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THE HAUSDORFF DIMENSION OF QUASI-ALL
BROWNIAN PATHS

TAKASHI KOMATSU AND KEIZO TAKASHIMA

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1. Introduction. Properties of Brownian paths have been main sub-
jects of many papers and various results have been already well known. The
size of sample path, for example, has been investigated by evaluating Lebesgue
measure and Hausdorff measure etc. of path. Taylor [7] determined the Haus-
dorff dimension of almost all Brownian paths. On the other hand, Fukushima
[2] recently studied some basic properties of path, such as nowhere-differenti-
ability, Lévy's Hölder continuity, the law of iterated logarithm etc. in connec-
tion with the Dirichlet space theory on the Wiener space. He proved that
these properties hold not only almost surely but also quasi-everywhere.

In this paper we shall present a refinement of Taylor's result from the
viewpoint of the Dirichlet space theory. It is easily derived from a combina-
tion of the definition of Hausdorff measure and Fukushima's result correspond-
ing to Lévy's Hölder continuity that the Hausdorff dimension of path is no
more than 2 quasi-everywhere. We shall prove that the Hausdorff dimension
is no less than 2 quasi-everywhere, by showing that a specific Wiener functional
used in [7] belongs not only to $L^2$-space but also to the Dirichlet space.

2. Dirichlet space. We first prepare some notations and definitions
on the Wiener space and a Dirichlet space on it (cf. Shigekawa [6] and Fuku-
shima [2]).

Let $W$ be the Banach space of all $\mathbb{R}^d$-valued continuous functions $w(t)$
on $[0, 1]$ satisfying $w(0) = 0$, with standard supremum norm, $H$ be the Hilbert
space of all absolutely continuous functions of $W$ having square integrable deri-
vatives, with inner product

$$\langle h, g \rangle_H = \int_0^1 \dot{h}(t) \cdot \dot{g}(t) dt,$$

and norm $\|h\|_H = \sqrt{\langle h, h \rangle_H}$, where $\dot{h}$ denotes the derivative of $h$ and $a \cdot b$ denotes
the inner product in $\mathbb{R}^d$. Let $P$ be the Wiener measure on $W$. A Wiener
functional is a real (or more generally Hilbert space) valued mapping defined
on $W$, measurable with respect to the Borel field of $W$. Let $L^2(W)$ be the
Hilbert space of real valued Wiener functionals, square integrable with respect to $P$, with inner product

$$(F, G)_{L^2(W)} = \mathbb{E}[F(w)G(w)],$$

where $\mathbb{E}$ denotes the integration with respect to $P$. Let $L^2(W; H)$ be the space of $H$-valued Wiener functionals $F$ satisfying $\mathbb{E}[\|F(w)\|^2_x] < \infty$.

We define a weak derivative of a Wiener functional after [6]. A real valued Wiener functional $F$ is called $H$-differentiable if for every $w$ of $W$ there exists $DF(w)$ of $H$ such that for any $h$ of $H$,

$$F(w+h) - F(w) = \langle DF(w), h \rangle_H + o(\|h\|_H), \quad \text{as } \|h\|_H \to 0.$$  

A real valued Wiener functional $F$ is called to have a weak derivative if there exists a sequence of real valued smooth (in the sense of [6]) functionals $F_k$ convergent to $F$ in $L^2(W)$ such that the sequence of $H$-derivatives $DF_k$ is a Cauchy sequence in $L^2(W; H)$. The weak derivative of $F$ is defined by $L^2(W; H)$-limit of $DF_k$ and denoted by $DF$ (see also Ikeda and Watanabe [3]).

