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EXTENDIBLE VECTOR BUNDLES OVER LENS SPACES MOD 3

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1. Introduction. In [5] Schwarzenberger investigated the problem of determining whether a real vector bundle over the real projective space RP^n can be extended to a real vector bundle over RP^m ($n < m$). In [3], he also investigated the case of the complex tangent bundle of the complex projective space.

The purpose of this note is to prove the non-extendibility of a bundle over lens spaces mod 3 by making use of Schwarzenberger's technique ([5]).

Let S^{2n+1} be the unit $(2n+1)$ -sphere. That is

$$S^{2n+1} = \{(z_0, \dots, z_n); \sum_{i=0}^n |z_i|^2 = 1, z_i \in C \text{ for all } i\}$$

Let γ be the rotation of S^{2n+1} defined by

$$\gamma(z_0, \dots, z_n) = (e^{2\pi i/p} z_0, \dots, e^{2\pi i/p} z_n).$$

Then γ generates the differentiable transformation group Γ of S^{2n+1} of order p , and lens space mod p is defined to be the orbit space $L^n(p) = S^{2n+1}/\Gamma$. It is a compact differentiable $(2n+1)$ -manifold without boundary and $L^n(2) = RP^{2n+1}$. The Grothendieck rings $\widetilde{KO}(L^n(p))$, $\widetilde{K}(L^n(p))$ were determined by T. Kambe [4]. We recall them in 2. Let $\{z_0, \dots, z_n\} \in L^n(p)$ denote the equivalence class of $(z_0, \dots, z_n) \in S^{2n+1}$. $L^n(p)$ is naturally embedded in $L^{n+1}(p)$ by identifying $\{z_0, \dots, z_n\}$ with $\{z_0, \dots, z_n, 0\}$. Hence $L^n(p)$ is embedded in $L^m(p)$ for $n < m$. Throughout this note we suppose $p=3$. Now we state our theorems which shall be proved in 3 and 4.

Let ζ be any t -dimensional real bundle over $L^n(3)$. Let $p(\zeta)$ be the mod 3 Pontryagin class of ζ

$$p(\zeta) = \sum_j p_j(\zeta) \text{ where } p_j(\zeta) = (-1)^j C_{2j}(\zeta \otimes C) \pmod{3}.$$

From the property of the cohomology algebra $H^*(L^n(3); Z_3)$, we have

$$p_j(\zeta) = d_j x^{2j},$$

where $d_j \in \mathbb{Z}_3$ and x is a generator of $H^2(L^n(3); \mathbb{Z})$. Then there exists an integer s such that

$$(1) \quad p(\zeta) = 1 + d_1 x^2 + \dots + d_s x^{2s} \quad \text{for } 0 \leq 2s \leq t.$$

Then we have the following

Theorem 1. *Let ζ be a t -dimensional real vector bundle over $L^n(3)$. If $2t < n + 1$, then we have*

$$p(\zeta) = (1 + x^2)^s \pmod 3 \text{ for some integer } s \quad 0 \leq 2s \leq t.$$

Corollary 2. *Under the assumptions of Theorem 1,*

$$p(\zeta) = p(\eta_{L^n} \oplus \dots \oplus (s) \dots \oplus \eta_{L^n}) \text{ for some } 0 \leq 2s \leq t,$$

where we denote by \oplus a Whitney sum of η_{L^n} . (See 2 for the definition of η_{L^n} .)

For a pair (X, Y) of compact spaces, a bundle ζ_Y over Y is said to be extendible to X provided there exists a bundle ζ_X over X such that

$$\zeta_X|_Y \cong \zeta_Y,$$

where we denote by $|_Y$ the restriction to Y .

Let a be a real number. We denote by $[a]$ the integral part of a . Let b be an integer. We denote by $\nu_3(b)$ an integer q such that

$$b = r \cdot 3^q, \quad \text{where } (r, 3) = 1.$$

For integers t and m , define

$$\beta_3(t, m) = \text{Min} \left[\left(i - \left[\frac{i}{2} \right] - 1 \right) - \nu_3 \left(i - \left[\frac{i}{2} \right] \right) + \nu_3 \left\{ \binom{i - [i/2]}{[i/2]} \right\} \right]$$

where $t < i < m$, $i \equiv 0 \pmod 2$ and $i \equiv 1 \pmod 6$.

