Title	On an algorithm for constructing multi-dimensional wavelets
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Citation	Osaka Journal of Mathematics. 1995, 32(2), p. 397-407
Version Type	VoR
URL	https://doi.org/10.18910/7200
rights	
Note	

Ashino, R., Nagase, M. and Vaillancourt, R.
Osaka, J. Math.
32 (1995), 397–407

ON AN ALGORITHM FOR CONSTRUCTING MULTI-DIMENSIONAL WAVELETS

Dedicated to Professor Tosinobu Muramatu on the occasion of his sixtieth birthday

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(Received October 18, 1993)

1. Introduction

Let $\{V_j\}_{j\in \boldsymbol{Z}}$ be an r-regular multiresolution analysis in $L^2(\boldsymbol{R}^n)$ and $\varphi(x)$ an r-regular father function. Denote by $\boldsymbol{T}$ the one-dimensional torus $\boldsymbol{R}/2\pi\boldsymbol{Z}$. Then there exists an isomorphism of Hilbert spaces between V_0 and $L^2(\boldsymbol{T}^n)$,

(1) $$V_0 \ni f \leftrightarrow m_f \in L^2(\boldsymbol{T}^n),$$

defined by the functional equations

(2) $$\hat{f}(2\xi)=m_f(\xi)\hat{\varphi}(\xi)$$

and

(3) $$m_f(\xi)=2^{-n}\sum_{k\in \boldsymbol{Z}^n}\left(f\left(\frac{x}{2}\right),\varphi(x-k)\right)e^{-ik\cdot\xi}.$$

Here $\hat{f}(\xi)$ denotes the Fourier transform of $f(x)$ and $(\cdot,\cdot)$ denotes the inner product of $L^2(\boldsymbol{R}^n)$. The function $m_f(\xi)$ will be called the *symbol* of $f(x)$.

Put $R=\{0,1\}^n$ and $E=R\backslash(0,\cdots,0)$. To construct (2^n-1) mother functions $\psi_\varepsilon(x)$, $\varepsilon\in E$, we need to construct $2\pi\boldsymbol{Z}^n$-periodic L^2-functions, m_{ψ_ε}, which satisfy conditions to be specified below. For simplicity, we write m_0 for m_φ and m_ε for $m_{\psi_\varepsilon}, \varepsilon\in E$.

To show that the mother functions $\psi_\varepsilon(x)$ are r-regular, it is sufficient to show that m_ε satisfy the same property as m_0. Therefore the simpler the construction of m_ε, the better it is.

As asserted by Meyer [2, Section 3.4, Corollary 2], the functions $\varphi(x-k)$ and $\psi_\varepsilon(x-k)$, $\varepsilon\in E$, $k\in \boldsymbol{Z}^n$ form an orthonornal basis of V_1 if and only if the $2^n\times 2^n$ matrix

This work was supported in part by the Oversea's Research Scholarship of Japan, the Natural Sciences and Engineering Research Council of Canada and the Centre de recherches mathématiques of the Université Montréal.

$$\tilde{U}(\xi)=(m_\varepsilon(\xi+\eta\pi))_{(\varepsilon,\eta)\in R\times R} \tag{4}$$

is unitary for almost all ξ.

In this paper, we obtain the eight possible independent sets of unitary matrices of the form (4) in dimension three for a particular ordering of the vertices of the unit cube. This ordering allows us to prove the non-existence of similar wavelets in dimensions higher than three. Our construction will be called a *simple construction*. We also remark that our result holds for any ordering of the vertices of the unit cube.

Our result is based on the construction method used by Mallat [1] for one-dimensional wavelets with a real- or complex-valued symbol m_0, and Meyer [2] for two-dimensional wavelets with a real-valuued symbol m_0.

Riemenschneider and Shen [3,4] and de Boor *et alii* [5] have used the machinery of box splines, which is well known to the approximation theorist but may not be familiar to the average analyst, to construct similar wavelets in dimensions two and three and used the fact, attributed to Hurwitz [6] by combinatorists, that no unitary matrices of the form (4) exist for $n>3$, to conclude to the non-existence of similar wavelets in dimensions higher than three. Jia and Micchelli [7], who are referred to in [3], have obtained similar results by a different menthod. The present paper is selfcontained and uses only elementary analysis tools.

