$C_{\cdot 0}$ - CONTRACTIONS

MITSURU UCHIYAMA

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Dedication

To Toshiko ,Shinichi,Nami and Takashi

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Introduction

In this thesis, I will make a study on operators of class $C_{\cdot 0}$ on a Hilbert space. When a bounded operator T on a Hilbert space satisfies $||T|| \leqq 1$ and $T^{*n} \to 0$ strongly as $n \to \infty$, T is said to belong to class $C_{\cdot 0}$. This particular class contains many non-normal operators. In particular, the unilateral shift S on the Hardy class H^2 on the unit disc D in the complex plane belongs to it. In [3] Beurling showed that the invariant subspaces for S are precisely those of the form ψH^2, where ψ is an inner function. For a Hilbert space E, we denote the E-valued Hardy class by $H^2(E)$. Lax [19] and Halmos [17] showed that the invariant subspaces for the unilateral shift S on $H^2(E)$ are precisely those of the form $\Theta H^2(F)$, where F is a Hilbert space with dim F $\leqq$ dim E and $\Theta(\lambda)$ is an arbitrary $B(F,E)$-valued inner function defined on D. In this case, if we set

$$H(\Theta) = H^2(E) \ominus \Theta H^2(F) \quad \text{and} \quad S(\Theta) = P_{H(\Theta)} S | H(\Theta),$$

then $S(\Theta)$ belongs to $C_{\cdot 0}$.

In [25] Rota showed that a contraction with norm < 1 is unitarily equivalent to $S(\Theta)$ for a suitable inner function $\Theta(\lambda)$.

Let T be a contraction on a Hilbert space H. Then Sz.-Nagy and Foias defined the characteristic function $\Theta_T(\lambda)$ of T by

$$\Theta_T(\lambda) = \{-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T\} | D_T H \quad \text{for } \lambda \in D,$$

where $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$. And they showed that T belongs to $C_{\cdot 0}$ if and only if $\Theta_T(\lambda)$ is inner. They also showed that in this case T is unitarily equivalent to $S(\Theta_T)$ (cf.[28]). Thus the theory of spaces of analytic functions (cf.[18]) and the corona theorem (]6],[24]) have come to play important roles in the study of $C_{\cdot 0}$.

A subspace of H is called hyper-invariant for an operator T on H if it is invariant for every bounded operator which commutes with T. In [20] Lomonosov proved a famous theorem : Every compact operator has a hyper-invariant subspace. The invariant subspace problem is an important subject in the actual study of operators.

Now ,I will give a few accounts of the contents of this thesis.

In chapter I ,we will characterize the hyper-invariant subspaces for a contraction T which belongs to $C_{\cdot 0}$ and satisfies $\dim D_T H < \infty$. Here the techniques introduced by Nordgren[22] is useful.

Chapter II is a study on the operators of the form $\phi(S(\psi))$. $\phi(S(\psi))$ is the general Toeplitz operator $PT_\phi|H(\psi)$. (For precise definitions, cf. the first few lines of Chapter II . These operators are considered to extend Toeplitz operators.) In [26],Sarason showed that , for ϕ in H^∞ and a scalar inner function $\psi,\phi(S(\psi))$ is compact if and only if $\bar{\psi}\phi$ belongs to $H^\infty + C$,where C is the Banach algebra of all continuous functions on the unit circle. In the first section of this chapter we will show that,for ϕ in $H^\infty + C$,this result is still true.

We then proceed to establish some results on the double commutant of the operator $S(\Theta)$. It is well-known that the double commutant of an arbitrary unilateral shift consists of multiplications by bounded scalar analytic functions. We extend this result to a wider class of operators of the form $S(\Theta)$. Indeed, we will show that the double commutant consists of $\phi(S(\Theta)), \phi \in H^{\infty}$.

Chapter III contains the main results of this thesis. A contraction T is called a weak contraction if $I-T^*T$ has a finite trace, and $\sigma(T) \neq D$. Weak contractions have nice properties and there are a good deal of studies (cf.[28]). My study concerns on the operators outside of this operator class. We will consider a contraction T which has following properties:

T belongs to $C_{\cdot 0}$,

$I-T^*T$ has a finite trace,

$\sigma(T)=D$ and $\sigma_p(T) \neq D$.

Every unilateral shift has these properties, and we will call such an operator a quasi unilateral shift. One of the B.D.F. theorems[4] implies that $T=S+\text{compact}$,where S is a unilateral shift with index S =index T . My contribution here is to show that there is an intertwining operator between T and S. This stronger result will make easier the analysis of the operators of this kind.

Chapter I. Hyperinvariant subspaces

1.1. $C_0(n)$-contractions.

Let T be a contraction on H belonging to $C._0$. Then it necessarily follows that

$$\delta_* = \dim \overline{D_{T^*}H} \geqq \dim \overline{D_T H} = \delta.$$

Suppose $\delta_*=\delta = n< \infty$, Then T is said to belong to $C_0(n)$ .Simply, we denote the characteristic function of T by $\Theta(\lambda)$ .In this case, we may regard $\Theta(\lambda)$ as an n×n matrix over H^∞ . Since $\Theta(\lambda)$ is inner , that is, $\Theta(e^{it})$ is isometry for almost all t, $\Theta(e^{it})$ is unitary for almost all t. And T on H is unitarily equivalent to $S(\Theta)$ on $H(\Theta)= H^2_n \ominus \Theta H^2_n$, Where H^2_n denotes $H^2(\mathbb{C}^n)$.

Definition 1.1. A *normal* n×n matrix Φ over H^∞ is of the form

$$\Phi = \begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_n \end{bmatrix}$$

, where ,for each i, ϕ_i is a scalar inner function and a divisor of ϕ_{i+1} . The operator $S(\Phi) = S(\phi_1) \oplus \dots \oplus S(\phi_n)$ induced by Φ is called a *Jordan* operator.

By the Sz.-Nagy and Foias theorem [29],every contraction in $C_0(n)$ is quasi-similar to a Jordan operator.

Theoreml.2. Let Θ be an n×n inner matrix over H^∞ and Φ an n×n normal one . If $S(\Theta)$ and $S(\Phi)$ are quasi similar ,then there exist quasi-affinities X from $H(\Theta)$ to $H(\Phi)$ and Y from

$H(\Theta)$ to $H(\Phi)$ and Y from $H(\Phi)$ to $H(\Theta)$ such that

(i) $X\,S(\Theta) = S(\Phi)\,X$ and $S(\Theta)Y = Y\,S(\Phi)$,

(ii) the correspondence τ: $L \to \overline{XL}$ and $\tau^*: M \to \overline{YM}$ establish an isomorphism from the lattice $\mathcal{J}_\Theta$ of hyperinvariant subspaces for $S(\Theta)$ onto the lattice $\mathcal{J}_\Phi$ for $S(\Phi)$, and its inverse, $\tau^* = \tau^{-1}$.

Proof. The hypothesis of quasi-similarity implies for $L \in \mathcal{J}_\Theta$

(1.1) $$\tau(L) = \bigvee_Z \{ZL;\ Z\,S(\Theta) = S(\Phi)\,Z\}$$

belongs to $\mathcal{J}_\Phi$ (c.f.[23]). By one of the Moore-Nordgren theorems ([21],[22]) the quasi-similarity of $S(\Theta)$ and $S(\Phi)$ implies that there exist matrices Δ, Δ', Λ and Λ' each of whose determinants is relatively prime to the determinants of Θ and Φ, and such that

(1.2) $$\Delta\,\Theta = \Phi\,\Lambda \text{ and } \Theta\,\Lambda' = \Delta'\,\Phi .$$

Define the operator X from $H(\Theta)$ to $H(\Phi)$ and Y from $H(\Phi)$ to $H(\Theta)$ by

(1.3) $$Xh = P_{H(\Phi)}\Delta h \text{ for } h \text{ in } H(\Theta),\quad Yg = P_{H(\Theta)}\Delta' g \text{ for } g \text{ in } H(\Phi).$$

Relation (1.2) guarantees condition (i), and X,Y are quasi-affinities. Take an arbitrary L in the lattice $\mathcal{J}_\Theta$ and let $L' = \tau(L)$. By a well-known theorem[28] the (hyper-)invariance of L and L' implies the existence of inner matrices $\Theta_1, \Theta_2, \Phi_1$ and Φ_2 over H^∞ satisfying

(1.4) $$\Theta = \Theta_2\Theta_1 \quad \text{and} \quad \Phi = \Phi_2\Phi_1 ,$$

and

(1.5) $$L = \Theta_2(H^2_n \ominus \Theta_1 H^2_n) \text{ and } L' = \Phi_2(H^2_n \ominus \Phi_1 H^2_n).$$

By the definition(1.1) of $\tau(L)$ we have $XL \subseteq \tau(L) = L'$. on the other hand, since YZ commutes with $S(\Theta)$ for every Z occuring in (1.1),hyper-invariance of L for $S(\Theta)$ implies $YZL \subseteq L$, and therefore $YL' = Y\tau(L) \subseteq L$. Now the inclusions $\overline{XL} \subseteq L'$ and $\overline{YL'} \subseteq L$, and relations (1.2)-(1.5) imply $\Delta\Theta_2 H^2_n \subseteq \Phi_2 H^2_n$ and $\Delta'\Phi_2 H^2_n \subseteq \Theta_2 H^2_n$; whence we deduce the existence of matrices A and B over H^∞ such that

(1.6) $\Delta\ \Theta_2 = \Phi_2\ A$ and $\Delta'\Phi_2 = \Theta_2 B$.

Thus it follows that $\Phi_2 AB = \Delta\ \Delta'\Phi_2$,and hence,

(1.7) $\det A \cdot \det B = \det\Delta \cdot \det \Delta'$.

Since $\det\Delta \cdot \det \Delta'$ is relatively prime to $\det\Phi$, (1.7) implies that det A is relatively prime to $\det\Phi$, hence to $\det\Phi_1$. To prove $L' = \overline{XL}$ suppose that $f \in L' \ominus \overline{XL}$. Then ,again using (1.2)-(1.5), we see that f is orthogonal to $\Delta\Theta_2 H^2_n$, and hence to $\Phi_2 AH^2_n$, by (1.6). Moreover,(1.5) implies $f = \Phi_2 g$ for some $g \in H^2_n \ominus \Phi_1 H^2_n$. Then for every $h \in H^2_n$

$$0 = (f,\Delta\Theta_2 h) = (\Phi_2\ g, \Phi_2 Ah) = (g,\ Ah).$$

Since $\det A$ is relatively prime to $\det\Phi_1$, AH^2_n and $\Phi_1 H^2_n$ span the whole H^2_n . This implies g=0, hence f=0, proving $L' = \overline{XL}$. The relation $L = \overline{YL'} = \overline{YXL}$ is proved in a similar way. This completes the proof.

Theorem 1.3. Let Φ be an n×n normal matrix over H^∞ .A subspace L of $H(\Phi)$ is hyper-invariant for $S(\Phi)$ if and only if there are n×n normal matrices Φ_1, Φ_2 satisfying

(1.8) $\Phi = \Phi_2 \Phi_1$ and $L = \Phi_2 (H^2_n \ominus \Phi_1 H^2_n)$.

Proof. By the lifting theorem ([28] p.258), for every operator X on $H(\Phi)$ commuting with $S(\Phi)$, there is a matrix Δ over H^∞ satisfying

(1.9) $Xh = P_{H(\Phi)} \Delta h$ $(h \in H(\Phi))$ and $\Delta\Phi\, H^2_n \subseteqq \Phi H^2_n$.

The latter condition is equivalent to the existence of a matrix Λ over H^∞ satisfying

(1.10) $\Delta\, \Phi = \Phi\, \Lambda$.

