

Title	On delta-unknotted operation
Author(s)	Uchida, Yoshiaki
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$\Delta$ -unknotting operations is infinite.

Proof. We consider the  $\Delta$ -unknotting operations  $\Delta_n (n \geq 0)$  as illustrated in Figure 2.

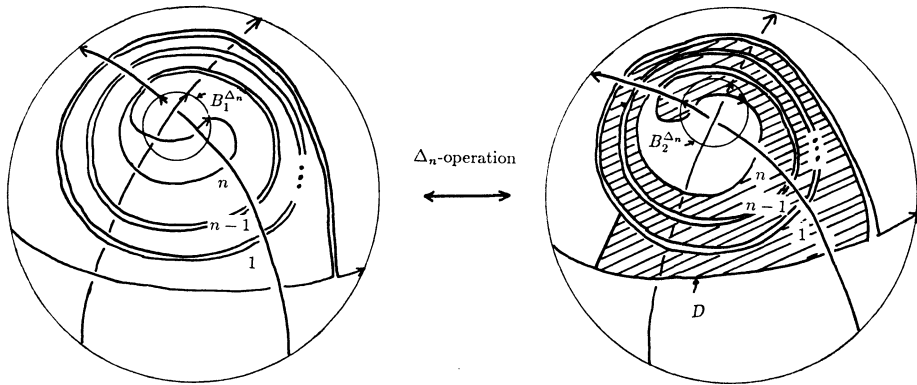


Figure 2

Considering the disk  $D$  in Figure 2, it is easy to show that  $k_{\Delta_n}$  is ambient isotopic to  $k_{\Delta_0}$ . Now we will prove that if  $n \neq m$  then  $\Delta_n$  is not homeomorphic to  $\Delta_m$ .

We consider the following graph. (See Figure 3(a).) It is an embedding of the graph indicated in Figure 3(b). If  $\Delta_1$  is homeomorphic to  $\Delta_2$ , then there is a homeomorphism of  $S^3$  such that  $h(k) = k, h(G_{\Delta_1}) = G_{\Delta_2}$ . To prove that  $G_{\Delta_n}$  is not equivalent to  $G_{\Delta_m}$ , it is sufficient to consider the three constituent knots, which spun all vertices, illustrated in Figure 3(c).

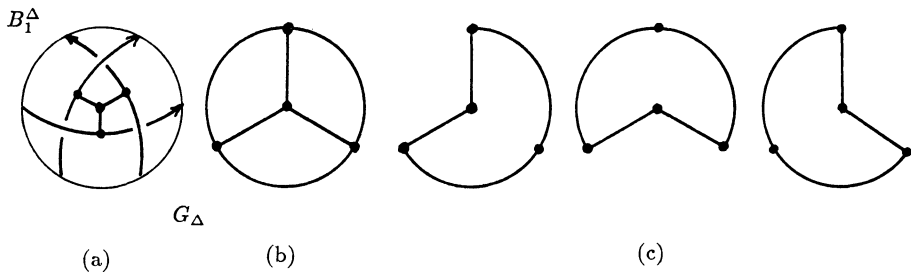


Figure 3

Since  $k$  is a knot, it is sufficient to consider two cases as indicated in Figure 4.

In the case (i), after moving by an ambient isotopy,  $G_{\Delta_n}$  and its three constituents knots are illustrated in Figure 5. It is easy to show that  $k_{n,1} \cong k_{m,1}$  and  $k_{n,2} \cong k_{m,2}$ . Now we will prove that  $k_{n,3} \not\cong k_{m,3}$ , if  $n \neq m$ . Let  $a_n$  be the second

