

Title	Principal circle actions on a product of spheres
Author(s)	Ozeki, Hideki; Uchida, Fuichi
Citation	Osaka Journal of Mathematics. 1972, 9(3), p. 379-390
Version Type	VoR
URL	https://doi.org/10.18910/7222
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

PRINCIPAL CIRCLE ACTIONS ON A PRODUCT OF SPHERES

HIDEKI OZEKI AND FUICHI UCHIDA

(Received February 4, 1972)

0. Introduction

A smooth circle action $\varphi : S^1 \times X \rightarrow X$ on a smooth manifold X is called *principal* if the isotropy subgroup

$$I(x) = \{z \in S^1 \mid \varphi(z, x) = x\}$$

consists of the identity element alone for each point x of X . For a principal smooth circle action on a smooth manifold X , the orbit space M is a smooth manifold, the natural projection $\pi : X \rightarrow M$ is a smooth principal S^1 -bundle, and in addition the manifold M is orientable if and only if the manifold X is orientable.

Two principal smooth circle actions (φ, X) and (φ', X') are called to be *equivalent* if there is an equivariant diffeomorphism of (φ, X) onto (φ', X') . A principal smooth circle action (φ, X) on a closed oriented smooth manifold X is called to *bord* if there is a principal smooth circle action (Φ, W) on a compact oriented smooth manifold W and there is an equivariant orientation preserving diffeomorphism of (φ, X) onto $(\Phi, \partial W)$, the boundary of W .

In this paper we consider principal smooth circle actions on a closed orientable smooth manifold which is cohomologically a product of spheres. We show that any principal circle action on a manifold which is cohomologically a product $S^{2m+1} \times S^{2n+1}$ of odd dimensional spheres bords but on a certain manifold which is cohomologically $S^{2m} \times S^{2n+1}$ ($n \geq m$) there is a principal circle action which does not bord. And the cohomology rings of orbit manifolds show that there are infinitely many (topologically) distinct principal circle actions on $S^{2m+1} \times S^{2n+1}$ ($m \neq n$). We can also show that the Pontrjagin classes of orbit manifolds well distinguish some of the circle actions on a product of spheres.

1. Cobordism of principal circle actions

Let E be a topological space whose integral cohomology group $H^*(E)$ is isomorphic to an integral cohomology group $H^*(S^{2m+1} \times S^{2n+1})$ of a product of

odd dimensional spheres with $0 \leq m \leq n$. Let $\pi : E \rightarrow M$ be a principal S^1 -bundle over an orientable closed smooth manifold M . Then,

Lemma 1.1.

(1) *The integral cohomology ring $H^*(M)$ of M is isomorphic to one of the truncated polynomial rings given under:*

- (a) $\mathbf{Z}[c, x]/(x^2, c^{n+1})$, where $\deg c = 2$ and $\deg x = 2m + 1$,
- (b) $\mathbf{Z}[c, y]/(y^2, c^{n+1}, kc^{m+1}, yc^{m+1})$, where $\deg c = 2$ and $\deg y = 2n + 1$

and k is a positive integer. Here the element c corresponds to the Euler class of the principal S^1 -bundle $\pi : E \rightarrow M$.

(2) *The each odd dimensional Stiefel-Whitney class of M vanishes.*

Proof. By the Thom-Gysin sequence ([5] p. 60, Theorem 21) for the principal S^1 -bundle $\pi : E \rightarrow M$, $H^{2m-1}(M) = 0$ and $H^{2m}(M)$ is an infinite cyclic group generated by c^m . Then $H^{2n+2}(M) = 0$ by the universal coefficient theorem and the Poincaré duality of M . Now the ring structure of $H^*(M)$ is obtained from the Thom-Gysin sequence by a routine calculation. Next, let $V_i \in H^i(M; \mathbf{Z}_2)$ be a class characterized by the equation

$$Sq^i \alpha = \alpha \cup V_i \quad \text{for all } \alpha \in H^{\dim M - i}(M; \mathbf{Z}_2),$$

and let $V = V_0 + V_1 + \dots + V_i + \dots$, then $SqV = W(M)$, the total Stiefel-Whitney class of M by the Wu's formula ([5] p. 55, Theorem 17). Then $W_{2i+1}(M) = 0$ follows from the ring structure of $H^*(M; \mathbf{Z}_2)$ and a property of the Steenrod operations ([6] p. 5, Lemma 2.5). *q.e.d.*

