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Osaka University

C.0 - CONTRACTIONS

MITSURU UCHIYAMA

1982

Dedication

To Toshiko ,Shinichi,Nami and Takashi

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Introduction

In this thesis, I will make a study on operators of class C_0 on a Hilbert space. When a bounded operator T on a Hilbert space satisfies $\|T\| \leq 1$ and $T^{*n} \rightarrow 0$ strongly as $n \rightarrow \infty$, T is said to belong to class C_0 . This particular class contains many non-normal operators. In particular, the unilateral shift S on the Hardy class H^2 on the unit disc D in the complex plane belongs to it. In [3] Beurling showed that the invariant subspaces for S are precisely those of the form ψH^2 , where ψ is an inner function. For a Hilbert space E , we denote the E -valued Hardy class by $H^2(E)$. Lax [19] and Halmos [17] showed that the invariant subspaces for the unilateral shift S on $H^2(E)$ are precisely those of the form $\Theta H^2(F)$, where F is a Hilbert space with $\dim F \leq \dim E$ and $\Theta(\lambda)$ is an arbitrary $B(F, E)$ -valued inner function defined on D . In this case, if we set

$$H(\Theta) = H^2(E) \ominus \Theta H^2(F) \quad \text{and} \quad S(\Theta) = P_{H(\Theta)} S|_{H(\Theta)},$$

then $S(\Theta)$ belongs to C_0 .

In [25] Rota showed that a contraction with norm < 1 is unitarily equivalent to $S(\Theta)$ for a suitable inner function $\Theta(\lambda)$.

Let T be a contraction on a Hilbert space H . Then Sz.-Nagy and Foias defined the characteristic function $\Theta_T(\lambda)$ of T by

$$\Theta_T(\lambda) = \{-T + \lambda D_{T^*} (I - \lambda T^*)^{-1} D_T\} |_{D_T H} \quad \text{for } \lambda \in D,$$

where $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$. And they showed

that T belongs to C_0 if and only if $\Theta_T(\lambda)$ is inner. They also

showed that in this case T is unitarily equivalent to $S(\theta_T)$ (cf. [28]). Thus the theory of spaces of analytic functions (cf. [18]) and the corona theorem ([6], [24]) have come to play important roles in the study of C_0 .

A subspace of H is called hyper-invariant for an operator T on H if it is invariant for every bounded operator which commutes with T . In [20] Lomonosov proved a famous theorem: Every compact operator has a hyper-invariant subspace. The invariant subspace problem is an important subject in the actual study of operators.

Now, I will give a few accounts of the contents of this thesis.

In chapter I, we will characterize the hyper-invariant subspaces for a contraction T which belongs to C_0 and satisfies $\dim D_T H^{<\infty}$. Here the techniques introduced by Nordgren [22] is useful.

Chapter II is a study on the operators of the form $\phi(S(\psi))$. $\phi(S(\psi))$ is the general Toeplitz operator $PT_\phi|_{H(\psi)}$. (For precise definitions, cf. the first few lines of Chapter II. These operators are considered to extend Toeplitz operators.) In [26], Sarason showed that, for ϕ in H^∞ and a scalar inner function ψ , $\phi(S(\psi))$ is compact if and only if $\bar{\psi}\phi$ belongs to $H^\infty + C$, where C is the Banach algebra of all continuous functions on the unit circle. In the first section of this chapter we will show that, for ϕ in $H^\infty + C$, this result is still true.

We then proceed to establish some results on the double commutant of the operator $S(\theta)$. It is well-known that the double commutant of an arbitrary unilateral shift consists of multiplications by bounded scalar analytic functions. We extend this result to a wider class of operators of the form $S(\theta)$. Indeed, we will show that the double commutant consists of $\phi(S(\theta)), \phi \in H^\infty$.

Chapter III contains the main results of this thesis. A contraction T is called a weak contraction if $I-T^*T$ has a finite trace, and $\sigma(T) \neq D$. Weak contractions have nice properties and there are a good deal of studies (cf. [28]). My study concerns on the operators outside of this operator class. We will consider a contraction T which has following properties:

T belongs to C_0 ,

$I-T^*T$ has a finite trace,

$\sigma(T) = D$ and $\sigma_p(T) \neq D$.

Every unilateral shift has these properties, and we will call such an operator a quasi unilateral shift. One of the B.D.F. theorems [4] implies that $T=S+\text{compact}$, where S is a unilateral shift with $\text{index } S = \text{index } T$. My contribution here is to show that there is an intertwining operator between T and S . This stronger result will make easier the analysis of the operators of this kind.

Chapter I. Hyperinvariant subspaces

1.1. $C_0(n)$ -contractions.

Let T be a contraction on H belonging to C_0 . Then it necessarily follows that

$$\delta_* = \dim \overline{D_{T^*}H} \geq \dim \overline{D_T H} = \delta.$$

Suppose $\delta_* = \delta = n < \infty$, Then T is said to belong to $C_0(n)$. Simply, we denote the characteristic function of T by $\theta(\lambda)$. In this case, we may regard $\theta(\lambda)$ as an $n \times n$ matrix over H^∞ . Since $\theta(\lambda)$ is inner, that is, $\theta(e^{it})$ is isometry for almost all t , $\theta(e^{it})$ is unitary for almost all t . And T on H is unitarily equivalent to $S(\theta)$ on $H(\theta) = H_n^2 \ominus \theta H_n^2$, Where H_n^2 denotes $H^2(\mathbb{C}^n)$.

Definition 1.1. A normal $n \times n$ matrix ϕ over H^∞ is of the form

$$\phi = \begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_n \end{bmatrix}, \text{ where, for each } i, \phi_i \text{ is a scalar}$$

inner function and a divisor of ϕ_{i+1} . The operator

$S(\phi) = S(\phi_1) \oplus \dots \oplus S(\phi_n)$ induced by ϕ is called a *Jordan operator*.

By the Sz.-Nagy and Foias theorem [29], every contraction in $C_0(n)$ is quasi-similar to a Jordan operator.

Theorem 1.2. Let θ be an $n \times n$ inner matrix over H^∞ and ϕ an $n \times n$ normal one. If $S(\theta)$ and $S(\phi)$ are quasi similar, then there exist quasi-affinities X from $H(\theta)$ to $H(\phi)$ and Y from

$H(\theta)$ to $H(\phi)$ and Y from $H(\phi)$ to $H(\theta)$ such that

(i) $X S(\theta) = S(\phi) X$ and $S(\theta)Y = Y S(\phi)$,

(ii) the correspondence $\tau: L \rightarrow \overline{XL}$ and $\tau^*: M \rightarrow \overline{YM}$ establish an isomorphism from the lattice \mathcal{U}_θ of hyperinvariant subspaces for $S(\theta)$ onto the lattice \mathcal{U}_ϕ for $S(\phi)$, and its inverse, $\tau^* = \tau^{-1}$.

Proof. The hypothesis of quasi-similarity implies for $L \in \mathcal{U}_\theta$

(1.1) $\tau(L) = \bigvee_Z \{ZL; Z S(\theta) = S(\phi) Z\}$

belongs to \mathcal{U}_ϕ (c.f. [23]). By one of the Moore-Nordgren theorems ([21], [22]) the quasi-similarity of $S(\theta)$ and $S(\phi)$ implies that there exist matrices Δ, Δ', Λ and Λ' each of whose determinants is relatively prime to the determinants of θ and ϕ , and such that

(1.2) $\Delta \theta = \phi \Lambda$ and $\theta \Lambda' = \Delta' \phi$.

Define the operator X from $H(\theta)$ to $H(\phi)$ and Y from $H(\phi)$ to $H(\theta)$ by

(1.3) $Xh = P_{H(\phi)} \Delta h$ for h in $H(\theta)$, $Yg = P_{H(\theta)} \Delta' g$ for g in $H(\phi)$.

Relation (1.2) guarantees condition (i), and X, Y are quasi-affinities. Take an arbitrary L in the lattice \mathcal{U}_θ and let

$L' = \tau(L)$. By a well-known theorem [28] the (hyper-)invariance of L and L' implies the existence of inner matrices $\theta_1, \theta_2, \phi_1$ and ϕ_2 over H^∞ satisfying

(1.4) $\theta = \theta_2 \theta_1$ and $\phi = \phi_2 \phi_1$,

and

(1.5) $L = \theta_2 (H_n^2 \ominus \theta_1 H_n^2)$ and $L' = \phi_2 (H_n^2 \ominus \phi_1 H_n^2)$.

By the definition (1.1) of $\tau(L)$ we have $XL \subseteq_{\tau(L)} L'$. On the other hand, since YZ commutes with $S(\theta)$ for every Z occurring in (1.1), hyper-invariance of L for $S(\theta)$ implies $YZL \subseteq L$, and therefore $YL' = Y\tau(L) \subseteq L$. Now the inclusions $\overline{XL} \subseteq L'$ and $\overline{YL'} \subseteq L$, and relations (1.2)-(1.5) imply $\Delta\theta_2 H_n^2 \subseteq \phi_2 H_n^2$ and $\Delta'\phi_2 H_n^2 \subseteq \theta_2 H_n^2$; whence we deduce the existence of matrices A and B over H^∞ such that

$$(1.6) \quad \Delta\theta_2 = \phi_2 A \quad \text{and} \quad \Delta'\phi_2 = \theta_2 B.$$

Thus it follows that $\phi_2 AB = \Delta\Delta'\phi_2$, and hence,

$$(1.7) \quad \det A \cdot \det B = \det \Delta \cdot \det \Delta'.$$

Since $\det \Delta \cdot \det \Delta'$ is relatively prime to $\det \phi$, (1.7) implies that $\det A$ is relatively prime to $\det \phi$, hence to $\det \phi_1$. To prove $L' = \overline{XL}$ suppose that $f \in L' \ominus \overline{XL}$. Then, again using (1.2)-(1.5), we see that f is orthogonal to $\Delta\theta_2 H_n^2$, and hence to $\phi_2 A H_n^2$, by (1.6). Moreover, (1.5) implies $f = \phi_2 g$ for some $g \in H_n^2 \ominus \phi_1 H_n^2$. Then for every $h \in H_n^2$

$$0 = (f, \Delta\theta_2 h) = (\phi_2 g, \phi_2 Ah) = (g, Ah).$$

Since $\det A$ is relatively prime to $\det \phi_1$, AH_n^2 and $\phi_1 H_n^2$ span the whole H_n^2 . This implies $g=0$, hence $f=0$, proving $L' = \overline{XL}$. The relation $L = \overline{YL'} = \overline{YXL}$ is proved in a similar way. This completes the proof.