Theorem 3. *Assume that n, m and t are the positive integers such that*

$$(2) \quad 2t < m + 1$$

$$(3) \quad n \not\equiv 0 \pmod 4$$

$$(4) \quad m \not\equiv 0 \pmod 4$$

$$(5) \quad \left[\frac{m}{2} \right] \geq \left[\frac{n}{2} \right] + \beta_3(t, m).$$

Let ζ be a t -dimensional real vector bundle over $L^n(3)$ which is extendible to $L^m(3)$ ($n < m$). Then ζ is stably equivalent to

$$\eta_{L^n} \oplus \cdots (s) \cdots \oplus \eta_{L^n} \text{ for some integer } s (0 \leq 2s \leq t).$$

As an application of Th. 3, we obtain the following

Theorem 4. *Let ζ be a t -dimensional real vector bundle over $L^n(3)$ ($n \not\equiv 0 \pmod 4$). Assume that ζ is stably equivalent to*

$$\eta_{L^n} \oplus \cdots (s) \cdots \oplus \eta_{L^n} \text{ for some } s > \left\lfloor \frac{t}{2} \right\rfloor.$$

Then ζ is not extendible to $L^{\phi(t,n)}(3)$, where

$$\phi(t, n) = \text{Min} \left\{ m \geq 2t; m \not\equiv 0 \pmod 4, \left\lfloor \frac{m}{2} \right\rfloor - \beta_3(t, m) \geq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Next we show

Theorem 5. *The tangent bundle $\tau(L^n(3))$ of $L^n(3)$ is not extendible to $L^{\phi(2n+1,n)}(3)$ for $n \not\equiv 0 \pmod 4$. And $\tau(L^n(3))$ is not extendible to $L^{n+2}(3)$ ($n \not\equiv 0 \pmod 4$).*

2. The structure of $\widetilde{KO}(L^n(p))$. The structure of $\widetilde{KO}(L^n(p))$ is stated as follows [4]. Let CP^n be the complex projective space of complex n -dimension. Let η be the canonical complex line bundle over CP^n , $r(\eta)$ the real restriction of η . Consider the natural projection

$$\pi: L^n(p) \rightarrow CP^n.$$

Define $\eta_{L^n} = \pi^*(r(\eta)) \in KO(L^n(p))$ where $\pi^*: KO(CP^n) \rightarrow KO(L^n(p))$ is the induced homomorphism of π . Let $\bar{\sigma}_n$ denote the stable class of η_{L^n} , i.e., $\bar{\sigma}_n = \eta_{L^n} - 2 \in \widetilde{KO}(L^n(p))$. We recall $\tau_{L^n} \oplus 1 = (n+1)\eta_{L^n}$ where τ_{L^n} is the tangent bundle of $L^n(p)$. The theorem of T. Kambe (Th. 2, [4]) is as follows:

Theorem (Kambe). *Let p be an odd prime, $q = (p-1)/2$ and $n = s(p-1) + r$ ($0 \leq r < p-1$). Then*

$$\widetilde{KO}(L^n(p)) \cong \begin{cases} (\mathbb{Z}_{p^{s+1}})^{\lfloor r/2 \rfloor} + (\mathbb{Z}_{p^s})^{q - \lfloor r/2 \rfloor} \dots & (\text{if } n \not\equiv 0 \pmod 4) \\ \mathbb{Z}_2 + (\mathbb{Z}_{p^{s+1}})^{\lfloor r/2 \rfloor} + (\mathbb{Z}_{p^s})^{q - \lfloor r/2 \rfloor} \dots & (\text{if } n \equiv 0 \pmod 4) \end{cases}$$

and the direct summand $(\mathbb{Z}_{p^{s+1}})^{\lfloor r/2 \rfloor}$ and $(\mathbb{Z}_{p^s})^{q - \lfloor r/2 \rfloor}$ are additively generated by $\bar{\sigma}_n, \dots, \bar{\sigma}_n^{\lfloor r/2 \rfloor}$ and $\bar{\sigma}_n^{\lfloor r/2 \rfloor + 1}, \dots, \bar{\sigma}_n^q$ respectively. Moreover its ring structure is given by

$$\bar{\sigma}_n^{q+1} = \sum_{i=1}^q \frac{-(2q+1)}{(2i-1)} \binom{q+i-1}{2i-2} \bar{\sigma}_n^i, \bar{\sigma}_n^{\lfloor n/2 \rfloor + 1} = 0.$$