Theorem 1. *Let E be an orientable closed smooth manifold. Assume that the integral cohomology group of E is isomorphic to one of a product $S^{2m+1} \times S^{2n+1}$ of odd dimensional spheres. Then any principal smooth circle action on E bords as an orientable principal smooth circle action.*

Proof. Let $\pi : E \rightarrow M$ be a principal S^1 -bundle associated with a given principal smooth circle action on E . Denote by \bar{c} the modulo 2 reduction of the Euler class c of the principal S^1 -bundle $\pi : E \rightarrow M$. Then the circle action on E bords as an orientable principal smooth circle action if and only if all bordism Stiefel-Whitney numbers vanish

$$\langle W_{i_1}(M) \cdots W_{i_r}(M) \bar{c}^k, [M]_2 \rangle = 0,$$

and all bordism Pontrjagin numbers vanish

$$\langle P_{i_1}(M) \cdots P_{i_r}(M) c^k, [M] \rangle = 0,$$

where $[M]_2$ is the modulo 2 reduction of the fundamental class $[M]$ of M ([3]

p. 49, Theorem 17.5). But the orbit manifold M is odd dimensional and each odd dimensional Stiefel-Whitney class of M vanishes by Lemma 1.1. Hence all bordism Stiefel-Whitney numbers and all bordism Pontrjagin numbers of $\pi : E \rightarrow M$ vanish. Therefore this principal smooth circle action boards as an orientable principal smooth circle action. q.e.d.

2. Principal circle actions on a product of spheres

For a sequence $a = (a_0, \dots, a_m)$ of integers, we define the circle action φ_a on C^{m+1} by

$$\varphi_a(z, (u_0, \dots, u_m)) = (z^a u_0, \dots, z^{a_m} u_m),$$

and denote by $S^{2m+1}(a_0, \dots, a_m)$ the unit sphere S^{2m+1} in C^{m+1} with this action φ_a .

Let $a = (a_0, \dots, a_m), b = (b_0, \dots, b_n)$ be sequences of integers. We also define the circle action $\varphi_{a,b}$ on $S^{2m+1} \times S^{2n+1}$ by

$$\varphi_{a,b}(z, (\vec{u}, \vec{v})) = (\varphi_a(z, \vec{u}), \varphi_b(z, \vec{v}))$$

where $\vec{u} = (u_0, \dots, u_m), \vec{v} = (v_0, \dots, v_n)$, and denote by

$$S^{2m+1}(a_0, \dots, a_m) \times S^{2n+1}(b_0, \dots, b_n)$$

the product $S^{2m+1} \times S^{2n+1}$ with the action $\varphi_{a,b}$. Then the circle action $\varphi_{a,b}$ is principal if and only if each a_i is relatively prime to each b_j . When the circle action $\varphi_{a,b}$ is principal, the orbit manifold is denoted by

$$M(a_0, \dots, a_m; b_0, \dots, b_n).$$

In particular, $M(a_0; b_0, \dots, b_n)$ is naturally diffeomorphic to the lens space obtained from S^{2n+1} by the identification $\vec{v} = \varphi_b(\lambda, \vec{v})$ for all $\lambda \in C, \lambda^{a_0} = 1$. The cohomology ring of $M(a_0, \dots, a_m; b_0, \dots, b_n)$ is determined as follows:

Theorem 2. *Suppose $0 \leq m \leq n$. Then the integral cohomology ring of $M(a_0, \dots, a_m; b_0, \dots, b_n)$ is isomorphic to*

(i) $Z[c, x]/(x^2, c^{n+1})$, where $\deg c = 2$ and $\deg x = 2m + 1$, if $m = n$ or if $a_i = 0$ for some i ,

(ii) $Z[c, y]/(y^2, c^{n+1}, kc^{m+1}, yc^{m+1})$, where $\deg c = 2, \deg y = 2n + 1$ and $k = \prod_i a_i$ if $m < n$ and $\prod_i a_i \neq 0$. Here the element c corresponds to the Euler class of the principal S^1 -bundle

$$\pi : S^{2m+1}(a_0, \dots, a_m) \times S^{2n+1}(b_0, \dots, b_n) \rightarrow M(a_0, \dots, a_m; b_0, \dots, b_n).$$

By virtue of Lemma 1.1, it is sufficient to determine the $(2m+2)$ -dimensional cohomology group of $M(a_0, \dots, a_m; b_0, \dots, b_n)$, and furthermore if

$m = n$ the cohomology ring is determined by Lemma 1.1 already.