Theorem 1.3. Let ϕ be an $n \times n$ normal matrix over H^∞ . A subspace L of $H(\phi)$ is hyper-invariant for $S(\phi)$ if and only if there are $n \times n$ normal matrices ϕ_1, ϕ_2 satisfying

$$(1.8) \quad \Phi = \Phi_2 \Phi_1 \quad \text{and} \quad L = \Phi_2 (H_n^2 \ominus \Phi_1 H_n^2).$$

Proof. By the lifting theorem ([28] p.258), for every operator X on $H(\Phi)$ commuting with $S(\Phi)$, there is a matrix Δ over H^∞ satisfying

$$(1.9) \quad Xh = P_{H(\Phi)} \Delta h \quad (h \in H(\Phi)) \quad \text{and} \quad \Delta \Phi H_n^2 \subseteq \Phi H_n^2.$$

The latter condition is equivalent to the existence of a matrix Λ over H^∞ satisfying

$$(1.10) \quad \Delta \Phi = \Phi \Lambda.$$

Suppose that L is of the form (1.8), and that $\Phi = \text{diag} (\phi_1, \dots, \phi_n)$. To prove the hyper-invariance of L for $S(\Phi)$, it suffices to show the invariance of L for the operator X defined by (1.9). The existence of Λ satisfying (1.10) implies that if $i > j$, then the inner function ϕ_i / ϕ_j is a divisor of the Δ_{ij} , that is, the (i, j) -th entry of Δ . Since Φ_2 and Φ_1 are normal matrices with $\Phi = \Phi_2 \Phi_1$, for $i > j$ the inner function u_i / u_j is a divisor of ϕ_i / ϕ_j , where u_i is the (i, i) -th entry of Φ_2 , hence a divisor of Δ_{ij} . This guarantees the existence of a matrix Λ' over H^∞ satisfying

$$(1.11) \quad \Delta \Phi_2 = \Phi_2 \Lambda',$$

and consequently the invariance of L for X .

Suppose conversely that L is hyper-invariant for $S(\Phi)$. Let P_i be the orthogonal Projection from $H(\Phi)$ onto the i -th component space. Since P_i commutes with $S(\Phi)$, the hyper-invariance of L implies that

$$L = P_1 L \oplus \dots \oplus P_n L$$

and each $P_i L$ is an invariant subspace for $S(\phi_i)$. By the Beurling theorem there are inner divisors u_i and v_i of ϕ_i satisfying

$$(1.12) \quad \phi_i = u_i v_i, \quad P_i L = u_i (H^2 \ominus v_i H^2).$$

Set $\Phi_2 = \text{diag} (u_1, \dots, u_n)$ and $\Phi_1 = \text{diag} (v_1, \dots, v_n)$, then Φ_2 and Φ_1 satisfy (1.8). It remains to prove the normality of Φ_2 and Φ_1 . To this end, take the matrix Δ over H^∞ whose (i, j) -th entry Δ_{ij} is defined by

$$\Delta_{ij} = 1 \quad (i \leq j) \quad \text{and} \quad \Delta_{ij} = \phi_i / \phi_j \quad (i > j).$$

Clearly there exists a matrix Λ over H^∞ satisfying (1.10).

The hyper-invariance of L implies the existence of a matrix Λ' satisfying (1.11). This means if $i < j$, then u_i is a divisor of u_j and u_j / u_i is a divisor of ϕ_j / ϕ_i . The former condition guarantees the normality of Φ_2 , while the latter does the normality of Φ_1 . This completes the proof.

Since every $C_0(n)$ -contraction is quasi-similar to its Jordan operator, by above theorems, we can characterize the hyper-invariant subspaces for it.

When ϕ is a scalar inner function, for the operator $S(\phi)$ the invariance of a subspace is equivalent to its hyper-invariance. The lattice \mathcal{L}_ϕ of all (hyper-)invariant subspaces is totally ordered if and only if ϕ is of the form

$$(1.13) \quad ((\lambda - \alpha) / (1 - \bar{\alpha}\lambda))^n \quad (|\alpha| < 1, n \text{ a positive integer})$$

or of the form

$$(1.14) \quad e_s(\lambda) \equiv \exp (s(\lambda+\alpha) / (\lambda-\alpha)) \quad (|\alpha|=1, s>0),$$

according as $\dim H(\phi)=n$ or $\dim H(\phi) = \infty$ (cf. [28] p.136).

This can be generalized to the case of inner matrices.

Theorem 1.4. Let Φ be an $n \times n$ normal matrix over H^∞ and $\dim H(\Phi) = \infty$. The lattice \mathcal{L}_Φ of hyper-invariant subspaces for $S(\Phi)$ is totally ordered if and only if ϕ_n is of the form (1.14) and each ϕ_i coincides with either 1 or ϕ_n , where ϕ_i is the (i, i) -th entry of Φ .

Proof. By theorem 1.3 the total orderdness of the lattice \mathcal{L}_Φ is equivalent to the condition that if normal matrices Φ_2 and Φ_2' are left divisors of Φ such that $\Phi_2^{-1}\Phi$ and $\Phi_2'^{-1}\Phi$ are normal too, then one of Φ_2 and Φ_2' is a left divisor of the other. Suppose that \mathcal{L}_Φ is totally ordered. Take arbitrary inner divisors u and v of ϕ_n , and set $u_i = u \wedge \phi_i$ and $v_i = v \wedge \phi_i$ ($a \wedge b$ denotes the gratest common inner divisor of a and b). Then the normal matrices Φ_2 and Φ_2' defined by

$\Phi_2 = \text{diag}(u_1, u_2, \dots, u_{n-1}, u)$ and $\Phi_2' = \text{diag}(v_1, v_2, \dots, v_{n-1}, v)$ are left divisor of Φ , and $\Phi_2^{-1}\Phi$ and $\Phi_2'^{-1}\Phi$ are normal matrices over H^∞ . The divisibility of Φ_2 by Φ_2' or Φ_2' by Φ_2 implies that one of u and v is a divisor of the other. The arbitrariness of u and v implies that ϕ_n is of the form (1.14)

because $\dim H(\Phi) = \infty$ implies $\dim H(\phi_n) = \infty$. There exists an ϕ_i such that $\phi_i/\phi_{i-1} = e_s$ ($1 \leq i \leq n$). In fact if any ϕ_i/ϕ_{i-1} is not equal to e_s , then there exists i and j such that $1 \leq i < j \leq n$, $\phi_i/\phi_{i-1} = e_a$ ($s > a > 0$), $\phi_j/\phi_{j-1} = e_b$ ($s > b > 0$) and $a + b \leq s$.

Now set c and d so that $0 < c \leq a$, $0 < d \leq b$ and $c < d$. Consider the normal matrices Ω_1 and Ω_2 defined by

$$\Omega_1 = \text{diag}(1, \dots, 1, e_c^{(i)}, \dots, e_c) \text{ and } \Omega_2 = \text{diag}(1, \dots, 1, e_d^{(j)}, \dots, e_d)$$

. Clearly Ω_i is a left divisor of Φ and $\Omega_i^{-1}\Phi$ is a normal matrix. By Theorem 1.3, the subspaces

$$\Omega_1 H_n^2 \ominus \Phi H_n^2 \quad \text{and} \quad \Omega_2 H_n^2 \ominus \Phi H_n^2$$

are hyper-invariant for $S(\Phi)$, but any one of them is not included in the other, a contradiction. Consequently $\Phi = \text{diag}(1, \dots, 1, e_s, \dots, e_s)$. The "only if" part is trivial. Therefore we omit the proof (see [33]).

1.2. C_0 - contractions.

In this section, we consider a contraction T in C_0 such that $m = \delta < \delta_* = n < \infty$. Firstly we decide the lattice of hyper-invariant subspaces for a Jordan operator in class C_0 . Next we establish a canonical isomorphism between the lattice of hyper-invariant subspaces for T and that for the Jordan model of T . Since $\delta = m$, $\delta_* = n$, the characteristic function $\theta(\lambda)$ of

T is regarded as an $n \times m$ matrix over H^∞ . Let d_k be the largest common inner divisor of all the minors of order k ($1 \leq k \leq m$). And set $\psi_k = d_k/d_{k-1}$ ($d_0=1$). Then ψ_k is a scalar inner function and a divisor of its successor. In this case, an $n \times m$ matrix;

$$\Phi = \begin{bmatrix} \psi_1 & & & 0 \\ & \psi_2 & & \\ & & \dots & \\ 0 & & & \psi_m \\ 0 & \dots & 0 & \end{bmatrix}$$

is called *normal*, and a corresponding operator;

$$S(\Phi) = S(\psi_1) \oplus \dots \oplus S(\psi_m) \oplus S,$$

where S is the unilateral shift with index $S = n-m$, is called *Jordan model* of T. Nordgren [22] has shown that there are pairs of matrices Δ_i, Λ_i and Δ_i', Λ_i' ($i=1,2$) satisfying

$$(2.1) \quad \Delta_i \Theta = \Phi \Lambda_i,$$

$$(2.1)' \quad \Theta \Lambda_i' = \Delta_i' \Phi,$$

$$(2.2) \quad (\det \Delta_i) (\det \Lambda_i') \wedge d_m = 1,$$

$$(2.3) \quad (\det \Delta_1) (\det \Delta_1') \wedge (\det \Delta_2) (\det \Delta_2') = 1,$$

$$(2.3)' \quad (\det \Lambda_1) (\det \Lambda_1') \wedge (\det \Lambda_2) (\det \Lambda_2') = 1.$$

Setting

$$(2.4) \quad X_i = P_\Phi \Delta_i | H(\Theta) \quad \text{and}$$

$$(2.4)' \quad Y_i = P_\Theta \Lambda_i' | H(\Phi) \quad \text{for } i=1,2,$$

where P_Φ simply denotes $P_{H(\Phi)}$,

$\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are injective families satisfying the following relations:

$$(2.5) \quad X_i S(\Theta) = S(\Phi) X_i,$$

$$(2.6) \quad S(\Theta) Y_i = Y_i S(\Phi),$$

$$(2.7) \quad X_1 H(\theta) \vee X_2 H(\theta) = H(\Phi) ,$$

$$(2.8) \quad Y_1 H(\Phi) \vee Y_2 H(\Phi) = H(\theta) .$$

This implies $S(\theta) \overset{i}{\sim} S(\Phi)$ [30].

Now set $\Psi = \text{diag} (\psi_1, \dots, \psi_m)$, that is, $\Phi = \begin{bmatrix} \Psi \\ 0 \end{bmatrix}$. Then $S(\Phi)$ on $H(\Phi)$ are identified with

$$S(\Psi) \oplus S \text{ on } H(\Psi) \oplus H_{n-m} .$$

Let N be a hyper-invariant subspace for $S(\Phi)$. Then it is clear that N is decomposed to the direct sum, $N = N_1 \oplus N_2$, where N_1 is a subspace of $H(\Psi)$, hyper-invariant for $S(\Psi)$, and N_2 is a subspace of H_{n-m} , hyper-invariant for S . In this case we have the following lemma.

Lemma 2.1. In order that $N = N_1 \oplus N_2$ is hyper-invariant for $S(\Phi)$, it is necessary and sufficient that $N_2 = \{0\}$ or there exists an inner function ϕ such that $N_2 = \phi H_{n-m}^2$ and $N_1 \cong \phi(S(\Psi))H(\Psi)$.

Proof. Simply set $k=n-m$. An operator $X = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ commutes

with $S(\Phi)$, if and only if Y_{ij} satisfy the following conditions:

$$Y_{11}S(\Psi) = S(\Psi) Y_{11} , \quad Y_{12}S = S(\Psi) Y_{12} ,$$

$$Y_{21}S(\Psi) = S Y_{21} , \quad Y_{22}S = S Y_{22} .$$

Since $S(\Psi)^n \rightarrow 0$ as $n \rightarrow \infty$ and S is isometry, we have $Y_{21} = 0$.