Next we define a Dirichlet space and a Dirichlet form after [2]. Let $\mathcal{F}$ be the space of Wiener functionals in $L^2(W)$ having weak derivatives in $L^2(W; H)$, and

$$\mathcal{E}(F, G) = \frac{1}{2} \mathbb{E}[\langle DF, DG \rangle_H],$$

and

$$\mathcal{E}_i(F, G) = \mathcal{E}(F, G) + (F, G)_{L^2(W)}, \quad F, G \in \mathcal{F}.$$

Then a capacity is defined by

$$\text{Cap}(A) = \inf \{\mathcal{E}_i(F, F): F \in \mathcal{F}, F \geq 1, P\text{-a.e. on } A\}$$

for an open set $A$ of $W$, and for any set $B$ of $W$

$$\text{Cap}(B) = \inf \{\text{Cap}(A): A \text{ is open, } A \supseteq B\}.$$  

It is well known that this capacity is a nonnegative, increasing set function on $W$ satisfying the following properties:

(I) $P(A) \leq \text{Cap}(A)$ for any Borel set $A$, in particular $\text{Cap}(W) = 1$.

(II) $\text{Cap}(\bigcup_{n=1}^\infty A_n) = \sup \{\text{Cap}(A_n): n \geq 1\}$, for an increasing sequence $A_n$.

(III) $\text{Cap}(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \text{Cap}(A_n)$.

(see also [1]). The term 'quasi-everywhere' or simply 'q.e.' means 'except on a set of zero capacity'.

The Hausdorff dimension $\dim(A)$ of a set of $\mathbb{R}^d$ is defined by
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\[ \dim(A) = \inf \{ \alpha > 0 : A^\alpha(A) = 0 \} , \]

where \( A^\alpha \) denotes the Hausdorff \( \alpha \)-measure, i.e. Hausdorff measure with respect to measure function \( f^\alpha \) (cf. Rogers [5]).

Let \( w[0, 1] = \{ w(t) : 0 \leq t \leq 1 \} \) for \( w \) of \( W \). Our result is the following.

**Theorem.** \( \dim(w[0, 1]) = 2 \), q.e. \( (d \geq 2) \).

3. **Proof of Theorem.** i) First we shall prove that \( \dim(w[0, 1]) \leq 2 \) q.e.

Let \( w'(t), 1 \leq i \leq d \), be the \( i \)-th component of \( w(t) \). Theorem 3 of [2] tells us that

\[
\lim_{\delta \to 0} \sup \{ |w'(t_2) - w'(t_1)| (2t \log 1/t)^{-1/\alpha} : 0 \leq t_1 < t_2 \leq 1, \ t = t_2 - t_1 < \delta \} = 1, \text{ q.e.}
\]

Let \( A_i \) be the set of \( w \) of \( W \) such that \( w' \) has the above continuity property and \( B \) be the set of \( w \) which has \( \lambda \)-Hölder continuity properties for all \( 0 < \lambda < 1/2 \). Then \( B \) includes \( \cap A_i \) and by (II) and (III),

\[
\text{Cap}(B^c) \leq \text{Cap}(\cup A_i) \leq \sum A_i \text{ Cap}(A_i) = 0.
\]

On the other hand, by the same argument as in [7], it is easily seen that for a \( \lambda \)-Hölder continuous \( w \), \( w[0, 1] \) has zero Hausdorff \( \beta \)-measure for \( \beta > 1/\lambda \). Thus for any \( w \) of \( B \), \( w[0, 1] \) has zero Hausdorff \( \beta \)-measure for any \( \beta > 2 \). This implies the desired result.

ii) We shall next prove that \( \dim(w[0, 1]) \geq 2 \) q.e. [7] showed that for any \( w \) of \( W \) and \( 1 < \alpha < 2 \),

\[
\int_0^1 \int_0^1 |w(s) - w(t)|^{-\alpha} \ ds \ dt < \infty
\]

implies that \( w[0, 1] \) has positive Hausdorff \( \alpha \)-measure. We define Wiener functionals \( F^{(\alpha)}, 1 \leq \alpha < 2 \), by

\[
F^{(\alpha)}(w) = \int_0^1 \int_0^1 |w(s) - w(t)|^{-\alpha} \ ds \ dt,
\]