2. Simple construction of wavelets

The construction of r-regular wavelets is reduced to the constructuction of an r-regular multiresolution analysis. More precisely, for a given r-regular multiresolution analysis $\{V\}_{j\in Z}$, we can construct an r-regular father function $\varphi(x)$. By using the general existence theorem as in [2, section 3.6], we can find $m_\varepsilon(\xi), \varepsilon\in E$, satisfying (4). Thus we can construct r-regular mother functions $\psi_\varepsilon(x), \varepsilon\in E$.

Our purpose is to show in which case it is possible to find $m_\varepsilon(\xi), \varepsilon\in E$, in the form $e^{i\beta_\varepsilon\cdot\xi}m_0(\xi+\alpha_\varepsilon\pi)$, $\alpha_\varepsilon\in R$, $\beta_\varepsilon\in R$. This form covers the form $\lambda_\varepsilon(\xi)e^{i\beta_\varepsilon\cdot\xi}m_0(\xi+\alpha_\varepsilon\pi)$, where $\lambda_\varepsilon(\xi)$ satisfies $|\lambda_\varepsilon(\xi)|\equiv 1$ for almost all ξ, because, if the former satisfies (4), then the latter form also satisfies (4). Since the orthonormality of $\{\varphi(x-k)\}_{k\in Z^n}$ implies the identity

$$\sum_{\eta\in R} m_0(\xi+\eta\pi)\overline{m_0(\xi+\eta\pi)}\equiv 1$$

for almost all ξ, it trivially follows, for this simple construction, that

$$\sum_{\eta\in R} e^{i\beta_\varepsilon\cdot(\xi+\eta\pi)}m_0(\xi+\eta\pi+\alpha_\varepsilon\pi)\overline{e^{i\beta_\varepsilon\cdot(\xi+\eta\pi)}m_0(\xi+\eta\pi+\alpha_\varepsilon\pi)}\equiv 1,$$

for $\varepsilon \in E$ and almost all ξ. Hence, in the case of our simple construction, we need only check the orthogonality relations:

$$\sum_{\eta \in R} e^{i\beta_\varepsilon \cdot (\xi+\eta\pi)} m_0(\xi+\eta\pi+\alpha_\varepsilon \pi) \overline{e^{i\beta_{\varepsilon'} \cdot (\xi+\eta\pi)} m_0(\xi+\eta\pi+\alpha_{\varepsilon'}\pi)} \equiv 0,$$

for $\varepsilon \neq \varepsilon', \varepsilon, \varepsilon' \in R$ and almost all ξ.

First we start with notation and definition. Let $J_n = \{0, 1, \cdots, 2^n - 1\}$. Then any $j \in J_n$ can be written uniquely, in the base two, as

$$j = c_{n-1}(j)2^{n-1} + c_{n-2}(j)2^{n-2} + \cdots + c_1(j)2^1 + c_0(j), \tag{5}$$

where each $c_k(j)$, $k = 0, \cdots, n-1$, is either 0 or 1. Without loss of generality, we use the lexicographic ordering of the vertices of the unit cube in $\boldsymbol{R}^n$,

$$\alpha_{n,j} = (c_{n-1}(j), c_{n-2}(j), \cdots, c_1(j), c_0(j)), \qquad j \in J_n.$$

We shall prove the following theorem.

Theorem. *For the lexicographic ordering of the vertices of the unit cube in $\boldsymbol{R}^3$, there exist eight independent simple constructions of wavelets in dimension three, and there are none in dimension greater than three.*

The following example is one of the eight simple constructions of wavelets in dimension three.

EXAMPLE. Let $\{V_j\}_{j \in \boldsymbol{Z}}$ be an r-regular multiresolution analysis in $L^2(\boldsymbol{R}^3)$ with the real-valued symbol $m_0(\xi)$ and $\varphi(x)$ be an r-regular father function. We define the symbol $m_0(\xi_1, \xi_2, \xi_3)$ by

$$m_0(\xi) = 2^{-3} \sum_{k \in \boldsymbol{Z}^3} \left(\varphi\left(\frac{x}{2}\right), \varphi(x-k) \right) e^{-ik\xi}$$