Denote by ξ_n the canonical complex line bundle over the complex projective n -space $P^n(C)$ obtained from $S^{2n+1} \times C$ by the identification $(\vec{u}, \rho) = (\lambda\vec{u}, \lambda\rho)$ for all $\lambda \in C, |\lambda| = 1$ ([5] p. 75). Then there is a mapping

$$p : M(a_0, \dots, a_m; \underbrace{1, \dots, 1}_{(n+1)\text{times}}) \rightarrow P^n(C)$$

given by the following commutative diagram:

$$\begin{array}{ccc} S^{2m+1}(a_0, \dots, a_m) \times S^{2n+1}(1, \dots, 1) & \xrightarrow{p_2} & S^{2n+1}(1, \dots, 1) \\ \downarrow \pi & & \downarrow \pi_0 \\ M(a_0, \dots, a_m; 1, \dots, 1) & \xrightarrow{p} & P^n(C) \end{array}$$

where p_2 is the projection to the second factor and π_0 is the projection of the principal S^1 -bundle associated with the canonical complex line bundle ξ_n .

Lemma 2.1.

(i) *The natural projection*

$$p : M(a_0, \dots, a_m; 1, \dots, 1) \rightarrow P^n(C)$$

is a sphere bundle associated with the complex $(m + 1)$ -plane bundle

$$\xi_n^{a_0} \oplus \dots \oplus \xi_n^{a_m}$$

where ξ^a is the a -fold tensor product of a complex line bundle ξ for $a \geq 0$ and the $(-a)$ -fold tensor product of the conjugate line bundle $\bar{\xi}$ of ξ for $a < 0$.

(ii) For $M = M(a_0, \dots, a_m; 1, \dots, 1)$, we have

$$H^{2m+2}(M) \cong \begin{cases} Z/(\prod_i a_i) \cdot Z & \text{if } m < n, \\ 0 & \text{if } m \geq n. \end{cases}$$

Proof. (i) is proved easily from the fact that the total space $E(\xi_n^a)$ of the complex line bundle ξ_n^a can be represented as the space obtained from $S^{2n+1} \times C$ by the identification $(\vec{u}, \rho) = (\lambda\vec{u}, \lambda^a\rho)$ for all $\lambda \in C, |\lambda| = 1$. Next the Euler class of the complex $(m + 1)$ -plane bundle $\zeta = \xi_n^{a_0} \oplus \dots \oplus \xi_n^{a_m}$ is

$$e(\zeta) = \left(\prod_{i=0}^m a_i\right) \cdot e(\xi_n)^{m+1}.$$

Then, by the Thom-Gysin sequence for the complex $(m + 1)$ -plane bundle ζ , there is an exact sequence:

$$H^0(P^n(C)) \xrightarrow{h} H^{2m+2}(P^n(C)) \xrightarrow{\hat{p}^*} H^{2m+2}(M) \xrightarrow{\hat{p}_*} H^1(P^n(C))$$

where the homomorphism h is given by $h(x) = x \cdot e(\zeta)$. And this implies (ii). q.e.d.

Lemma 2.2. *We have*

$$H^{2m+2}(M(a_0, \dots, a_m; b_0, \dots, b_n)) \cong Z / (\prod_i a_i) \cdot Z$$

for $m < n$.

Proof. Consider the following commutative diagram:

$$\begin{CD} S^{2m+1}(a) \times S^{2n+1}(b) @>i_1>> S^{2m+1}(a) \times S^{4n+3}(b, c) @<i_2<< S^{2m+1}(a) \times S^{2n+1}(c) \\ @VV\pi V @VV\pi V @VV\pi V \\ M(a; b) @>f_1>> M(a; b, c) @<f_2<< M(a; c) \end{CD}$$

where $a = (a_0, \dots, a_m)$, $b = (b_0, \dots, b_n)$, $c = (c_0, \dots, c_n)$, $i_1((\vec{u}, \vec{v})) = (\vec{u}, (\vec{v}, 0))$, $i_2((\vec{u}, \vec{v})) = (\vec{u}, (0, \vec{v}))$ and f_1, f_2 are induced mappings. Then f_1, f_2 induce isomorphisms of $(2m+2)$ -dimensional cohomology groups if $m < n$, and we have

$H^{2m+2}(M(a_0, \dots, a_m; b_0, \dots, b_n)) \cong H^{2m+2}(M(a_0, \dots, a_m; c_0, \dots, c_n))$. Thus Lemma 2.2 follows from this isomorphism and Lemma 2.1 (ii). q.e.d.