In the theorem, $(\mathbb{Z}_a)^b$ indicates the direct sum of b -copies of cyclic group of order a . Let $p=3$ in the above theorem. If $n \not\equiv 0 \pmod 4$ then

$$\widetilde{KO}(L^n(3)) \cong Z_3^s, \quad s = \left[\frac{n}{2} \right].$$

and Z_3^s is generated by $\bar{\sigma}_n$. Its ring structure is given by

$$\bar{\sigma}_n^2 = (-3)\bar{\sigma}_n, \quad \bar{\sigma}^{s+1} = 0.$$

3. The proofs of Theorem 1 and Corollary 2. From Th. 11.3 in [2], we obtain the following equality. (For the proof, see Proposition 5, in the last part of this section.) Let $\mathcal{P}_3^k: H^q(L^n(3): Z_3) \rightarrow H^{q+4k}(L^n(3): Z_3)$ the k -th reduced power operation mod 3. Then we have

$$(6) \quad \mathcal{P}_3^k(p_s(\zeta)) = \left(\sum_{n+m=k} p_n(\zeta)p_m(\zeta) \right) p_s(\zeta) + \sum_{j>s} p_j(\zeta) (\dots)$$

for $0 \leq k \leq s$.

Let s be an integer such as (1) in 1. Since $d_j \equiv 0$ for all $j > s$ and $d_s \equiv 0$, then (6) gives

$$(7) \quad \mathcal{P}_3^k(p_s(\zeta)) = \left(\sum_{n+m=k} p_n(\zeta)p_m(\zeta) \right) p_s(\zeta).$$

For an element x^{2s} of $H^s(L^n(3): Z_3)$, we have

$$\mathcal{P}_3^k(x^{2s}) = \binom{2s}{k} x^{2s+2k} \quad \text{and} \quad d_s \binom{2s}{k} x^{2s+2k} = \left(\sum_{n+m=k} d_n d_m \right) d_s x^{2s+2k}.$$

From $2s+2k \leq 4s \leq 2t < n+1$, $x^{2s+2k} \neq 0$. Hence $\binom{2s}{k} = \sum_{n+m=k} d_n d_m$.

By induction, we obtain $d_j \equiv \binom{s}{j} \pmod{3}$. Therefore

$$\begin{aligned} p(\zeta) &= 1 + \binom{s}{1} x^2 + \dots + \binom{s}{s} x^{2s} \pmod{3} \\ &= (1+x^2)^s \pmod{3}. \end{aligned}$$

The proof of Theorem 1 is completed if we prove Proposition 5. Now, it is well known that the bundle η_{L^n} over $L^n(3)$ has the total Pontryagin class mod 3 $p(\eta_{L^n}) = 1+x^2$. Thus the proof of Corollary 2 is completed.

Now, in order to prove the formula (6) in the proof of Theorem 1, we consider a following symmetric polynomial. Let $\sum x_1^p x_2^p \dots x_k^p x_{k+1} \dots x_s$ be a homogeneous symmetric polynomial in variables x_1, x_2, \dots, x_t of degree $N = (p-1)k + s$ where p, k and N are positive integers.

To prove (6), we show the following propositions.

Proposition 1.

$$\sum x_1^2 x_2^2 \cdots x_k^2 x_{k+1} \cdots x_s = \sum_{i=0}^k A(i) \sigma_{k-i} \sigma_{s+i}$$

where $A(i) = (-1)^i \sum_{j=1}^i \binom{s-k+2i}{j} A(i-j)$, $1 \leq i \leq k$ and $A(0) = 1$.