and \( F_n = F^{(2-1/n)} \), \( n = 1, 2, \ldots \). If we can show that for any \( n \), \( \text{Cap} \{ F_n = \infty \} = 0 \), then by (III)

\[
\text{Cap}(\cup_{n=1} F_n = \infty) \leq \sum_{n=1} \text{Cap}(F_n = \infty) = 0,
\]

and for any \( \omega \) of \( \cap_{n=1} F_n = \infty \), \( w[0, 1] \) has positive Hausdorff \((2-1/n)\)-measure. This shows that \( \dim(w[0, 1]) \geq 2 \) q.e.

Therefore we have only to prove that \( \text{Cap}(\{ F^{(\alpha)} = \infty \}) = 0 \) for any \( 1 \leq \alpha < 2 \). Fix \( \alpha, 1 \leq \alpha < 2 \). Let

\[
F_\varepsilon(w) = \int_0^1 \int_0^1 (|w(s) - w(t)|^\varepsilon + \varepsilon)^{-\alpha/\varepsilon} \ ds \ dt, \ \varepsilon > 0.
\]
Then \( F_\varepsilon \) is continuous on \( W \) and for any \( w, F_\varepsilon(w) \uparrow F(w) = F(w) \) as \( \varepsilon \downarrow 0 \). If we can verify that \( F_\varepsilon \in \mathcal{F} \) and \( K = \sup_{t > 0} \mathcal{E}(F_\varepsilon, F_t) \) is finite, then the next well-known inequality (cf. [1])

\[
\text{Cap}(\{F_\varepsilon > \lambda\}) \leq \lambda^{-2} \mathcal{E}(F_\varepsilon, F_0), \quad \lambda > 0,
\]

and (II) imply that for \( \lambda > 0 \),

\[
\text{Cap}(\{F > \lambda\}) \leq \sup_{t > 0} \text{Cap}(\{F_\varepsilon > \lambda\}) \leq \lambda^{-2} K.
\]

Letting \( \lambda \to \infty \), we have \( \text{Cap}(\{F = \infty\}) = 0 \).

The finiteness of \( K \) is derived from the following two lemmas and the proof of Theorem is completed.

**Lemma 1.** \( \sup_{t > 0} (F_\varepsilon, F_t)_1^2(\omega) < \infty \).

**Lemma 2.** \( \sup_{t > 0} \mathcal{E}(F_\varepsilon, F_t) < \infty \).

**Proof of Lemma 1.** Let \( \phi_\varepsilon(x) = (|x|^d + \varepsilon)^{-\alpha/d} \) and

\[
f_\varepsilon(x) = \int_0^{[\cdot]} r^{-d+1} \int_0^{r} s^{d-1}(s^d + \varepsilon)^{-\alpha/d} ds
\]

\[
= d^{-1} \int_0^{[\cdot]} r^{-d+1} \int_0^{r} (s^d + \delta)^{-\alpha/d} ds
\]

\[
= (d - \alpha)^{-1} \int_0^{[\cdot]} r^{-d+1}(r^d + \varepsilon)^{-\alpha/d} - \varepsilon^{1-\alpha/d} ) dr, \quad x \in \mathbb{R}^d.
\]

Then \( \Delta f_\varepsilon(x) = \phi_\varepsilon(x) \), where \( \Delta \) is the Laplacian on \( \mathbb{R}^d \).

\[ (F_\varepsilon, F_t)_1^2(\omega) = E[(F_\varepsilon)^2] \]

\[ = 4E[\int_0^1 dt \int_0^t \phi_\varepsilon(\omega(s) - \omega(t)) ds]^2] \]

\[ \leq 4E[\int_0^1 dt (\int_0^t \phi_\varepsilon(\omega(s) - \omega(t)) ds)^2] \]

\[ = 4\int_0^1 dt E[(\int_0^{1-t} \phi_\varepsilon(\omega(s)) ds)^2] \]

\[ \leq 4E[\int_0^1 \phi_\varepsilon(\omega(s)) ds]^2]. \]