The proof of Theorem 2 completes.

Corollary. *There are infinitely many (topologically) distinct principal smooth circle actions on $S^{2m+1} \times S^{2n+1}$ for each $m \neq n$.*

This follows directly from Lemma 2.2.

3. Pontrjagin classes of orbit manifolds

For a given principal smooth circle action on E in our examples, the Pontrjagin classes of the orbit manifold M can be expressed by the Euler class c of the principal S^1 -bundle $\pi : E \rightarrow M$.

Let E be a smooth submanifold of an N -dimensional euclidean space \mathbf{R}^N . For each point p of E , the tangent space $\tau_p(E)$ of E at p can be canonically imbedded into the tangent space $\tau_p(\mathbf{R}^N)$ of \mathbf{R}^N at p . If we denote by $\nu_p(E)$ the orthogonal complement of $\tau_p(E)$ in $\tau_p(\mathbf{R}^N)$, then $\nu(E) = \bigcup_{p \in E} \nu_p(E)$ is the normal bundle of E in \mathbf{R}^N . Let t be an isometry of \mathbf{R}^N such that $t(E) \subset E$. Then the differential $(dt)_p$ of t at p in E maps $\tau_p(E)$ onto $\tau_{t(p)}(E)$, and $\nu_p(E)$ onto $\nu_{t(p)}(E)$. Suppose the normal bundle $\nu(E)$ is trivial, i.e.

$$\nu(E) \cong E \times \mathbf{R}^k.$$

If dt on $\nu(E)$ satisfies:

$$(dt)_p(p, v) = (t(p), v) \text{ for } p \in E, v \in \mathbf{R}^k,$$

then we say the action of t on $\nu(E)$ is compatible with the trivialization, or simply t acts on $\nu(E)$ trivially.

Lemma 3.1. *Let E be a smooth submanifold of an N -dimensional euclidean space \mathbf{R}^N , and T a circle subgroup of $\mathbf{SO}(N, \mathbf{R})$ acting principally on E . Suppose the normal bundle $\nu(E)$ of E in \mathbf{R}^N is trivial and the action of T on $\nu(E)$ is compatible with the trivialization. Then the tangent bundle $\tau(M)$ of the orbit manifold M is stably equivalent to the vector bundle obtained from $E \times \mathbf{R}^N$ by the identification $(p, v) = (t(p), t(v))$, for all $t \in T$.*

Proof. At each point p of \mathbf{R}^N , we have the usual identification of $\tau_p(\mathbf{R}^N)$ with \mathbf{R}^N , which is denoted by h_p . First we remark that for any element t in $\mathbf{GL}(N, \mathbf{R})$, the differential dt of t is compatible with the above identifications, i.e.

$$(3.1) \quad h_{t(p)} \circ (dt)_p = t \circ h_p \quad \text{at each } p \in \mathbf{R}^N.$$

Denote by $\lambda(E)$ the restriction of $\tau(\mathbf{R}^N)$ over E given by

$$\lambda_p(E) = \tau_p(\mathbf{R}^N), \text{ for } p \in E.$$

Consider the equivalence relation on $\lambda(E)$ as follows: $X \sim Y$ if and only if $Y = (dt)(X)$ for some t in T . Now (3.1) shows that the bundle over M obtained from $\lambda(E)$ by the above relation is isomorphic with the vector bundle stated in Lemma 3.1. Let $\gamma_p(E)$ be the kernel of $(d\pi)_p : \tau_p(E) \rightarrow \tau_p(M)$, and $\tau'_p(E)$ the orthogonal complement of $\gamma_p(E)$ in $\tau_p(E)$. We have the decomposition:

$$\lambda(E) = \tau'(E) \oplus \gamma(E) \oplus \nu(E),$$

where T acts on each factor. From the assumption in Lemma 3.1, $\nu(E)$ is trivial and T acts trivially on $\nu(E)$. The bundle $\gamma(E)$ is trivial and the action of T on $\gamma(E)$ is compatible with this trivialization since T is abelian. Thus the bundle over M obtained from $\lambda(E)$ by the above equivalence relation is stably equivalent to the bundle over M obtained from $\tau'(E)$ by the same relation. The differential $d\pi$ of the projection $\pi : E \rightarrow M$ gives an isomorphism $\tau'_p(E)$ with $\tau_{\pi(p)}(M)$ and $d\pi$ is compatible with the action of T , i.e.