Proof. Put $f(k, s) = \sum x_1^2 \cdots x_k^2 x_{k+1} \cdots x_s$. By an easy calculation,

$$(8) \quad f(k, s) = (\sum x_1 \cdots x_k)(\sum x_1 \cdots x_s) - \sum_{j_1=1}^k \binom{s-k+2j_1}{j_1} f(k-j_1, s+j_1).$$

By making use of (8) repeatedly, we have

$$f(k, s) = \sigma_k \sigma_s + \sum_{l=1}^k F_l$$

where $F_l = (-1)^l \sum_{j_1=1}^k \sum_{j_2=1}^{k-j_1} \cdots \sum_{j_{l-1}=1}^{k-\sum_{i=1}^{l-1} j_i} \binom{s-k+2j_1}{j_1} \cdots \binom{s-k+2\sum_{i=1}^l j_i}{j_l} \sigma_{k-\sum_{i=1}^l j_i} \sigma_{s+\sum_{i=1}^l j_i}$ and

$1 \leq j_l \leq k - \sum_{i=1}^{l-1} j_i \leq k - (l-1)$. If $l=k$, then $k = \sum_{i=1}^k j_i$.

Let $A_l(i)$ be the coefficient of $\sigma_{k-i} \sigma_{s+i}$ in F_l , then

$$A_l(i) = (-1)^l \sum_{n=1}^{i-1} \sum_{j_1+\dots+j_{l-1}=i-n} \binom{s-k+2j_1}{j_1} \cdots \binom{s-k+2\sum_{i=1}^{l-1} j_i}{j_{l-1}} \binom{s-k+2i}{n}.$$

Put $A(i) = A_1(i) + \cdots + A_i(i)$. Then

$$A(i) = (-1)^i \sum_{n=1}^{i-1} \binom{s-k+2i}{n} \sum_{l=2}^{i-n+1} A_{l-1}(i-n) - \binom{s-k+2i}{i}.$$

Since $\sum_{j=2}^{i+1} A_{l-1}(j)$ is a coefficient $A(j)$ of $\sigma_{k-j} \sigma_{s+j}$ in $f(k, s)$, we have

$$A(i) = (-1)^i \sum_{j=1}^i \binom{s-k+2i}{j} A(i-j).$$

This completes the proof of Proposition 1.

Proposition 2.

$$\sum x_1^2 \cdots x_k^2 = \sum_{i=0}^k A(i) \sigma_{k-i} \sigma_{k+i}$$

where (a) $\sum_{j=0}^i A(i-j) \binom{2i}{j} = 0$ and $A(0) = 1$

(b) $A(i) \equiv (-1)^{i+1} \pmod{3}$ for $i \neq 0$.

Proof. The part of (a) is completed by Proposition 1. The proof of (b) is obtained by induction. For $i=1, 2, A(1)=(-1)\binom{2}{1}\equiv 1$ and $A(2)=2$. Assuming the equation (b) for integers $i \leq 2q$, we have

$$A(i+1) = 2 \sum_{j=1}^i \binom{2i+2}{j} (-1)^{i-j+2} + 2 \binom{2i+2}{i+1}$$

By making use of $\sum_{j=1}^i \binom{2i+2}{j} (-1)^{j+1} \equiv \binom{2j+2}{i+1} + 2 \binom{2i+2}{2i+2} + 2 \binom{2i+2}{0} + 2 \binom{2i+2}{i+1}$, we have

$$A(i+1) \equiv 1 \pmod{3}.$$

Assuming the equation (b) for integers $i \leq 2q+1$, we can obtain $A(i+1) \equiv 2 \pmod{3}$. Thus Proposition 2 is obtained by induction.