Suppose that L is of the form (1.8), and that $\Phi = \operatorname{diag}(\phi_1, \ldots, \phi_n)$. To prove the hyper-invariance of L for $S(\Phi)$, it suffices to show the invariance of L for the operator X defined by (1.9). The existence of Λ satisfying (1.10) implies that if $i>j$, then the inner function ϕ_i/ϕ_j is a divisor of the Δ_{ij}, that is, the (i,j)-th entry of Δ. Since Φ_2 and Φ_1 are normal matrices with $\Phi = \Phi_2 \Phi_1$, for $i>j$ the inner function u_i/u_j is a divisor of ϕ_i/ϕ_j, where u_i is the (i,i)-th entry of Φ_2, hence a divisor of Δ_{ij}. This guarantees the existence of a matrix Λ' over H^∞ satisfying

(1.11) $\Delta\, \Phi_2 = \Phi_2\, \Lambda'$,

and consequently the invariance of L for X.

Suppose conversely that L is hyper-invariant for $S(\Phi)$. Let P_i be the orthogonal Projection from $H(\Phi)$ onto the i-th component space . Since P_i commutes with $S(\Phi)$, the hyper-invariance of L implies that

$$L = P_1 L \oplus, \ldots, \oplus P_n L$$

and each $P_i L$ is an invariant subspace for $S(\phi_i)$. By the Beurling theorem there are inner divisors u_i and v_i of ϕ_i satisfying

(1.12) $\quad \phi_i = u_i v_i \quad , \; P_i L = u_i (H^2 \ominus v_i \; H^2)$.

Set $\Phi_2 = \operatorname{diag}(u_1, \ldots, u_n)$ and $\Phi_1 = \operatorname{diag}(v_1, \ldots, v_n)$, then Φ_2 and Φ_1 satisfy (1.8). It remains to prove the normality of Φ_2 and Φ_1. To this end, take the matrix Δ over H^∞ whose (i,j)-th entry Δ_{ij} is defined by

$$\Delta_{ij} = 1 \;(\; i \leqq j \;) \quad \text{and} \quad \Delta_{ij} = \phi_i / \phi_j \;(\; i>j).$$

Clearly there exists a matrix Λ over H^∞ satisfying (1.10). The hyper-invariance of L implies the existence of a matrix Λ' satisfying (1.11). This means if $i<j$, then u_i is a divisor of u_j and u_j/u_i is a divisor of ϕ_j/ϕ_i. The former condition guarantees the normality of Φ_2, while the latter does the normality of Φ_1. This completes the proof.

Since every $C_0(n)$-contraction is quasi-similar to its Jordan operator ,by above theorems, we can characterize the hyper-invariant subspaces for it.

When ϕ is a scalar inner function, for the operator $S(\phi)$ the invariance of a subspace is equivalent to its hyper-invariance . The lattice $\mathcal{S}_\phi$ of all (hyper-)invariant subspaces is totally ordered if and only if ϕ is of the form

(1.13) $\quad ((\lambda-\alpha) / (1 - \bar{\alpha}\lambda))^n \quad (|\alpha| < 1, n \text{ a positive integer})$

or of the form

(1.14) $\quad e_s(\lambda) \equiv \exp\ (s(\lambda+\alpha)/(\lambda-\alpha))$ $\quad (|\alpha|=1, s>0)$,

according as $\dim H(\phi)=n$ or $\dim H(\phi) = \infty$ (cf.[28] p.136). This can be generarized to the case of inner matrices.

Theorem 1.4. Let Φ be an $n\times n$ normal matrix over H^∞ and $\dim H(\Phi) = \infty$. The lattice $\mathcal{J}_\Phi$ of hyper-invariant subspaces for $S(\Phi)$ is totally ordered if and only if ϕ_n is of the form (1.14) and each ϕ_i coincides with either 1 or ϕ_n, where ϕ_i is the (i,i)-th entry of Φ.

Proof. By theorem 1.3 the total orderdness of the lattice $\mathcal{J}_\Phi$ is equivalent to the condition that if normal matrices Φ_2 and Φ_2' are left divisors of Φ such that $\Phi_2^{-1}\Phi$ and $\Phi_2'^{-1}\Phi$ are normal too, then one of Φ_2 and Φ_2' is a left divisor of the other. Suppose that $\mathcal{J}_\Phi$ is totally ordered. Take arbitrary inner divisors u and v of ϕ_n, and set $u_i = u \wedge \phi_i$ and $v_i = v \wedge \phi_i$ ($a \wedge b$ denotes the gratest common inner divisor of a and b). Then the normal matrices Φ_2 and Φ_2' defined by

$$\Phi_2 = \mathrm{diag}(u_1, u_2, \ldots, u_{n-1}, u) \quad \text{and} \quad \Phi_2' = \mathrm{diag}(v_1, v_2, \ldots, v_{n-1}, v)$$

are left divisor of Φ, and $\Phi_2^{-1}\Phi$ and $\Phi_2'^{-1}\Phi$ are normal matrices over H^∞. The divisibility of Φ_2 by Φ_2' or Φ_2' by Φ_2 implies that one of u and v is a divisor of the other. The arbitrariness of u and v implies that ϕ_n is of the form (1.14) because $\dim H(\Phi)=\infty$ implies $\dim H(\phi_n)=\infty$. There exists an ϕ_i such that $\phi_i/\phi_{i-1}=e_s$ $(1\leqq i\leqq n)$. If fact if any ϕ_i/ϕ_{i-1} is not equal to e_s , then there exists i and j such that $1\leqq i<j\leqq n$, $\phi_i/\phi_{i-1}=e_a$ $(s>a>0)$, $\phi_j/\phi_{j-1}=e_b$ $(s>b>0)$ and $a+b\leqq s$.

Now set c and d so that $0<c\leqq a$, $0<d\leqq b$ and $c<d$. Consider the normal matrices Ω_1 and Ω_2 defined by

$$\Omega_1 = \mathrm{diag}(1,..,1,\overset{(i)}{e_c},...,e_c) \text{ and } \Omega_2 = \mathrm{diag}(1,..,1,\overset{(j)}{e_d},..,e_d)$$

. Clearly Ω_i is a left divisor of Φ and $\Omega_i^{-1}\Phi$ is a normal matrix. By Theorem 1.3, the subspaces

$$\Omega_1 H^2_n \ominus \Phi H^2_n \quad \text{and} \quad \Omega_2 H^2_n \ominus \Phi H^2_n$$

are hyper-invariant for $S(\Phi)$, but any one of them is not included in the other, a contadiction. Consequently $\Phi=\mathrm{diag}(1,..,1,e_s,..,e_s)$. The "only if" part is trivial. Therefore we omit the proof (see[33]).

1.2. $C_{\cdot 0}$ - contractions.

In this section, we consider a contraction T in $C_{\cdot 0}$ such that $m=\delta<\delta_*=n<\infty$. Firstly we decide the lattice of hyper-invariant subspaces for a Jordan operator in class $C_{\cdot 0}$. Next we establish a canonical isomorphism between the lattice of hyper-invariant subspaces for T and that for the Jordan model of T. Since $\delta=m$, $\delta_*=n$, the characteristic function $\Theta(\lambda)$ of

T is regarded as an $n\times m$ matrix over H^{∞}. Let d_k be the largest common inner divisor of all the minors of order k ($1\leq k\leq m$). And set $\psi_k = d_k/d_{k-1}$ ($d_0=1$). Then ψ_k is a scalar inner function and a divisor of its succesor. In this case, an $n\times m$ matrix;

$$\Phi = \begin{bmatrix} \psi_1 & & 0 \\ & \psi_2 \ddots & \\ 0 & & \psi_m \\ 0 & \dots & 0 \end{bmatrix}$$

is called *normal*, and a corresponding operator;

$$S(\Phi)=S(\psi_1) \oplus \dots \oplus S(\psi_m) \oplus S,$$

where S is the unilateral shift with index S = n-m, is called *Jordan model* of T. Nordgren [22] has shown that there are pairs of matrices Δ_i, Λ_i and Δ_i', Λ_i' ($i=1,2$) satisfying

(2.1) $\quad \Delta_i\Theta = \Phi\Lambda_i$,

(2.1)' $\quad \Theta\Lambda_i' = \Delta_i'\Phi$,

(2.2) $\quad (\det \Delta_i)(\det \Lambda_i') \wedge d_m = 1$,

(2.3) $\quad (\det \Delta_1)(\det \Delta_1') \wedge (\det \Delta_2)(\det \Delta_2') = 1$,

(2.3)' $\quad (\det \Lambda_1)(\det \Lambda_1') \wedge (\det \Lambda_2)(\det \Lambda_2') = 1$.

Setting

(2.4) $\quad X_i = P_\Phi \Delta_i | H(\Theta)$ and

(2.4)' $\quad Y_i = P_\Theta \Delta_i' | H(\Phi)$ for $i=1,2$,

where P_Φ simply denotes $P_{H(\Phi)}$,

$\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are injective families satisfying the following relations:

(2.5) $\quad X_i S(\Theta) = S(\Phi) X_i$,

(2.6) $\quad S(\Theta) Y_i = Y_i S(\Phi)$, (2.7) $\quad X_1 H(\Theta) \bigvee X_2 H(\Theta) = H(\Phi)$,

(2.8) $\quad Y_1 H(\Phi) \bigvee Y_2 H(\Phi) = H(\Theta)$.

This implies $S(\Theta) \overset{i}{\sim} S(\Phi)$ [30].

Now set $\Psi = \text{diag}\,(\psi_1, \ldots, \psi_m)$, that is, $\Phi = \begin{bmatrix} \Psi \\ 0 \end{bmatrix}$. Then $S(\Phi)$ on $H(\Phi)$ are identified with

$$S(\Psi) \oplus S \quad \text{on} \quad H(\Psi) \oplus H_{n-m} .$$

Let N be a hyper-invariant subspace for $S(\Phi)$. Then it is clear that N is decomposed to the direct sum, $N = N_1 \oplus N_2$, where N_1 is a subspace of $H(\Psi)$, hyper-invariant for $S(\Psi)$, and N_2 is a subspace of H_{n-m} , hyper-invariant for S. In this case we have the following lemma.

Lemma 2.1. In order that $N = N_1 \oplus N_2$ is hyper-invariant for $S(\Phi)$, it is necessary and sufficient that $N_2 = \{0\}$ or there exists an inner function ϕ such that $N_2 = \phi H^2_{n-m}$ and $N_1 \supseteqq \phi(S(\Psi))H(\Psi)$.

Proof. Simply set $k=n-m$. An operator $X = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ commutes with $S(\Phi)$, if and only if Y_{ij} satisfy the following conditions:

$$Y_{11} S(\Psi) = S(\Psi)\, Y_{11} , \quad Y_{12} S = S(\Psi)\, Y_{12} ,$$

$$Y_{21} S(\Psi) = S\, Y_{21} , \quad Y_{22} S = S\, Y_{22} .$$

Since $S(\Psi)^n \to 0$ as $n \to 0$ and S is isometry, we have $Y_{21} = 0$. Thus if $N_2 = \{0\}$, then it follows that $XN \subseteqq N$ for every X commuting $S(\Phi)$. By the lifting theorem ([26],[28]), a bounded operator Y_{12} from H_k^2 to $H(\Psi)$ intertwines S and $S(\Psi)$, if and only if there is an $m\times k$ matrix Ω over H^∞ such that $Y_{12} = P_\Psi \Omega$. Thus, if $\mathcal{N}_2 = \phi H_k^2$ and $\mathcal{N}_1 \supseteq \phi(S(\Psi))H(\Psi)$ for some inner function ϕ, then we have

$$\begin{aligned}
X\,\mathcal{N} &= (Y_{11}\mathcal{N}_1 + Y_{12}\ \phi H_k^2) \oplus Y_{22}\phi H_k^2 \\
&\subseteq (\mathcal{N}_1 + P_\Psi \Omega\ \phi H_k^2\ ) \oplus \phi H_k^2 \\
&\subseteq (\mathcal{N}_1 + P_\Psi\ \phi H_m^2\ ) \oplus \phi H_k^2 \\
&= (\mathcal{N}_1 + \phi(S(\Psi))H(\Psi)\ ) \oplus \phi H_k^2 \\
&\subseteq \mathcal{N}_1 \oplus \phi H_k^2 = \mathcal{N} ,
\end{aligned}$$

where $\phi(S(\Psi))h = P_\Psi \phi h$ for $h \in H(\Psi)$. Thus $\mathcal{N}$ is hyper-invariant for $S(\Phi)$.

