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Osaka University
1. Introduction. In [5] Schwarzenberger investigated the problem of determining whether a real vector bundle over the real projective space $RP^n$ can be extended to a real vector bundle over $RP^m (n < m)$. In [3], he also investigated the case of the complex tangent bundle of the complex projective space.

The purpose of this note is to prove the non-extendibility of a bundle over lens spaces mod 3 by making use of Schwarzenberger's technique ([5]).

Let $S^{2m+1}$ be the unit $(2n+1)$-sphere. That is

$$S^{2m+1} = \{ (z_0, \ldots, z_n); \sum_{i=0}^n |z_i|^2 = 1, z_i \in C \text{ for all } i \}$$

Let $\gamma$ be the rotation of $S^{2m+1}$ defined by

$$\gamma(z_0, \ldots, z_n) = (e^{2\pi i/p} z_0, \ldots, e^{2\pi i/p} z_n).$$

Then $\gamma$ generates the differentiable transformation group $\Gamma$ of $S^{2m+1}$ of order $p$, and lens space mod $p$ is defined to be the orbit space $L^n(p) = S^{2m+1}/\Gamma$ It is a compact differentiable $(2n+1)$-manifold without boundary and $L^n(2) = RP^{2m+1}$.

The Grothendieck rings $KO(L^n(p))$, $K(L^n(p))$ were determined by T. Kambe [4]. We recall them in 2. Let $\{z_0, \ldots, z_n\} \in L^n(p)$ denote the equivalence class of $(z_0, \ldots, z_n) \in S^{2m+1}$. $L^n(p)$ is naturally embedded in $L^{n+1}(p)$ by identifying $\{z_0, \ldots, z_n\}$ with $\{z_0, \ldots, z_n, 0\}$. Hence $L^n(p)$ is embedded in $L^m(p)$ for $n < m$. Throughout this note we suppose $p = 3$. Now we state our theorems which shall be proved in 3 and 4.

Let $\zeta$ be any $t$-dimensional real bundle over $L^n(3)$. Let $p(\zeta)$ be the mod 3 Pontryagin class of $\zeta$

$$p(\zeta) = \sum \rho_j(\zeta) \text{ where } \rho_j(\zeta) = (-1)^j C_{\zeta} \otimes C \mod 3.$$ 

From the property of the cohomology algebra $H^*(L^n(3); Z_3)$, we have

$$p_j(\zeta) = d_j x^j,$$
where \( d_j \in \mathbb{Z} \) and \( x \) is a generator of \( H^2(L^n(3); \mathbb{Z}) \). Then there exists an integer \( s \) such that

\[
\rho(\zeta) = 1 + d_1x^2 + \cdots + d_sx^{2s} \quad \text{for} \quad 0 \leq 2s \leq t.
\]

Then we have the following

**Theorem 1.** Let \( \zeta \) be a t-dimensional real vector bundle over \( L^n(3) \). If \( 2t < n+1 \), then we have

\[
\rho(\zeta) = (1 + x^s)^s \mod 3 \quad \text{for some integer} \quad s \quad 0 \leq 2s \leq t.
\]

**Corollary 2.** Under the assumptions of Theorem 1,

\[
\rho(\zeta) = \rho(\eta_{L^n} \oplus \cdots (s) \cdots \oplus \eta_{L^n}) \quad \text{for some} \quad 0 \leq 2s \leq t,
\]

where we denote by \( \oplus \) a Whitney sum of \( \eta_{L^n} \). (See 2 for the definition of \( \eta_{L^n} \).)

For a pair \( (X, Y) \) of compact spaces, a bundle \( \zeta_Y \) over \( Y \) is said to be extendible to \( X \) provided there exists a bundle \( \zeta_X \) over \( X \) such that

\[
\zeta_X |_Y \cong \zeta_Y,
\]

where we denote by \( |_Y \) the restriction to \( Y \).

Let \( a \) be a real number. We denote by \( [a] \) the integral part of \( a \). Let \( b \) be an integer. We denote by \( \nu_3(b) \) an integer \( q \) such that

\[
b = r \cdot 3^q, \quad \text{where} \quad (r, 3) = 1.
\]

For integers \( t \) and \( m \), define

\[
\beta_3(t, m) = \text{Min} \left[ \left( \left( i - \left[ \frac{i}{2} \right] - 1 \right) - \nu_3 \left( i - \left[ \frac{i}{2} \right] \right) \right) + \nu_3 \left( \left( i - \left[ \frac{i}{2} \right] \right) \right) \right]
\]

where \( t < i < m \), \( i \equiv 0 \mod 2 \) and \( i \equiv 1 \mod 6 \).