and put

$$\begin{aligned}
m_1(\xi_1, \xi_2, \xi_3) &= e^{i\xi_1 + i\xi_3} m_0(\xi_1, \xi_2, \xi_3 + \pi), \\
m_2(\xi_1, \xi_2, \xi_3) &= e^{i\xi_1 + i\xi_2 + i\xi_3} m_0(\xi_1, \xi_2 + \pi, \xi_3), \\
m_3(\xi_1, \xi_2, \xi_3) &= e^{i\xi_1 + i\xi_2} m_0(\xi_1, \xi_2 + \pi, \xi_3 + \pi), \\
m_4(\xi_1, \xi_2, \xi_3) &= e^{i\xi_1} m_0(\xi_1 + \pi, \xi_2, \xi_3), \\
m_5(\xi_1, \xi_2, \xi_3) &= e^{i\xi_2 + i\xi_3} m_0(\xi_1 + \pi, \xi_2, \xi_3 + \pi), \\
m_6(\xi_1, \xi_2, \xi_3) &= e^{i\xi_2} m_0(\xi_1 + \pi, \xi_2 + \pi, \xi_3), \\
m_7(\xi_1, \xi_2, \xi_3) &= e^{i\xi_3} m_0(\xi_1 + \pi, \xi_2 + \pi, \xi_3 + \pi).
\end{aligned}$$

Then the functions $\psi_j(x), j=1,\cdots,7$, defined by

$$\hat{\psi}_j(2\xi)=m_j(\xi)\hat{\varphi}(\xi), \qquad j=1,\cdots,7,$$

are r-regular mother functions of wavelets in dimension three.

REMARK. It will be observed later that this theorem is in dependent of the ordering of the vertices of the unit cube in $\boldsymbol{R}^n$.

Let $x_1,x_2,\cdots,x_{2^n}$ be real variables. Then we put $x'=(x_1,x_2,\cdots,x_{2^{n-1}})$, $x''=(x_{2^{n-1}+1},\cdots,x_{2^n})$ and $x=(x',x'')$.

DEFINITION 1. The symmetric matric $F_n(x)$ defined inductively by the follwing recurrence,

$$F_1(x_1,x_2)=\begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix},$$

$$F_n(x)=\begin{bmatrix} F_{n-1}(x') & F_{n-1}(x'') \\ F_{n-1}(x'') & F_{n-1}(x') \end{bmatrix}, \qquad n\geq 2,$$

is called a function matrix of order n.

Put

$$F_n(x)=(f_{jk}(x);\ j\downarrow 1,\cdots,2^n, k\rightarrow 1,\cdots,2^n),$$

where $x=(x_1,\cdots,x_{2^n})$. Put

$$\tilde{m}_{\alpha_{n,j}}(\xi)=m_0(\xi+\alpha_{n,j}\pi), \qquad j\in J_n.$$

Since $m_0(\xi)$ is a $2\pi\boldsymbol{Z}^n$-periodic function, for every $j,k\in J_n$, there exists $l\in J_n$ such that

$$m_0(\xi+(\alpha_{n,j}+\alpha_{n,k})\pi)=m_0(\xi+\alpha_{n,l}\pi).$$

Then the following lemma is obvious.

Lemma 1.

$$(6) \qquad (\tilde{m}_{\alpha_{n,k}}(\xi+\alpha_{n,j}\pi);\ j\downarrow 0,\cdots,2^n-1, k\rightarrow 0,\cdots,2^n-1)$$
$$=F_n(m_0(\xi+\alpha_{n,0}\pi),m_0(\xi+\alpha_{n,1}\pi),\cdots,m_0(\xi+\alpha_{n,2^n-1}\pi)).$$

Since, in general, we cannot say anything on the relation among the $m_0(\xi+\alpha_{n,j}\pi)$, $j\in J_n$, we may regard $m_0(\xi+\alpha_{n,j}\pi)$, $j\in J_n$, as variables denoted by x_j, $j\in J_n$. Hence we need only consider the matrix

$$(e^{i\beta_{n,k}\cdot\xi}\tilde{m}_{\alpha_{n,k}}(\xi+\alpha_{n,j}\pi);\ j\downarrow 0,\cdots,2^n-1, k\rightarrow 0,\cdots,2^n-1)$$
$$=(e^{i\beta_{n,k}\cdot\xi}\tilde{m}_0(\xi+(\alpha_{n,j}+\alpha_{n,k})\pi);\ j\downarrow 1,\cdots,2^n, k\rightarrow 0,\cdots,2^n-1),$$