In the above we have used the time-homogeneity of Brownian motion. To estimate the right-hand side, we apply the Ito formula to \( f_\varepsilon(x) \),

\[ f_\varepsilon(\omega(1)) = \int_0^1 \nabla f_\varepsilon(\omega(s)) \cdot d\omega(s) + 1/2 \int_0^1 \phi_\varepsilon(\omega(s)) ds, \]

where

\[ \nabla f_\varepsilon(x) = \left( \frac{\partial}{\partial x_j} f_\varepsilon(x) \right)_{1 \leq j \leq d} \]
\[= ((d-\alpha)^{-1}|x|^{-d} \{(|x|^d + \varepsilon)^{1-\alpha/d} - \varepsilon^{1-\alpha/d} \} x_j)_{1 \leq j \leq d}.\]

Since

\[|f_\alpha(x)| \leq (d-\alpha)^{-1}(2-\alpha)^{-1}|x|^{2-\alpha} \quad \text{and} \quad |\nabla f_\alpha(x)| \leq (d-\alpha)^{-1}|x|^{1-\alpha},\]

\[E[(\int_0^1 \phi_\alpha(w(s))\,ds)^2] \leq 4E[(f_\alpha(w(1)) - \int_0^1 \nabla f_\alpha(w(s)) \cdot dw(s))^2] \leq 8E[f_\alpha(w(1))^2] + 8E[\int_0^1 |\nabla f_\alpha(w(s))|^2\,ds] \leq 8(d-\alpha)^{-2}(2-\alpha)^{-2}E[|w(1)|^{2(2-\alpha)}] + 8(d-\alpha)^{-2}\int_0^1 E[|w(s)|^{2(1-\alpha)}]\,ds < \infty.\]

This completes the proof of Lemma 1.

**Corollary.** The Ito formula is applicable to \(|x|^{2-\alpha}, \alpha < 2, i.e.

\[|w(t)|^{2-\alpha} = \int_0^t (2-\alpha)|w(s)|^{-\alpha}w(s) \cdot dw(s) + \int_0^t (d-\alpha)(2-\alpha)|w(s)|^{-\alpha}\,ds.\]

Furthermore, \(\int_0^1 |w(s)|^{-\alpha}\,ds \in L^2(W).\)

**Proof of Lemma 2.** We use the same notations as in the proof of Lemma 1. By the mean value theorem, for \(w\) of \(W\) and \(h\) of \(H\),

\[F_\alpha(w+h) - F_\alpha(w) = 2\int_0^t du \left( \int_t^u \nabla \phi_\alpha(w(s)-w(t)) \, ds + o(||h||_H) \right) \cdot \left( \int_t^u h(u) \, du \right) ds + o(||h||_H)\]

as \(||h||_H \to 0\). This implies the \(H\)-differentiability of \(F_\alpha\) and its \(H\)-derivative \(DF_\alpha\) is given by

\[\langle DF_\alpha(w), h \rangle_H = \int_0^t du \hat{h}(u) \cdot \left( \int_t^u \nabla \phi_\alpha(w(s)-w(t)) \, ds \right).\]

Since the boundedness of \(||DF_\alpha(w)||_H\) on \(W\) is easily derived from that of \(|\nabla \phi_\alpha(x)|\), by the same argument as in Section 2 and 3 of Kusuoka [4], it is verified that the \(H\)-derivative of \(F_\alpha\) coincides with the weak derivative of \(F_\alpha\) a.s. Thus

\[2\mathcal{E}(F_\alpha, F_\alpha) = E[||DF_\alpha||_H^2] = 4\int_0^1 du \sum_{j=1}^d E[\left( \int_0^u dt \int_t^u \psi_{\phi_j}(w(s)-w(t)) \, ds \right)^2],\]