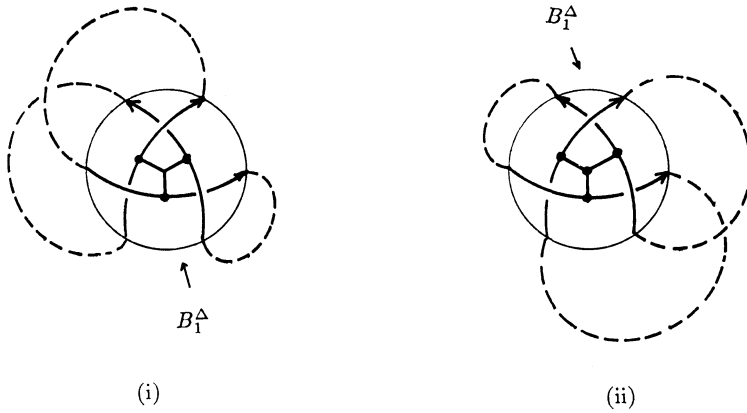


Figure 4

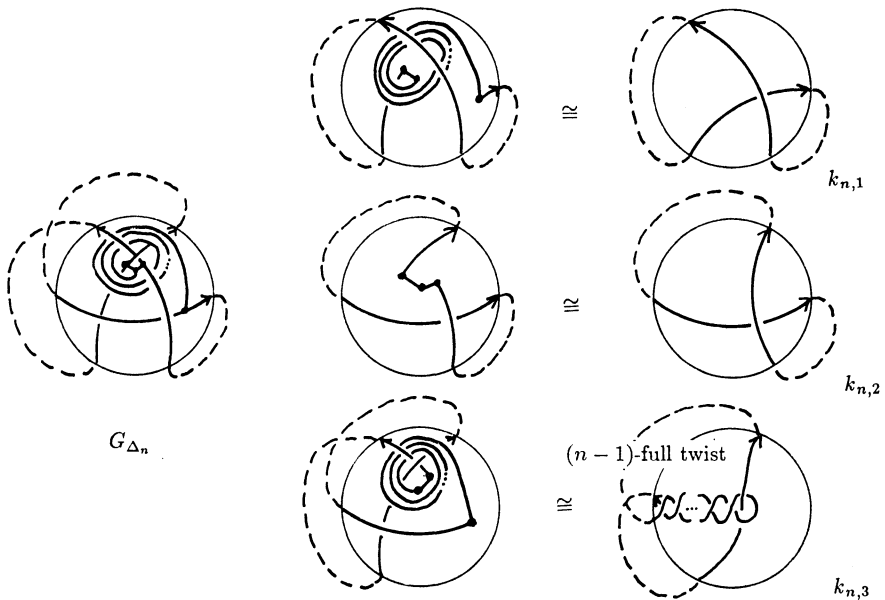


Figure 5

coefficient of the Conway polynomial of  $k_{n,3}$ . We have  $a_n - a_{n-1} = 1$  i.e.  $a_n = a_0 + (n-1)$ . Then  $k_{n,3} \not\cong k_{m,3}$  if  $n \neq m$ .

In the case (ii), we can prove that similarly. This completes the proof.

**2. Note.** In this section, we consider a  $\Delta$ -unknotting operation as a local move on a knot diagram, ignoring the orientations [3]. Furthermore, we consider the mirror image of a  $\Delta$ -unknotting operation as a  $\Delta$ -unknotting operation, too. Suppose that  $\Delta_l$  and  $\Delta_r$  are like as illustrated in Figure 6, then

$\Delta_l$  and  $\Delta_r$  are said to be twin-equivalent. The performances of  $\Delta$ -unknotting operations on  $\Delta_l$  and  $\Delta_r$  are equivalent. Let  $k$  and  $k'$  be diagrams of a knot  $K$ ,  $\Delta(\Delta', \text{resp.})$   $\Delta$ -unknotting operation for  $k(k', \text{resp.})$ .  $\Delta$  and  $\Delta'$  are equivalent, write  $\Delta \cong \Delta'$ , if there exists a finite sequence  $\{k_i, \Delta_i\}_{i=1,2,\dots,n}$  such that

- (1)  $\Delta_i$  and  $\Delta_{i+1}$  are  $\Delta$ -unknotting operations of  $k_{i+1}$ ,
- (2)  $k_{i+1}$  is obtained from  $k_i$  by a combination of Reidemeister moves which fix  $\Delta_i$ ,
- (3)  $\Delta_i$  is twin-equivalent to  $\Delta_{i+1}$  on  $k_{i+1}$ ,
- (4)  $(k, \Delta) \cong (k_1, \Delta_1)$  and  $(k', \Delta') \cong (k_n, \Delta_n)$ ,
- (5)  $(k_{i+1}, \Delta_{i+1})$  is obtained from  $(k_i, \Delta_i)$  by the move illustrated as in Figure 7.

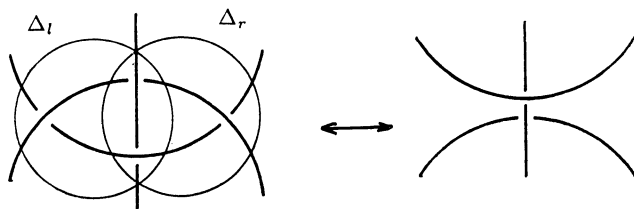


Figure 6

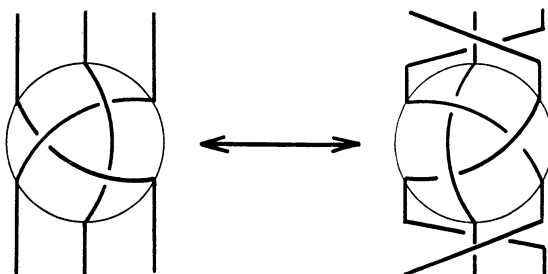


Figure 7

EXAMPLE 1. The knots as in Figure 8 have  $\Delta$ -unknotting number one. The triangle regions marked by  $\blacktriangle$  are places to be performed by  $\Delta$ -unknotting operations. For each knot, these  $\Delta$ -unknotting operations are equivalent in the

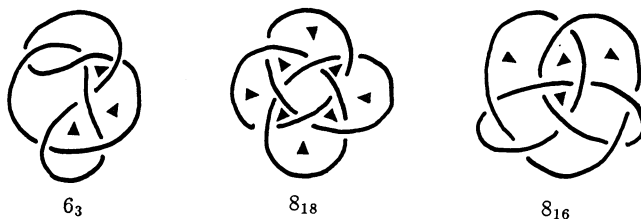


Figure 8

above sense.

EXAMPLE 2. Each  $\Delta_n$  in the proof of Theorem is equivalent in the above sense.

Here, we raise the following problem.

**Problem.** *Let  $K$  be a knot with  $\Delta$ -unknotting number one. Suppose that  $\Delta$  and  $\Delta'$  are  $\Delta$ -unknotting operations which deform  $K$  into a trivial knot. Are  $\Delta$  and  $\Delta'$  equivalent in the above sense?*

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Department of Mathematics  
Kobe University  
Nada, Kobe 657  
Japan