Here $\beta_{n,k}\in\{\alpha_{n,k}\}_{k\in J_n}$ and we allow equality: $\beta_{n,k}=\beta_{n,k'}$ for some $k\neq k'$. If $\beta_{n,k}=\beta_{n,k'}$ for some $k\neq k'$, then the scalar product of the kth and k'th columns cannot be identically zero. Hence we need only consider the case

$$\beta_{n,k}=\alpha_{n,\sigma(k)}, \qquad \sigma\in\mathfrak{S}_{2^n},$$

where $\mathfrak{S}_n$ denotes the symmetric group of order n and $\mathfrak{S}_{2^n}$ acts on $J_n=(0,\cdots,2^n-1)$.

Now we want to evaluate

$$e^{i\alpha_{n;\sigma(k)}\cdot(\xi+\alpha_{n;j}\pi)}=e^{i\alpha_{n,\sigma(k)}\cdot\alpha_{n,j}\pi}e^{i\alpha_{n;\sigma(k)}\cdot\xi}.$$

It is enough to construct the table of signs, + or −, corresponding to the values + 1 or −1, of the exponential

$$e^{i\pi[\alpha_{n,\sigma(k)}\cdot\alpha_{n,j}\,(\mathrm{mod}\ 2)]}, \qquad k,j\in J_n.$$

Define

$$a_{j+1,k+1}=(-1)^{\alpha_{n,j}\cdot\alpha_{n,k}}, \qquad j,k\in J_n.$$

For $\sigma\in\mathfrak{S}_{2^n}$, put

$$s_{jk}=a_{j,\sigma(k)}, \qquad j,k=1,\cdots,2^n,$$

where $\mathfrak{S}_{2^n}$ acts on $(1,\cdots,2)$.

DEFINITION 2. For $\sigma\in\mathfrak{S}_{2^n}$, the matrix

$$S_{n,\sigma}=(s_{jk};\ j\downarrow 1,\cdots,2^n, k\rightarrow 1,\cdots,2^n)$$

is called a sign matrix.

Then $S_{n,\sigma}$ can be regarded as the table of signs coresponding to

$$e^{i\pi[\alpha_{n,\sigma(k)}\cdot\alpha_{n,j}\,(\mathrm{mod}\ 2)]}, \qquad k,j\in J_n.$$

For $\sigma\in\mathfrak{S}_{2^n}$, write

$$u_{\sigma,k}(x)=(s_{jk}f_{jk};\ j\downarrow 1,\cdots,2^n),$$
$$U_{n,\sigma}(x)=(u_{\sigma,k}(x);\ k\rightarrow 1,\cdots,2^n).$$

Denote by $O(n)$ the orthogonal group of order n, by $U(n)$ the unitary group of order n, and by S^{n-1} the unit sphere in $\boldsymbol{R}^n$. Since

$$\tilde{U}(\xi)=(e^{i\alpha_{n,\sigma(k)}\cdot\xi}u_{\sigma,k}(x);\ k\to 1,\cdots,2^n),$$

where $x_j=m_0(\xi+\alpha_{n,j}\pi)$, $j\in J_n$, then we have $\tilde{U}(\xi)\in U(2^n)$ if and only if $U_{n,\sigma}(x)\in O(2^n)$. Hence the simple construction of wavelets reduces to the following problem.

Problem. Are there any $\sigma\in\mathfrak{S}_{2^n}$ satisfying $\sigma(1)=1$ such that $U_{n,\sigma}(x)$ belongs to $O(2^n)$ for all $x\in S^{2^n-1}$?

To solve this problem we need several lemmas.

3. Preliminaries

Denote

$$\begin{aligned}
A_{n,11}&=(a_{jk};\ j\downarrow 1,\cdots,2^{n-1},k\to 1,\cdots,2^{n-1}),\\
A_{n,12}&=(a_{jk};\ j\downarrow 1,\cdots,2^{n-1},k\to 2^{n-1}+1,\cdots,2^n),\\
A_{n,21}&=(a_{jk};\ j\downarrow 2^{n-1}+1,\cdots,2^n,k\to 1,\cdots,2^{n-1}),\\
A_{n,22}&=(a_{jk};\ j\downarrow 2^{n-1}+1,\cdots,2^n,k\to 2^{n-1}+1,\cdots,2^n),\\
A_n&=(a_{jk};\ j\downarrow 1,\cdots,2^n,k\to 1,\cdots,2^n),\\
&=(A_{n,jk};\ j\downarrow 1,2,k\to 1,2).
\end{aligned}$$

The lexicographic ordering (5) of the vertices of the unit cube will give a particular interesting form to matrices (8) and (9) of the following lemma.