**Theorem 3.** Assume that \( n, m \) and \( t \) are the positive integers such that

1. \( 2t < m+1 \)
2. \( n \equiv 0 \mod 4 \)
3. \( m \equiv 0 \mod 4 \)
4. \( \left[ \frac{m}{2} \right] \geq \left[ \frac{n}{2} \right] + \beta_3(t, m) \).

Let \( \zeta \) be a t-dimensional real vector bundle over \( L^n(3) \) which is extendible to \( L^m(3) \) \((n < m)\). Then \( \zeta \) is stably equivalent to
EXTENDIBLE VECTOR BUNDLES

\( \eta_{L^n} \oplus \cdots \oplus \eta_{L^n} \) for some integer \( s \) \((0 \leq 2s \leq t)\).

As an application of Th. 3, we obtain the following

**Theorem 4.** Let \( \zeta \) be a \( t \)-dimensional real vector bundle over \( L^s(3) \) \((n \equiv 0 \mod 4)\). Assume that \( \zeta \) is stably equivalent to

\[ \eta_{L^n} \oplus \cdots \oplus \eta_{L^n} \] for some \( s \geq \left\lfloor \frac{t}{2} \right\rfloor \).

Then \( \zeta \) is not extendible to \( L^{t+\eta_n}(3) \), where

\[ \phi(t, n) = \min \{ m \geq 2t; m \equiv 0 \mod 4, \left\lfloor \frac{m}{2} \right\rfloor - \beta_s(t, m) \geq \left\lfloor \frac{n}{2} \right\rfloor \} . \]

Next we show

**Theorem 5.** The tangent bundle \( \tau(L^s(3)) \) of \( L^s(3) \) is not extendible to \( L^{t+\eta_n}(3) \) for \( n \equiv 0 \mod 4 \). And \( \tau(L^s(3)) \) is not extendible to \( L^{t+\eta_n}(3) \) \((n \equiv 0 \mod 4)\).

2. The structure of \( \widetilde{KO}(L^s(p)) \). The structure of \( \widetilde{KO}(L^s(p)) \) is stated as follows \([4]\). Let \( CP^n \) be the complex projective space of complex \( n \)-dimension. Let \( \eta \) be the canonical complex line bundle over \( CP^n \), \( r(\eta) \) the real restriction of \( \eta \). Consider the natural projection

\( \pi: L^s(p) \to CP^n \).

Define \( \eta_{L^n} = \pi^*(r(\eta)) \subseteq KO(L^s(p)) \) where \( \pi^*: KO(CP^n) \to KO(L^s(p)) \) is the induced homomorphism of \( \pi \). Let \( \sigma_n \) denote the stable class of \( \eta_{L^n} \), i.e., \( \sigma_n = \eta_{L^n} - 2 \in KO(L^s(p)) \). We recall \( \tau_{L^n} \oplus 1 = (n + 1) \eta_{L^n} \) where \( \tau_{L^n} \) is the tangent bundle of \( L^s(p) \). The theorem of T. Kambe (Th. 2, \([4]\)) is as follows:

**Theorem (Kambe).** Let \( p \) be an odd prime, \( q = (p - 1)/2 \) and \( n = s(p - 1) + r \) \((0 \leq r < p - 1)\). Then

\[ \widetilde{KO}(L^s(p)) \cong \begin{cases} (Z_{p^{i+1}})^{r_{/2}} + (Z_{p^4})^{q - r_{/2}} \cdots \text{if } n \equiv 0 \mod 4 \\ Z_s + (Z_{p^{i+1}})^{r_{/2}} + (Z_{p^4})^{q - r_{/2}} \cdots \text{if } n \equiv 0 \mod 4 \end{cases} \]

and the direct summand \((Z_{p^{i+1}})^{r_{/2}}\) and \((Z_{p^4})^{q - r_{/2}}\) are additively generated by \( \sigma_n, \ldots, \sigma_n^{r_{/2}} \) and \( \sigma_n^{r_{/2}+1}, \ldots, \sigma_n^q \) respectively. Moreover its ring structure is given by

\[ \sigma_n^{q+1} = \sum_{i=0}^{q} \frac{-(2q+1)}{(2i-1)} \binom{q+i-1}{2i-2} \sigma_n^i, \sigma_n^{r_{/2}+1} = 0 \, . \]