Conversely suppose $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ is hyper-invariant for $S(\Phi)$, and $\mathcal{N}_2 = \{0\}$. Then by [10], there is an inner function ϕ such that $\mathcal{N}_2 = \phi H_k^2$. Let Ω_i $(i=1,2,..,m)$ be the $m\times(n-m)$ matrix such that the $(i,1)$-th entry of Ω_i is 1 and the other entry is 0. Setting

$$X_i = \begin{bmatrix} 0 & Y_i \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y_i = P_\Psi \Omega_i ,$$

each X_i commutes with $S(\Phi)$, hence we have

$$\mathcal{N}_1 = \sum_{i=1}^{n} Y_i\ \phi H_k^2 = P_\Psi\ \phi H_m^2 = \phi(S(\Psi))H(\Psi).$$

This completes the proof.

Theorem 2.2. In order that a factorization $\Phi = \Phi_2 \Phi_1$ of Φ into the product of an $n\times l$ inner matrix Φ_2 and an $l\times m$ inner matrix Φ_1 $(n \geqq l \geqq m)$ corresponds to a hyper-invariant subspace

$\mathcal{N}$ for $S(\Phi)$,it is necessary and sufficient that $\Phi_1$and Φ_2 are normal matrices satisfying (i) or (ii):

(i) $l=m$,

(ii) $l=n$ and Φ_2 has the form $\begin{bmatrix} \Psi_2 & 0 \\ 0 & \phi I_k \end{bmatrix}$.

Proof. First, assume that $l=m$, and both $\Phi_1$and Φ_2 are normal inner matrices. Then,setting $\Phi_2 = \begin{bmatrix} \Psi_2' \\ 0 \end{bmatrix}$,it follows that

$\Phi_2 H(\Phi_1) = \Psi_2' H(\Phi_1)$ is hyper-invariant for $S(\Psi)$ (see Sec.1.1). Therefore , by Lemma 2.1, it is hyper-invariant for $S(\Phi)$.

Next, assume that $\Phi_1$and Φ_2 are normal matrices satisfying (ii). Set $\Phi_1 = \begin{bmatrix} \Psi_1 \\ 0 \end{bmatrix}$. Then we have

$$\mathcal{N} = \Phi_2 \{ H^2_n \ominus \Phi_1 H^2_m \} = \Psi_2 H(\Psi_1) \oplus \phi H^2_k .$$

Normality of Ψ_1 and Ψ_2 implies that $\Psi_2 H(\Psi_1)$ is hyper-invariant for $S(\Psi)$. On the other hand ,normality of Φ_2 implies $\Psi_2 H^2_m \supseteqq \phi H^2_m$, and hence we have

$$\Psi_2 H^2_m \ominus \Psi H^2_m \supseteqq \phi(S(\Psi))H(\Psi) .$$

Thus ,from Lemma 2.1, we deduce that $\mathcal{N}$ is hyper-invarinat for $S(\Phi)$.

Conversely,first assume that $\mathcal{N} = \mathcal{N}_1 \oplus \{0\}$ is hyper-invariant for $S(\Phi)$, and $\Phi = \Phi_2 \Phi_1$ is the factorization corresponding to $\mathcal{N}$. Since $S(\Phi)|\mathcal{N} = S(\Psi)|\mathcal{N}_1$is of class C_0 , S(is of class C_0 (about notation C_0 see [28]). This implies that $\Phi_1$is an $m\times m$ inner matrix, that is , $l=m$. Setting $\Phi_2 = \begin{bmatrix} \Psi_2 \\ \Gamma \end{bmatrix}$, where Ψ_2 is an $m\times m$ matrix and Γ an $k\times m$ matrix $(k=n-m)$, we have

$$\Psi = \Psi_2 \Phi_1 , \quad N_1 = \Psi_2 H(\Phi_1) \text{ and } \Gamma H^2_m = \{0\} .$$

Since $\Gamma = 0$ and Φ_2 is inner, also Ψ_2 is inner. Thus the hyper-invariance of N_1 corresponding to $\Psi = \Psi_2\Phi_1$ implies that Ψ_2 and Φ_1 are m×m normal matrices. Next assume that $N = N_1 \oplus \phi H^2_k$ and $N_1 \supseteqq \phi(S(\Psi))H(\Psi)$.

Clearly we have

$$P_{N^\perp} S(\Phi)|_{N^\perp} = P_{N_1^\perp} S(\Psi)|N_1^\perp \oplus S(\phi I_k) .$$

Since the right hand operator is of class C_0 , $S(\Phi_2)$ is of class C_0 . This implies Φ_2 is an n×n matrix; i.e.,l=n. To the hyper-invariant subspace N_1 for $S(\Psi)$ there corresponds a factorization $\Psi = \Psi_2\Psi_1$,where Ψ_1 and Ψ_2 are m×m normal matrices . Thus setting $\Phi_2' = \begin{bmatrix} \Psi_2 & 0 \\ 0 & \phi I_k \end{bmatrix}$ and $\Phi_1' = \begin{bmatrix} \Psi_1 \\ 0 \end{bmatrix}$,

it is clear that

$$\Phi = \Phi_2' \Phi_1' \text{ and } N = \Phi_2' \{H^2_n \ominus \Phi_1' H^2_m \} .$$

From the uniqueness of the factorization of Φ into product of two inner matrices corresponding to invariant subspace N , only this factorization $\Phi = \Phi_2'\Phi_1'$ corresponds to N , that is, $\Phi_2 = \Phi_2'$ and $\Phi_1 = \Phi_1'$. Since

$$\Psi_2 H(\Psi_1) = N_1 \supseteqq \phi(S(\Psi))H(\Psi) = P_\Psi \phi H^2_m ,$$

we have $\Psi_2 H^2_m \geqq \phi H^2_m$; this implies that every entry of Ψ_2 is a divisor of ϕ . Therefore Φ_2 is an n×n normal matrix. Hence Φ_1 and Φ_2 are normal matrices satisfying (ii). Q.E.D.

Set $\quad \tau(L) = \bigvee_Z \{ZL: \ ZS(\Theta)=S(\Phi)Z\}$

and $\quad \tau^*(N) = \bigvee_W \{WN: \ WS(\Phi)=S(\Theta)W\}$

for each subspace L and N hyper-invariant for $S(\Theta)$ and $S(\Phi)$, respectively. Since $S(\Theta) \overset{ci}{\sim} S(\Phi)$, it is clear that $\tau(L)$ is the nontrivial hyper-invariant subspace for $S(\Phi)$, if L is non-trivial.

Lemma 2.3. If $\Theta=\Theta_2\Theta_1$ is the factorization corresponding to a non -trivial hyper-invariant subspace L for $S(\Theta)$, then Θ_1 is an $m\times m$ inner matrix, or Θ_2 is an $n\times n$ inner matrix.

Proof. Let $S(\Theta) = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ and $S(\Phi) = \begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix}$ be the triangulations corresponding to

$$H(\Theta) = L \oplus L^{\perp} \quad \text{and} \quad H(\Phi) = \tau(L) \oplus \tau(L)^{\perp}$$,respectively.

Theorem 2.2. implies that S_1or S_2 is in C_0 . First, suppose $u(S_1)=0$ for some u in H^{∞} . For the bounded operator X_1 given by (2.4) and every f in L , in virtue of (2.1), it follows that $X_1\ u(T_1)f = X_1 u(S(\Theta))f = P_{\Phi}\Delta_1 P_{\Theta} uf = P_{\Phi}\Delta_1 uf = P_{\Phi} u\Delta_1 f =$

$=u(S(\Phi))X_1 f = 0$.

Since X_1 is an injection, we have $u(T_1)f=0$,which implies that T_1 belongs to C_0 , that is, Θ_1 is an $m\times m$ inner matrix. Next suppose S_2 belong to C_0 , hence so does $S_2{}^*$. For Y_i given by (2.4)' and every Z such that $ZS(\Theta)=S(\Phi)Z$, in virtue of (2.6), $Y_i Z$ commutes with $S(\Theta)$, this implies $Y_i ZL \subseteqq L$ and hence $Y_i\tau(L) \subseteqq L$. Thus we have $Y_i{}^* L^{\perp} \subseteqq \tau(L)^{\perp}$. From this and (2.6), for each h in L, it follows that

$$Y_i{}^*T_2{}^*h = S_2{}^*Y_i{}^*h \quad \text{for} \quad i=1,2.$$

From this, we can deduce that

$$Y_i{}^*u(T_2{}^*)h = u(S_2{}^*)Y_i{}^*h \quad \text{for every } u \text{ in } H^\infty.$$

Since $Y_1H(\Phi) \bigvee Y_2H(\Phi) = H(\Theta)$, we have $u(T_2{}^*)=0$ for u satisfying $u(S_2{}^*)=0$. Therefore Θ_2 is an $n\times n$ inner matrix. This completes the proof.

A following theorem implies that the mapping τ is isomorphism from the lattice $\mathcal{J}_\Theta$ onto the lattice $\mathcal{J}_\Phi$, and its inverse is given by τ^*.

Theorem 2.4. For X_i and Y_i given by (2.4),(2.4)',

(2.9) $\tau(L) = X_1L \bigvee X_2L$ and $\tau^*(\tau(L))=L$,

(2.9)' $\tau^*(N) = Y_1N \bigvee Y_2N$ and $\tau(\tau^*(N))=N$,

where $L\in\mathcal{J}_\Theta$ and $N\in\mathcal{J}_\Phi$.

Proof. Let $\Theta=\Theta_2\Theta_1$ and $\Phi=\Phi_2\Phi_1$ be the factorizations of Θ and Φ corresponding to L and $\tau(L)$, respectively. Then the proof of Lemma 2.3 implies that both Θ_1 and Φ_1 are $l\times m$ matrices and both Θ_2 and Φ_2 are $n\times l$ matrices, where $l=n$ or $l=m$. Since $X_iL \subseteqq \tau(L)$ and $Y_i\tau(L) \subseteqq L$, it clearly follows that

$$\Delta_i\Theta_2H_l^2 \subseteqq \Phi_2H_l^2 \quad \text{and} \quad \Delta_i{}'\Phi_2H_l^2 \subseteqq \Theta_2H_l^2,$$

which guarantee the existence of $l\times l$ matrices A_i and B_i over H^∞ satisfying

(2.10) $\quad \Delta_i\Theta_2 = \Phi_2 A_i$ and $\Delta_i'\Phi_2 = \Theta_2 B_i$.

This and (2.1) implies that

(2.10)' $\quad A_i\Theta_1 = \Phi_1\Lambda_i$ and $B_i\Phi_1 = \Theta_1\Lambda_i'$.

By (2.10) we have

(2.11) $\quad \Delta_i'\Delta_i\Theta_2 = \Theta_2 B_i A_i$,

and by (2.10)'

(2.11)' $\quad B_i A_i\Theta_1 = \Theta_1\Lambda_i'\Lambda_i$.

Thus if $l=n$, then $\det A_i$ is a divisor of $\det \Delta_i \cdot \det \Delta_i'$, and if $l=m$ then $\det A_i$ is a divisor of $\det \Lambda_i \cdot \det \Lambda_i'$. To prove the first relation of (2.9) suppose that

$$f \in \tau(L) \ominus \{X_1 L \vee X_2 L\}.$$

Then f is orthogonal to $\Delta_1\Theta_2 H^2_l \vee \Delta_2\Theta_2 H^2_l$. On the other hand $f \in \tau(L)$ implies the existence of g belonging to $H^2_l \ominus \Phi_1 H^2_m$ such that $f = \Phi_2 g$. Thus for every h in H^2_k ,we have

$$0 = (f, \Delta_i\Theta_2 h) = (\Phi_2 g, \Phi_2 A_i h) = (g, A_i h) \qquad (i=1,2)$$

Thus if $l=n$, then , by (2.3) and Beurling's theorem

$$A_i H^2_n \supseteq (\det A_i) H^2_n \supseteq (\det \Delta_i)(\det \Delta_i') H^2_n$$

induce $A_1 H^2_n \vee A_2 H^2_n = H^2_n$ and hence $g=0$.