Proposition 3.

$$\sum x_1^3 x_1^3 \cdots x_k^3 x_{k+1} \cdots x_s = (\sum x_1^2 \cdots x_k^2) \sigma_s - \sum_{l>s} \sigma_l(\cdots).$$

Proof. Put $f(a, b) = \sum x_1^3 \cdots x_a^3 x_{a+1}^2 \cdots x_{a+b}^2 x_{a+b+1} \cdots x_c$ with $c=N-2a-b$. By calculation, we have the following equality;

$$(9) \quad f(k, 0) = (\sum x_1 \cdots x_k) \sigma_s - \sum_{\alpha_1=1}^k f(k-\alpha_1, \alpha_1).$$

Define $a_0=k, b_0=0$ and $c_0=N-2a_0-b_0$. Then (9) is reformed as follows:

$$(10) \quad f(a_0, b_0) = f(0, a_0) \sigma_{c_0} - \sum_{\alpha_1=1}^{a_0} f(a_1, b_1) \text{ where } a_1=a_0-\alpha_1, b_1=\alpha_1+\beta_1(\beta_1=0)$$

Now for each term $f(a_1, b_1)$ in (10), we obtain

$$(11) \quad f(a_1, b_1) = f(0, a_1) \sigma_{N-2a_1-b_1} - \sum_{\alpha_2=0}^{a_1} \sum_{\beta_2=0}^{b_1} A(\alpha_2, \beta_2) f(a_1-\alpha_2, \alpha_2+\beta_2)$$

for some integers $A(\alpha_2, \beta_2)$ and $A(0, b_1)=0$. We can inductively define two sequences $\{a_i\}, \{b_i\}$ satisfying the followings

$$(12) \quad f(a_{i-1}, b_{i-1}) = f(0, a_{i-1}) \sigma_{N-2a_{i-1}-b_{i-1}} - \sum_{\alpha_i=0}^{a_{i-1}} \sum_{\beta_i=0}^{b_{i-1}} A(\alpha_i, \beta_i) f(a_i, b_i)$$

$$(13) \quad a_i = a_{i-1} - \alpha_i, b_i = \alpha_i + \beta_i$$

with some integers $A(\alpha_i, \beta_i)$ and $A(0, b_{i-1}) = 0$.

Put $c_{i-1}=N-2a_{i-1}-b_{i-1}$. Then we have $s < c_1 < c_2 < \cdots < c_i < \cdots$. From (13), $a_{i+1} \leq a_i$ for all i . Hence consider the following cases:

(14) there exists an integer n such as $a_{i+1} < a_i$ for all $i \geq n$,

(15) there exists an integer m such as $a_m = \cdots = a_i = \cdots$ for all $i \geq m$.

If (14) is satisfied, then $a_q=0$ for some integer q . From (12) and Proposition 1, we have

$$(16) \quad \begin{aligned} f(a_{q-1}, b_{q-1}) &= f(0, a_{q-1})\sigma_{c_{q-1}} \sum_{\beta_q=1}^{b_{q-1}} A(0, \beta_q) f(0, b_q) \\ &= f(0, a_{q-1})\sigma_{c_{q-1}} - \sum_{\beta_q=0}^{b_{q-1}} \sum_{i=0}^{b_q} A(0, \beta_q) A(i) \sigma_{b_q-i} \sigma_{c_{q+i}} \end{aligned}$$

If (15) is satisfied, then $b_i > b_{i+1}$ for all $i \geq m$.

Therefore $b_r=0$ for some integer r . From (12) we have

$$f(a_{r-1}, b_{r-1}) = f(0, a_{r-1})\sigma_{a_{r-2}} - \sum_{\alpha_r=0}^{a_{r-1}} A(\alpha_r, 0) f(a_r, 0).$$

Since $a_r < a_0$, the above discussion is also applied to $f(a_r, 0)$ in this case. Hence, by making use of (9) repeatedly, we have finally

$$f(k, 0) = f(0, k)\sigma_s - \sum_{l>s} (\cdots)\sigma_l.$$

From Proposition 2 and Proposition 3 we have the following

Proposition 4.

$$\sum x_1^3 \cdots x_k^3 x_{k+1} \cdots x_s = (\sigma_k^2 + \sum_{j=1}^k (-1)^{j+1} \sigma_{k-j} \sigma_{k+j}) \sigma_s + \sum_{l>s} (\cdots) \sigma_l.$$

Now, we can prove the formula (6) in the proof of Theorem 1.

