sec\phi]\,.$$

We next consider a Rayleigh wave propagating on the boundary $\partial\boldsymbol{R}^n_+$ of $\overline{\boldsymbol{R}^n_+}$ in the direction $(\eta'\sin\theta, \cos\theta, 0)$ where $\eta'=(\eta_1, \cdots, \eta_{n-2})$, $|\eta'|=1$ and $0\leqq\theta<\pi/2$. Denote by $c_R(\eta'; \theta)$ the slowest Rayleigh speed in this direction. Then, the c_Σ is characterized as

$$c_\Sigma=\begin{cases} c_A & \text{if the } c_R \text{ smaller than } c_A \text{ does not exist}\,,\\ \inf\limits_{\substack{|\eta'|=1\\ 0\leqq\pi<\theta/2}} c_R(\eta'; \theta)\sec\theta & \text{otherwise}\end{cases}$$

with the infimum taken over all $(\eta'; \theta)$, $|\eta'|=1$ and $0\leqq\theta<\pi/2$, such that $c_R(\eta'; \theta)$ exists.

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References

[1] D.M. Barnett & J. Lothe: *The existence of Rayleigh (surface) wave solutions in anisotropic linear elastic half spaces,* Problems in elastic wave propagation, ed. J.M. Miklowitz & J.D. Achenbach, Wiley-Interscience, 1978, 445–457.

[2] V.I. Čehlov: *A mixed problem with a discontinuous boundary operator for a hyperbolic equation,* Proc. Steklov Inst. Math. **126** (1973), 189–256.

[3] P. Chadwick & G.D. Smith: *Foundations of the theory of surface waves in anisotropic elastic materials,* Advances in Applied Mechanics **17**, Academic Press (1977), 303–375.

[4] G. Duvaut & J.L Lions: Les inéquations en mécanique et en physique, Dunod, Paris, 1972 (Inequalities in mechanics and physics, Springer, 1976).

[5] G. Eskin: *Mixed initial-boundary value problems for second order hyperbolic equations,* Comm. P.D.E. **12** (1987), 503–587.

[6] K. Hayashida: *On a mixed problem for hyperbolic equations with discontinuous boundary conditions,* Publ. RIMS, Kyoto Univ. **7** (1971/72), 57–67.

[7] K. Ibuki: *On the regularity of solutions of a mixed problem for hyperbolic equations of second order in a domain with corners,* J. Math. Kyoto Univ. **16** (1976), 167–153.

[8] M. Ikawa: *Mixed problem for hyperbolic equations of second order,* J. Math. Soc. Japan **20** (1968), 580–608.

[9] M. Ikawa: *A mixed problem for hyperbolic equations of second order with non-homogeneous Neumann type boundary condition,* Osaka J. Math. **6** (1969), 339–374.

[10] M. Ikawa: *On the mixed problem for hyperbolic equations of second order with Neumann boundary condition,* Osaka J. Math. **7** (1970), 203–223.

[11] A. Inoue: *Sur* $\square u+u^3=f$ *dans un domaine non-cylindrique,* J. Math. Anal. Appl. **46** (1973), 777–819.

[12] A. Inoue: *On a mixed problem for d'Alembertian with a mixed boundary condition—an example of moving boundary,* Publ. RIMS, Kyoto Univ. **11** (1976), 339–401.

[13] A. Inoue: *On a mixed problem for* $\square$ *with a discontinuous boundary condition* (II)—*an example of moving boundary,* J. Math. Soc. Japan **30** (1978), 633–651.

[14] A. Inoue & M. Wakimoto: *On existence of solutions of the Navier-Stokes equation in a time dependent domain,* J. Fac. Sci. Univ. Tokyo Sect. IA **24** (1977), 303–320.

[15] H. Ito: *Extended Korn's inequalities and the associated best possible constants,* to appear in J. Elasticity.

[16] H. Ito: *On certain mixed-type boundary-value problems of elastostatics—with a simple example of Melin's inequality for a system,* to appear in Tsukuba J. Math.

[17] J.L. Lions & E. Magenes: Non-homogeneous boundary value problems and applications I, Springer, 1972.

[18] J.E. Marsden & T.J.R. Hughes: Mathematical foundation of elasticity, Princes-Hall, 1983.

[19] G. Nakamura: *Existence and propagation of Rayleigh waves and Rayleigh pulses in an inhomogeneous medium with curved boundary,* preprint.

[20] Y. Shibata: *On the Neumann problem for some linear hyperbolic system of second order,* Tsukuba J. Math. **12** (1988), 149–209.

[21] H.C. Simpson & S.J. Spector: *On the positivity of the second variation in finite elasticity,* Arch. Rational Anal. Appl. **98** (1987), 1–30.

[22] H. Tanabe: Equations of evolution, Pitman, 1979.

[23] M.E. Taylor: *Rayleigh waves in linear elasticity as a propagation of singularity phenomenon,* Partial differential equations and geometry, ed. C.I. Byrnes, Marcel Dekker, 1979, 273–292.

[24] K. Yamamoto: *Singularities of solutions to the boundary value problems for elastic and Maxwell's equations,* Japan J. Math. **14** (1988), 119–163.

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