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ON DELTA-UNKNOTTING OPERATION

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1. Statement of Theorem. In this paper we study oriented knots in the oriented 3-sphere S^3. In [3], H. Murakami and Y. Nakanishi defined a Δ-unknotting operation and proved that any knot can be transformed into a trivial knot by a finite sequence of Δ-unknotting operations. Let k be a knot in S^3 and B_1^Δ a 3-ball which intersects k as illustrated in Figure 1(a). Then k_Δ denotes the knot in S^3 obtained from k by changing B_1^Δ to B_2^Δ as illustrated in Figure 1(b). k_Δ is said to be obtained from k by a Δ-*unknotting operation.*

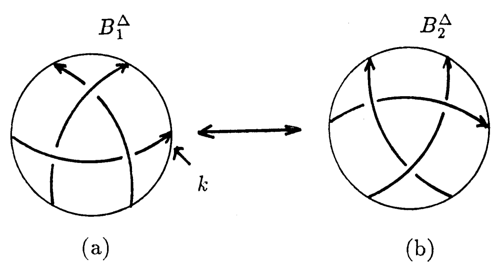


Figure 1

Let Δ_1 and Δ_2 be two Δ-unknotting operations for k such that $k_{\Delta_1} \cong k_{\Delta_2}$. Then Δ_1 and Δ_2 are said to be *homeomorphic,* if there is a homeomorphism h: $S^3 \to S^3$ such that $h(k)=k$, $h(k_{\Delta_1})=k_{\Delta_2}$, $h(B_1^{\Delta_1})=B_1^{\Delta_2}$, and $h(B_2^{\Delta_1})=B_2^{\Delta_2}$.

REMARK. For an ordinary unknotting operation, the following results are known. If the image of an ordinary unknotting operation is unknot, then T. Kobayashi [2], Scharlemann and A. Thompson [4] proved that the number of homeomorphism classes for a non-trivial doubled knot is one. K. Taniyama [5] proved for two-bridge knots, the number is at most two. In constract to such knots, Y. Nakanishi conjectured that for any natural number n, there exist knots such that the number of homeomorphism classes is at least n. A. Kawauchi proved that affirmatively by using imitation theory [1].

Theorem. *Let k be a knot in S^3. Suppose that k_Δ is obtained from k by a Δ-unknotting operation. Then the number of the homeomorphism classes of*

Δ-unknotting operations is infinite.

Proof. We consider the Δ-unknotting operations $\Delta_n (n \geq 0)$ as illustrated in Figure 2.

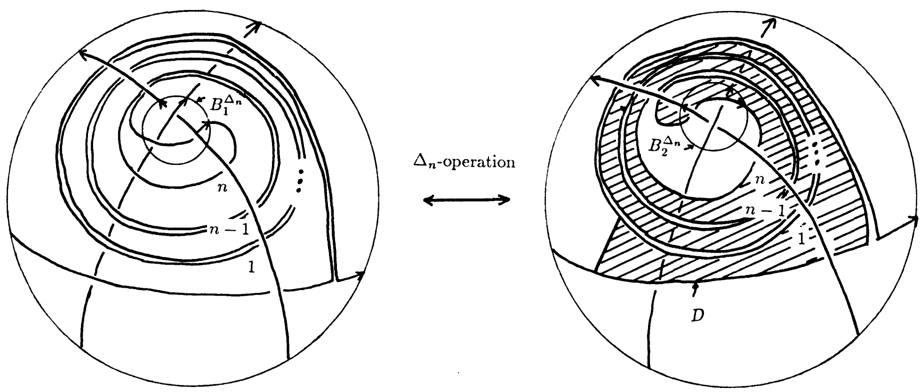
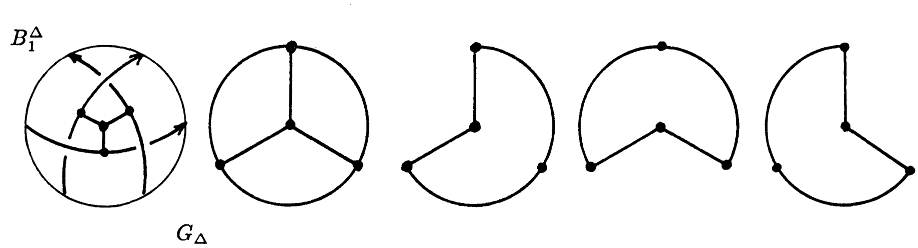


Figure 2

Considering the disk D in Figure 2, it is easy to show that k_{Δ_n} is ambient isotopic to k_{Δ_0}. Now we will prove that if $n \neq m$ then Δ_n is not homeomorphic to Δ_m.