If $l=m$, then, by (2.3)' and Beurling's theorem

$$A_i H^2_m \supseteq (\det A_i) H^2_m \supseteq (\det \Lambda_i)(\det \Lambda_i') H^2_m$$

induce $A_1 H^2_m \vee A_2 H^2_m = H^2_m$ and hence $g=0$. Thus we showed $\tau(L) = X_1 L \vee X_2 L$. The rest is proved in a similar way. Q.E.D.

Chapter II . Commutants and double commutants

2.1. Generalized Toeplitz operator.

Let L^2 be the Hilbert space of all square Lebesgue integrable functions defined on the unit circle, and L^∞ the Banach algebra of all essentially bounded functions defined on the unit circle. Given ϕ in L^∞ , $M(\phi)$ denotes the multiplication of ϕ on L^2 . Let P' be the projection from L^2 onto H^2 . Then a Toeplitz operator T_ϕ is defined by $T_\phi = P'M(\phi) \mid H^2$. Let ψ be a scalar inner function. Then ,for ϕ in L^∞ , we define the general Toeplitz operator $\phi(S(\psi))$ in the sense of [7] by $\phi(S(\psi)) = P\, T_\phi | H(\psi)$, where $P=P_\psi$.We denote the inner products in $H(\psi)$, H^2 and L^2 by (,),(,)' and (,)",respectvely, and the identical operators in them by I, I' and I".

Lemma 1.1. For ϕ in $H^\infty + C$, $(I''-P')M(\phi)P'$ is a compact operator on L^2 ,where C is a space of all continuous functions on the unit circle.

Proof. Let $\phi = \phi_1 + \phi_2$ be a decomposition of ϕ such that ϕ_1 is in H^∞ and ϕ_2 in C. Then it follows that

$$(I''-P')M(\phi)P' = (I''-P')M(\phi_2)P'.$$

Take trigonometric polynomials $g_n (n=1,2,..)$ whose sequence uniformly converges to ϕ_2 . Then,since

$$\|(I''-P')M(g_n)P' - (I'' - P')M(\phi_2)P'\| \leqq \| M(g_n) - M(\phi_2)\|$$
$$\leqq \| g_n - \phi_2 \|_\infty \to 0 \quad \text{as } n\to \infty ,$$

finiteness of the rank of $(I'' - P')M(g_n)P'$ implies that

$(I'' - P')M(\phi_2)P'$ is compact.

Lemma 1.2. For ϕ in $H^\infty + C$, $PT_\phi(I'-P)$ is compact.

Proof. This lemma follows from Lemma1.1 and next relations;

$$PT_\phi(I'-P) = P\,P'M(\phi)(I'-P) = P\,P'M(\phi)M(\psi)M(\bar\psi)(I'-P)$$
$$= PP'M(\psi)M(\phi)M(\bar\psi)(I'-P) = PP'M(\psi)(I''-P')M(\phi)P'M(\bar\psi)(I'-P).$$

Lemma 1.3. If ϕ is in $H^\infty + C$, then there exists a compact operator K from H^2 to $\bar H^2_0$,which is the conjugate space of H^2_0 , such that

$$\frac{1}{2\pi}\int_0^{2\pi}\phi\bar\psi\ f\ dt = (Kf_1,\ f_2)'' + (\phi(S(\psi))Pf_1,\ P'\psi\bar f_2)$$

for every f in H^1_0 , f_1 in H^2 and f_2 in H^2_0 such that $f = f_1 f_2$.

Proof. $\psi\bar f_2$ is orthogonal to $\psi H^{2'}$, and $P'\psi\bar f_2$ belongs to $H(\psi)$. Therefore we have

$$\frac{1}{2\pi}\int_0^{2\pi}\phi\bar\psi\ f\ dt = (\phi f_1,\psi\bar f_2)'' = (P'\phi Pf_1,\psi\bar f_2)'' +$$
$$+(P'\phi(I'-P)f_1,\psi\bar f_2)'' + ((I''-P')\phi f_1,\psi\bar f_2)''$$
$$= (P'\phi Pf_1,P'\psi\bar f_2)'' + (\bar\psi\ PP'\phi(I'-P)f_1,\bar f_2)'' + (\bar\psi(I''-P')\phi f_1,\bar f_2)''$$
$$= (\phi(S(\psi))Pf_1,P'\psi\bar f_2) + (\bar\psi PT_\phi(I'-P)f_1,\ \bar f_2)'' +$$
$$(\bar\psi\ (I''-P')M(\phi)f_1,\bar f_2)'' \quad .$$

Thus $K = M(\bar\psi)PT_\phi(I'-P) + M(\bar\psi)(I''-P')M(\phi)|H^2$ satisfies the conditions of this lemma.

The proof of the next theorem deeply depends on [26].

Proposition 1.4. Let ϕ be a function in $H^\infty + C$.Then $\phi(S(\psi))$

is compact if and only if $\bar{\psi}\phi$ belongs to $H^{\infty}+C$.

Proof. "Only if " part is obvious. Suppose $\phi(S(\psi))$ be compact. We wish to show that the kernel of functional of $\bar{\psi}\phi + H^{\infty}$ on H_0^1 is sequentially weak star closed. Let f_n be a sequence in its kernel and converge weak star to f. Let $f_n = f_{1n} f_{2n}$ be the factorization of f_n such that f_{1n} and f_{2n} belong to H^2 and H_0^2 ,respectively, and $|f_n| = |f_{1n}|^2 = |f_{2n}|^2$.

Then, since $\{f_{1n}\}$ and $\{f_{2n}\}$ are bounded in L^2, we may assume that they converge weakly to f_1 and f_2 in L^2, respectively, and $f = f_1 f_2$. It is clear that f_1 is in H_0^2 and f_2 is in H^2 . From Lemma1.3, there is a compact operator K such that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\bar{\psi}f_n \, dt = (Kf_{1n} , \bar{f}_{2n})'' + (\phi(S(\psi))Pf_{1n}, P'\psi\bar{f}_{2n})$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\bar{\psi}f \, dt = (Kf_1, \bar{f}_2)'' + (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2) .$$

Since both K and $\phi(S(\psi))$ are compact, it follows that

$$(Kf_{1n} , \bar{f}_{2n})'' \to (Kf_1, \bar{f}_2)'' \quad (n\to \infty)$$

and

$$(\phi(S(\psi))Pf_{1n} ,P'\psi\bar{f}_{2n}) \to (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2) \quad (n\to\infty).$$

Thus we have $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} \phi\bar{\psi} f \, dt = 0.$

The proof is complete.

Theorem 1.5. If ϕ is in H^∞, then next conditions are equivalent;

(a) $\phi(S(\psi))$ is a Fredholm operator ,

(b) there are $\varepsilon>0$ and $1>\delta \geqq 0$ such that

$$|\phi(\lambda)| + |\psi(\lambda)| \geqq \varepsilon \quad \text{for} \quad 1>|\lambda| \geqq \delta ,$$

(c) $\phi(H^\infty + C) + \psi(H^\infty + C) = H^\infty + C$.

Proof. First assume (a). Then there is a factorization $\phi = \phi_1\phi_2$, where $\phi_1(S(\psi))$ is invertible and ϕ_2 is a finite Blashke function. By [12] and [13], there is an $\varepsilon_1>0$ such that

$$|\phi_1(\lambda)| + |\psi(\lambda)| \geqq \varepsilon_1 \text{ for } |\lambda| < 1 .$$

Since ϕ_2 is a finite Blashke function, we can easily show (b).

Next assume (b). Setting $\eta = \phi \wedge \psi$, there is an $\varepsilon_1 > 0$ such that $|\eta(\lambda)| \geqq \varepsilon_1$ for $1>|\lambda| \geqq \delta$.

Consequently $1/\eta$ belongs to $H^\infty + C$ [8]. Set $\phi' = \phi/\eta$ and $\psi' = \psi/\eta$. Then it is clear that there is an $\varepsilon_2 > 0$ such that

$$|\phi'(\lambda)| + |\psi'(\lambda)| \geqq \varepsilon_2 \qquad \text{for } |\lambda| < 1.$$

Hence ,by corona theorem [6] [24], we have $\phi' H^\infty + \psi' H^\infty = H^\infty$, which yields (c). It is clear that (c) implies (a). Thus the theorem is established.

2.2. Double commutants.

When T is a special $C_{\cdot 0}$-contraction, the A_T and $\{T\}''$ were investigated by several authors (for unilateral shift see

[5], for C_0-contraction [1], [31] and [40]), where $\mathcal{A}_T$ is a weakly closed algebra generated by T and I. In place of $C_{\cdot 0}$-contraction T with $\delta=m$, $\delta_*=n$ (necessarily $n \geqq m$) we may consider $S(\Theta)$, where $\Theta(\lambda)$ is the characteristic function of T, $n\times m$ matrix of H^∞ and $\|\Theta(\lambda)\|\leqq 1$ for every λ in D. In this section we assume $\infty \geqq n > m$. In this case there is an $n\times m$ normal matrix;

$$\Phi = \begin{bmatrix} \psi_1 & & 0 \\ & \ddots & \psi_m \\ 0 & & \\ 0 & \cdots & 0 \end{bmatrix},$$

and injective families $\{X, X'\}$ and $\{Y, Y'\}$ such that

$$XS(\Theta) = S(\Phi)X, \quad S(\Theta)Y = YS(\Phi),$$

$$X'S(\Theta) = S(\Phi)X', \quad S(\Theta)Y' = Y'S(\Phi),$$

$$XY = \eta(S(\Phi)), \quad YX = \eta(S(\Theta))$$

$$X'Y' = \eta'(S(\Phi)), \quad Y'X' = \eta'(S(\Phi)),$$

and $\eta' \wedge \eta \cdot \psi_m = 1$ ([21], [22], [27]). Next two lemmas are obvious.

Lemma 2.1. $\phi(S(\Theta))$ is injective if and only if $\phi \wedge \psi_m = 1$, and $\phi(S(\Theta))H(\Theta)$ is dense in $H(\Theta)$ if and only if ϕ is outer.

Lemma 2.2. $\{S(\Phi)\}'' = \{\phi(S(\Phi)) : \phi \in H^\infty\}$.

For a bounded operator T, we denote the lattice of invariant subspaces for T by Lat T.

Lemma 2.3. $\{A: \text{Lat } A \supseteqq \text{Lat } S(\Phi)\} = \{\phi(S(\Phi)) : \phi \in H^\infty\}$.

Proof. Suppose Lat $A \supseteqq$ Lat $S(\Phi)$. Since each component space of $H(\Phi)$ reduces $S(\Phi)$, it also reduce A, that is, A has the form $A=\sum_{i=1}^{n} \oplus A_i$. $\psi_{i+1}/\psi_i \in H^\infty$ implies that $H(\psi_i) \subseteqq H(\psi_{i+1}) \subseteqq H^2$. Let P_i be the projection from $H(\Phi)$ onto i-th component space. Then $L_{ij} \equiv \{(P_i x \oplus P_j x : x \in H^\infty\}$ is invariant for $S(\Phi)$. If $i,j \geqq m+1$, then $A L_{ij} \subseteqq L_{ij}$ implies $\phi_i = \phi_j$. If $i \leqq m < j$, then $AL_{ij} \subseteqq L_{ij}$ implies that for every x in $H(\psi_i)$ there is a y in H^2 such that

$$A_i x \oplus \phi_j x = P_i y \oplus y ,$$

which implies $A_i = \phi_j(S(\psi_i))$ and hence $A=\phi(S(\Phi))$ for some ϕ in H^∞ . The converse assertion is trivial.

Lemma 2.4. $\{S(\Theta)\}'' = \{N : \eta(S(\Theta))N=\phi(S(\Theta))$ for some ϕ in $H^\infty\}$.