$$d\pi \circ dt = d\pi \quad \text{for any } t \in T.$$

Now it is easy to see that the bundle obtained from $\tau'(E)$ is isomorphic with the tangent bundle $\tau(M)$. q.e.d.

Theorem 3. Under the same notations as in section 2, the total Pontrjagin class of the orbit manifold $M = M(a_0, \dots, a_m; b_0, \dots, b_n)$ is

$$P(M) = \prod_{i=0}^m (1 + a_i^2 c^2) \cdot \prod_{j=0}^n (1 + b_j^2 c^2),$$

where c is the Euler class of the principal S^1 -bundle associated with the circle action $\varphi(a_0, \dots, a_m; b_0, \dots, b_n)$.

Proof. For the unit $(2m+1)$ -sphere S^{2m+1} in $\mathbf{C}^{m+1} = \mathbf{R}^{2m+2}$, we choose a unit normal vector field X on S^{2m+1} . Then each element in $\mathbf{SO}(2m+2, \mathbf{R})$ fixes X , thus the action of $\mathbf{SO}(2m+2, \mathbf{R})$ is trivial on the normal bundle of S^{2m+1} . S^{2n+1} in \mathbf{C}^{n+1} has the same property. It is easy to see that our action $\varphi(a_0, \dots, a_m; b_0, \dots, b_n)$ on $E = S^{2m+1} \times S^{2n+1}$ in $\mathbf{C}^{m+1} \times \mathbf{C}^{n+1} = \mathbf{R}^N$ satisfies the required assumption in Lemma 3.1, where $N = 2m + 2n + 2$ and $S^1 = T$ is expressed as

$$S^1 = \left\{ \begin{pmatrix} z^{a_0} & & & & 0 \\ & \ddots & & & \\ & & z^{a_m} & & \\ & & & z^{b_0} & \\ & & & & \ddots \\ 0 & & & & & z^{b_n} \end{pmatrix} \right\}$$

by the complex coordinates. On the other hand, if we denote by ξ the complex line bundle over M associated with the principal S^1 -bundle $\pi : E \rightarrow M$, then the bundle constructed in Lemma 3.1. is isomorphic with

$$\zeta = \xi^{a_0} \oplus \dots \oplus \xi^{a_m} \oplus \xi^{b_0} \oplus \dots \oplus \xi^{b_n},$$

where ξ^a denotes the a -fold tensor product of ξ . Thus, by Lemma 3.1, the tangent bundle $\tau(M)$ of the orbit manifold is stably equivalent to the real restriction of the complex vector bundle ζ . Now the conclusion of Theorem 3 follows from properties of Pontrjagin classes ([5], Chapter XII). q.e.d.

Corollary. If two principal circle actions $\varphi_{a,b}$ and $\varphi_{c,d}$ are equivalent, where $a = (a_0, \dots, a_m)$, $b = (b_0, \dots, b_n)$, $c = (c_0, \dots, c_m)$ and $d = (d_0, \dots, d_n)$, then

$$(1) \quad \sigma_k(a_0^2, \dots, a_m^2, b_0^2, \dots, b_n^2) = \sigma_k(c_0^2, \dots, c_m^2, d_0^2, \dots, d_n^2)$$

for $2k \leq m \leq n$,

$$(2) \quad \left| \prod_{i=0}^m a_i \right| = \left| \prod_{i=0}^m c_i \right| \quad \text{for } m < n,$$

and

$$(3) \quad \sigma_k(a_0^2, \dots, a_m^2, b_0^2, \dots, b_n^2) = \sigma_k(c_0^2, \dots, c_m^2, d_0^2, \dots, d_n^2)$$

mod $|\prod_{i=0}^m a_i|$ for $m < 2k \leq n$. Here σ_k is the k -th elementary symmetric function on $(m+n+2)$ -variables.