In the theorem, \((Z_a)^b\) indicates the direct sum of \( b \)-copies of cyclic group of order \( a \). Let \( p = 3 \) in the above theorem. If \( n \equiv 0 \mod 4 \) then
and $Z_s$ is generated by $\sigma_n$. Its ring structure is given by

$$\sigma_n^2 = (-3)\sigma_n, \quad \sigma^{s+1} = 0.$$ 

3. The proofs of Theorem 1 and Corollary 2. From Th. 11.3 in [2], we obtain the following equality. (For the proof, see Proposition 5, in the last part of this section.) Let $\partial_k^s \colon H^s(L^n(3); Z_3) \to H^{s+k}(L^n(3); Z_3)$ the $k$-th reduced power operation mod 3. Then we have

$$\partial_k^s(p_s(\zeta)) = \left( \sum_{n+m=k} p_n(\zeta)p_m(\zeta) \right) p_s(\zeta) + \sum_{i>0} p_i(\zeta) \cdots$$

for $0 \leq k \leq s$.

Let $s$ be an integer such as (1) in 1. Since $d_j \equiv 0$ for all $j > s$ and $d_s \equiv 0$, then (6) gives

$$\partial_k^s(p_s(\zeta)) = \left( \sum_{n+m=k} p_n(\zeta)p_m(\zeta) \right) p_s(\zeta).$$

For an element $x^{2s}$ of $H^s(L^n(3); Z_3)$, we have

$$\partial_k^s(x^{2s}) = \binom{2s}{k} x^{2s+2k} \text{ and } d_s \binom{2s}{k} x^{2s+2k} = \left( \sum_{n+m=k} d_n d_m \right) d_s x^{2s+2k}.$$ 

From $2s+2k \leq 4s \leq 2t < n+1$, $x^{2s+2k} \equiv 0$. Hence $\binom{2s}{k} = \sum_{n+m=k} d_n d_m$.

By induction, we obtain $d_j \equiv \binom{s}{j} \mod 3$. Therefore

$$p(\zeta) = 1 + \binom{s}{1} x^2 + \cdots + \binom{s}{s} x^{2s} \mod 3$$

$$= (1+x^2)^s \mod 3.$$

The proof of Theorem 1 is completed if we prove Proposition 5. Now, it is well known that the bundle $\eta_{L^n}$ over $L^n(3)$ has the total Pontryagin class mod 3 $p(\eta_{L^n}) = 1+x^2$. Thus the proof of Corollary 2 is completed.

Now, in order to prove the formula (6) in the proof of Theorem 1, we consider a following symmetric polynomial. Let $\sum x_1^i x_2^j \cdots x_t^k$ be a homogeneous symmetric polynomial in variables $x_1, x_2, \ldots, x_t$ of degree $N = (p-1)k+s$ where $p, k$ and $N$ are positive integers.

To prove (6), we show the following propositions.
Proposition 1.

\[ \sum x_1^2x_2^2\cdots x_k^2 = \sum_{i=0}^{k} A(i)\sigma_{k-i}\sigma_{k+i} \]

where \( A(i) = (-1)^i \left( \begin{array}{c} s-k+2i \\ j \end{array} \right) A(i-j), \quad 1 \leq i \leq k \) and \( A(0) = 1 \).

Proof. Put \( f(k, s) = \sum x_1^2\cdots x_k^2 \). By an easy calculation,

\[ f(k, s) = (\sum x_i\cdots x_k)(\sum x_i\cdots x_s) - \sum_{l=1}^{k} \left( \begin{array}{c} s-k+2j_l \\ j_l \end{array} \right) f(k-j_l, s+j_l). \]

By making use of (8) repeatedly, we have

\[ f(k, s) = \sigma_k\sigma_s + \sum_{i=1}^{k} F_i \]

where \( F_i = (-1)^i \sum_{j_1=1}^{k-i} \cdots \sum_{j_i=1}^{l-i} \left( \begin{array}{c} s-k+2j_1 \\ j_1 \end{array} \right) \cdots \left( \begin{array}{c} s-k+2j_i \\ j_i \end{array} \right) \sigma_{k-j_1-j_2}\cdots \sigma_{k-j_i}. \]

If \( l = k \), then \( k = \sum_{i=1}^{k} j_i \).