We consider the following graph. (See Figure 3(a).) It is an embedding of the graph indicated in Figure 3(b). If Δ_1 is homeomorphic to Δ_2, then there is a homeomorphism of S^3 such that $h(k)=k$, $h(G_{\Delta_1})=G_{\Delta_2}$. To prove that G_{Δ_n} is not equivalent to G_{Δ_m}, it is sufficient to consider the three constituent knots, which spun all vertices, illustrated in Figure 3(c).

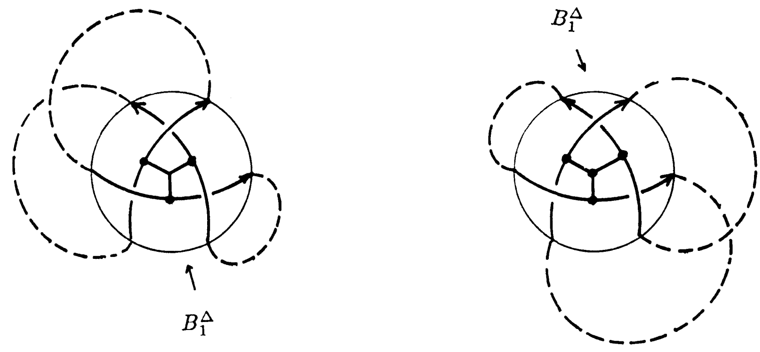


(a) (b) (c)

Figure 3

Since k is a knot, it is sufficient to consider two cases as indicated in Figure 4.

In the case (i), after moving by an ambient isotopy, G_{Δ_n} and its three constituents knots are illustrated in Figure 5. It is easy to show that $k_{n,1} \simeq k_{m,1}$ and $k_{n,2} \simeq k_{m,2}$. Now we will prove that $k_{n,3} \not\simeq k_{m,3}$, if $n \neq m$. Let a_n be the second



(i) (ii)

Figure 4

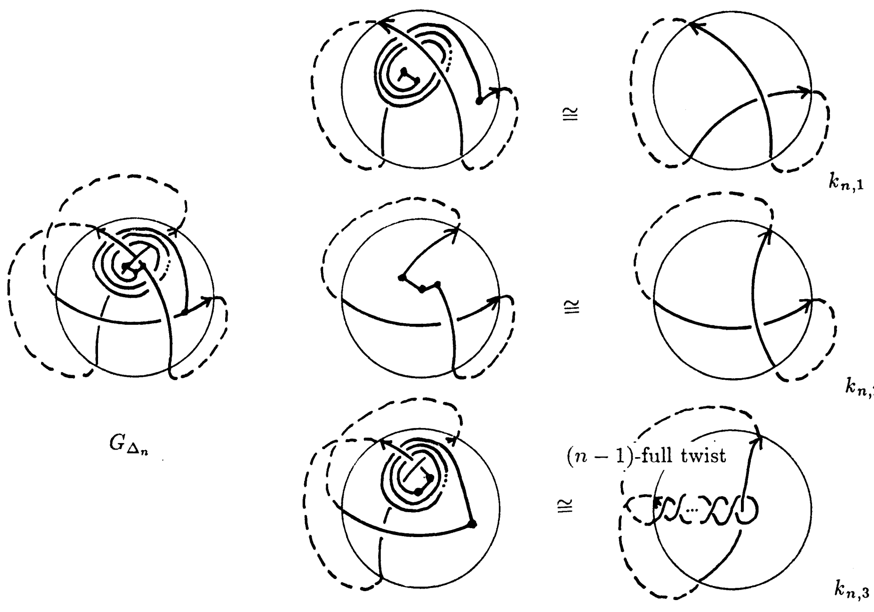


Figure 5

coefficient of the Conway polynomial of $k_{n,3}$. We have $a_n - a_{n-1} = 1$ i.e. $a_n = a_0 + (n-1)$. Then $k_{n,3} \not\cong k_{m,3}$ if $n \neq m$.