where \(\psi_{\phi_j}(x)\) is the \(j\)-th component of \(\nabla \phi_\alpha(x)\). Let \(g_{\phi_j}(x)\) be the \(j\)-th component of \(\nabla f_\alpha(x)\). Then \(g_{\phi_j}(x)\) is twice continuously differentiable and \(\Delta g_{\phi_j}(x) = \psi_{\phi_j}(x)\). To estimate the above, we again apply the Ito formula to \(g_{\phi_j}(x),\)
\[ g_{x_{ij}}(w(1) - w(t)) g_{x_{ij}}(w(u) - w(t)) \]
\[ = \int_0^1 \nabla g_{x_{ij}}(w(s) - w(t)) \cdot dw(s) + 1/2 \int_0^1 \phi_{x_{ij}}(w(s) - w(t)) ds. \]

Hence we have
\[
(*) \quad E\left[ \left( \int_0^1 dt \int_0^1 \phi_{x_{ij}}(w(s) - w(t)) ds \right)^2 \right] 
\leq 8 E\left[ \left( \int_0^1 \{ g_{x_{ij}}(w(1) - w(t)) - g_{x_{ij}}(w(u) - w(t)) \} dt \right)^2 \right] 
+ 8 E\left[ \left( \int_0^1 \nabla g_{x_{ij}}(w(s) - w(t)) \cdot dw(s) \right)^2 \right] 
\]

As for the first term in the right-hand side, it is less than
\[
16 E\left[ \int_0^1 \{ g_{x_{ij}}(w(1) - w(t))^2 + g_{x_{ij}}(w(u) - w(t))^2 \} dt \right] 
\leq 32 E\left[ \int_0^1 g_{x_{ij}}(w(t))^2 dt \right] 
\leq 32 (d - \alpha)^2 E\left[ \int_0^1 |w(t)|^{2(1 - \alpha)} dt \right],
\]

since \(|g_{x_{ij}}(x)| \leq (d - \alpha)^{-1} |x|^{1 - \alpha}, \ |x| \neq 0. \) Thus the first term is bounded independently of \( \varepsilon \) by Corollary of Lemma 1. As for the second term, by standard calculation we have
\[
E\left[ \left( \int_0^1 dt \int_0^1 \nabla g_{x_{ij}}(w(s) - w(t)) \cdot dw(s) \right)^2 \right] 
= E\left[ \int_0^1 dt \int_0^1 \nabla g_{x_{ij}}(w(s) - w(t)) \cdot dw(s) \right] 
= E\left[ \int_0^1 \nabla g_{x_{ij}}(w(t)) \cdot dt \right]^2 
= E\left[ \int_0^1 |\nabla g_{x_{ij}}(w(t))| dt \right]^2. 
\]

Note that
\[
\nabla g_{x_{ij}}(x) = \nabla((d - \alpha)^{-1} x_j |x|^{-d} \{(|x|^d + \varepsilon)^{(1-\alpha)/d} - \varepsilon^{1-\alpha/d}\}) 
= ((d - \alpha)^{-1} x_j |x|^{-d} \{(|x|^d + \varepsilon)^{(1-\alpha)/d} - \varepsilon^{1-\alpha/d}\} (-d x_j x_k |x|^{-2 - \delta_{jk}}) 
+ x_j x_k |x|^{-2} \phi(x))_{1 \leq k \leq d},
\]

where \( \delta_{jk} \) is Kronecker's symbol. Hence we have
\[
|\nabla g_{x_{ij}}(x)| \leq ((1 + d)/(d - \alpha) + 1) |x|^{-\alpha}.
\]
Therefore the second term in the right-hand side of (*) is less than
\[ 8\{(1+d)/(d-\alpha)+1\}^2E[\int_0^t |v(t)|^{-\alpha}dt]^2 \]
and is bounded independently of \( \varepsilon \) by Corollary of Lemma 1. This completes the proof of Lemma 2.

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References