Proof. For each N in $\{S(\Theta)\}''$ and each B in $\{S(\Phi)\}'$, set $K=XNYB - BXNY$. Then, since $YBX \in \{S(\Theta)\}'$ and $XY \in \{S(\Phi)\}''$, it follows that $YK=YXNYB-YBXNY=NYXYB-NYBXY=0$, which implies $K=0$. Consequently, from Lemma 2.2, there is a ϕ in H^∞ such that $XNY=\phi(S(\Phi))$. Since $YX=\eta(S(\Theta))$ is injective, from $YX\eta(S(\Theta))N=YXN\eta(S(\Theta))=YXNYX=Y\phi(S(\Phi))X=YX\phi(S(\Theta))$, we have $\eta(S(\Theta))N=\phi(S(\Theta))$. The converse assertion is trivial.

Lemma 2.5. If $XNY=\phi(S(\Phi))$ and $X'NY'=\phi'(S(\Phi))$ for ϕ,ϕ' in H^∞, then N belongs to $\{S(\Theta)\}''$.

Proof. Clearly we have

$N\eta(S(\Theta))=\phi(S(\Theta))$ and $N\eta'(S(\Theta))=\phi'(S(\Theta))$.

Hence, for each M in $\{S(\Theta)\}'$, we have

$$NM\eta(S(\Theta))= N\eta(S(\Theta))M =\phi(S(\Theta))M= M\phi(S(\Theta))=MN\eta(S(\Theta)),$$

and similarly $NM\eta'(S(\Theta))= MN\eta'(S(\Theta))$. Since $\eta\wedge\eta'=1$, the ranges of $\eta(S(\Theta))$ and $\eta'(S(\Theta))$ span a dense set in $H(\Theta)$. Thus we have $NM=MN$.

Theorem 2.6. If N belongs to $\{S(\Theta)\}''$, then there is a unique ϕ in H^∞ such that $N=\phi(S(\Theta))$. In this case $\|N\|=\|\phi\|_\infty$.

Proof. Let N belong to $\{S(\Theta)\}''$. Then from Lemma 2.5 and Lemma 2.1 we have $\phi_1(S(\Theta))\, N =\phi_2(S(\Theta))$, where $\phi_1=\eta/\eta\wedge\phi$ and $\phi_2=\phi/\eta\wedge\phi$. Thus from the lifting theorem, there are an $n\times n$ bounded matrix $\Gamma = (\gamma_{ij}')$ over H^∞, and an $m\times n$ bounded matrix $\Omega = (\omega_{ij})$ over H^∞ such that

$$(2.1)\qquad \Gamma\Theta H^2_m \subseteq \Theta H^2_m,\quad N= P_\Theta\Gamma|H(\Theta),\ \|N\| = \|\Gamma\|_\infty=\sup_\lambda \|\Gamma(\lambda)\|,$$

and

$$(2.2)\qquad \phi_2 I_n - \phi_1\Gamma = \Theta\Omega .$$

Since Θ is inner, $1=\det(\Theta^*(e^{it})\Theta(e^{it}))= \sum_a \det|\Theta_a(e^{it})|^2$, where Θ_a denotes an $m\times m$ submatrix. Therefore there is a Θ_a such that $\det\Theta_a\neq 0$. We may assume that the first minor is not 0.

Let θ_{ij} and $\theta_{a(i)j}$ be the (i,j)-th component of Θ and Θ_a, respectively. Let $\Theta_a'=(\theta'_{a(i)j})$ be the classical adjoint matrix of Θ_a. Then, for $k(a)\neq a(i)$ $(1\leq i \leq m)$, by the same technique as the proof of Theorem 1 of [35], from (2.2), we have

$$-\phi_1\Theta_a' \begin{bmatrix}\gamma_{a(1)k(a)}\\ \vdots \\ \gamma_{a(m)k(a)}\end{bmatrix} = \det\Theta_a \begin{bmatrix}\omega_{1k(a)}\\ \vdots \\ \omega_{mk(a)}\end{bmatrix},$$

and hence

$$-\phi_1(\theta_{k(a)1},\dots,\theta_{k(a)m})\Theta_a' \begin{bmatrix}\gamma_{a(1)k(a)}\\ \vdots \\ \gamma_{a(m)k(a)}\end{bmatrix} = \det\Theta_a(\phi_2-\phi_1\gamma_{k(a)k(a)})$$

Thus ,by simple calculations , we have

$$(2.3)\quad \phi_1 \det \begin{bmatrix}\theta_{a(1)1}\cdots & \theta_{a(1)m} & \gamma_{a(1)k(a)}\\ \vdots & \vdots & \vdots\\ \theta_{a(m)1}\cdots & \theta_{a(m)m} & \gamma_{a(m)k(a)}\\ \theta_{k(a)1}\cdots & \theta_{k(a)m} & \gamma_{k(a)k(a)}\end{bmatrix} = \phi_2 \det \Theta_a$$

This implies that the inner factor of ϕ_1 is a divisor of $\bigwedge_a \det\Theta_a$ which is equal to ψ_m ([21],[27]). Thus $\phi_1 \wedge \psi_m = 1$ deduce that ϕ_1 is outer. For a submatrix Θ_a satisfying $1 \leqq a(1) < \cdots < a(m) \leqq m+1$, there is a unique $k(a)$ such that $1 \leqq k(a) \leqq m+1$ and $k(a) \neq a(i)$. Conversely , for every $1 \leqq k \leqq m+1$, there is a unique Θ_a such that $1 \leqq a(1) < \cdots < a(m) \leqq m+1$ and $k(a)=k$. Thus setting

$\xi_{k(a)}(\lambda) = \det \Theta_a(\lambda)$,from (2.3), we have

$$|\phi_2(\lambda)|^2 |\xi_k(\lambda)|^2 = |\phi_1(\lambda)|^2 \left|\det \begin{bmatrix}\theta_{11}\cdots & \theta_{1m} & \gamma_{1k}\\ \vdots & \vdots & \vdots\\ \theta_{m1}\cdots & \theta_{mm} & \gamma_{mk}\\ \theta_{m+11} & \theta_{m+1m+1} & \gamma_{m+1k}\end{bmatrix}\right|^2$$

for every k; $1 \leqq k \leqq m+1$. Hence it follows that

$$|\phi_2(\lambda)|^2 \sum_{k=1}^{m+1} |\xi_k(\lambda)|^2 = |\phi_1(\lambda)|^2 \left\| \begin{bmatrix}\gamma_{11}(\lambda)\cdots & \gamma_{m+11}(\lambda)\\ \vdots & \vdots\\ \gamma_{1m+1}(\lambda)\cdots & \gamma_{m+1m+1}(\lambda)\end{bmatrix} \begin{bmatrix}\xi_1(\lambda)\\ \vdots \\ (-1)^m \xi_{m+1}(\lambda)\end{bmatrix}\right\|^2$$

$$\leqq |\phi_1(\lambda)|^2 \| {}^t\Gamma_{m+1}(\lambda)\|^2 \left(\sum_{k=1}^{m+1} |\xi_k(\lambda)|^2\right),$$

where $\Gamma_{m+1}(\lambda)$ is the first submatrix of $\Gamma(\lambda)$ of order m+1, and ${}^t\Gamma_{m+1}(\lambda)$ is the transposed matrix of $\Gamma_{m+1}(\lambda)$. Since by the assumption $\xi_{m+1}(\lambda)\neq 0$, it follows that

$$|\phi_2(\lambda)|^2 \leqq |\phi_1(\lambda)|^2 \|{}^t\Gamma_{m+1}(\lambda)\|^2 \leqq |\phi_1(\lambda)|^2 \|\Gamma\|_\infty^2 .$$

Thus there is a ϕ in H^∞ such that $\phi_2 = \phi\phi_1$ and $\|\phi\|_\infty \leqq \|\Gamma\|_\infty = \|N\|$ (cf. [8]). Hence we have $N=\phi(S(\Theta))$. Since $\|N\| \leqq \|\phi\|_\infty$ is clear, we have $\|N\| = \|\phi\|_\infty$. Assume that $\phi(S(\Theta))=\Psi(S(\Theta))$ for ϕ and ψ in H^∞. From $X\, S(\Theta) = S(\Phi)\, X$ and $X'\, S(\Theta) = S(\Phi) X'$, we have

$$\phi(S(\Phi))X = \psi(S(\Phi))X \quad \text{and} \quad \phi(S(\Phi))X' = \psi(S(\Phi))X' .$$

By $X\, H(\Theta) \vee X'\, H(\Theta) = H(\Phi)$, we deduce $\phi(S(\Phi))=\psi(S(\Phi))$, from which $\phi = \psi$ follows.

Theorem 2.7. $\mathcal{A}_{S(\Theta)} = \{ N: \text{Lat } N \supseteqq \text{Lat } S(\Theta)\} = \{S(\Theta)\}'' = \{\phi(S(\Theta)) : \phi \in H^\infty\}$.

Proof. From Theorem 2.6, it follows that

$$\{S(\Theta)\}'' = \{\phi(S(\Theta)) : \phi\in H^\infty\} \subseteqq \mathcal{A}_{S(\Theta)} \subseteqq \{N: \text{Lat } N \supseteqq \text{Lat } S(\Theta)\} .$$

Therefore we must only show that if Lat $N \supseteqq$ Lat $S(\Theta)$, then N belongs to $\{S(\Theta)\}''$. Let $\mathcal{L}$ be an arbitrary subspace in Lat $S(\Phi)$. Then, since $\overline{Y\mathcal{L}}$ is in Lat $S(\Theta)$,

$$XNY\mathcal{L} \subseteqq XN\overline{Y\mathcal{L}} \subseteqq X\overline{Y\mathcal{L}} \subseteqq \overline{XY\mathcal{L}} = \overline{\eta(S(\Phi))\mathcal{L}} \subseteqq \mathcal{L} .$$

From Lemma 2.3, we have $XNY=\phi(S(\Phi))$ for some ϕ in H^{∞} . Similarly we have $X'NY'=\phi'(S(\Phi))$. Thus by Lemma 2.5, we can conclude the theorem .

Chapter III. C_{10}- contraction

We determine $C_{1\cdot}$,C_{10} and C_{11} by

$$C_{1\cdot} = \{T: T^n x \to 0 \text{ as } n\to \infty \quad \text{for all } x\},$$

$$C_{10} = C_{1\cdot} \cap C_{\cdot 0} \qquad \text{and}$$

$$C_{11} = \{T: T \in C_{1\cdot}\ ,\ T^* \in C_{1\cdot}\}.$$

It is well-known that there is a C_0- C_{11}decomposition for a weak contraction. Therefore we can easily show that if T is of class C_{10} and $I-T^*T \in (\tau,c)$, where (τ,c) denotes the trace class , then $\sigma_p(T^*) = D$ and $\sigma_p(T) \cap D = \phi$.

In this chapter , we shall investigate a contraction T such that $I-T^*T \in (\tau,c)$ and $\sigma(T) = \bar{D}$. The main tool is the theory of infinite determinant [15]. About C_{10} see [11],[14] and [41].