In the case (ii), we can prove that similarly. This completes the proof.

2. Note. In this section, we consider a Δ-unknotting operation as a local move on a knot diagram, ignoring the orientations [3]. Furthemore, we consider the mirror image of a Δ-unknotting operation as a Δ-unknotting operation, too. Suppose that Δ_l and Δ_r are like as illustrated in Figure 6, then

Δ_l and Δ_r are said to be twin-equivalent. The performances of Δ-unknotting operations on Δ_l and Δ_r are equivalent. Let k and k' be diagrams of a knot K, $\Delta(\Delta'$, resp.) Δ-unknotting operation for $k(k'$, resp.). Δ and Δ' are equivalent, write $\Delta \cong \Delta'$, if there exists a finite sequence $\{k_i, \Delta_i\}_{i=1,2,\cdots,n}$ such that

(1) Δ_i and Δ_{i+1} are Δ-unknotting operations of k_{i+1},

(2) k_{i+1} is obtained from k_i by a combination of Reidemeister moves which fix Δ_i,

(3) Δ_i is twin-equivalent to Δ_{i+1} on k_{i+1},

(4) $(k, \Delta) \cong (k_1, \Delta_1)$ and $(k', \Delta') \cong (k_n, \Delta_n)$,

(5) (k_{i+1}, Δ_{i+1}) is obtained from (k_i, Δ_i) by the move illustrated as in Figure 7.

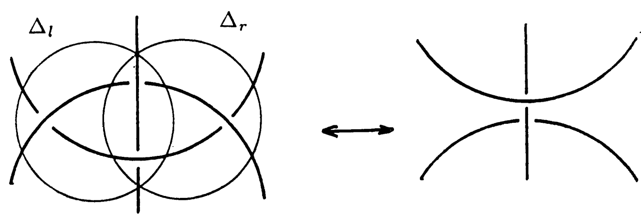


Figure 6

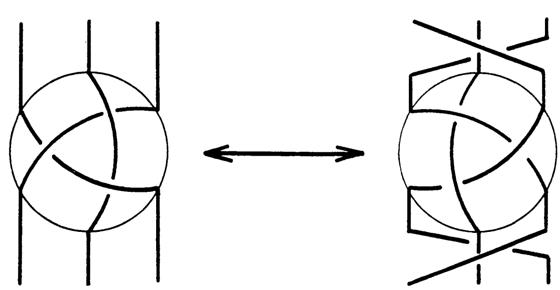


Figure 7

EXAMPLE 1. The knots as in Figure 8 have Δ-unknotting number one. The triangle regions marked by ▲ are places to be performed by Δ-unknotting operations. For each knot, these Δ-unknotting operations are equivalent in the

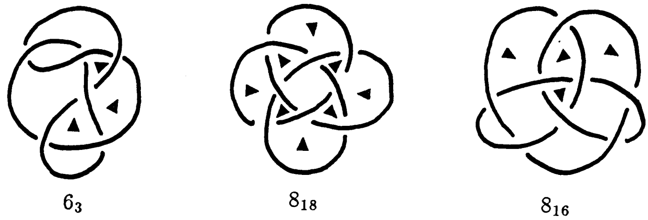


Figure 8

above sense.

EXAMPLE 2. Each Δ_n in the proof of Theorem is equivalent in the above sense.

Here, we raise the following problem.

Problem. *Let K be a knot with Δ-unknotting number one. Suppose that Δ and Δ' are Δ-unknotting operations which deform K into a trivial knot. Are Δ and Δ' equivalent in the above sense?*

References

[1] A. Kawauchi: *Introduction to topological imitations of* (3,1)*-dimensional manifold pairs*, "Topics in Knot Theory" Proc. NATO-ASI(Erzurum/Turkey, 1992) to appear.

[2] T. Kobayashi: *Minimal genus Seifert surfaces for unknotting number* 1 *knots*, Kobe J. Math. **6** (1989), 53–62.

[3] H. Murakami and Y. Nakanishi: *On a certain move generating link-homology*, Math. Ann. **284** (1989), 75–89.

[4] M. Sharlemann and A. Thompson: *Link genus and the Conway moves*, Comm. Math. Helv. **64** (1989), 527–535.

[5] K. Taniyama: *On unknotting operations of two-bridge knots*, Math. Ann. **291** (1991), 579–589.

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