Thus if $N_2 = \{0\}$, then it follows that $XN \subseteq N$ for every X

commuting $S(\Phi)$. By the lifting theorem ([26],[28]), a bounded

operator Y_{12} from H_k^2 to $H(\Psi)$ intertwines S and $S(\Psi)$, if and only if there is an $m \times k$ matrix Ω over H^∞ such that $Y_{12} = P_\Psi \Omega$. Thus, if $N_2 = \phi H_k^2$ and $N_1 \supseteq \phi(S(\Psi))H(\Psi)$ for some inner function ϕ , then we have

$$\begin{aligned} X N &= (Y_{11}N_1 + Y_{12} \phi H_k^2) \oplus Y_{22} \phi H_k^2 \\ &\subseteq (N_1 + P_\Psi \Omega \phi H_k^2) \oplus \phi H_k^2 \\ &\subseteq (N_1 + P_\Psi \phi H_m^2) \oplus \phi H_k^2 \\ &= (N_1 + \phi(S(\Psi))H(\Psi)) \oplus \phi H_k^2 \\ &\subseteq N_1 \oplus \phi H_k^2 = N, \end{aligned}$$

where $\phi(S(\Psi))h = P_\Psi \phi h$ for $h \in H(\Psi)$. Thus N is hyper-invariant for $S(\Phi)$.

Conversely suppose $N = N_1 \oplus N_2$ is hyper-invariant for $S(\Phi)$, and $N_2 = \{0\}$. Then by [10], there is an inner function ϕ such that $N_2 = \phi H_k^2$. Let Ω_i ($i=1,2,\dots,m$) be the $m \times (n-m)$ matrix such that the $(i,1)$ -th entry of Ω_i is 1 and the other entry is 0. Setting

$$X_i = \begin{bmatrix} 0 & Y_i \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y_i = P_\Psi \Omega_i,$$

each X_i commutes with $S(\Phi)$, hence we have

$$N_1 = \sum_{i=1}^n Y_i \phi H_k^2 = P_\Psi \phi H_m^2 = \phi(S(\Psi))H(\Psi).$$

This completes the proof.

Theorem 2.2. In order that a factorization $\Phi = \Phi_2 \Phi_1$ of Φ into the product of an $n \times 1$ inner matrix Φ_2 and an $1 \times m$ inner matrix Φ_1 ($n \geq 1 \geq m$) corresponds to a hyper-invariant subspace

N for $S(\Phi)$, it is necessary and sufficient that Φ_1 and Φ_2 are normal matrices satisfying (i) or (ii):

(i) $l=m$,

(ii) $l=n$ and Φ_2 has the form
$$\begin{bmatrix} \Psi_2 & 0 \\ 0 & \phi I_k \end{bmatrix} .$$

Proof. First, assume that $l=m$, and both Φ_1 and Φ_2 are normal inner matrices. Then, setting $\Phi_2 = \begin{bmatrix} \Psi_2 \\ 0 \end{bmatrix}$, it follows that

$\Phi_2 H(\Phi_1) = \Psi_2' H(\Phi_1)$ is hyper-invariant for $S(\Psi)$ (see Sec.1.1).

Therefore, by Lemma 2.1, it is hyper-invariant for $S(\Phi)$.

Next, assume that Φ_1 and Φ_2 are normal matrices satisfying

(ii). Set $\Phi_1 = \begin{bmatrix} \Psi_1 \\ 0 \end{bmatrix}$. Then we have

$$N = \Phi_2 \{ H_n^2 \ominus \Phi_1 H_m^2 \} = \Psi_2 H(\Psi_1) \oplus \phi H_k^2 .$$

Normality of Ψ_1 and Ψ_2 implies that $\Psi_2 H(\Psi_1)$ is hyper-invariant for $S(\Psi)$. On the other hand, normality of Φ_2 implies $\Psi_2 H_m^2 \supseteq \phi H_m^2$, and hence we have

$$\Psi_2 H_m^2 \ominus \Psi H_m^2 \supseteq \phi (S(\Psi)) H(\Psi) .$$

Thus, from Lemma 2.1, we deduce that N is hyper-invariant for $S(\Phi)$.

Conversely, first assume that $N = N_1 \oplus \{0\}$ is hyper-invariant for $S(\Phi)$, and $\Phi = \Phi_2 \Phi_1$ is the factorization corresponding to N . Since $S(\Phi)|_N = S(\Psi)|_{N_1}$ is of class C_0 , $S(\Psi)$ is of class C_0 (about notation C_0 see [28]). This implies that Φ_1 is an $m \times m$ inner matrix, that is, $l=m$. Setting $\Phi_2 = \begin{bmatrix} \Psi_2 \\ \Gamma \end{bmatrix}$, where Ψ_2 is an $m \times m$ matrix and Γ an $k \times m$ matrix ($k=n-m$), we have

$$\Psi = \Psi_2 \Phi_1, \quad N_1 = \Psi_2 H(\Phi_1) \quad \text{and} \quad \Gamma H_m^2 = \{0\}.$$

Since $\Gamma = 0$ and Φ_2 is inner, also Ψ_2 is inner. Thus the hyper-invariance of N_1 corresponding to $\Psi = \Psi_2 \Phi_1$ implies that Ψ_2 and Φ_1 are $m \times m$ normal matrices. Next assume that

$$N = N_1 \oplus \phi H_k^2 \quad \text{and} \quad N_1 \supseteq \phi(S(\Psi))H(\Psi).$$

Clearly we have

$$P_N^\perp S(\Phi) |_{N^\perp} = P_{N_1^\perp} S(\Psi) |_{N_1^\perp} \oplus S(\phi I_k).$$

Since the right hand operator is of class C_0 , $S(\Phi_2)$ is of class C_0 . This implies Φ_2 is an $n \times n$ matrix; i.e., $l=n$. To the hyper-invariant subspace N_1 for $S(\Psi)$ there corresponds a

factorization $\Psi = \Psi_2 \Psi_1$, where Ψ_1 and Ψ_2 are $m \times m$ normal matrices. Thus setting $\Phi_2' = \begin{bmatrix} \Psi_2 & 0 \\ 0 & \phi I_k \end{bmatrix}$ and $\Phi_1' = \begin{bmatrix} \Psi_1 \\ 0 \end{bmatrix}$,

it is clear that

$$\Phi = \Phi_2' \Phi_1' \quad \text{and} \quad N = \Phi_2' \{H_n^2 \ominus \Phi_1' H_m^2\}.$$

From the uniqueness of the factorization of Φ into product of two inner matrices corresponding to invariant subspace N ,

only this factorization $\Phi = \Phi_2' \Phi_1'$ corresponds to N , that

is, $\Phi_2 = \Phi_2'$ and $\Phi_1 = \Phi_1'$. Since

$$\Psi_2 H(\Psi_1) = N_1 \supseteq \phi(S(\Psi))H(\Psi) = P_\Psi \phi H_m^2,$$

we have $\Psi_2 H_m^2 \supseteq \phi H_m^2$; this implies that every entry of Ψ_2 is a divisor of ϕ . Therefore Φ_2 is an $n \times n$ normal matrix. Hence

Φ_1 and Φ_2 are normal matrices satisfying (ii). Q.E.D.

Set $\tau(L) = \bigvee_Z \{ZL: ZS(\theta) = S(\phi)Z\}$
 and $\tau^*(N) = \bigvee_W \{WN: WS(\phi) = S(\theta)W\}$

for each subspace L and N hyper-invariant for $S(\theta)$ and $S(\phi)$, respectively. Since $S(\theta) \overset{c}{\sim} S(\phi)$, it is clear that $\tau(L)$ is the nontrivial hyper-invariant subspace for $S(\phi)$, if L is non-trivial.

Lemma 2.3. If $\theta = \theta_2 \theta_1$ is the factorization corresponding to a non-trivial hyper-invariant subspace L for $S(\theta)$, then θ_1 is an $m \times m$ inner matrix, or θ_2 is an $n \times n$ inner matrix.

Proof. Let $S(\theta) = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ and $S(\phi) = \begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix}$ be the

triangulations corresponding to

$$H(\theta) = L \oplus L^\perp \quad \text{and} \quad H(\phi) = \tau(L) \oplus \tau(L)^\perp, \text{ respectively.}$$

Theorem 2.2. implies that S_1 or S_2 is in C_0 . First, suppose $u(S_1) = 0$ for some u in H^∞ . For the bounded operator X_1 given by (2.4) and every f in L , in virtue of (2.1), it follows that $X_1 u(T_1)f = X_1 u(S(\theta))f = P_\phi \Delta_1 P_\theta u f = P_\phi \Delta_1 u f = P_\phi u \Delta_1 f = u(S(\phi)) X_1 f = 0$.

Since X_1 is an injection, we have $u(T_1)f = 0$, which implies that T_1 belongs to C_0 , that is, θ_1 is an $m \times m$ inner matrix. Next suppose S_2 belong to C_0 , hence so does S_2^* . For Y_1 given by (2.4)' and every Z such that $ZS(\theta) = S(\phi)Z$, in virtue of (2.6), $Y_1 Z$ commutes with $S(\theta)$, this implies $Y_1 ZL \subseteq L$ and hence $Y_1 \tau(L) \subseteq L$. Thus we have $Y_1^* L^\perp \subseteq \tau(L)^\perp$. From this and (2.6),

for each h in L , it follows that

$$Y_i^* T_2^* h = S_2^* Y_i^* h \quad \text{for } i=1,2.$$

From this, we can deduce that

$$Y_i^* u(T_2^*)h = u(S_2^*)Y_i^* h \quad \text{for every } u \text{ in } H^\infty.$$

Since $Y_1 H(\Phi) \vee Y_2 H(\Phi) = H(\Theta)$, we have $u(T_2^*)=0$ for u satisfying $u(S_2^*)=0$. Therefore Θ_2 is an $n \times n$ inner matrix. This completes the proof.

A following theorem implies that the mapping τ is isomorphism from the lattice \mathcal{J}_Θ onto the lattice \mathcal{J}_Φ , and its inverse is given by τ^* .

Theorem 2.4. For X_i and Y_i given by (2.4), (2.4)',

$$(2.9) \quad \tau(L) = X_1 L \vee X_2 L \quad \text{and} \quad \tau^*(\tau(L)) = L,$$

$$(2.9)' \quad \tau^*(N) = Y_1 N \vee Y_2 N \quad \text{and} \quad \tau(\tau^*(N)) = N,$$

where $L \in \mathcal{J}_\Theta$ and $N \in \mathcal{J}_\Phi$.