Proposition 5. $\mathcal{O}_3^k(p_s(\zeta)) = (\sum_{n+m=k} p_n(\zeta) p_m(\zeta)) p_s(\zeta) + \sum_{l>s} p_l(\zeta) (\cdots).$

Proof. Let $C_i \in H^{2i}(B_{U(d)} : Z_3)$ be the i -th Chern class mod 3. By Th. 11.3 ([2]) and Proposition. 4, we have

$$(17) \quad \mathcal{O}_3^k(C_i) = (C_k^2 + \sum_{j=1}^k (-1)^{j+1} C_{k-j} C_{k+j}) C_i + \sum_{l>i} (\cdots) C_l.$$

Let $p_s(\zeta)$ be the s -th Pontrjagin class mod 3 of a real bundle ζ . Then $p_s(\zeta) = (-1)^s C_{2s}(\zeta \otimes C)$ mod 3 where $C_{2s}(\zeta \otimes C)$ is a $2s$ -th Chern class of $\zeta \otimes C$. From (17) we obtain

$$\mathcal{O}_3^{2i}(p_s(\zeta)) = (p_i^2(\zeta) + 2 \sum_{l=1}^i p_{i+l}(\zeta) p_{i-l}(\zeta)) p_s(\zeta) + \sum_{l>s} (\cdots) p_l(\zeta)$$

and $\mathcal{O}_3^{2i+1}(p_s(\zeta)) = 2(\sum_{l=1}^i p_{i-l}(\zeta) p_{i+l}(\zeta)) p_s(\zeta) + \sum_{l>s} (\cdots) p_l(\zeta)$. This completes the proof of Proposition 5.

4. Proofs of Theorem 3, 4 and 5. To prove Theorem 3, we discuss the following lemmas. The proofs of Lemma 1, 2 and 3 are omitted.

Lemma 1. Let A_0, A_1, \dots, A_n be integers with $v_3(A_j) > 0$ for all $j \neq n$ and $v_3(A_n) \geq 0$. If $v_3(A_n) < v_3(A_j)$ for all $j \neq n$, then

$$v_3\left(\sum_{j=0}^n A_j\right) = v_3(A_n).$$

Lemma 2. If r, s, a and u are positive integers with $s < a < 3^u$ and $(r, 3) = 1$ then following hold.

$$(18) \quad v_3\left\{\binom{rs^u+s}{a}\right\} = v_3\left\{\binom{3^u+s}{a}\right\} \leq v_3\left\{\binom{3^u}{a}\right\}.$$

$$(19) \quad v_3\left\{\binom{3^u}{a}\right\} = u - v_3(a).$$

Lemma 3. If u and n are positive integers, then

$$(20) \quad v_3((3^u)!) = \frac{3^u - 1}{2}$$

$$(21) \quad v_3((2n+1)!) \leq n$$

$$(22) \quad v_3((2n)!) < n.$$

Put $A_j = (-1)^j (-3)^{i-j-1} \binom{q}{i-j} \binom{i-j}{j}$ ($j = 0, 1, \dots, \left[\frac{i}{2}\right]$) for some positive integer $q, i > 2$ with $q > i - j$.

Lemma 4. Let A_j be above integers. Then

$$v_3\left(\sum_{j=0}^{\lfloor i/2 \rfloor} A_j\right) = v_3(A_{\lfloor i/2 \rfloor}) \quad \text{for } i \equiv 1 \pmod 6 \text{ and } i \equiv 0 \pmod 2.$$

Proof. If $i = 2n$, then for each $l = 1, 2, \dots, n - 1$

$$v_3(A_{n-l}) = (n+l-1) - v_3((2l)!) - v_3((n-l)!) + v_3(q) + \dots + v_3(q-n-l+1).$$

From Lemma 3 (22) $v_3(A_{n-l}) - v_3(A_n) > l - v_3((2l)!)$. Then we have

$$v_3(A_j) > v_3(A_n) \text{ and } v_3(A_j) > 0 \quad \text{for all } j \neq n.$$

Therefore by Lemma 1 we obtain $v_3\left(\sum_{j=0}^{\lfloor i/2 \rfloor} A_j\right) = v_3(A_{\lfloor i/2 \rfloor})$ for $i \equiv 0 \pmod 2$.