4. Gysin homomorphism

Let ξ be an oriented n -plane bundle over a topological space X with a Thom class $U \in H^n(D(\xi), S(\xi))$, where $p : D(\xi) \rightarrow X$ and $\pi : S(\xi) \rightarrow X$ are the associated disk bundle and the associated sphere bundle respectively. Then there is a commutative diagram:

$$\begin{array}{ccccccc}
 H^*(S(\xi)) & \xrightarrow{\delta} & H^*(D(\xi), S(\xi)) & \longrightarrow & H^*(D(\xi)) & \xrightarrow{i^*} & H^*(S(\xi)) \\
 \parallel & & \cong \uparrow \phi_\xi & & \cong \uparrow p^* & & \parallel \\
 H^*(S(\xi)) & \xrightarrow{\pi_*} & H^*(X) & \xrightarrow{\cdot e(\xi)} & H^*(X) & \xrightarrow{\pi_*} & H^*(S(\xi))
 \end{array}$$

where the homomorphism ϕ_ξ is a Thom isomorphism defined by $\phi_\xi(x) = p^*(x)U$, $e(\xi)$ is a Euler class of ξ and the homomorphism π_* is a Gysin homomorphism. The lower horizontal line is a Thom-Gysin sequence for the oriented n -plane bundle ξ ([5] p. 60).

Lemma 4.1.

(1) $\pi_*(\pi^*x \cup y) = (-1)^{\deg x} x \cup \pi_*y$ for $x \in H^*(X)$ and $y \in H^*(S(\xi))$, ([4] p. 71; [7] p. 121)

(2) $\pi_*(Sq^i u) = \sum_{j+k=i} Sq^j \pi_* u \cup W_k(\xi)$ for $u \in H^*(S(\xi); Z_2)$ where $W_k(\xi)$ is a k -th Stiefel-Whitney class of ξ , ([5] p. 35; [7] p. 137)

(3) $\pi_*(P^i v) = \sum_{j+k=i} P^j \pi_* v \cup Q_k(\xi)$ for $v \in H^*(S(\xi); Z_p)$ where p is an odd prime, P^i is a reduced power operation and $Q_k(\xi) \in H^{2k(p-1)}(X; Z_p)$ is a k -th Wu class defined by $Q_k(\xi) = \phi_\xi^{-1} P^k U$, ([5] p. 120).

Proof.

$$\begin{aligned}
 \phi_\xi \pi_*(\pi^*x \cup y) &= \delta(\pi^*x \cup y) \\
 &= \delta(i^* p^* x \cup y) \\
 &= (-1)^{\deg x} p^* x \cup \delta y \\
 &= (-1)^{\deg x} p^* x \cup (p^* \pi_* y \cup U) \\
 &= (-1)^{\deg x} \phi_\xi(x \cup \pi_* y).
 \end{aligned}$$

This implies (1), since ϕ_ξ is an isomorphism. Next

$$\begin{aligned}
 \phi_\xi \pi_*(Sq^i u) &= \delta(Sq^i u) \\
 &= Sq^i(\delta u) \\
 &= Sq^i(p^* \pi_* u \cup U) \\
 &= \sum_{j+k=i} Sq^j p^* \pi_* u \cup Sq^k U \quad (\text{Cartan formula})
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j+k=i} p^* Sq^j \pi_* u \cup (p^* W_k(\xi) \cup U) \quad ([5]p. 35; [7]p. 137) \\
 &= \phi_{\xi} \left(\sum_{j+k=i} Sq^j \pi_* u \cup W_k(\xi) \right).
 \end{aligned}$$

This implies (2). The relation (3) is proved similarly by the Cartan formula of reduced power operations ([6] p. 76) and the definition of Wu classes ([5] p. 120). q.e.d.

5. Miscellaneous principal circle actions

In this section we give some examples of principal circle actions on a closed orientable smooth manifold E which is cohomologically a product $S^{2m} \times S^{2n+1}$.