Proof. Let $\Theta = \Theta_2 \Theta_1$ and $\Phi = \Phi_2 \Phi_1$ be the factorizations of Θ and Φ corresponding to L and $\tau(L)$, respectively. Then the proof of Lemma 2.3 implies that both Θ_1 and Φ_1 are $l \times m$ matrices and both Θ_2 and Φ_2 are $n \times l$ matrices, where $l=n$ or $l=m$. Since $X_i L \subseteq \tau(L)$ and $Y_i \tau(L) \subseteq L$, it clearly follows that

$$\Delta_i \Theta_2 H_1^2 \subseteq \Phi_2 H_1^2 \quad \text{and} \quad \Delta_i' \Phi_2 H_1^2 \subseteq \Theta_2 H_1^2,$$

which guarantee the existence of $l \times l$ matrices A_i and B_i over H^∞ satisfying

$$(2.10) \quad \Delta_i \theta_2 = \phi_2 A_i \quad \text{and} \quad \Delta_i' \phi_2 = \theta_2 B_i .$$

This and (2.1) implies that

$$(2.10)' \quad A_i \theta_1 = \phi_1 \Lambda_i \quad \text{and} \quad B_i \phi_1 = \theta_1 \Lambda_i' .$$

By (2.10) we have

$$(2.11) \quad \Delta_i' \Delta_i \theta_2 = \theta_2 B_i A_i ,$$

and by (2.10)'

$$(2.11)' \quad B_i A_i \theta_1 = \theta_1 \Lambda_i' \Lambda_i .$$

Thus if $l=n$, then $\det A_i$ is a divisor of $\det \Delta_i \cdot \det \Delta_i'$, and if $l=m$ then $\det A_i$ is a divisor of $\det \Lambda_i \cdot \det \Lambda_i'$. To prove the first relation of (2.9) suppose that

$$f \in \tau(L) \ominus \{X_1 L V X_2 L\}.$$

Then f is orthogonal to $\Delta_1 \theta_2 H_1^2 \vee \Delta_2 \theta_2 H_1^2$. On the other hand $f \in \tau(L)$ implies the existence of g belonging to $H_1^2 \ominus \phi_1 H_m^2$ such that $f = \phi_2 g$. Thus for every h in H_k^2 , we have

$$0 = (f, \Delta_i \theta_2 h) = (\phi_2 g, \phi_2 A_i h) = (g, A_i h) \quad (i=1, 2)$$

Thus if $l=n$, then, by (2.3) and Beurling's theorem

$$A_i H_n^2 \supseteq (\det A_i) H_n^2 \supseteq (\det \Delta_i) (\det \Delta_i') H_n^2$$

induce $A_1 H_n^2 \vee A_2 H_n^2 = H_n^2$ and hence $g=0$.

If $l=m$, then, by (2.3)' and Beurling's theorem

$$A_i H_m^2 \supseteq (\det A_i) H_m^2 \supseteq (\det \Lambda_i) (\det \Lambda_i') H_m^2$$

induce $A_1 H_m^2 \vee A_2 H_m^2 = H_m^2$ and hence $g=0$. Thus we showed

$\tau(L) = X_1 L V X_2 L$. The rest is proved in a similar way. Q.E.D.

Chapter II. Commutants and double commutants

2.1. Generalized Toeplitz operator.

Let L^2 be the Hilbert space of all square Lebesgue integrable functions defined on the unit circle, and L^∞ the Banach algebra of all essentially bounded functions defined on the unit circle. Given ϕ in L^∞ , $M(\phi)$ denotes the multiplication of ϕ on L^2 . Let P' be the projection from L^2 onto H^2 . Then a Toeplitz operator T_ϕ is defined by $T_\phi = P'M(\phi)|_{H^2}$. Let ψ be a scalar inner function. Then, for ϕ in L^∞ , we define the general Toeplitz operator $\phi(S(\psi))$ in the sense of [7] by $\phi(S(\psi)) = P T_\phi |_{H(\psi)}$, where $P = P_\psi$. We denote the inner products in $H(\psi)$, H^2 and L^2 by (\cdot, \cdot) , $(\cdot, \cdot)'$ and $(\cdot, \cdot)''$, respectively, and the identical operators in them by I , I' and I'' .

Lemma 1.1. For ϕ in $H^\infty + C$, $(I'' - P')M(\phi)P'$ is a compact operator on L^2 , where C is a space of all continuous functions on the unit circle.

Proof. Let $\phi = \phi_1 + \phi_2$ be a decomposition of ϕ such that ϕ_1 is in H^∞ and ϕ_2 in C . Then it follows that

$$(I'' - P')M(\phi)P' = (I'' - P')M(\phi_2)P'.$$

Take trigonometric polynomials g_n ($n=1, 2, \dots$) whose sequence uniformly converges to ϕ_2 . Then, since

$$\begin{aligned} \|(I'' - P')M(g_n)P' - (I'' - P')M(\phi_2)P'\| &\leq \|M(g_n) - M(\phi_2)\| \\ &\leq \|g_n - \phi_2\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

finiteness of the rank of $(I'' - P')M(g_n)P'$ implies that

$(I'' - P')M(\phi_2)P'$ is compact.

Lemma 1.2. For ϕ in $H^\infty + C$, $PT_\phi(I'-P)$ is compact.

Proof. This lemma follows from Lemmal.1 and next relations;

$$\begin{aligned} PT_\phi(I'-P) &= P P' M(\phi) (I'-P) = P P' M(\phi) M(\psi) M(\bar{\psi}) (I'-P) \\ &= PP' M(\psi) M(\phi) M(\bar{\psi}) (I'-P) = PP' M(\psi) (I'' - P') M(\phi) P' M(\bar{\psi}) (I'-P). \end{aligned}$$

Lemma 1.3. If ϕ is in $H^\infty + C$, then there exists a compact operator K from H^2 to \bar{H}_0^2 , which is the conjugate space of H_0^2 , such that

$$\frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f dt = (Kf_1, f_2)'' + (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2)$$

for every f in H_0^1 , f_1 in H^2 and f_2 in H_0^2 such that $f=f_1f_2$.

Proof. $\psi\bar{f}_2$ is orthogonal to ψH^2 , and $P'\psi\bar{f}_2$ belongs to $H(\psi)$. Therefore we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f dt &= (\phi f_1, \psi\bar{f}_2)'' = (P'\phi Pf_1, \psi\bar{f}_2)'' + \\ &+ (P'\phi(I'-P)f_1, \psi\bar{f}_2)'' + ((I''-P')\phi f_1, \psi\bar{f}_2)'' \\ &= (P'\phi Pf_1, P'\psi\bar{f}_2)'' + (\bar{\psi} PP'\phi(I'-P)f_1, \bar{f}_2)'' + (\bar{\psi}(I''-P')\phi f_1, \bar{f}_2)'' \\ &= (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2)'' + (\bar{\psi}PT_\phi(I'-P)f_1, \bar{f}_2)'' + \\ &(\bar{\psi}(I''-P')M(\phi)f_1, \bar{f}_2)'' . \end{aligned}$$

Thus $K= M(\bar{\psi})PT_\phi(I'-P) + M(\bar{\psi})(I''-P')M(\phi)|_{H^2}$ satisfies the conditions of this lemma.

The proof of the next theorem deeply depends on [26].

Proposition 1.4. Let ϕ be a function in $H^\infty + C$. Then $\phi(S(\psi))$

is compact if and only if $\bar{\psi}\phi$ belongs to $H^\infty + C$.

Proof. "Only if " part is obvious. Suppose $\phi(S(\psi))$ be compact. We wish to show that the kernel of functional of $\bar{\psi}\phi + H^\infty$ on H_0^1 is sequentially weak star closed. Let f_n be a sequence in its kernel and converge weak star to f . Let $f_n = f_{1n} f_{2n}$ be the factorization of f_n such that f_{1n} and f_{2n} belong to H^2 and H_0^2 , respectively, and $|f_n| = |f_{1n}|^2 = |f_{2n}|^2$.

Then, since $\{f_{1n}\}$ and $\{f_{2n}\}$ are bounded in L^2 , we may assume that they converge weakly to f_1 and f_2 in L^2 , respectively, and $f = f_1 f_2$. It is clear that f_1 is in H_0^2 and f_2 is in H^2 . From Lemmal.3, there is a compact operator K such that

$$\frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f_n dt = (Kf_{1n}, \bar{f}_{2n}) + (\phi(S(\psi))Pf_{1n}, P'\psi\bar{f}_{2n})$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f dt = (Kf_1, \bar{f}_2) + (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2).$$

Since both K and $\phi(S(\psi))$ are compact, it follows that

$$(Kf_{1n}, \bar{f}_{2n}) \rightarrow (Kf_1, \bar{f}_2) \quad (n \rightarrow \infty)$$

and

$$(\phi(S(\psi))Pf_{1n}, P'\psi\bar{f}_{2n}) \rightarrow (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2) \quad (n \rightarrow \infty).$$

Thus we have $\frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f dt = 0$.

The proof is complete.

Theorem 1.5. If ϕ is in H^∞ , then next conditions are equivalent;

- (a) $\phi(S(\psi))$ is a Fredholm operator ,
- (b) there are $\epsilon > 0$ and $1 > \delta \geq 0$ such that

$$|\phi(\lambda)| + |\psi(\lambda)| \geq \epsilon \quad \text{for } 1 > |\lambda| \geq \delta ,$$

- (c) $\phi(H^\infty + C) + \psi(H^\infty + C) = H^\infty + C$.

Proof. First assume (a). Then there is a factorization $\phi = \phi_1 \phi_2$, where $\phi_1(S(\psi))$ is invertible and ϕ_2 is a finite Blaschke function. By [12] and [13], there is an $\epsilon_1 > 0$ such that

$$|\phi_1(\lambda)| + |\psi(\lambda)| \geq \epsilon_1 \quad \text{for } |\lambda| < 1 .$$

Since ϕ_2 is a finite Blaschke function, we can easily show (b).

Next assume (b). Setting $\eta = \phi \wedge \psi$, there is an $\epsilon_1 > 0$ such that $|\eta(\lambda)| \geq \epsilon_1$ for $1 > |\lambda| \geq \delta$.

Consequently $1/\eta$ belongs to $H^\infty + C$ [8]. Set $\phi' = \phi/\eta$ and $\psi' = \psi/\eta$. Then it is clear that there is an $\epsilon_2 > 0$ such that

$$|\phi'(\lambda)| + |\psi'(\lambda)| \geq \epsilon_2 \quad \text{for } |\lambda| < 1.$$

Hence, by corona theorem [6] [24], we have $\phi'H^\infty + \psi'H^\infty = H^\infty$, which yields (c). It is clear that (c) implies (a). Thus the theorem is established.

2.2. Double commutants.

When T is a special C_0 -contraction, the A_T and $\{T\}''$ were investigated by several authors (for unilateral shift see

[5], for C_0 -contraction [1], [31] and [40]), where A_T is a weakly closed algebra generated by T and I . In place of C_0 -contraction T with $\delta=m$, $\delta_*=n$ (necessarily $n \geq m$) we may consider $S(\theta)$, where $\theta(\lambda)$ is the characteristic function of T , $n \times m$ matrix of H^∞ and $|\theta(\lambda)| \leq 1$ for every λ in D . In this section we assume $\infty \geq n > m$. In this case there is an $n \times m$ normal matrix;

$$\Phi = \begin{bmatrix} \psi_1 & \dots & 0 \\ 0 & \dots & \psi_m \\ 0 & \dots & 0 \end{bmatrix},$$

and injective families $\{X, X'\}$ and $\{Y, Y'\}$ such that

$$XS(\theta) = S(\Phi)X, \quad S(\theta)Y = YS(\Phi),$$

$$X'S(\theta) = S(\Phi)X', \quad S(\theta)Y' = Y'S(\Phi),$$

$$XY = \eta(S(\Phi)), \quad YX = \eta(S(\theta))$$

$$X'Y' = \eta'(S(\Phi)), \quad Y'X' = \eta'(S(\theta)),$$

and $\eta \wedge \eta \cdot \psi_m = 1$ ([21], [22], [27]). Next two lemmas are obvious.