From Lemma 3 (21), we obtain

$$v_3(A_{n-l}) - v_3(A_n) > v_3(n!) - v_3((n-l)!) > 0$$

under the condition $i \equiv 1 \pmod 6, \left[\frac{i}{2}\right] = n = 3m$.

Now we prove the theorems.

Proof of Theorem 3. Let ζ' be the extension over $L^m(3)$ of ζ . By the structure of \widetilde{KO} -ring of the lens space ([4]), ζ' is stably equivalent to $q\eta_{L^m}$, for some $q \in \mathbb{Z}_3 \lfloor m/2 \rfloor$. Since $\zeta' - t = q\bar{\sigma}_m \in \widetilde{KO}(L^m(3))$, we have

$$(23) \quad \zeta - t = q(i^*\eta_{L^m} - 2) \in \widetilde{KO}(L^n(3))$$

where $i^*: \widetilde{KO}(L^m(3)) \rightarrow \widetilde{KO}(L^n(3))$ is the induced homomorphism of natural embedding $i: L^n(3) \rightarrow L^m(3)$. If $2q \leq t$, then ζ is stably equivalent to $\eta_{L^n} \oplus \dots \oplus (q) \dots \oplus \eta_{L^n}$ for some integer q ($0 \leq 2q \leq t$). If $2q > t$, $\gamma^i(q\bar{\sigma}_m) = 0$ for all $i > g. \dim(q\bar{\sigma}_m)$ ([1] Prop. 2.3). Since $t \geq g. \dim(q\bar{\sigma}_m)$, we have

$$(24) \quad \gamma^i(q\bar{\sigma}_m) = 0 \quad \text{for all } i > t.$$

According to the Theorem of Kambe ([4] Lemma 4.8),

$$\begin{aligned} \gamma_t(q\bar{\sigma}_m) &= (1 + \bar{\sigma}_m(t - t^2))^q \\ &= \sum_{\alpha=0}^{2q} \left(\sum_{j=0}^{\lfloor \alpha/2 \rfloor} A_j \right) \bar{\sigma}_m t^\alpha \end{aligned}$$

where $A_j = (-1)^j (-3)^{\alpha-j-1} \binom{q}{\alpha-j} \binom{\alpha-j}{j}$.

Then we have $\gamma^i(q\bar{\sigma}_m) = \sum_{j=0}^{\lfloor \alpha/2 \rfloor} A_j \bar{\sigma}_m$. From (23),

$$\left(\sum_{j=0}^{\lfloor \alpha/2 \rfloor} A_j \right) \bar{\sigma}_m = 0 \in \widetilde{KO}(L^m(3)) = \mathbb{Z}_3 \lfloor m/2 \rfloor \quad \text{for all } i > t. \quad \text{Therefore}$$

$$(24) \quad \nu_3 \left(\sum_{j=0}^{\lfloor \alpha/2 \rfloor} A_j \right) \geq \left\lfloor \frac{m}{2} \right\rfloor \quad \text{for all } i > t.$$

Now, according to Lemma 4, we have

$$\nu_3 \left(\sum_{j=0}^{\lfloor \alpha/2 \rfloor} A_j \right) = \nu_3(A_{\lfloor i/2 \rfloor}) \quad \text{for } i > t \quad (i \equiv 0 \pmod 2 \text{ and } i \equiv 1 \pmod 6)$$

And so we have

$$(25) \quad \left(i - \left\lfloor \frac{i}{2} \right\rfloor - 1 \right) + \nu_3 \left\{ \binom{q}{i - \lfloor i/2 \rfloor} \right\} + \nu_3 \left\{ \binom{i - \lfloor i/2 \rfloor}{\lfloor i/2 \rfloor} \right\} \geq \left\lfloor \frac{m}{2} \right\rfloor \quad \text{for } i > t, i \equiv 0 \pmod 2$$

and $i \equiv 1 \pmod 6$.

