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<th><strong>Title</strong></th>
<th>On delta-unknotting operation</th>
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<td><strong>Author(s)</strong></td>
<td>Uchida, Yoshiaki</td>
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<td><strong>Note</strong></td>
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1. Statement of Theorem. In this paper we study oriented knots in the oriented 3-sphere $S^3$. In [3], H. Murakami and Y. Nakanishi defined a $\Delta$-unknotting operation and proved that any knot can be transformed into a trivial knot by a finite sequence of $\Delta$-unknotting operations. Let $k$ be a knot in $S^3$ and $B_1^\Delta$ a 3-ball which intersects $k$ as illustrated in Figure 1(a). Then $k^\Delta_1$ denotes the knot in $S^3$ obtained from $k$ by changing $B_1^\Delta$ to $B_2^\Delta$ as illustrated in Figure 1(b). $k_\Delta$ is said to be obtained from $k$ by a $\Delta$-unknotting operation.

Let $\Delta_1$ and $\Delta_2$ be two $\Delta$-unknotting operations for $k$ such that $k^{\Delta_1} \approx k^{\Delta_2}$. Then $\Delta_1$ and $\Delta_2$ are said to be homeomorphic, if there is a homeomorphism $h: S^3 \rightarrow S^3$ such that $h(k) = k$, $h(k^{\Delta_1}) = k^{\Delta_2}$, $h(B_1^{\Delta_1}) = B_1^{\Delta_2}$, and $h(B_2^{\Delta_1}) = B_2^{\Delta_2}$.

**Remark.** For an ordinary unknotting operation, the following results are known. If the image of an ordinary unknotting operation is unknot, then T. Kobayashi [2], Scharlemann and A. Thompson [4] proved that the number of homeomorphism classes for a non-trivial doubled knot is one. K. Taniyama [5] proved for two-bridge knots, the number is at most two. In contrast to such knots, Y. Nakanishi conjectured that for any natural number $n$, there exist knots such that the number of homeomorphism classes is at least $n$. A. Kawauchi proved that affirmatively by using imitation theory [1].

**Theorem.** Let $k$ be a knot in $S^3$. Suppose that $k^{\Delta_1}$ is obtained from $k$ by a $\Delta$-unknotting operation. Then the number of the homeomorphism classes of
\( \Delta \)-unknotting operations is infinite.

Proof. We consider the \( \Delta \)-unknotting operations \( \Delta_n(n \geq 0) \) as illustrated in Figure 2.

Considering the disk \( D \) in Figure 2, it is easy to show that \( k_n \) is ambient isotopic to \( \Delta_0 \). Now we will prove that if \( n \neq m \) then \( \Delta_n \) is not homeomorphic to \( \Delta_m \).

We consider the following graph. (See Figure 3(a).) It is an embedding of the graph indicated in Figure 3(b). If \( \Delta_1 \) is homeomorphic to \( \Delta_2 \), then there is a homeomorphism of \( S^3 \) such that \( h(k) = k, h(G_{\Delta_1}) = G_{\Delta_2} \). To prove that \( G_{\Delta_n} \) is not equivalent to \( G_{\Delta_m} \), it is sufficient to consider the three constituent knots, which span all vertices, illustrated in Figure 3(c).

Since \( k \) is a knot, it is sufficient to consider two cases as indicated in Figure 4.

In the case (i), after moving by an ambient isotopy, \( G_{\Delta_n} \) and its three constituents knots are illustrated in Figure 5. It is easy to show that \( k_{n,1} \cong k_{m,1} \) and \( k_{n,2} \cong k_{m,2} \). Now we will prove that \( k_{n,3} \not\cong k_{m,3} \), if \( n \neq m \). Let \( a_n \) be the second
coefficient of the Conway polynomial of \(k_{n,3}\). We have \(a_n-a_{n-1} = 1\) i.e. \(a_n = a_0 + (n-1)\). Then \(k_{n,3} \not= k_{m,3}\) if \(n \neq m\).

In the case (ii), we can prove that similarly. This completes the proof.

2. **Note.** In this section, we consider a \(\Delta\)-unknotting operation as a local move on a knot diagram, ignoring the orientations [3]. Furthermore, we consider the mirror image of a \(\Delta\)-unknotting operation as a \(\Delta\)-unknotting operation, too. Suppose that \(\Delta_l\) and \(\Delta_r\) are like as illustrated in Figure 6, then
\( \Delta_i \) and \( \Delta_r \) are said to be twin-equivalent. The performances of \( \Delta \)-unknotting operations on \( \Delta_i \) and \( \Delta_r \) are equivalent. Let \( k \) and \( k' \) be diagrams of a knot \( K, \Delta(\Delta', \text{resp.}) \)-unknotting operation for \( k(k', \text{resp.}). \) \( \Delta \) and \( \Delta' \) are equivalent, write \( \Delta \cong \Delta' \), if there exists a finite sequence \( \{k_i, \Delta_i\}_{i=1,2, \ldots, n} \) such that

1. \( \Delta_i \) and \( \Delta_{i+1} \) are \( \Delta \)-unknotting operations of \( k_{i+1} \),
2. \( k_{i+1} \) is obtained from \( k_i \) by a combination of Reidemeister moves which fix \( \Delta_i \),
3. \( \Delta_i \) is twin-equivalent to \( \Delta_{i+1} \) on \( k_{i+1} \),
4. \( (k, \Delta) \cong (k_1, \Delta_i) \) and \( (k', \Delta') \cong (k_n, \Delta_n) \),
5. \( (k_{i+1}, \Delta_{i+1}) \) is obtained from \( (k_i, \Delta_i) \) by the move illustrated as in Figure 7.

**Example 1.** The knots as in Figure 8 have \( \Delta \)-unknotting number one. The triangle regions marked by \( \blacktriangle \) are places to be performed by \( \Delta \)-unknotting operations. For each knot, these \( \Delta \)-unknotting operations are equivalent in the

![Figure 6](image)

![Figure 7](image)

![Figure 8](image)
above sense.

**Example 2.** Each $\Delta_s$ in the proof of Theorem is equivalent in the above sense.

Here, we raise the following problem.

**Problem.** Let $K$ be a knot with $\Delta$-unknotting number one. Suppose that $\Delta$ and $\Delta'$ are $\Delta$-unknotting operations which deform $K$ into a trivial knot. Are $\Delta$ and $\Delta'$ equivalent in the above sense?

**References**


Department of Mathematics
Kobe University
Nada, Kobe 657
Japan