Lemma 2.1. $\phi(S(\theta))$ is injective if and only if $\phi \wedge \psi_m = 1$, and $\phi(S(\theta))H(\theta)$ is dense in $H(\theta)$ if and only if ϕ is outer.

Lemma 2.2. $\{S(\Phi)\}'' = \{\phi(S(\Phi)) : \phi \in H^\infty\}$.

For a bounded operator T , we denote the lattice of invariant subspaces for T by $\text{Lat } T$.

Lemma 2.3. $\{A : \text{Lat } A \supseteq \text{Lat } S(\Phi)\} = \{\phi(S(\Phi)) : \phi \in H^\infty\}$.

Proof. Suppose $\text{Lat } A \supseteq \text{Lat } S(\Phi)$. Since each component space of $H(\Phi)$ reduces $S(\Phi)$, it also reduces A , that is, A has the form $A = \sum_{i=1}^n \oplus A_i$. $\psi_{i+1}/\psi_i \in H^\infty$ implies that $H(\psi_i) \subseteq H(\psi_{i+1}) \subseteq H^2$. Let P_i be the projection from $H(\Phi)$ onto i -th component space. Then $L_{ij} \equiv \{(P_i x \oplus P_j x : x \in H^\infty)\}$ is invariant for $S(\Phi)$. If $i, j \geq m+1$, then $A L_{ij} \subseteq L_{ij}$ implies $\phi_i = \phi_j$. If $i \leq m < j$, then $A L_{ij} \subseteq L_{ij}$ implies that for every x in $H(\psi_i)$ there is a y in H^2 such that

$$A_i x \oplus \phi_j x = P_i y \oplus y,$$

which implies $A_i = \phi_j(S(\psi_i))$ and hence $A = \phi(S(\Phi))$ for some ϕ in H^∞ . The converse assertion is trivial.

Lemma 2.4. $\{S(\theta)\}'' = \{N : \eta(S(\theta))N = \phi(S(\theta))\}$ for some ϕ in H^∞ .

Proof. For each N in $\{S(\theta)\}''$ and each B in $\{S(\Phi)\}'$, set $K = XNYB - BXNY$. Then, since $YBX \in \{S(\theta)\}'$ and $XY \in \{S(\Phi)\}''$, it follows that $YK = YXNYB - YBXNY = NYXYB - NYBXY = 0$, which implies $K = 0$. Consequently, from Lemma 2.2, there is a ϕ in H^∞ such that $XNY = \phi(S(\Phi))$. Since $YX = \eta(S(\theta))$ is injective, from $YX\eta(S(\theta))N = YXN\eta(S(\theta)) = YXNYX = Y\phi(S(\Phi))X = YX\phi(S(\theta))$, we have $\eta(S(\theta))N = \phi(S(\theta))$. The converse assertion is trivial.

Lemma 2.5. If $XNY = \phi(S(\Phi))$ and $X'NY' = \phi'(S(\Phi))$ for ϕ, ϕ' in H^∞ , then N belongs to $\{S(\theta)\}''$.

Proof. Clearly we have

$N\eta(S(\theta)) = \phi(S(\theta))$ and $N\eta'(S(\theta)) = \phi'(S(\theta))$.

Hence, for each M in $\{S(\theta)\}'$, we have

$$NM\eta(S(\theta)) = N\eta(S(\theta))M = \phi(S(\theta))M = M\phi(S(\theta)) = MN\eta(S(\theta)),$$

and similarly $NM\eta'(S(\theta)) = MN\eta'(S(\theta))$. Since $\eta \wedge \eta' = 1$, the ranges of $\eta(S(\theta))$ and $\eta'(S(\theta))$ span a dense set in $H(\theta)$. Thus we have $NM = MN$.

Theorem 2.6. If N belongs to $\{S(\theta)\}''$, then there is a unique ϕ in H^∞ such that $N = \phi(S(\theta))$. In this case $\|N\| = \|\phi\|_\infty$.

Proof. Let N belong to $\{S(\theta)\}''$. Then from Lemma 2.5 and Lemma 2.1 we have $\phi_1(S(\theta))N = \phi_2(S(\theta))$, where $\phi_1 = \eta/\eta \wedge \phi$ and $\phi_2 = \phi/\eta \wedge \phi$. Thus from the lifting theorem, there are an $n \times n$ bounded matrix $\Gamma = (\gamma_{ij})$ over H^∞ , and an $m \times n$ bounded matrix $\Omega = (\omega_{ij})$ over H^∞ such that

$$(2.1) \quad \Gamma \Theta H_m^2 \subseteq \Theta H_m^2, \quad N = P_\Theta \Gamma|_{H(\theta)}, \quad \|N\| = \|\Gamma\|_\infty = \sup_\lambda \|\Gamma(\lambda)\|,$$

and

$$(2.2) \quad \phi_2 I_n - \phi_1 \Gamma = \Theta \Omega.$$

Since Θ is inner, $1 = \det(\Theta^*(e^{it})\Theta(e^{it})) = \sum_a \det|\theta_a(e^{it})|^2$, where

θ_a denotes an $m \times m$ submatrix. Therefore there is a θ_a such

that $\det \theta_a \neq 0$. We may assume that the first minor is not 0.

Let θ_{ij} and $\theta_{a(i)j}$ be the (i,j) -th component of θ and θ_a ,

respectively. Let $\theta_a' = (\theta_{a(i)j})$ be the classical adjoint matrix

of θ_a . Then, for $k(a) \neq a(i)$ ($1 \leq i \leq m$), by the same technique

as the proof of Theorem 1 of [35], from (2.2), we have

$$-\phi_1 \theta_a' \begin{bmatrix} \gamma_{a(1)k(a)} \\ \vdots \\ \gamma_{a(m)k(a)} \end{bmatrix} = \det \theta_a \begin{bmatrix} \omega_{1k(a)} \\ \vdots \\ \omega_{mk(a)} \end{bmatrix} ,$$

and hence

$$-\phi_1 (\theta_{k(a)1} \dots \theta_{k(a)m}) \theta_a' \begin{bmatrix} \gamma_{a(1)k(a)} \\ \vdots \\ \gamma_{a(m)k(a)} \end{bmatrix} = \det \theta_a (\phi_2 - \phi_1 \gamma_{k(a)k(a)})$$

Thus ,by simple calculations , we have

$$(2.3) \phi_1 \det \begin{bmatrix} \theta_{a(1)1} \dots \theta_{a(1)m} & \gamma_{a(1)k(a)} \\ \vdots & \vdots \\ \theta_{a(m)1} \dots \theta_{a(m)m} & \gamma_{a(m)k(a)} \\ \theta_{k(a)1} \dots \theta_{k(a)m} & \gamma_{k(a)k(a)} \end{bmatrix} = \phi_2 \det \theta_a$$

This implies that the inner factor of ϕ_1 is a divisor of $\Delta \det \theta_a$ which is equal to ψ_m ([21],[27]). Thus $\phi_1 \wedge \psi_m = 1$ deduce that ϕ_1 is outer. For a submatrix θ_a satisfying $1 \leq a(1) < \dots < a(m) \leq m+1$, there is a unique $k(a)$ such that $1 \leq k(a) \leq m+1$ and $k(a) \neq a(i)$. Conversely , for every $1 \leq k \leq m+1$, there is a unique θ_a such that $1 \leq a(1) < \dots < a(m) \leq m+1$ and $k(a)=k$. Thus setting

$\xi_{k(a)}(\lambda) = \det \theta_a(\lambda)$,from (2.3), we have

$$|\phi_2(\lambda)|^p |\xi_k(\lambda)|^2 = |\phi_1(\lambda)|^2 \left| \det \begin{bmatrix} \theta_{11} \dots \theta_{1m} & \gamma_{1k} \\ \vdots & \vdots \\ \theta_{m1} \dots \theta_{mm} & \gamma_{mk} \\ \theta_{m+11} & \theta_{m+1m+1} & \gamma_{m+1k} \end{bmatrix} \right|^2$$

for every $k; 1 \leq k \leq m+1$. Hence it follows that

$$|\phi_2(\lambda)|^2 \prod_{k=1}^{m+1} |\xi_k(\lambda)|^2 = |\phi_1(\lambda)|^2 \left\| \begin{bmatrix} \gamma_{11}(\lambda) \dots \gamma_{m+11}(\lambda) \\ \vdots \\ \gamma_{1m+1}(\lambda) \dots \gamma_{m+1m+1}(\lambda) \end{bmatrix} \begin{bmatrix} \xi_1(\lambda) \\ \vdots \\ (-1)^m \xi_{m+1}(\lambda) \end{bmatrix} \right\|^2$$

$$\leq |\phi_1(\lambda)|^2 \left\| \Gamma_{m+1}(\lambda) \right\|^2 \left(\prod_{k=1}^{m+1} |\xi_k(\lambda)|^2 \right) ,$$

where $\Gamma_{m+1}(\lambda)$ is the first submatrix of $\Gamma(\lambda)$ of order $m+1$, and

${}^t\Gamma_{m+1}(\lambda)$ is the transposed matrix of $\Gamma_{m+1}(\lambda)$. Since by the assumption $\xi_{m+1}(\lambda) \neq 0$, it follows that

$$|\phi_2(\lambda)|^2 \leq |\phi_1(\lambda)|^2 \|{}^t\Gamma_{m+1}(\lambda)\|^2 \leq |\phi_1(\lambda)|^2 \|\Gamma\|_\infty^2.$$

Thus there is a ϕ in H^∞ such that $\phi_2 = \phi\phi_1$ and

$$\|\phi\|_\infty \leq \|\Gamma\|_\infty = \|N\| \quad (\text{cf. [8]}). \text{ Hence we have } N = \phi(S(\theta)). \text{ Since}$$

$$\|N\| \leq \|\phi\|_\infty \text{ is clear, we have } \|N\| = \|\phi\|_\infty. \text{ Assume that}$$

$\phi(S(\theta)) = \psi(S(\theta))$ for ϕ and ψ in H^∞ . From $X S(\theta) = S(\phi) X$

and $X' S(\theta) = S(\phi) X'$, we have

$$\phi(S(\phi))X = \psi(S(\phi))X \quad \text{and} \quad \phi(S(\phi))X' = \psi(S(\phi))X'.$$

By $X H(\theta) \vee X' H(\theta) = H(\phi)$, we deduce

$\phi(S(\phi)) = \psi(S(\phi))$, from which $\phi = \psi$ follows.

$$\text{Theorem 2.7. } A_{S(\theta)} = \{N: \text{Lat } N \supseteq \text{Lat } S(\theta)\} = \{S(\theta)\}'' = \{\phi(S(\theta)): \phi \in H^\infty\}.$$

Proof. From Theorem 2.6, it follows that

$$\{S(\theta)\}'' = \{\phi(S(\theta)): \phi \in H^\infty\} \subseteq A_{S(\theta)} \subseteq \{N: \text{Lat } N \supseteq \text{Lat } S(\theta)\}.$$

Therefore we must only show that if $\text{Lat } N \supseteq \text{Lat } S(\theta)$, then N belongs to $\{S(\theta)\}''$. Let L be an arbitrary subspace in $\text{Lat } S(\phi)$

. Then, since \overline{YL} is in $\text{Lat } S(\theta)$,

$$XNYL \subseteq XN\overline{YL} \subseteq X\overline{YL} \subseteq \overline{XYL} = \overline{\eta(S(\phi))L} \subseteq L.$$

From Lemma 2.3, we have $XNY = \phi(S(\Phi))$ for some ϕ in H^∞ .
Similarly we have $X'NY' = \phi'(S(\Phi))$. Thus by Lemma 2.5, we can
conclude the theorem .