3.1. Operator valued functions.

For $T\in I + (\tau,c)$,Bercovici and Voiculescu defined the algebraic adjoint T^a, which satisfies

$$T^a T = TT^a = \det T \quad .$$

They showed that if $\Theta(\lambda)$ is a contractive holomorphic function and if $\Theta(\lambda)\in I + (\tau,c)$ for every $\lambda\in D$, then $\Theta(\lambda)^a$ is a contractive holomorphic function. In this case , if $\det \Theta(e^{it})\neq 0$ a.e. , then $\Theta(e^{it})$ is invertible and its inverse is

$$\Theta(e^{it})^a \big/ \det\Theta(e^{it}) \quad \text{a.e. .}$$

Theorem 1.1. Let $\Theta(\lambda)$ be an inner function (that is, $\Theta(\lambda)$ is a contractive holomorphic function defined on D and $\Theta(e^{it})$ is isometric a.e.) with values in $L(E,E')$, where E, E' are separable Hilbert space. If there is an isometry V in $L(E,E')$ such that for every $\lambda \in D$

(1.1) $$I_E - V^*\Theta(\lambda) \in (\tau, C),$$

(1.2) $$\det V^*\Theta(\lambda) \not\equiv 0,$$

then there is a bounded holomorphic function $\Delta(\lambda)$ with values in $L(E',F)$ for a suitable Hilbert space F such that

(1.3) $$\Theta(e^{it})E \oplus \Delta^*(e^{it})F = E' \quad \text{a.e..}$$

Proof. If V is a unitary, then $\Theta(e^{it})$ is invertible a.e.. Hence we may assume that V is not a unitary. Set $F = E' \ominus VE$. Let $E_0 = E \oplus F$ be the direct summation of E and F. For $\lambda \in D$, define $\Theta'(\lambda) \in L(E_0,E')$ by

$$\Theta'(\lambda)|_E = \Theta(\lambda) \text{ and } \Theta'(\lambda)|_F = I_F.$$

For simplicity, set $d(\lambda) = \det V^*\Theta(\lambda)$ and $A(\lambda) = (V^*\Theta(\lambda))^a$. Determine $\Delta(\lambda) \in L(E',F)$ by

(1.4) $$\Delta(\lambda) = -P_F\Theta(\lambda)A(\lambda)V^* + d(\lambda)P_F$$

and $\Delta'(\lambda) \in L(E',E_0)$ by

$$\Delta'(\lambda) = A(\lambda)V^* + \Delta(\lambda).$$

Then we have

$$\Delta'(\lambda)\Theta'(\lambda)|_E = \Delta'(\lambda)\Theta(\lambda) = A(\lambda)V^*\Theta(\lambda) + \Delta(\lambda)\Theta(\lambda)$$
$$= d(\lambda)I_E - P_F\Theta(\lambda)d(\lambda)I_E + d(\lambda)P_F\Theta(\lambda) = d(\lambda)I_E \quad ,$$

$$\Delta'(\lambda)\Theta'(\lambda)|_F = A(\lambda)V^*I_F + \Delta(\lambda)I_F = d(\lambda)I_F,$$

and

$$\Theta'(\lambda)\Delta'(\lambda) = \Theta(\lambda)A(\lambda)V^* + \Delta(\lambda) = (I-P_F)\Theta(\lambda)A(\lambda)V^* + d(\lambda)I_F$$

$$= VV^*\Theta(\lambda)A(\lambda)V^* + d(\lambda)I_F = V\,d(\lambda)V^* + d(\lambda)I_F = d(\lambda)I_{E'}.$$

Thus we have

$$\Delta'(\lambda)\Theta'(\lambda) = d(\lambda)I_{E_0}\ ,\ \Theta'(\lambda)\Delta'(\lambda) = d(\lambda)I_{E'}\ .$$

Since the inverse of $\Theta'(e^{it})$ is $\Delta'(e^{it})/d(e^{it})$ a.e., the orthogonal complement of $\Theta(e^{it})E = \Theta'(e^{it})E$ is

$$\frac{\Delta'(e^{it})^*}{\overline{d(e^{it})}}(E_0 \ominus E) = \Delta(e^{it})^*F.$$

It is clear that $\Delta(\lambda)$ is a bounded holomorphic function. Q.E.D.

Cambern showed that the orthogonal complement of a finite dimensional holomorphic range function is conjugate holomorphic (c.f. p.94 of [16]). Now, we can show this result as a corollary.

Corollary 12. Let $\Theta(\lambda)$ be an inner function with values in $\mathcal{L}(E,E')$. Suppose $\dim E = m<\infty$. Then there is an bounded holomorphic function $\Delta(\lambda)$ satisfying (1.3).

Proof. We may assume that $E\subset E'$ and $\Theta(e^{it})$ is a matrix. Since $1 = \det(\Theta^*(e^{it})\Theta(e^{it})) = \sum_\sigma |\det\Theta_\sigma(e^{it})|^2$, a.e., where $\sum_\sigma$ is taken over all $m\times m$ submatrices of $\Theta(e^{it})$, there is at least one σ such that $\det\Theta_\sigma(e^{it}) \neq 0$ a.e.. Thus there is an isometry V such that

$$\det V^*\Theta(e^{it}) = \det \Theta_\sigma(e^{it}) \neq 0 \text{ a.e.} \quad (\text{see}[30]).$$

Hence V and $\Theta(\lambda)$ satisfy (1.1),(1.2). Q.E.D.

3.2.Quasi unilateral shifts.

We begin with a short review about the canonical model theory of Sz,Nagy and C.Foiaş. Let T be a contraction of class $C_{\cdot 0}$ on a separable Hilbert space H. Set $D_T = (I - T^*T)^{1/2}$,and let E and E' be the closures of $D_T H$ and $D_{T^*}H$, respectively. Then the characteristic function $\Theta(\lambda)$ of T determined by

(2.1) $\Theta(\lambda) = \{-T + \lambda D_{T^*}(I-\lambda T^*)^{-1} D_T\}|_E$ for $\lambda \in D$

is an inner function with values in $L(E,E')$. Therefore

$$\dim E \leq \dim E'.$$

Moreover T is unitarily equivalent to $S(\Theta)$ on $H(\Theta)$ defined by

(2.2) $H(\Theta) = H^2(E') \ominus \Theta H^2(E)$, $S(\Theta)^*h = \overline{\lambda}h$ for h in $H(\Theta)$.

T is of class $C_{1\cdot}$ if and only if $\Theta(\overline{\lambda})^*H^2(E')$ is dense in $H^2(E)$ (that is,Θ is $*$-outer).

In this thesis,for simplicity, we call T a <u>quasi unilateral shift</u> if T is a contraction of class $C_{\cdot 0}$ such that

$$I - T^*T \in (\tau, C),\ K(T) = \{0\} \text{ and } K(T^*) \neq \{0\}.$$

Theorem 2.1. If T is a quasi unilateral shift on H,then there is a bounded operator X with dense range satisfying

(2.3) $$X\,T = S\,X,$$

where S is a unilateral shift satisfying

$$0 > \text{index } S = \text{index } T \geqq -\infty .$$

Proof. We may assume $I-T^*T \neq 0$. From $T(I-T^*T)=(I-TT^*)T$, it follows that $TE \subset E'$, $T(H \ominus E)=H \ominus E'$, where E and E' are the spaces defined above. Thus we have

(2.4) $$H \ominus TH = E' \ominus TE \neq \{0\}.$$

Let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be the C.O.N.B. of E such that $(I-T^*T)e_n=\mu_n e_n$, $\mu_n \geqq 0$. Then $f_n=(1-\mu_n)^{-\frac{1}{2}} Te_n$ $(n=1,2,\ldots)$ is a C.O.N.B. of TE and $T^*f_n=(1-\mu_n)^{\frac{1}{2}} e_n$ (see [28]). Setting $Ve_n=-f_n (n=1,2,\ldots)$, V is an isometry from E to E', and

(2.5) $$V + T|_E \in (\tau, C) \qquad \text{(see [2])} .$$

Setting $F=E' \ominus VE$, from (2.4), it follows that

(2.6) $$\dim F = -\text{ index } T.$$

$I-T^*T \in (\tau,C)$ implies $D_T \in (\sigma,C)$ which denotes the Hilbert Schmidt class. Since $(I-TT^*)|_{TE}$ is unitarily equivalent to $I-T^*T$, we have $D_{T^*}|_{TE} \in (\sigma,C)$. Thus

$$\lambda V^* D_{T^*}(I-\lambda T^*)^{-1} D_T = \lambda V^*(D_{T^*}|_{TE})(I-\lambda T^*)^{-1} D_T \qquad (\lambda \in D)$$

belongs to (τ,C). Thus, from (2.1), (2.5), we have

$$I - V^*\Theta(\lambda) \in (\tau, C) \text{ for each } \lambda.$$

Since

$$|\det(V^*\Theta(0))|^2 = \det(\Theta(0)^* VV^*\Theta(0)) = \det(T^*VV^*T|_E)$$
$$= \det(T^*T|_E) = 0 ,$$

We have $\det V^*\Theta(\lambda) \not\equiv 0$. Thus V and $\Theta(\lambda)$ satisfy the conditions of Theorem 1.1. Hence $\Delta(\lambda)$ defined by (1.4) satisfy (1.3). Since $\Delta(\lambda)\Theta(\lambda) = 0$, setting

(2.7) $$X_0 h = \Delta h \quad \text{for} \quad h \text{ in } H(\Theta),$$

we have $X_0 \in \mathcal{L}(H(\Theta), H^2(F))$ and $X_0 S(\Theta) = S_0 X_0$, where S_0 is the unilateral shift on $H^2(F)$. Since

$$H^2(F) \supset X_0 H(\Theta) = \Delta H^2(E') \supset \Delta H^2(F) = (\det V^*\Theta(\lambda)) H^2(F),$$

it follows that $S = S_0|_{\overline{X_0 H(\Theta)}}$ is unitarily equivalent to S_0. Thus, from (2.6), we have

$$\text{index } S = \text{index } S_0 = -\dim F = \text{index } T.$$

Consequently an operator X from $H(\Theta)$ to $\overline{X_0 H(\Theta)}$ defined by

(2.8) $$X h = X_0 h \quad \text{for} \quad h \text{ in } H(\Theta)$$

satisfy (2.3). Q.E.D.

Corollary 2.2. Let T be a contraction of class C_{00} such that $I-T^*T$ and $I-TT^*$ belong to (τ, C). Then, for $a \in D$, $\mathcal{K}(T-aI) = \{0\}$ if and only if $\mathcal{K}(T^*-\bar{a}I) = \{0\}$.

Proof. Set $T_a = (T-aI)(1-\bar{a}T)^{-1}$ and $A = (1-|a|^2)^{1/2}(1-\bar{a}T)^{-1}$. Then we have $I-T_a^*T_a = A^*(I-T^*T)A$, $I-T_aT_a^* = A(I-TT^*)A^*$, and T_a is of class C_{00} (see p.240 and P.257 of [28]). Suppose $\mathcal{K}(T-aI) = \{0\}$ and $\mathcal{K}(T^* - \bar{a}I) \neq \{0\}$. Then T_a is a quasi unilateral shift. Therefore, there is an X satisfying

$X T_a = S X$, which implies that T_a is not of class C_{00} . This is a contradiction. Thus $K(T-aI) =\{0\}$ implies $K(T^*-\bar{a}I)=\{0\}$. Similarly we can proove the converse assertion. Q.E.D.

For a contraction T on H, we have

(2.9) $\quad \|I-T^*T\|_p + \dim K(T^*) = \|I-TT^*\|_p + \dim K(T)$,

where $\| \quad \|_p$ denotes the p-Schatten norm.

Indeed, from $T(I-T^*T) = (I-TT^*)T$, $(I-T^*T)|_{\overline{T^*H}}$ and $(I-TT^*)|_{\overline{TH}}$ are unitarily equivalent. $(I-T^*T)|_{K(T)} = I_{K(T)}$ and $(I-TT^*)|_{K(T^*)} = I_{K(T^*)}$ imply that

$$\|I-T^*T\|_p = \|(I-T^*T)|_{\overline{T^*H}}\|_p + \dim K(T),$$
$$\|I-TT^*\|_p = \|(I-TT^*)|_{\overline{TH}}\|_p + \dim K(T^*).$$

Thus we have (2.9). Similarly we have

(2.9)' $\quad \mathrm{rank}(I-T^*T) + \dim K(T^*) = \mathrm{rank}(I-TT^*) + \dim K(T)$.

Proposition 2.3. Let T be a Fredholm quasi unilateral shift. Suppose X with dense range satisfies $XT = SX$, where S is a unilateral shift with index S = index T. Then $T|_{K(X)}$ is of class C_0 .

Proof. Let $T = \begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix}$ be a decomposition of T corresponding to $H= K(X) \oplus K(X)^{\perp}$. Then T_1 is injective and ,from (2.3),also T_2 is injective. From the assumption and

(2.9), it follows that $I-T^*T \in (\tau,C)$ and $I-TT^* \in (\tau,C)$, which imply

(2.10) $I - T_1{}^* T_1 \in (\tau,C)$,

(2.11) $I - (T_1 T_1{}^* + T_{12} T_{12}{}^*) \in (\tau,C)$,

(2.12) $I - (T_{12}{}^* T_{12} + T_2{}^* T_2) \in (\tau,C)$,

(2.13) $I - T_2 T_2{}^* \in (\tau,C)$.