Now the total Pontrjagin class mod 3 of $q\eta_{L^m}$ is given by the equation $p(q\eta_{L^m}) = (1 + x^2)^q$. Since $m > 2t - 1$, Theorem 1 implies that there exists an integer s such that

$$p(\zeta') = (1 + x^2)^s, \quad 0 \leq 2s \leq t.$$

Hence we have

$$(1+x^2)^q \equiv (1+x^2)^s \pmod{3}, \text{ i.e.,}$$

$$1 + \binom{q-s}{1}x^2 + \dots + \binom{q-s}{[m/2]}x^{2[m/2]} \equiv 1 \pmod{3}.$$

This implies that there exists an integer u such that

$$(26) \quad q-s = 3^u r, (r, 3) = 1 \quad \text{and} \quad 3^u > [m/2].$$

Then we obtain the following

$$\begin{aligned} \nu_3 \left\{ \binom{q}{i-[i/2]} \right\} &= \nu_3 \left\{ \binom{r3^u+s}{i-[i/2]} \right\} \\ &\leq \nu_3 \left\{ \binom{3^u}{i-[i/2]} \right\} \quad \text{for } t < i < m \text{ (by Lemma 2)} \\ &= u - \nu_3(i-[i/2]). \end{aligned}$$

Hence from (25) $u + (i-[i/2]-1) - \nu_3(i-[i/2]) + \nu_3 \left\{ \binom{i-[i/2]}{[i/2]} \right\} \geq \left\lceil \frac{m}{2} \right\rceil$ for $t < i < m$ and $i \equiv 0 \pmod{2}, i \equiv 1 \pmod{6}$. By the assumption (5) of Theorem 3, we have

$$(27) \quad \begin{aligned} u &\geq [m/2] - \text{Min} \left[(i-[i/2]-1) - \nu_3(i-[i/2]) + \nu_3 \left\{ \binom{i-[i/2]}{[i/2]} \right\} \right] \\ &= [m/2] - \beta_3(t, m) \geq [n/2]. \end{aligned}$$

According to (23), (26) and (27), there exists an integer s such that

$$\begin{aligned} 0 &\leq 2s \leq t, \\ \zeta - t &= (r3^u + s)\bar{\sigma}_n \\ &= s\bar{\sigma}_n. \end{aligned}$$

This completes the proof of Theorem 3.

Proof of Theorem 4. By the contraposition of Theorem 3 and the main theorem of Kambe ([4] Th. 2), it is clear.

Proof of Theorem 5. Since $\tau(L^n(3)) \oplus 1 = (n+1)\eta_{L^n}$ and $n+1 > n = \left\lceil \frac{2n+1}{2} \right\rceil = [1/2 \dim \tau(L^n(3))]$, Theorem 4 implies that the tangent bundle τ is not extendible to $L^{\phi(2n+1, n)}(3)$. For every $m > 2n+1, \beta_3(2n+1, m) \leq n$ whenever $n \equiv 0 \pmod{3}, n \equiv 1 \pmod{3} \beta_3(2n+1, m) < n$ whenever $n \equiv 2 \pmod{3}$. Then $\phi(2n+1, n) = 2(2n+1)$.

This completes the proof of Theorem 5.

REMARK. The following table shows the value of $\phi(t, n)$ where $1 \leq t \leq 10$ and $1 \leq n \leq 16$.

$n \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	3	3	4	4	6	6	8	8	10	10	12	12	14	14	16
2	5	5	5	6	6	8	8	10	10	12	12	14	14	16	16	18
3	6	6	6	6	6	8	8	10	10	12	12	14	14	16	16	18
4	8	8	8	8	8	8	8	10	10	12	12	14	14	16	16	18
5	10	10	10	10	10	10	10	10	10	12	12	14	14	16	16	18
6	12	12	12	12	12	12	12	14	14	16	16	18	18	20	20	22
7	14	14	14	14	14	14	14	14	14	16	16	18	18	20	20	22
8	16	16	16	16	16	16	16	16	16	18	18	20	20	22	22	24
9	18	18	18	18	18	18	18	18	18	18	18	20	20	22	22	24
10	20	20	20	20	20	20	20	20	20	20	20	20	20	22	22	24

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