Chapter III. C_{10} - contraction

We determine $C_{1.}$, C_{10} and C_{11} by

$$C_{1.} = \{T: T^n x \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } x\},$$

$$C_{10} = C_{1.} \cap C_{.0} \quad \text{and}$$

$$C_{11} = \{T: T \in C_{1.}, T^* \in C_{1.}\}.$$

It is well-known that there is a C_0 - C_{11} decomposition for a weak contraction. Therefore we can easily show that if T is of class C_{10} and $I - T^*T \in (\tau, c)$, where (τ, c) denotes the trace class, then $\sigma_p(T^*) = D$ and $\sigma_p(T) \cap D = \emptyset$.

In this chapter, we shall investigate a contraction T such that $I - T^*T \in (\tau, c)$ and $\sigma(T) = \bar{D}$. The main tool is the theory of infinite determinant [15]. About C_{10} see [11], [14] and [41].

3.1. Operator valued functions.

For $T \in I + (\tau, c)$, Bercovici and Voiculescu defined the algebraic adjoint T^a , which satisfies

$$T^a T = T T^a = \det T.$$

They showed that if $\theta(\lambda)$ is a contractive holomorphic function and if $\theta(\lambda) \in I + (\tau, c)$ for every $\lambda \in D$, then $\theta(\lambda)^a$ is a contractive holomorphic function. In this case, if $\det \theta(e^{it}) \neq 0$ a.e., then $\theta(e^{it})$ is invertible and its inverse is

$$\theta(e^{it})^a / \det \theta(e^{it}) \text{ a.e. .}$$

Theorem 1.1. Let $\theta(\lambda)$ be an inner function (that is, $\theta(\lambda)$ is a contractive holomorphic function defined on D and $\theta(e^{it})$ is isometric a.e.) with values in $L(E, E')$, where E, E' are separable Hilbert space. If there is an isometry V in $L(E, E')$ such that for every $\lambda \in D$

$$(1.1) \quad I_E - V^* \theta(\lambda) \in (\tau, C),$$

$$(1.2) \quad \det V^* \theta(\lambda) \neq 0,$$

then there is a bounded holomorphic function $\Delta(\lambda)$ with values in $L(E', F)$ for a suitable Hilbert space F such that

$$(1.3) \quad \theta(e^{it})|_E \oplus \Delta^*(e^{it})|_F = E' \quad \text{a.e.}$$

Proof. If V is a unitary, then $\theta(e^{it})$ is invertible a.e.. Hence we may assume that V is not a unitary. Set $F = E' \ominus VE$. Let $E_0 = E \oplus F$ be the direct summation of E and F . For $\lambda \in D$, define $\theta'(\lambda) \in L(E_0, E')$ by

$$\theta'(\lambda)|_E = \theta(\lambda) \quad \text{and} \quad \theta'(\lambda)|_F = I_F.$$

For simplicity, set $d(\lambda) = \det V^* \theta(\lambda)$ and $A(\lambda) = (V^* \theta(\lambda))^a$.

Determine $\Delta(\lambda) \in L(E', F)$ by

$$(1.4) \quad \Delta(\lambda) = -P_F \theta(\lambda) A(\lambda) V^* + d(\lambda) P_F$$

and $\Delta'(\lambda) \in L(E', E_0)$ by

$$\Delta'(\lambda) = A(\lambda) V^* + \Delta(\lambda).$$

Then we have

$$\begin{aligned} \Delta'(\lambda) \theta'(\lambda)|_E &= \Delta'(\lambda) \theta(\lambda) = A(\lambda) V^* \theta(\lambda) + \Delta(\lambda) \theta(\lambda) \\ &= d(\lambda) I_E - P_F \theta(\lambda) d(\lambda) I_E + d(\lambda) P_F \theta(\lambda) = d(\lambda) I_E \end{aligned}$$

$$\Delta'(\lambda)\theta'(\lambda)|_F = A(\lambda)V^*I_F + \Delta(\lambda)I_F = d(\lambda)I_F,$$

and

$$\begin{aligned} \theta'(\lambda)\Delta'(\lambda) &= \theta(\lambda)A(\lambda)V^* + \Delta(\lambda) = (I - P_F)\theta(\lambda)A(\lambda)V^* + d(\lambda)I_F \\ &= VV^*\theta(\lambda)A(\lambda)V^* + d(\lambda)I_F = V d(\lambda)V^* + d(\lambda)I_F = d(\lambda)I_{E'}. \end{aligned}$$

Thus we have

$$\Delta'(\lambda)\theta'(\lambda) = d(\lambda)I_{E_0}, \quad \theta'(\lambda)\Delta'(\lambda) = d(\lambda)I_{E'}.$$

Since the inverse of $\theta'(e^{it})$ is $\Delta'(e^{it})/d(e^{it})$ a.e., the orthogonal complement of $\theta(e^{it})E = \theta'(e^{it})E$ is

$$\frac{\Delta'(e^{it})^*}{d(e^{it})} (E_0 \ominus E) = \Delta(e^{it})^*F.$$

It is clear that $\Delta(\lambda)$ is a bounded holomorphic function. Q.E.D.

Cambern showed that the orthogonal complement of a finite dimensional holomorphic range function is conjugate holomorphic (c.f. p.94 of [16]). Now, we can show this result as a corollary.

Corollary 12. Let $\theta(\lambda)$ be an inner function with values in $L(E, E')$. Suppose $\dim E = m < \infty$. Then there is an bounded holomorphic function $\Delta(\lambda)$ satisfying (1.3).

Proof. We may assume that $E \subset E'$ and $\theta(e^{it})$ is a matrix. Since $1 = \det(\theta^*(e^{it})\theta(e^{it})) = \sum_{\sigma} |\det \theta_{\sigma}(e^{it})|^2$, a.e., where \sum_{σ} is taken over all $m \times m$ submatrices of $\theta(e^{it})$, there is at least one σ such that $\det \theta_{\sigma}(e^{it}) \neq 0$ a.e.. Thus there is an isometry V such that

$$\det v^* \theta(e^{it}) = \det \theta_{\sigma}(e^{it}) \neq 0 \text{ a.e. (see [30])}.$$

Hence V and $\theta(\lambda)$ satisfy (1.1), (1.2). Q.E.D.

3.2. Quasi unilateral shifts.

We begin with a short review about the canonical model theory of Sz, Nagy and C. Foias. Let T be a contraction of class C_0 on a separable Hilbert space H . Set $D_T = (I - T^*T)^{1/2}$, and let E and E' be the closures of $D_T H$ and $D_{T^*} H$, respectively.

Then the characteristic function $\theta(\lambda)$ of T determined by

$$(2.1) \quad \theta(\lambda) = \{-T + \lambda D_{T^*} (I - \lambda T^*)^{-1} D_T\} \Big|_E \quad \text{for } \lambda \in D$$

is an inner function with values in $L(E, E')$. Therefore

$$\dim E \leq \dim E'.$$

Moreover T is unitarily equivalent to $S(\theta)$ on $H(\theta)$ defined by

$$(2.2) \quad H(\theta) = H^2(E') \ominus \theta H^2(E), \quad S(\theta)^* h = \bar{\lambda} h \text{ for } h \text{ in } H(\theta).$$

T is of class C_1 if and only if $\theta(\bar{\lambda})^* H^2(E')$ is dense in $H^2(E)$ (that is, θ is $*$ -outer).

In this thesis, for simplicity, we call T a quasi unilateral shift if T is a contraction of class C_0 such that

$$I - T^*T \in (\tau, C), \quad K(T) = \{0\} \text{ and } K(T^*) \neq \{0\}.$$

Theorem 2.1. If T is a quasi unilateral shift on H , then there is a bounded operator X with dense range satisfying

$$(2.3) \quad X T = S X,$$

where S is a unilateral shift satisfying

$$0 > \text{index } S = \text{index } T \geq -\infty.$$

Proof. We may assume $I-T^*T \neq 0$. From $T(I-T^*T) = (I-TT^*)T$, it follows that $TE \subset E'$, $T(H \ominus E) = H \ominus E'$, where E and E' are the spaces defined above. Thus we have

$$(2.4) \quad H \ominus TH = E' \ominus TE \neq \{0\}.$$

Let $\{e_1, e_2, \dots, e_n, \dots\}$ be the C.O.N.B. of E such that

$(I-T^*T)e_n = \mu_n e_n$, $\mu_n \geq 0$. Then $f_n = (1-\mu_n)^{-1/2} T e_n$ ($n=1, 2, \dots$) is a C.O.N.B. of TE and $T^*f_n = (1-\mu_n)^{1/2} e_n$ (see [28]). Setting

$V e_n = -f_n$ ($n=1, 2, \dots$), V is an isometry from E to E' , and

$$(2.5) \quad V + T|_E \in (\tau, C) \quad (\text{see [2]}) .$$

Setting $F = E' \ominus VE$, from (2.4), it follows that

$$(2.6) \quad \dim F = -\text{index } T.$$

$I-T^*T \in (\tau, C)$ implies $D_T \in (\sigma, C)$ which denotes the Hilbert Schmidt class. Since $(I-TT^*)|_{TE}$ is unitarily equivalent to $I-T^*T$, we have $D_{T^*}|_{TE} \in (\sigma, C)$. Thus

$$\lambda V^* D_{T^*} (I-\lambda T^*)^{-1} D_T = \lambda V^* (D_{T^*}|_{TE}) (I-\lambda T^*)^{-1} D_T \quad (\lambda \in D)$$

belongs to (τ, C) . Thus, from (2.1), (2.5), we have

$$I - V^* \theta(\lambda) \in (\tau, C) \text{ for each } \lambda.$$

Since

$$\begin{aligned} |\det(V^* \theta(0))|^2 &= \det(\theta(0)^* V V^* \theta(0)) = \det(T^* V V^* T|_E) \\ &= \det(T^* T|_E) = 0, \end{aligned}$$

We have $\det V^*\theta(\lambda) \neq 0$. Thus V and $\theta(\lambda)$ satisfy the conditions of Theorem 1.1. Hence $\Delta(\lambda)$ defined by (1.4) satisfy (1.3).