Lemma 2. *The following relations hold:*

$$\begin{cases} a_{jk}=a_{j+2^{n-1},k}, & j=1,\cdots,2^{n-1},\quad k=1,\cdots,2^{n-1},\\ a_{jk}=-a_{j+2^{n-1},k}, & j=1,\cdots,2^{n-1},\quad k=2^{n-1}+1,\cdots,2^n,\end{cases}\tag{7}$$

$$A_{n,11}=A_{n,12}=A_{n,21}=-A_{n,22}=A_{n-1},\tag{8}$$

$$A_{3,11}=\begin{bmatrix} +1 & +1 & +1 & +1\\ +1 & -1 & +1 & -1\\ +1 & +1 & -1 & -1\\ +1 & -1 & -1 & +1\end{bmatrix},\tag{9}$$

and

(10) $$A_{4,11}=\begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{bmatrix}$$

Proof. For every $j\in J_{n-1}$, there exist two natural injections ι_0 and $\iota_1$from $\{\alpha_{n-1,j}\}_{j\in J_{n-1}}$ to $\{\alpha_{n,j}\}_{j\in J_n}$ defined by

$$\iota_0(\alpha_{n-1,j})=(0,\alpha_{n-1,j})=\alpha_{n,j},$$
$$\iota_1(\alpha_{n-1,j})=(1,\alpha_{n-1,j})=\alpha_{n,j+2^{n-1}},$$

respectively. Then for $j,k\in J_{n-1}$,

$$\alpha_{n,j}\cdot\alpha_{n,k}=\alpha_{n-1,j}\cdot\alpha_{n-1,k}$$
$$\alpha_{n,j+2^{n-1}}\cdot\alpha_{n,k}=\alpha_{n-1,j}\cdot\alpha_{n-1,k},$$
$$\alpha_{n,j+2^{n-1}}\cdot\alpha_{n,k+2^{n-1}}=1+\alpha_{n-1,j}\cdot\alpha_{n-1,k}.$$

This implies (7).
By definition, $a_{jk}=a_{kj}$ $j,k\in J_n$. This symmetry and (7) imply (8).
Sinece $A_{2,11}=\begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}$, then (9) and (10) follow inductively from (7). □

4. Necessary condition for the existence of the simple construction

The aim of this section is to seek a necessary condition on the sign matrix $S_{n,\sigma}$ so that $U_{n,\sigma}(x)\in O(2^n)$. Put

$$S_{n,11}=(s_{jk};\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2,\cdots,2^{n-1}),$$
$$S_{n,12}=(s_{jk};\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2^{n-1}+2,\cdots,2^n),$$
$$S_{n,21}=(s_{jk};\ j\downarrow 2^{n-1}+2,\cdots,2^n,k\rightarrow 2,\cdots,2^{n-1}),$$
$$S_{n,22}=(s_{jk};\ j\downarrow 2^{n-1}+2,\cdots,2^n,k\rightarrow 2^{n-1}+2,\cdots,2^n),$$

Denote by δ_{jk} the Kronecker delta and by I_n the unit matrix of order n.

Lemma 3. *If $U_{n,\sigma}(x)\in O(2^n)$ for all $x\in S^{2^n-1}$, then*

$$(11)\qquad S_{n,21}=(s_{2^{n-1}+1,k}s_{jk};\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2,\cdots,2^{n-1}),$$

$$(12)\qquad S_{n,12}=(-s_{2^{n-1}+1,j}s_{kj};\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2,\cdots,2^{n-1}),$$

$$=(s_{2^{n-1}+1,j}(s_{jk}+2\delta_{jk});\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2,\cdots,2^{n-1}),$$

$$(13)\qquad S_{n,22}=(s_{2^{n-1}+1,j}s_{2^{n-1}+1,k}s_{kj};\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2,\cdots,2^{n-1}),$$

$$=(-s_{2^{n-1}+1,j}s_{2^{n-1}+1,k}(s_{jk}+2\delta_{jk});\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2,\cdots,2^{n-1}).$$