From $K(T_2{}^*) \subset K(T^*)$, it follows that

$$\text{index } T = -\dim K(T^*) \leqq -\dim K(T_2{}^*) \leqq -\dim K(S^*) = \text{index } T,$$

which implies index T = index T_2 . From (2.9) and (2.13), we have $I-T_2{}^*T_2 \in (\tau,C)$, which, by (2.12), implies $T_{12} \in (\sigma,C)$. Therefore, from (2.10) and (2.11), T_1 is a Fredholm operator. Since

$$\text{index } T = \text{index} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \text{index } T_1 + \text{index } T_2 ,$$

we have index $T_1 = 0$. Thus T_1 is invertible. Hence T_1 is a weak contraction of class $C_{\cdot 0}$. Consequently T_1 is of class C_0 . Q.E.D.

Corollary 2.4. Let T be a Fredholm quasi unilateral shift of class C_{10} . Then, if $AT=TA$ and $K(A^*)=\{0\}$, $K(A)=\{0\}$ (c.f.[42]).

Proof. For X defined in Theorem 2.1, we have $(XA)T = S(XA)$. From Proposition 2.3, we have $K(XA) = \{0\}$. Q.E.D.

Proposition 2.5. Let T be of class $C_{\cdot 0}$. Then T is of class C_{10} if and only if

$$\Theta L^2(E) \cap H^2(E') = \Theta H^2(E). \tag{2.14}$$

Proof. Since

$$(\Theta(\bar{\lambda})^* h(\lambda), f(\lambda))_{H^2(E)} = \frac{1}{2\pi}\int_0^{2\pi} (\Theta(e^{-it})^* h(e^{it}), f(e^{it}))_E dt$$

$$= -\frac{1}{2\pi}\int_0^{-2\pi} (\Theta(e^{it})^* h(e^{-it}), f(e^{-it}))_E \, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\Theta(e^{it})^* h(e^{-it}), f(e^{-it}))_E \, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\Theta(e^{it})^* e^{-it} h(e^{-it}), e^{-it} f(e^{-it}))_E \, dt$$

$$= (\Theta(\lambda)^* \bar{\lambda} h(\bar{\lambda}), \bar{\lambda} f(\bar{\lambda}))_{L^2(E)},$$

$\Theta(\bar{\lambda})^* H^2(E')$ is dense in $H^2(E)$ if and only if

$\Theta(\lambda)^* (H^2(E'))^{\perp}$ is dense in $(H^2(E))^{\perp}$, where $\perp$ denotes the orthogonal complement. We have always

$$\Theta L^2(E) \cap H^2(E') \supset \Theta H^2(E).$$

At first, assume that T is of class C_{10}. Suppose

$$\Theta g \in \{\Theta L^2(E) \cap H^2(E')\} \ominus \Theta H^2(E).$$

Then $\Theta g \in H^2(E')$ and $g \perp H^2(E)$, because Θ is an isometry from $L^2(E)$ to $L^2(E')$. Thus $g \perp \Theta^*(H^2(E'))^{\perp}$ and $g \in (H^2(E))^{\perp}$. Since $\Theta(\lambda)$ is $*$-outer, we have $g = 0$. Consequently (2.14) follows.

Conversely assume (2.14). Suppose $f \perp \Theta(\lambda)^*(H^2(E'))^{\perp}$ and

$f \in (H^2(E))^{\perp}$. Then $\Theta f \in H^2(E')$ and $\Theta f \perp \Theta H^2(E)$. Thus from (2.14), we have $\Theta f = 0$ and hence $f=0$. Consequently $\Theta(\lambda)$ is $*$-outer.

Q.E.D.

Theorem 2.6. Let T be a quasi unilateral shift. Then $T \prec S$ (that is, there is an X such that $K(X) = K(X^*) = \{0\}$, $XT=SX$), where S is a unilateral shift with index S = index T, if and only if T is of class C_{10}.

Proof. Assume that T is of class C_{10}. Then, from Theorem 2.1, there is an X with dense range satisfying (2.3). If $Xh=0$ for h in $H(\Theta)$, then, from (2.7) and (2.8), $\Delta(e^{it})h(e^{it})=0$ a.e.. Thus, from (1.3), $h \in \Theta L^2(E)$, so that, from (2.14), $h \in \Theta H^2(E)$. Consequently $h=0$. Thus we have $T \prec S$.
Conversely, assume $XT=SX$ and $K(X) = K(X^*) = \{0\}$. From $XT^n = S^n X$ $(n=1,2,\ldots)$ it follows that T is of class C_{10}. Q.E.D.

Remark 1. If T is a Fredholm operator, then, from Theorem 2.1 and Proposition 2.3, it is clear that $T \prec S$ if T is of class C_{10}.

Remark 2. Theorem 2.6. implies that the Jordan model of a quasi unilateral shift of class C_{10} is a unilateral shift.

Corollary 2.7. Let T be a quasi unilateral shift of class C_{10}. Then T^* has a cyclic vector.

Proof. $T \prec S$ imlies that $S^* \prec T^*$. Since S^* has a cyclic vector, also T^* does. Q.E.D.

Proposition 2.8. Let T be a quasi unilateral shift. Then there is an injection Y such that

(2.15) $$Y S = T Y,$$

where S is a unilateral shift such that index S = index T.

Proof. Consider $S(\Theta)$ defined by (2.2) instead of T. Let V be an isometry defined in the proof of Theorem 2.1, Then

$$E' = V E \oplus F \quad \text{and} \quad \det V^*\Theta(e^{it}) \neq 0 \quad \text{a.e..}$$

Define an operator Y from $H^2(F)$ to $H(\Theta)$ by

$$Y h = P_{H(\Theta)} h \quad \text{for } h \text{ in } H^2(F).$$

Then we have

$$YS\, h = P_{H(\Theta)} S\, h = P_{H(\Theta)} S\, P_{H(\Theta)} h = S(\Theta) Y\, h\, ,$$

which implies (2.15). Suppose $Yh=0$. Then $h=\Theta f$ for some $f \in H^2(E)$. Thus $0 = V^*h(e^{it}) = V^*\Theta(e^{it}) f(e^{it})$ a.e.. Since $V^*\Theta(e^{it})$ is invertible a.e. , $f(e^{it})=0$ a.e.. Consequently Y is injective . Q.E.D.

Proposition 2.9. Let T be a quasi unilateral shift of class C_{10}. Then, if $T \prec S'$, where S' is a unilateral shift, then index S' = index T.

Proof. From $S'^* \prec T^*$, $\dim \mathcal{K}(S'^*) \leqq \dim \mathcal{K}(T^*)$. Above proposition implies that there is an injection Y' such that

$$Y' S = S' Y', \text{ index } S = \text{index } T,$$

which implies that $0 >$ index $S \geqq$ index S' (c.f. [30]). we have

$$\text{index } T = \text{index } S \geqq \text{index } S' \geqq \text{index } T,$$

from which index T = index S' follows. Q.E.D.

Remark 3. In [42], P.Y.Wu showed that if $I-T^*T$ is a finite rank operator, and if $T \prec S'$, then

$$\operatorname{rank}(I-TT^*)-\operatorname{rank}(I-T^*T)=-\text{index } S'.$$

From (2.9)', our proposition is a extension of this result.

3.3. Cyclic vector.

In this section, we consider a quasi unilateral shift of class C_{10} which has a cyclic vector. Next proposition is a partial extension of Proposition 2 of [30] and Theorem 3.1 of [41].

Proposition 3.1. Let T be a quasi unilateral shift of class C_{10}. Then next conditions are equivalent:

(a) T has a cyclic vector ;

(b) there is a bounded operator Y satisfying

$$Y S_1 = T Y \ , \quad K(Y^*) = \{0\} \ , \tag{3.1}$$

where S_1 is a unilateral shift with index $S_1 = -1$;

(c) $S_1 \prec T$;

(d) $S_1 \prec T$ and $T \prec S_1$;

(e) $\| I-TT^* \|_1 - \| I-T^*T \|_1 = 1$, and there is a holomorphic function Γ from $H^2(\mathbb{C})$ to $H^2(E')$ satisfying

$$\| \Gamma(e^{it}) \|_{E'} \leqq 1 \text{ a.e. } , \tag{3.2}$$

$$\Gamma H^2(\mathbb{C}) \vee \Theta H^2(E) = H^2(E'), \tag{3.3}$$

where Θ is a characteristic function of T defined by (2.1).

Proof. (a) $\to$ (e). From Theorem 2.6, for a unilateral shift S with index S = index$T$, we have $T \prec S$. That T has a cyclic vector implies that also S does. Thus index $S = -1$. Consequently, from (2.9), we have

$$\| I-TT^* \|_1 - \| I-T^*T \|_1 = 1.$$

We can construct a function Γ in the same way as [30].

(e) $\to$ (b). A contraction Y defined by $Yh = P_{H(\Theta)} \Gamma h$ for h in $H^2(\mathbb{C})$ satisfies (3.1).

(b) $\to$ (c). Suppose $K(Y) \neq \{0\}$. Since $S_1 K(Y) \subset K(Y)$, there is a scalar inner function ψ such that $K(Y) = \psi H^2(\mathbb{C})$. Thus

$$K(Y)^{\perp} = H(\psi) \ (= H^2(\mathbb{C}) \ominus \psi H^2(\mathbb{C})),$$

$$Y|_{H(\psi)} S(\psi) = T\, Y|_{H(\psi)},$$

where $S(\psi) = P_{H(\psi)} S|_{H(\psi)}$. Since $S(\psi)$ is of class C_0, T must be of class C_0. This is a contradiction. Consequently $K(Y) = \{0\}$.

(c) $\to$ (d). $S_1 \prec T$ implies $T^* \prec S_1^*$, from which it follows that $\dim K(T^*) \leqq \dim K(S_1^*) = 1$. That T is of class C_{10} implies index $T < 0$. Thus index $T = -1$. By theorem 2.6, we have $T \prec S_1$

(d) $\to$ (a). This is obvious. Q.E.D.

(3.3) implies that $[\Gamma, \Theta]$ is an outer function from $H^2(\mathbb{C}) \oplus H^2(E)$ to $H^2(E')$. Generally $[\Gamma, \theta]$ is not contractive. Therefore $d(\lambda) = \det[\Gamma(\lambda), \Theta(\lambda)] \in H^{\infty}$ and $d(\lambda) \leqq 1$ are not obvious. We shall show these results.

Let $A \in L(E, E')$ be a contraction and $V \in L(E, E')$ an isometry with index $V = -1$. Let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be a C.O.N.B. in E. Then, setting $d_n = Ve_n$ $(n = 1, 2, \ldots)$, $\{d_0, d_1, \ldots, d_n, \ldots\}$ is a C.O.N.B. in E', where d_0 is a unit vector in $K(V^*)$. For $i = 1, 2, \ldots$, define an isometry $V_i \in L(E, E')$ by

$$V_i e_1 = d_0, \ldots, V_i e_i = d_{i-1}, V_i e_{i+1} = d_{i+1}, V_i e_{i+2} = d_{i+2}, \ldots \quad .$$

Let $a_{ij} = (Ae_j, d_i)$ $(i \geqq 0, j \geqq 1)$. Then, by base $\{e_1, e_2, \ldots\}$, we have

$$V_i^* A = \begin{bmatrix} a_{01} & ,\ldots, & a_{0j} & ,\ldots \\ \vdots & & \vdots & \\ a_{i-1\,1} & ,\ldots, & a_{i-1\,j} & ,\ldots \\ a_{i+1\,1} & ,\ldots, & a_{i+1\,j} & ,\ldots \\ \vdots & & \vdots & \end{bmatrix} \qquad (i=1,2,\ldots)$$

Let $E_0 = \mathbb{C} \oplus E$ be a direct sum of $\mathbb{C}$ and E, and e_0 a unit vector in $\mathbb{C}$. Let $x_n (n=0,1,2,\ldots)$ be a scalar number such that $\sum_{n=0}^{\infty} |x_n|^2 \leqq 1$. Let $B \in \mathcal{L}(E_0, E')$ be an operator defined by

$$(Be_0, d_i) = x_i \ , \quad (Be_j, d_i) = a_{ij} \qquad (i \geqq 0, j \geqq 1).$$

Determine a unitary $U \in \mathcal{L}(E_0, E')$ by $Ue_i = d_i$ $(i \geqq 0)$. Then by base $\{e_0, e_1, \ldots, e_i, \ldots\}$ of E_0 we have

$$U^* B = \begin{bmatrix} x_0, a_{01}, \ldots, a_{0j}, \ldots \\ x_1, a_{11}, \ldots, a_{1j}, \ldots \\ \vdots \quad \vdots \qquad\quad \vdots \\ x_i, a_{i1}, \ldots, a_{ij}, \ldots \\ \vdots \quad \vdots \qquad\quad \vdots \end{bmatrix}$$

Let $I_E - V^*A \in (\tau, C)$. Then, since $(V_i^*Ae_j, e_k) = (V^*Ae_j, e_k)$ for $j \geqq 0$ and $k \geqq i+1$, $I_E - V_i^*A \in (\tau, C)$ for every i.

$$P_E(I_{E_0} - U^*B)|_E = I_E - V^*A$$

implies $I_{E_0} - U^*B \in (\tau, C)$.