Since $\Delta(\lambda)\theta(\lambda) = 0$, setting

$$(2.7) \quad X_0 h = \Delta h \quad \text{for } h \text{ in } H(\theta),$$

we have $X_0 \in L(H(\theta), H^2(F))$ and $X_0 S(\theta) = S_0 X_0$, where S_0 is the unilateral shift on $H^2(F)$. Since

$$H^2(F) \supset X_0 H(\theta) = \Delta H^2(E') \supset \Delta H^2(F) = (\det V^*\theta(\lambda)) H^2(F),$$

it follows that $S = S_0 \Big|_{\overline{X_0 H(\theta)}}$ is unitarily equivalent to S_0 .

Thus, from (2.6), we have

$$\text{index } S = \text{index } S_0 = - \dim F = \text{index } T.$$

Consequently an operator X from $H(\theta)$ to $\overline{X_0 H(\theta)}$ defined by

$$(2.8) \quad X h = X_0 h \quad \text{for } h \text{ in } H(\theta)$$

satisfy (2.3).

Q.E.D.

Corollary 2.2. Let T be a contraction of class $C_{0,0}$ such that $I - T^*T$ and $I - TT^*$ belong to (τ, C) . Then, for $a \in D$, $K(T - aI) = \{0\}$ if and only if $K(T^* - \bar{a}I) = \{0\}$.

Proof. Set $T_a = (T - aI)(1 - \bar{a}T)^{-1}$ and $A = (1 - |a|^2)^{\frac{1}{2}}(1 - \bar{a}T)^{-1}$.

Then we have $I - T_a^* T_a = A^*(I - T^*T)A$, $I - T_a T_a^* = A(I - TT^*)A^*$, and T_a is of class $C_{0,0}$ (see p.240 and P.257 of [28]).

Suppose $K(T - aI) = \{0\}$ and $K(T^* - \bar{a}I) \neq \{0\}$. Then T_a is a quasi unilateral shift. Therefore, there is an X satisfying

$X T_a = S X$, which implies that T_a is not of class C_{00} . This is a contradiction. Thus $K(T-aI) = \{0\}$ implies $K(T^*-\bar{a}I) = \{0\}$. Similarly we can prove the converse assertion. Q.E.D.

For a contraction T on H , we have

$$(2.9) \quad \|I-T^*T\|_p + \dim K(T^*) = \|I-TT^*\|_p + \dim K(T),$$

where $\| \cdot \|_p$ denotes the p -Schatten norm.

Indeed, from $T(I-T^*T) = (I-TT^*)T$, $(I-T^*T)|_{\overline{T^*H}}$ and $(I-TT^*)|_{\overline{TH}}$ are unitarily equivalent. $(I-T^*T)|_{K(T)} = I_{K(T)}$ and $(I-TT^*)|_{K(T^*)} = I_{K(T^*)}$ imply that

$$\begin{aligned} \|I-T^*T\|_p &= \|(I-T^*T)|_{\overline{T^*H}}\|_p + \dim K(T), \\ \|I-TT^*\|_p &= \|(I-TT^*)|_{\overline{TH}}\|_p + \dim K(T^*). \end{aligned}$$

Thus we have (2.9). Similarly we have

$$(2.9)' \quad \text{rank}(I-T^*T) + \dim K(T^*) = \text{rank}(I-TT^*) + \dim K(T).$$

Proposition 2.3. Let T be a Fredholm quasi unilateral shift. Suppose X with dense range satisfies $XT = SX$, where S is a unilateral shift with $\text{index } S = \text{index } T$. Then $T|_{K(X)}$ is of class C_0 .

Proof. Let $T = \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix}$ be a decomposition of T corresponding to $H = K(X) \oplus K(X)^\perp$. Then T_1 is injective and, from (2.3), also T_2 is injective. From the assumption and

(2.9), it follows that $I - T^*T \in (\tau, C)$ and $I - TT^* \in (\tau, C)$, which imply

$$(2.10) \quad I - T_1^* T_1 \in (\tau, C),$$

$$(2.11) \quad I - (T_1 T_1^* + T_{12} T_{12}^*) \in (\tau, C),$$

$$(2.12) \quad I - (T_{12}^* T_{12} + T_2^* T_2) \in (\tau, C),$$

$$(2.13) \quad I - T_2 T_2^* \in (\tau, C).$$

From $K(T_2^*) \subset K(T^*)$, it follows that

$$\text{index } T = -\dim K(T^*) \leq -\dim K(T_2^*) \leq -\dim K(S^*) = \text{index } T,$$

which implies $\text{index } T = \text{index } T_2$. From (2.9) and (2.13), we

have $I - T_2^* T_2 \in (\tau, C)$, which, by (2.12), implies $T_{12} \in (\sigma, C)$.

Therefore, from (2.10) and (2.11), T_1 is a Fredholm operator.

Since

$$\text{index } T = \text{index} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \text{index } T_1 + \text{index } T_2,$$

we have $\text{index } T_1 = 0$. Thus T_1 is invertible. Hence T_1 is a weak contraction of class C_0 . Consequently T_1 is of class C_0 . Q.E.D.

Corollary 2.4. Let T be a Fredholm quasi unilateral shift of class C_{10} . Then, if $AT=TA$ and $K(A^*)=\{0\}, K(A)=\{0\}$ (c.f.[42]).

Proof. For X defined in Theorem 2.1, we have $(XA)T = S(XA)$

. From Proposition 2.3, we have $K(XA) = \{0\}$. Q.E.D.

Proposition 2.5. Let T be of class C_0 . Then T is of class C_{10} if and only if

$$(2.14) \quad \theta L^2(E) \cap H^2(E') = \theta H^2(E).$$

Proof. Since

$$\begin{aligned} (\theta(\bar{\lambda}) * h(\lambda), f(\lambda))_{H^2(E)} &= \frac{1}{2\pi} \int_0^{2\pi} (\theta(e^{-it}) * h(e^{it}), f(e^{it}))_E dt \\ &= -\frac{1}{2\pi} \int_0^{-2\pi} (\theta(e^{it}) * h(e^{-it}), f(e^{-it}))_E dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\theta(e^{it}) * h(e^{-it}), f(e^{-it}))_E dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\theta(e^{it}) * e^{-it} h(e^{-it}), e^{-it} f(e^{-it}))_E dt \\ &= (\theta(\lambda) * \bar{\lambda} h(\bar{\lambda}), \bar{\lambda} f(\bar{\lambda}))_{L^2(E)}, \end{aligned}$$

$\theta(\bar{\lambda}) * H^2(E')$ is dense in $H^2(E)$ if and only if

$\theta(\lambda) * (H^2(E'))^\perp$ is dense in $(H^2(E))^\perp$, where \perp denotes the orthogonal complement. We have always

$$\theta L^2(E) \cap H^2(E') \supset \theta H^2(E).$$

At first, assume that T is of class C_{10} . Suppose

$$\theta g \in \{\theta L^2(E) \cap H^2(E')\} \ominus \theta H^2(E).$$

Then $\theta g \in H^2(E')$ and $g \perp H^2(E)$, because θ is an isometry from $L^2(E)$ to $L^2(E')$. Thus $g \perp \theta * (H^2(E'))^\perp$ and $g \in (H^2(E))^\perp$. Since $\theta(\lambda)$ is $*$ -outer, we have $g = 0$. Consequently (2.14) follows.

Conversely assume (2.14). Suppose $f \perp \theta(\lambda) * (H^2(E'))^\perp$ and

$f \in (H^2(E))^\perp$. Then $\theta f \in H^2(E')$ and $\theta f \perp \theta H^2(E)$. Thus from (2.14), we have $\theta f = 0$ and hence $f=0$. Consequently $\theta(\lambda)$ is $*$ -outer.
Q.E.D.

Theorem 2.6. Let T be a quasi unilateral shift. Then $T \prec S$ (that is, there is an X such that $K(X) = K(X^*) = \{0\}$, $XT = SX$), where S is a unilateral shift with index $S = \text{index } T$, if and only if T is of class C_{10} .

Proof. Assume that T is of class C_{10} . Then, from Theorem 2.1, there is an X with dense range satisfying (2.3). If $Xh=0$ for h in $H(\theta)$, then, from (2.7) and (2.8), $\Delta(e^{it})h(e^{it})=0$ a.e.. Thus, from (1.3), $h \in \theta L^2(E)$, so that, from (2.14), $h \in \theta H^2(E)$. Consequently $h=0$. Thus we have $T \prec S$.
Conversely, assume $XT = SX$ and $K(X) = K(X^*) = \{0\}$. From $XT^n = S^n X$ ($n=1,2,\dots$) it follows that T is of class C_{10} . Q.E.D.

Remark 1. If T is a Fredholm operator, then, from Theorem 2.1 and Proposition 2.3, it is clear that $T \prec S$ if T is of class C_{10} .

Remark 2. Theorem 2.6 implies that the Jordan model of a quasi unilateral shift of class C_{10} is a unilateral shift.

Corollary 2.7. Let T be a quasi unilateral shift of class C_{10} . Then T^* has a cyclic vector.

Proof. $T \prec S$ implies that $S^* \prec T^*$. Since S^* has a cyclic vector, also T^* does. Q.E.D.

Proposition 2.8. Let T be a quasi unilateral shift. Then there is an injection Y such that

$$(2.15) \quad Y S = T Y,$$

where S is a unilateral shift such that $\text{index } S = \text{index } T$.

Proof. Consider $S(\theta)$ defined by (2.2) instead of T . Let V be an isometry defined in the proof of Theorem 2.1, Then

$$E' = V E \oplus F \quad \text{and} \quad \det V^* \theta(e^{it}) \neq 0 \quad \text{a.e..}$$

Define an operator Y from $H^2(F)$ to $H(\theta)$ by

$$Y h = P_{H(\theta)} h \quad \text{for } h \text{ in } H^2(F).$$

Then we have

$$Y S h = P_{H(\theta)} S h = P_{H(\theta)} S P_{H(\theta)} h = S(\theta) Y h,$$

which implies (2.15). Suppose $Yh=0$. Then $h=\theta f$ for some $f \in H^2(E)$

. Thus $0 = V^* h(e^{it}) = V^* \theta(e^{it}) f(e^{it}) \quad \text{a.e..}$ Since $V^* \theta(e^{it})$

is invertible a.e., $f(e^{it})=0$ a.e.. Consequently Y is injective

Q.E.D.

Proposition 2.9. Let T be a quasi unilateral shift of class C_{10} . Then, if $T \prec S'$, where S' is a unilateral shift, then $\text{index } S' = \text{index } T$.

Proof. From $S'^* \prec T^*$, $\dim K(S'^*) \leq \dim K(T^*)$. Above proposition implies that there is an injection Y' such that

$$Y' S = S' Y', \text{ index } S = \text{index } T,$$

which implies that $0 > \text{index } S \geq \text{index } S'$ (c.f. [30]).
we have

$$\text{index } T = \text{index } S \geq \text{index } S' \geq \text{index } T,$$

from which $\text{index } T = \text{index } S'$ follows.

Q.E.D.

Remark 3. In [42], P.Y.Wu showed that if $I - T^*T$ is a finite rank operator, and if $T \prec S'$, then

$$\text{rank}(I - TT^*) - \text{rank}(I - T^*T) = -\text{index } S'.$$

From (2.9)', our proposition is an extension of this result.

3.3. Cyclic vector.

In this section, we consider a quasi unilateral shift of class C_{10} which has a cyclic vector. Next proposition is a partial extension of Proposition 2 of [30] and Theorem 3.1 of [41].