Proof. Assume that there exists $\sigma\in\mathfrak{S}_{2n}$ satisfying $\sigma(1)=1$ such that $U_{n,\sigma}(x)\in O(2^n)$ for all $x\in S^{2^n-1}$. Since, for $k=2,\cdots,2^n$,

$$u_{\sigma,1}(x)\cdot u_{\sigma,k}(x)\equiv 0,$$

then $s_{kk}=-1$. Hence, by the orthogonality of the jth and kth columns, we have

$$(14)\qquad s_{jk}=-s_{kj},\qquad j\neq k,\qquad j,k=2,\cdots,2^n.$$

In particular, by (7) of Lemma 2, for $j,k=2,\cdots,2^{n-1}$, we have

$$s_{2^{n-1}+1,k}=-s_{2^{n-1}+1,k+2^{n-1}}=-s_{j,2^{n-1}+1}=s_{j+2^{n-1},2^{n-1}+1}.$$

Thus (7) of Lemma 2 implies (11) and

$$S_{n,22}=(-s_{2^{n-1}+1,k}s_{jk};\ j\downarrow 2,\cdots,2^{n-1},k\rightarrow 2^{n-1}+2,\cdots,2^n).$$

By (14) we have $S_{n,12}=-{}^tS_{n,21}$. Hence we have the first equalities of (12) and (13). Since by (14), $S_{n,11}+I_{2^{n-1}-1}$ is an alternating matrix, then

$${}^tS_{n,11}=-S_{n,11}-2I_{2^{n-1}-1}.$$

Thus we have the second equalities of (12) and (13), respectively. This completes the proof. □

Lemma 4. *If $U_{n,\sigma}(x)\in O(2^n)$ for all $x\in S^{2^n-1}$, then*

$$\{(s_{jk};\ j\downarrow 1,\cdots,2^{n-1})\}_{k=1,\cdots,2^{n-1}}=\{(s_{jk};\ j\downarrow 1,\cdots,2^{n-1})\}_{k=2^{n-1}+1,\cdots,2^n}$$
$$=\{(a_{jk};\ j\downarrow 1,\cdots,2^{n-1})\}_{k=1,\cdots,2^{n-1}}.$$

Proof. By its matrix structure, $U_{n,\sigma}(x)\in O(2^n)$ only if the $2^{n-1}\times 2^{n-1}$ upper-left block $U_{n-1,\sigma'}(x')$ belongs to $O(2^{n-1})$. In particular, every two columns of the sign matrix $S_{n-1,\sigma'}$ of $U_{n-1,\sigma'}(x')$ are different. On the other hand, by (8) of Lemma 2, $(A_{n,11},A_{n,12})$ consists of 2^{n-1} pairs of columns of A_{n-1}. This completes the proof. □

5. Proof of the Theorem

First, we consider the three-diminsional case. Since $s_{55} = -1$, there are four possibilities for the choice of $(s_{5,k};\ k \to 2,3,4)$, that is,

$$(+1,+1,+1),\quad (-1,+1,-1),\quad (+1,-1,-1),\quad (-1,-1,+1).$$

Here, we check only the case corresponding to the first choice: $(+1,+1,+1)$. Since $S_{3,11}$ is a permutation of the columns of the matrix

$$\begin{bmatrix} -1 & +1 & -1 \\ +1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$$

and every diagonal element of $S_{3,11}$ is -1, there are two possible forms for $S_{3,11}$, namely,

$$\begin{bmatrix} -1 & -1 & +1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & +1 & -1 \\ -1 & -1 & +1 \\ +1 & -1 & -1 \end{bmatrix}$$

Corresponding to these two matrices, the permutations of the columns of the sign matrix are $\sigma = (0,5,7,6,4,3,2,1)$ and $\sigma = (0,7,6,5,4,1,3,2)$, respectively. We can easily check that $U_{3,\sigma}(x) \in O(3)$ in these two cases. By the same argument, we have the following table of all the possible column permutations of the sign matrix.

Table 1. Column permutations of the sign matrix producing orthogonal matrices in $\boldsymbol{R}^3$.