Lemma 3.2. Let $I_E - V^*A \in (\tau, C)$. Set $V_0 = V$. Then

$$\det U^*B = \sum_{i=0}^{\infty} x_i \cdot (-1)^i \det(V_i^*A),$$

and

$$\sum_{i=0}^{\infty} |x_i \cdot (-1)^i \det(V_i^*A)| \leqq 1 .$$

Proof. For simplicity, let $[A]_n$ denote the first $n \times n$

submatrix of A, and A_n the $A|_{E_n}$,where $E_n=\langle e_1,\ldots,e_n\rangle$. For any k and n as $n \geqq k$, we have

(3.4) $$\sum_{i=1}^{k} |\det[V_i{}^*A]_n|^2 \leqq \det(A_n{}^*A_n) = \det[A^*A]_n \leqq 1,$$

because A is a contraction. Since for each i

$$\det[V_i{}^*A]_n \longrightarrow \det(V_i{}^*A) \qquad (n\longrightarrow\infty),$$

we have $\sum_{i=0}^{k} |\det(V_i{}^*A)|^2 \leqq 1$, which implies

(3.5) $$\sum_{i=0}^{\infty} |\det(V_i{}^*A)|^2 \leqq 1$$

Consequently $\sum_{i=0}^{\infty} |x_i\cdot(-1)^i\det(V_i{}^*A)| \leqq 1.$

For any $\varepsilon>0$, take an m such that

(3.6) $$\sum_{i=m+1}^{\infty}|x_i|^2<\varepsilon^2 .$$

Since $\det[U^*B]_n \rightarrow \det(U^*B)$,and $\det[V_i{}^*A]_n \rightarrow \det(V_i{}^*A)$ as $n\rightarrow\infty$, we can take an N such that

(3.7) $n \geqq N \rightarrow |\det[U^*B]_n - \det(U^*B)| < \varepsilon,$

and

(3.8) $n \geqq N \rightarrow \sum_{i=0}^{m} |\det[V_i{}^*A]_n - \det(V_i{}^*A)|^2 < \varepsilon^2$.

Fix a k as $k \geqq N+1$ and $k \geqq m+1$ .Then it follows that

$$|\det(U^*B) - \sum_{i=0}^{\infty} x_i\cdot(-1)^i\det(V_i{}^*A)|$$

$$\leqq|\det(U^*B) - \det[U^*B]_k| + |\det[U^*B]_k - \sum_{i=0}^{m} x_i\cdot(-1)^i\det[V_i{}^*A]_{k-1}$$

$$+ |\sum_{i=0}^{m} x_i\cdot(-1)^i \{\det[V_i{}^*A]_{k-1} - \det(V_i{}^*A)\}|$$

$$+ |\sum_{i=m+1}^{\infty} x_i\cdot(-1)^i\det(V_i{}^*A)| \quad .$$

From (3.7) $\quad |\det(U^*B) - \det[U^*B]_k| < \varepsilon$, and from (3.8)

$$\Big|\sum_{i=0}^{m} x_i \cdot (-1)^i \{\det[V_i^*A]_{k-1} - \det(V_i^*A)\}\Big|$$
$$\leqq \Big(\sum_{i=0}^{m} |x_i|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=0}^{m} |\det[V_i^*A]_{k-1} - \det(V_i^*A)|^2\Big)^{\frac{1}{2}} < \varepsilon .$$

(3.5) and (3.6) implies that

$$\Big|\sum_{i=m+1}^{\infty} x_i \cdot (-1)^i \det(V_i^*A)\Big| < \varepsilon \quad .$$

By the finite matrix theory

$$\Big|\det[U^*B]_k - \sum_{i=0}^{m} x_i \cdot (-1)^i \det[V_i^*A]_{k-1}\Big|$$
$$= \Big|\sum_{i=m+1}^{k-1} x_i \cdot (-1)^i \det[V_i^*A]_{k-1}\Big| < \varepsilon \quad ,$$

because the last inequality follows from (3.4),(3.6). Consequently, for any $\varepsilon > 0$ we have

$$\Big|\det(U^*B) - \sum_{i=0}^{\infty} x_i \cdot (-1)^i \det(V_i^*A)\Big| < 4\varepsilon \text{ .Q.E.D.}$$

In (e) of Proposition 3.1, set $(\Gamma(\lambda)e_0, d_i) = h_i(\lambda)$ for $i \geqq 0$.

Then we have:

Proposition 3.3. $|\det(U^*[\Gamma(\lambda),\Theta(\lambda)])| \leqq 1$, and

(3.9) $\quad \det(U^*[\Gamma(\lambda),\Theta(\lambda)]) = \sum_{i=0}^{\infty} h_i(\lambda) \cdot (-1)^i \det(V_i^*\Theta(\lambda))$

is holomorphic on D.

Proof. From (3.2), we have $\sum_{i=0}^{\infty} |h_i(\lambda)|^2 \leqq 1$. Since $V_i^*\Theta(\lambda)$ is a contractive holomorphic function, $\det(V_i^*\Theta(\lambda)) \in H^\infty$.

Since $\Theta(\lambda)$ is a contraction for every $\lambda \in D$, it follows that

$$\sum_{i=0}^{\infty} |h_i(\lambda) \cdot (-1)^i \det(V_i{}^*\Theta(\lambda))| \leqq 1,$$

which implies $\sum_{i=0}^{\infty} h_i(\lambda) \cdot (-1)^i \det(V_i{}^*\Theta(\lambda))$ is holomorphic.

Equality (3.9) follows from Lemma. Q.E.D.

Problem. Is $\det(U^*[\Gamma(\lambda),\Theta(\lambda)])$ outer?

BIBLIOGRAPHY

1. H.Bercovici,C.Foias and B.Sz.-Nagy, Complements a letude des operateurs de class C_0 .III, Acta Sci.Math.37(1975),313-322.

2. H.Bercovici and D.Voiculescu, Tensor operations on characteristic functions of C_0 contractions,ibidem,39(1977),205-231.

3. A.Beurling, On two problems concerning linear transformations in Hilbert space,Acta Math.,81(1949),239-255.

4. L.A.Brown,R.G.Douglas and P.Fillmore,Unitary equivalence module the compact operators and extensions of C*-algebras, Proc.Conference on Operator theory,Springer,(1973).

5. L.A.Brown and P.R.Halmos,Algebraic properties of Toeplitz operators,J.Reine Angew.Math.,213(1964),89-102.

6. L.Carleson,Interpolation by bounded analytic functions and the corona problem, Ann.of Math.,76(1962),547-559.

7. A.Devinatz and M.Shinbrot,General Wiener-Hopf operators,Trans. Amer.Math.Soc.,149(1969),467-494.

8. R.G.Douglas,Banach algebra techniques in operator theory, Academic press,New York,(1972).

9. ————————,On the hyperinvariant subspaces for isometries , Math.Z.,107(1968),297-300.

10. R.G.Douglas and C.Peacy,On a topology for invariant subspaces ,J.Func.Anal.,2(1968),323-341.

11. G.Eckstein,On the spectrum of contractions of class $C_{\cdot 1}$, Acta Sci.Math.,39(1977),251-254.

12. P.A.Fuhrmann, On the corona theorem and its application to spectral problems in Hilbert space,Trans.Amer.Math.Soc.,132 (1968),55-66.

13. —————————, Linear systems and operators in Hilbert space, Mcgraw-Hill,New York,1981.

14. F.Gilfeather, Weighted bilateral shifts of class C_{01},Acta Sci.Math.,32(1971),251-254.

15. I.Gohberg and M.G.Krein, Introduction to the theory of linear non-selfadjoint operators,Nauka,Moskwa,(1965).

16. H.Helson, Lectures on invariant subspaces,New York,(1964).

17. P.R.Halmos,Shifts on Hilbert spaces, J.reine angew.Math., 208(1961),102-112.

18. K.Hoffman, Banach spaces of analytic functions,Englewood Cliffs,N.J.,1962.

19. P.Lax, Translaton invariant spaces, Acta Math.,101(1959), 163-178.

20. V.Lomonosov, Invariant subspaces for operators commuting with compact operators, Func.Anal.and its appl. (1973), 55-56.

21. B.Moore and E.A.Nordgren, On quasi-equivalence and quasi-similarity, Acta Sci.Math.,34(1973),311-316.

22. E.A.Nordgren, On quasi-equivalence of matrices over H^∞, ibidem,34(1973),301-310.

23. H.Radjavi and P.Rosenthal, Invariant subspaces, Springer, Berlin,1973.

24. M.Rosenblum, A corona theorem for countably many functions, Integral Equations and Operator theory,3(1980),125-137.

25. G.C.Rota, On models for linear operators,Comm.Pure Appl. Math. 13(1960),468-472.

26. D.Sarason, Generalized interpolation in H^{∞},Trans.Amer.Math. Soc.,127(1967),179-203.

27. B.Sz.-Nagy, Diagonalization of matrices over H^{∞} ,Acta Sci. Math.,38(1976),223-238.

28. B.Sz-Nagy and C.Foias, Harmonic analysis of operators on Hilber space, Academiai Kiado, Budapest,1970.

29. ————————, Modele de Jordan pour une classe d'operateurs de l'espace de Hilbert , Acta Sci.Math.61(1970),91-115.

30. ————————, Jordan model for contractions of class C_0 , ibidem, 36(1974),305-322.

31. ————————, Commutants and bicommutants of class C_0 , ibidem, 38(1976),311-315.

32. ————————, On injections, intertwining operators of class C_0 ,ibidem,40(1978),163-167.

33. M.Uchiyama, Hyperinvariant subspaces of operators of class $C_0(N)$, ibidem,39(1977),179-184.

34. —————— , Hyperinvariant subspaces for contractions of class $C_{\cdot 0}$, Hokkaido M.J.,6(1977),260-272.

35. —————— , Double commutants of $C_{\cdot 0}$ contractions, Proc. A.M.S.,69(1978),283-288.

36. —————— , Double commutants of $C_{\cdot 0}$ contractions.II ,ibidem, 74(1979),271-277.

37. M.Uchiyama, Some generalized Toeplitz operators, Bull.Fukuoka Univ. of Education, 28(1979),29-34.

38. ———— , Quasi-similarity of restricted C_0 contractions, Acta Sci.Math. 41(1979),429-433.

39. ———— , $C._0$ contractions and unilateral shifts, to appear,

40. P.Y.Wu, Commutants of $C_0(N)$ contractions, Acta Sci.Math.38 (1976),1973-202.

41. ——— , On contractions of class $C_1.$, ibidem, 42(1980), 205-210.

42. ——— , On the quasi-similarity of hyponormal contractions, Illinois J.M. to appear.