Let \( A_i(i) \) be the coefficient of \( \sigma_{k-i}\sigma_{s+i} \) in \( F_i \), then

\[ A_i(i) = (-1)^i \sum_{j_1=1}^{i} \cdots \sum_{j_i=1}^{l-i} \left( \begin{array}{c} s-k+2j_1 \\ j_1 \end{array} \right) \cdots \left( \begin{array}{c} s-k+2j_i \\ j_i \end{array} \right) \left( \begin{array}{c} s-k+2i \\ n \end{array} \right). \]

Put \( A(i) = A_i(i) + \cdots + A_{i+1}(i) \). Then

\[ A(i) = (-1)^i \sum_{j=1}^{i} \left( \begin{array}{c} s-k+2j \\ n \end{array} \right) \sum_{i=1}^{j} \sigma_{i+j} \left( \begin{array}{c} s-k+2i \\ j \end{array} \right). \]

This completes the proof of Proposition 1.

Proposition 2.

\[ \sum x_1^2\cdots x_k^2 = \sum_{i=0}^{k} A(i)\sigma_{k-i}\sigma_{k+i} \]

where

(a) \[ \sum_{i=0}^{k} A(i-j)\left( \begin{array}{c} 2i \\ j \end{array} \right) = 0 \quad \text{and} \quad A(0) = 1 \]

(b) \[ A(i) \equiv (-1)^{i+1} \mod 3 \quad \text{for} \quad i \neq 0. \]
Proof. The part of (a) is completed by Proposition 1. The proof of (b) is obtained by induction. For \( i=1, 2, A(1)=(-1)^2 \equiv 1 \) and \( A(2)=2 \). Assuming the equation (b) for integers \( i \leq 2q \), we have

\[
A(i+1) = 2 \sum_{j=1}^{i} \binom{2i+2}{j} (-1)^{i-j+1} + 2 \binom{2i+2}{i+1}
\]

By making use of \( \binom{2i+2}{j} (-1)^{i-j+1} \equiv \binom{2j+2}{i+1} + 2 \binom{2j+2}{2i+2} + 2 \binom{2j+2}{i+1} \), we have

\[
A(i+1) \equiv 1 \mod 3
\]

Assuming the equation (b) for integers \( i \leq 2q+1 \), we can obtain \( A(i+1) \equiv 2 \mod 3 \). Thus Proposition 2 is obtained by induction.

**Proposition 3.**

\[
\sum x_1 x_2 \cdots x_{k+1} x_1 x_2 \cdots x_n = (\sum x_1 x_2 \cdots x_n) \sigma_n - \sum_{j \neq n} \sigma_j(\cdots).
\]

Proof. Put \( f(a, b) = \sum x_1 x_2 \cdots x_{k+1} x_1 x_2 \cdots x_n \) with \( c=N-2a-b \). By calculation, we have the following equality;

\[
f(k, 0) = (\sum x_1 x_2 \cdots x_n) \sigma_n - \sum_{\alpha=1}^{k} f(k-\alpha, \alpha) \cdot \sum_{\beta=1}^{k} f(k-\alpha, \beta) - f(k, 0)
\]

Define \( a_0=k, b_0=0 \) and \( c_0=N-2a_0-b_0 \). Then (9) is reformed as follows:

\[
f(a_0, b_0) = f(0, a_0) \sigma_{c_0} - f(a_0, b_0)
\]

Now for each term \( f(a_i, b_i) \) in (10), we obtain

\[
f(a_i, b_i) = f(0, a_i) \sigma_{c_{i-1}} - \sum_{\alpha=0}^{a_i} \sum_{\beta=0}^{b_i} A(\alpha, \beta) f(a_i - \alpha, \beta - \beta_2)
\]

for some integers \( A(\alpha_2, \beta_2) \) and \( A(0, 0) = 0 \). We can inductively define two sequences \( \{a_i\}, \{b_i\} \) satisfying the followings

\[
f(a_{i-1}, b_{i-1}) = f(0, a_{i-1}) \sigma_{c_{i-1}} - \sum_{\alpha=0}^{a_i} \sum_{\beta=0}^{b_i} A(\alpha, \beta) f(a_i - \alpha, \beta)
\]

\[
a_i = a_{i-1} - \alpha_i, \quad b_i = \alpha_i + \beta_i
\]

with some integers \( A(\alpha_i, \beta_i) \) and \( A(0, b_i) = 0 \). Put \( c_{i-1} = N-2a_{i-1}-b_{i-1} \). Then we have \( s<c_s<c_{s+1}<\cdots<c_t<\cdots \). From (13), \( a_i \leq a_i \) for all \( i \). Hence consider the following cases:

(14) there exists an integer \( n \) such as \( a_{i+1} < a_i \) for all \( i \geq n \),

(15) there exists an integer \( m \) such as \( a_m = \cdots = a_i = \cdots \) for all \( i \geq m \).
If (14) is satisfied, then \( a_q = 0 \) for some integer \( q \). From (12) and Proposition 1, we have

\[ f(a_{q-1}, b_{q-1}) = f(0, a_{q-1}) \sigma_{e_q - 1} \sum_{\beta_q = 1}^{-1} A(0, \beta_q) f(0, b_q) \]
\[ = f(0, a_{q-1}) \sigma_{e_q - 1} \sum_{\beta_q = 0}^{-1} \sum_{i=0}^{b_q} A(0, \beta_q) A(i) \sigma_{b_q - i} \sigma_{e_q + i} \]

If (15) is satisfied, then \( b_i > b_{i+1} \) for all \( i \geq m \). Therefore \( b_r = 0 \) for some integer \( r \). From (12) we have

\[ f(a_{r-1}, b_{r-1}) = f(0, a_{r-1}) \sigma_{a_{r-2}} \sum_{\alpha_r = 0}^{-1} A(\alpha_r, 0) f(a_r, 0). \]

Since \( a_r < a_0 \), the above discussion is also applied to \( f(a_r, 0) \) in this case. Hence, by making use of (9) repeatedly, we have finally

\[ f(k, 0) = f(0, k) \sigma_x \sum_{i=1}^{x} (\cdots) \sigma_i. \]

From Proposition 2 and Proposition 3 we have the following

**Proposition 4.**

\[ \sum x_1^a x_2^b \cdots x_k^e = (\sigma_x^2 + \sum_{j=1}^{k} (-1)^{j+1} \sigma_{k-j} \sigma_{k+j}) \sigma_x + \sum_{i=1}^{x} (\cdots) \sigma_i. \]

Now, we can prove the formula (6) in the proof of Theorem 1.

**Proposition 5.** \( \partial_3^a \left( p_s(\zeta) \right) = \left( \sum_{n+m=s} p_n(\zeta) p_m(\zeta) \right) p_s(\zeta) + \sum_{i=1}^{s} p_i(\zeta)(\cdots). \)

Proof. Let \( C_i \in H^i(B_{U(2)} : Z_3) \) be the \( i \)-th Chern class mod 3. By Th. 11.3 ((2)) and Proposition 4, we have

\[ \partial_3^a(C_i) = (C_i^2 + \sum_{j=1}^{k} (-1)^{j+1} C_{k-j} C_{k+j}) C_i + \sum_{i=1}^{x} (\cdots) C_i. \]

Let \( p_s(\zeta) \) be the \( s \)-th Pontrjagin class mod 3 of a real bundle \( \zeta \). Then \( p_s(\zeta) = (-1)^s C_{2s}(\zeta \otimes C) \) mod 3 where \( C_{2s}(\zeta \otimes C) \) is a 2s-th Chern class of \( \zeta \otimes C \). From (17) we obtain

\[ \partial_3^a(p_s(\zeta)) = (p_s^2(\zeta) + 2 \sum_{i=1}^{s} p_{i+1}(\zeta) p_{i-1}(\zeta)) p_s(\zeta) + \sum_{i=1}^{s} (\cdots) p_i(\zeta), \]

and \( \partial_3^{2s+1}(p_s(\zeta)) = 2(\sum_{i=1}^{s} p_{i-1}(\zeta) p_{i+1}(\zeta)) p_s(\zeta) + \sum_{i=1}^{s} (\cdots) p_i(\zeta) \). This completes the proof of Proposition 5.