5.1. Given a sequence $a = (a_1, \dots, a_m)$ of integers, let ψ_a be a principal smooth circle action on $S^{2m} \times S^{2n+1}$ given by

$$\psi_a(z, ((u_0, \dots, u_m), (v_0, \dots, v_n))) = ((u_0, z^{a_1} u_1, \dots, z^{a_m} u_m), (z v_0, \dots, z v_n))$$

in complex coordinates, where u_0 is a real number. Denote by M_a the orbit manifold. Then there is a mapping $p : M_a \rightarrow P^n(\mathbb{C})$ given by the following commutative diagram as in section 2:

$$\begin{array}{ccc}
 S^{2m} \times S^{2n+1} & \xrightarrow{p_2} & S^{2n+1} \\
 \downarrow \pi & & \downarrow \pi_0 \\
 M_a & \xrightarrow{p} & P^n(\mathbb{C})
 \end{array}$$

where p_2 is a projection to the second factor, π and π_0 are natural projections.

The projection $p : M_a \rightarrow P^n(\mathbb{C})$ is a sphere bundle associated with a real $(2m + 1)$ -plane bundle

$$\zeta = \theta_{\mathbb{R}}^1 \oplus \xi_n^{a_1} \oplus \dots \oplus \xi_n^{a_m}$$

where ξ_n is the canonical complex line bundle over $P^n(\mathbb{C})$ and $\theta_{\mathbb{R}}^1$ is a trivial real line bundle (see Lemma 2.1), and there is a cross-section $s : P^n(\mathbb{C}) \rightarrow M_a$ defined by $s([v_0, \dots, v_n]) = \pi((1, 0, \dots, 0), (v_0, \dots, v_n))$, so the Euler class $e(\zeta) = 0$. Then, by the Thom-Gysin sequence for ζ , there is a short exact sequence:

$$(5.1.1) \quad 0 \longrightarrow H^k(P^n(\mathbb{C})) \xrightarrow{p^*} H^k(M_a) \xrightarrow{p^*} H^{k-2m}(P^n(\mathbb{C})) \longrightarrow 0.$$

Proposition 5.1.

(1) *The integral cohomology ring of M_a is*

$$H^*(M_a) = \mathbb{Z}[c, x] / (c^{n+1}, x^2 - (\prod_i a_i) \cdot xc^m)$$

where $c = p^*e(\xi_n)$, $\deg x = 2m$, $p_*x = 1$ and $s^*x = 0$.

(2) The total Pontrjagin class of M_a is

$$P(M_a) = (1 + c^2)^{n+1} \cdot \prod_{i=1}^m (1 + a_i^2 c^2).$$

Proof. The module structure of $H^*(M_a)$ and the relation $c^{n+1} = 0$ are obtained directly by the exact sequence (5.1.1). And the total Pontrjagin class of M_a is calculated similarly as Theorem 3. Finally the relation $x^2 = (\prod_i a_i) \cdot xc^m$ is obtained from Lemma 4.1 (2), (3), a property of the reduced power operations ([6] p. 1, p. 76) and a property of Wu classes ([5] p. 120), so we leave it to the reader. q.e.d.

Corollary. *If the corresponding actions ϕ_a and ϕ_b are equivalent for sequences $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$ of integers. Then*

$$\sigma_p(a_1^2, \dots, a_m^2) = \sigma_p(b_1^2, \dots, b_m^2)$$

for any positive integer p with $2p \leq n$, where σ_p is the p -th elementary symmetric function on m -variables.

5.2. Let ξ_1 be the canonical complex line bundle over $P^1(C)$. Given a sequence $a = (a_0, \dots, a_n)$ of integers, denote by

$$S(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})$$

the total space of a sphere bundle associated with the complex $(n+1)$ -plane bundle $\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n}$ over $P^1(C)$. Then there is a natural principal circle action φ on $S(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})$ whose orbit space is $CP(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})$, the total space of a projective space bundle.

Proposition 5.2.

(1) $H^*(CP(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})) \cong Z[c, x]/(x^2, c^{n+1} + (a_0 + \dots + a_n)xc^n)$, where $\deg c = \deg x = 2$, and c is the Euler class of the canonical line bundle over $CP(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})$,

(2) $H^*(S(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})) \cong H^*(S^2 \times S^{2n+1})$ if $n > 0$,

(3) If $a_0 + \dots + a_n = 1 \pmod{2}$, then the principal circle action φ on $S(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})$ does not bord even as unoriented principal smooth circle action.