Proposition 3.1. Let T be a quasi unilateral shift of class C_{10} . Then next conditions are equivalent:

(a) T has a cyclic vector ;

(b) there is a bounded operator Y satisfying

$$(3.1) \quad Y S_1 = T Y, \quad K(Y^*) = \{0\},$$

where S_1 is a unilateral shift with index $S_1 = -1$;

(c) $S_1 \prec T$;

(d) $S_1 \prec T$ and $T \prec S_1$;

(e) $\|I - TT^*\|_1 - \|I - T^*T\|_1 = 1$, and there is a holomorphic function Γ from $H^2(\mathbb{C})$ to $H^2(E')$ satisfying

$$(3.2) \quad \|\Gamma(e^{it})\|_{E'} \leq 1 \text{ a.e.},$$

$$(3.3) \quad \Gamma H^2(\mathbb{C}) \vee \theta H^2(E) = H^2(E'),$$

where θ is a characteristic function of T defined by (2.1).

Proof. (a) \rightarrow (e). From Theorem 2.6, for a unilateral shift S with index $S = \text{index } T$, we have $T \prec S$. That T has a cyclic vector implies that also S does. Thus $\text{index } S = -1$. Consequently, from (2.9), we have

$$\|I - TT^*\|_1 - \|I - T^*T\|_1 = 1.$$

We can construct a function Γ in the same way as [30].

(e) \rightarrow (b). A contraction Y defined by $Yh = P_{H(\theta)} \Gamma h$ for h in $H^2(\mathbb{C})$ satisfies (3.1).

(b) \rightarrow (c). Suppose $K(Y) \neq \{0\}$. Since $S_1 K(Y) \subset K(Y)$, there is a scalar inner function ψ such that $K(Y) = \psi H^2(\mathbb{C})$. Thus

$$K(Y)^\perp = H(\psi) (= H^2(\mathbb{C}) \ominus \psi H^2(\mathbb{C})),$$

$$Y|_{H(\psi)} S(\psi) = T Y|_{H(\psi)},$$

where $S(\psi) = P_{H(\psi)} S|_{H(\psi)}$. Since $S(\psi)$ is of class C_0 , T must be of class C_0 . This is a contradiction. Consequently $K(Y) = \{0\}$.

(c) \rightarrow (d). $S_1 \prec T$ implies $T^* \prec S_1^*$, from which it follows that $\dim K(T^*) \leq \dim K(S_1^*) = 1$. That T is of class C_{10} implies index $T < 0$. Thus index $T = -1$. By theorem 2.6, we have $T \prec S_1$.

(d) \rightarrow (a). This is obvious. Q.E.D.

(3.3) implies that $[\Gamma, \theta]$ is an outer function from $H^2(\mathbb{C}) \oplus H^2(E)$ to $H^2(E')$. Generally $[\Gamma, \theta]$ is not contractive. Therefore $d(\lambda) = \det[\Gamma(\lambda), \theta(\lambda)] \in H^\infty$ and $d(\lambda) \leq 1$ are not obvious. We shall show these results.

Let $A \in L(E, E')$ be a contraction and $V \in L(E, E')$ an isometry with index $V = -1$. Let $\{e_1, e_2, \dots, e_n, \dots\}$ be a C.O.N.B. in E . Then, setting $d_n = V e_n$ ($n=1, 2, \dots$), $\{d_0, d_1, \dots, d_n, \dots\}$ is a C.O.N.B. in E' , where d_0 is a unit vector in $K(V^*)$. For $i=1, 2, \dots$, define an isometry $V_i \in L(E, E')$ by

$$V_i e_1 = d_0, \dots, V_i e_i = d_{i-1}, V_i e_{i+1} = d_{i+1}, V_i e_{i+2} = d_{i+2}, \dots$$

Let $a_{ij} = (Ae_j, d_i)$ ($i \geq 0, j \geq 1$). Then, by base $\{e_1, e_2, \dots\}$, we have

$$V_i^* A = \begin{bmatrix} a_{01} & \dots & a_{0j} & \dots \\ \vdots & & \vdots & \\ a_{i-1,1} & \dots & a_{i-1,j} & \dots \\ a_{i+1,1} & \dots & a_{i+1,j} & \dots \\ \vdots & & \vdots & \end{bmatrix} \quad (i=1,2,\dots)$$

Let $E_0 = \mathbb{C} \oplus E$ be a direct sum of \mathbb{C} and E , and e_0 a unit vector in \mathbb{C} . Let x_n ($n=0,1,2,\dots$) be a scalar number such that $\sum_{n=0}^{\infty} |x_n|^2 \leq 1$. Let $B \in L(E_0, E')$ be an operator defined by

$$(Be_0, d_i) = x_i, \quad (Be_j, d_i) = a_{ij} \quad (i \geq 0, j \geq 1).$$

Determine a unitary $U \in L(E_0, E')$ by $Ue_i = d_i$ ($i \geq 0$). Then by base $\{e_0, e_1, \dots, e_i, \dots\}$ of E_0 we have

$$U^* B = \begin{bmatrix} x_0, a_{01}, \dots, a_{0j}, \dots \\ x_1, a_{11}, \dots, a_{1j}, \dots \\ \vdots & \vdots & \vdots & \\ x_i, a_{i1}, \dots, a_{ij}, \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Let $I_E - V^*A \in (\tau, \mathbb{C})$. Then, since $(V_i^*Ae_j, e_k) = (V^*Ae_j, e_k)$ for $j \geq 0$ and $k \geq i+1$, $I_E - V_i^*A \in (\tau, \mathbb{C})$ for every i .

$$P_E(I_{E_0} - U^*B)|_E = I_E - V^*A$$

implies $I_{E_0} - U^*B \in (\tau, \mathbb{C})$.

Lemma 3.2. Let $I_E - V^*A \in (\tau, \mathbb{C})$. Set $V_0 = V$. Then

$$\det U^*B = \prod_{i=0}^{\infty} x_i \cdot (-1)^i \det(V_i^*A),$$

and

$$\prod_{i=0}^{\infty} |x_i \cdot (-1)^i \det(V_i^*A)| \leq 1.$$

Proof. For simplicity, let $[A]_n$ denote the first $n \times n$

submatrix of A, and A_n the $A|_{E_n}$, where $E_n = \langle e_1, \dots, e_n \rangle$. For any k and n as $n \geq k$, we have

$$(3.4) \quad \sum_{i=1}^k |\det[V_i^*A]_n|^2 \leq \det(A_n^*A_n) = \det[A^*A]_n \leq 1,$$

because A is a contraction. Since for each i

$$\det[V_i^*A]_n \rightarrow \det(V_i^*A) \quad (n \rightarrow \infty),$$

we have $\sum_{i=0}^k |\det(V_i^*A)|^2 \leq 1$, which implies

$$(3.5) \quad \sum_{i=0}^{\infty} |\det(V_i^*A)|^2 \leq 1$$

Consequently $\sum_{i=0}^{\infty} |x_i \cdot (-1)^i \det(V_i^*A)| \leq 1$.

For any $\epsilon > 0$, take an m such that

$$(3.6) \quad \sum_{i=m+1}^{\infty} |x_i|^2 < \epsilon^2.$$

Since $\det[U^*B]_n \rightarrow \det(U^*B)$, and $\det[V_i^*A]_n \rightarrow \det(V_i^*A)$ as $n \rightarrow \infty$, we can take an N such that

$$(3.7) \quad n \geq N \rightarrow |\det[U^*B]_n - \det(U^*B)| < \epsilon,$$

and

$$(3.8) \quad n \geq N \rightarrow \sum_{i=0}^m |\det[V_i^*A]_n - \det(V_i^*A)|^2 < \epsilon^2.$$

Fix a k as $k \geq N+1$ and $k \geq m+1$. Then it follows that

$$\begin{aligned} & |\det(U^*B) - \sum_{i=0}^{\infty} x_i \cdot (-1)^i \det(V_i^*A)| \\ & \leq |\det(U^*B) - \det[U^*B]_k| + |\det[U^*B]_k - \sum_{i=0}^m x_i \cdot (-1)^i \det[V_i^*A]_{k-1}| \\ & \quad + \left| \sum_{i=0}^m x_i \cdot (-1)^i \{ \det[V_i^*A]_{k-1} - \det(V_i^*A) \} \right| \\ & \quad + \left| \sum_{i=m+1}^{\infty} x_i \cdot (-1)^i \det(V_i^*A) \right|. \end{aligned}$$

From (3.7) $|\det(U^*B) - \det[U^*B]_k| < \varepsilon$, and from (3.8)

$$\begin{aligned} & \left| \sum_{i=0}^m x_i \cdot (-1)^i \{ \det[V_i^*A]_{k-1} - \det(V_i^*A) \} \right| \\ & \leq \left(\sum_{i=0}^m |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=0}^m | \det[V_i^*A]_{k-1} - \det(V_i^*A) |^2 \right)^{\frac{1}{2}} < \varepsilon . \end{aligned}$$

(3.5) and (3.6) implies that

$$\left| \sum_{i=m+1}^{\infty} x_i \cdot (-1)^i \det(V_i^*A) \right| < \varepsilon .$$

By the finite matrix theory

$$\begin{aligned} & \left| \det[U^*B]_k - \sum_{i=0}^m x_i \cdot (-1)^i \det[V_i^*A]_{k-1} \right| \\ & = \left| \sum_{i=m+1}^{k-1} x_i \cdot (-1)^i \det[V_i^*A]_{k-1} \right| < \varepsilon , \end{aligned}$$

because the last inequality follows from (3.4), (3.6). Consequently, for any $\varepsilon > 0$ we have

$$\left| \det(U^*B) - \sum_{i=0}^{\infty} x_i \cdot (-1)^i \det(V_i^*A) \right| < 4\varepsilon . \text{Q.E.D.}$$

In (e) of Proposition 3.1, set $(\Gamma(\lambda)e_0, d_i) = h_i(\lambda)$ for $i \geq 0$.

Then we have:

Proposition 3.3. $|\det(U^*[\Gamma(\lambda), \theta(\lambda)])| \leq 1$, and

$$(3.9) \quad \det(U^*[\Gamma(\lambda), \theta(\lambda)]) = \sum_{i=0}^{\infty} h_i(\lambda) \cdot (-1)^i \det(V_i^*\theta(\lambda))$$

is holomorphic on D.

Proof. From (3.2), we have $\sum_{i=0}^{\infty} |h_i(\lambda)|^2 \leq 1$. Since $V_i^*\theta(\lambda)$ is a contractive holomorphic function, $\det(V_i^*\theta(\lambda)) \in H^{\infty}$.

Since $\theta(\lambda)$ is a contraction for every $\lambda \in D$, it follows that

$$\sum_{i=0}^{\infty} |h_i(\lambda) \cdot (-1)^i \det(V_i * \theta(\lambda))| \leq 1,$$

which implies $\sum_{i=0}^{\infty} h_i(\lambda) \cdot (-1)^i \det(V_i * \theta(\lambda))$ is holomorphic.

Equality (3.9) follows from Lemma.

Q.E.D.

Problem. Is $\det(U^*[\Gamma(\lambda), \theta(\lambda)])$ outer?

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