$\sigma(0)$	$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\sigma(4)$	$\sigma(5)$	$\sigma(6)$	$\sigma(7)$
0	1	3	6	7	4	5	2
0	1	7	2	5	6	3	4
0	3	2	5	7	6	4	1
0	3	6	1	5	4	2	7
0	5	3	2	6	1	4	7
0	5	7	6	4	3	2	1
0	7	2	1	6	3	5	4
0	7	6	5	4	1	3	2

Next, we consider the case of dimension four. To prove the non-existence of wavelets obtained by the simple construction, it will suffice to show that Lemma 3 contradicts Lemma 4. By Lemma 4, every column vector

$$(s_{9,j}(s_{jk}+2\delta_{jk});\ j\downarrow 2,\cdots,8), \qquad k=2,\cdots,8,$$

of $S_{n,12}$ coincides with one of

$$(a_{jk};\ j\downarrow 2,\cdots,8), \qquad k=1,\cdots,8,$$

that is, one of

$$(s_{jk};\ j\downarrow 2,\cdots,8), \qquad k=2,\cdots,8,$$

or

$$(a_{j,1};\ j\downarrow 2,\cdots,8).$$

But this condition cannot be satisfied, because each column

$$(\pm a_{jk};\ j\downarrow 2,\cdots,8), \qquad k=1,\cdots,8,$$

differs from each column

$$(a_{jk'};\ j\downarrow 2,\cdots,8), \qquad k'=1,\cdots,8,$$

by at least three elements, except in the case where $k=k'$ and $s_{9,j}=1, j=2,\cdots,8$. But, even in this exceptional case, the construction is impossible besause

$$(s_{jk}+2\delta_{jk};\ j\downarrow 2,\cdots,8)\neq(s_{jk};\ j\downarrow 2,\cdots,8).$$

Finally, we consider the case of dimension greater than four. By the structure of $U_{n,\sigma}(x)$, which is induced by the ordering (5) of the vertices of the unit cube in $\boldsymbol{R}^n$, it is necessary for $U_{n,\sigma}(x)\in O(2^n)$ that the $2^{n-1}\times 2^{n-1}$ upper-left block of the sign matrix $S_{n,\sigma}$ satisfies the condition of the Theorem for dimension $n-1$. By induction, there is no such simple construction. $\square$

To generalize the result of the above Theorem to any ordering of the vertices, α_k, of the unit cube in $\boldsymbol{R}^3$, we interpret each line of Table 1 as a permutation, σ, of these vertices, that is, $\beta_k=\sigma(\alpha_k)$. Thus, if σ is one of these permutations, then the 8×8 matrix $((-1)^{\sigma(\alpha_k)\cdot\alpha_j}m_0(\xi+\pi(\alpha_k+\alpha_j));\ j\downarrow 0,1,\cdots,7, k\rightarrow 0,1,\cdots,7)$ is orthogonal. It thus follows that the permutation σ satisfies the relation

$$((\sigma(\alpha_k)+\sigma(\alpha_j))\cdot(\alpha_k+\alpha_j) \qquad \text{is odd if } k\neq j.$$

Since this relation remains true for any permutation P applied to $(\alpha_0,\cdots,\alpha_7)$ and $(\sigma(\alpha_0),\cdots,\sigma(\alpha_7))$, then the above Theorem remains valid in $\boldsymbol{R}^3$ for any ordering of the vertices of the unit cube, and hence in $\boldsymbol{R}^n$, for $n>3$.

As a closing remark, we mention that the wavelets obtained by Riemenschneider and Shen [3] with the following ordering of the vertices of the unit cube in $\boldsymbol{R}^3$:

$$(0,0,0),\quad (1,1,0),\quad (0,1,1),\quad (1,0,0),\quad (1,0,1),\quad (0,0,1),\quad (0,1,0),\quad (1,1,1),$$

can be obtained from the lexicographic ordering (5):

$$(0,0,0),\quad (0,0,1),\quad (0,1,0),\quad (0,1,1),\quad (1,0,0),\quad (1,0,1),\quad (1,1,0),\quad (1,1,1),$$

with the permutation σ given by the fifth line of the sign matrix of Table 1, simply by mapping the latter ordering onto the former one.

ACKNOWLEDGEMENT. The authors would like to express their special thanks to the referee for letting them know of references [3] and [5].

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