Proof. (1), (2) are clear from the cohomology ring structure of the projective space bundle ([2] p. 8, Proposition 3.1, 3.2). Next, assume $a_0 + \dots + a_n = 1 \pmod{2}$, then

$$\langle \bar{c}^{n+1}, [CP(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})]_2 \rangle \neq 0$$

where \bar{c} is a modulo 2 reduction of the Euler class c . Thus the action φ does

not bord as unoriented principal smooth circle action ([3] p. 47, Theorem 17.2).
q.e.d.

REMARK. If $a_0 + \dots + a_n = 1 \pmod{2}$, $S(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n})$ is not the same homotopy type as a product $S^{2m} \times S^{2n+1}$, since $Sq^2 u \neq 0$ for non-zero element $u \in H^{2n+1}(S(\xi_1^{a_0} \oplus \dots \oplus \xi_1^{a_n}); \mathbb{Z}_2)$ by Lemma 4.1 (2).

5.3. There is a complex $(n+1)$ -plane bundle ξ over S^{2m} with $\langle c_m(\xi), [S^{2m}] \rangle = (m-1)!$ for any $n+1 \geq m$ ([1]p. 349, Theorem 26.5 (a)).

Proposition 5.3. *Assume $n \geq m > 0$, then*

$$(1) \quad H^*(CP(\xi)) \cong \mathbb{Z}[c, x]/(x^2, c^{n+1} + (m-1)! xc^{n-m+1}),$$

where $\deg c = 2$, $\deg x = 2m$ and c is the Euler class of the canonical line bundle over $CP(\xi)$,

$$(2) \quad H^*(S(\xi)) \cong H^*(S^{2m} \times S^{2n+1}),$$

(3) *the natural principal circle action on $S(\xi)$ does not bord as an orientable principal smooth circle action.*

Proof. (1), (2) are clear. And

$$c^{n+m} = -(m-1)! xc^n \neq 0.$$

Hence (3) is obtained (see the proof of Theorem 1). q.e.d.

5.4. Let E be a topological space which is cohomologically a product $S^{2m} \times S^{2n+1}$ with $m > n \geq 0$. Let $\pi : E \rightarrow M$ be a principal S^1 -bundle over an orientable closed smooth manifold M . Then,

Proposition 5.4.

(1) *The integral cohomology ring $H^*(M)$ of M is*

$$H^*(M) = \mathbb{Z}[c, x]/(c^{n+1}, x^2), \text{ where } \deg c = 2, \deg x = 2m$$

and the element c is the Euler class of the principal S^1 -bundle $\pi : E \rightarrow M$

(2) *The Stiefel-Whitney classes of M are*

$$W_{2i+1}(M) = 0, W_{2i}(M) = a_i \bar{c}^i (a_i = 0, 1)$$

where \bar{c} is a modulo 2 reduction of the Euler class c .

Proof. This is proved similarly as Lemma 1.1, but it makes Lemma 4.1 (1) necessary to determine the ring structure of $H^*(M)$. We leave it to the reader.
q.e.d.

Proposition 5.5. *Let E be an orientable closed smooth manifold which is cohomologically a product $S^{2m} \times S^{2n+1}$ with $m > n \geq 0$. Then any principal smooth circle action on E bords as unoriented principal smooth circle action.*

Proof. By Proposition 5.4, all bordism Stiefel-Whitney numbers of an associated principal S^1 -bundle vanish (see Theorem 1). Thus the result is obtained ([3] p. 47, Theorem 17.2). q.e.d.

REMARK. There is no principal smooth circle action on a compact smooth manifold whose each odd dimensional integral cohomology group is zero.

OSAKA UNIVERSITY

References

- [1] A. Borel and F. Hirzebruch: *Characteristic classes and homogeneous spaces* I, II, Amer. J. Math. **80** (1958), 458–538; **81** (1959), 315–382.
- [2] R. Bott: *Lectures on $K(X)$* , Benjamin Inc., 1969.
- [3] P. Conner and E. Floyd: *Differentiable Periodic Maps*, Springer-Verlag, 1964.
- [4] F. Hirzebruch: *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.
- [5] J. Milnor: *Lectures on Characteristic Classes*, Princeton University, (mimeographed), 1957.
- [6] N. Steenrod and D. Epstein: *Cohomology Operations*, Princeton University Press, 1962.
- [7] R. Thom: *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Sci. École Norm. Sup. **69** (1952), 109–181.