**4. Proofs of Theorem 3, 4 and 5.** To prove Theorem 3, we discuss the following lemmas. The proofs of Lemma 1, 2 and 3 are omitted.
**Lemma 1.** Let $A_0, A_1, \ldots, A_n$ be integers with $\nu_3(A_j) > 0$ for all $j \neq n$ and $\nu_3(A_n) \geq 0$. If $\nu_3(A_0) \neq \nu_3(A_j)$ for all $j \neq n$, then

$$\nu_3\left(\sum_{j=0}^{n} A_j\right) = \nu_3(A_n).$$

**Lemma 2.** If $r, s, a$ and $u$ are positive integers with $s < a < 3^u$ and $(r, 3) = 1$ then following hold.

\[(18) \quad \nu_3\left(\binom{rs + s}{a}\right) = \nu_3\left(\binom{3^u + s}{a}\right) \leq \nu_3\left(\binom{3^u}{a}\right).\]

\[(19) \quad \nu_3\left(\binom{3^u}{a}\right) = u - \nu_3(a).\]

**Lemma 3.** If $u$ and $n$ are positive integers, then

\[(20) \quad \nu_3((3^u)! \neq \frac{3^u - 1}{2}.\]

\[(21) \quad \nu_3((2n + 1)!) \leq n\]

\[(22) \quad \nu_3((2n)! < n.\]

Put $A_j = (-1)^i(-3)^{-i-1}\left(\frac{q}{i-j}\right)^{i-j} \left(\begin{array}{c} i-j \\hline j \end{array}\right) (j = 0, 1, \ldots, \left[\frac{i}{2}\right])$ for some positive integer $q$, $i > 2$ with $q > i - j$.

**Lemma 4.** Let $A_j$ be above integers. Then

$$\nu_3\left(\sum_{j=0}^{i/3} A_j\right) = \nu_3(A_{i/3}) \quad \text{for} \quad i \equiv 1 \text{ mod } 6 \text{ and } i \equiv 0 \text{ mod } 2.$$

**Proof.** If $i = 2n$, then for each $l = 1, 2, \ldots, n - 1$

$$\nu_3(A_{n-l}) = (n+l-1) - \nu_3((2l)! - \nu_3((n-l)!) + \nu_3(q) + \cdots + \nu_3(q-n-l-1).$$

From Lemma 3 (22) $\nu_3(A_{n-l}) - \nu_3(A_n) > l - \nu_3((2l)!).$ Then we have

$$\nu_3(A_j) > \nu_3(A_n) \quad \text{and} \quad \nu_3(A_j) > 0 \quad \text{for all} \quad j \neq n.$$ 

Therefore by Lemma 1 we obtain $\nu_3\left(\sum_{j=0}^{i/3} A_j\right) = \nu_3(A_{i/3})$ for $i \equiv 0 \text{ mod } 2$.

From Lemma 3 (21), we obtain

$$\nu_3(A_{n-l}) - \nu_3(A_n) > \nu_3(n!) - \nu_3((n-l)!) > 0$$

under the condition $i \equiv 1 \text{ mod } 6, \left[\frac{i}{2}\right] = n = 3m.$

Now we prove the theorems.
Proof of Theorem 3. Let $\zeta'$ be the extension over $L^m(3)$ of $\zeta$. By the structure of $\tilde{KO}$-ring of the lens space ([4]), $\zeta'$ is stably equivalent to $q_n L^m$, for some $q \in \mathbb{Z}_{t \equiv m \equiv m/3}$. Since $\zeta' - t = q_n L^m, \tilde{KO}(L^m(3))$, we have

\begin{equation}
(23) \quad \zeta - t = q(i* L^m - 2) \in \tilde{KO}(L^n(3))
\end{equation}

where $i^*: \tilde{KO}(L^m(3)) \to \tilde{KO}(L^n(3))$ is the induced homomorphism of natural embedding $i: L^m(3) \to L^n(3)$. If $2q \leq t$, then $\zeta$ is stably equivalent to $n Q + \cdots (q) \cdots L^n$ for some integer $q (0 \leq 2q \leq t)$. If $2q > t$, $\gamma'(q_n L^m) = 0$ for all $i > g \dim (q_n L^m)$ ([1] Prop. 2). Since $i \geq g \dim (q_n L^m)$, we have

\begin{equation}
(24) \quad \gamma'(q_n L^m) = 0 \quad \text{for all } i > t.
\end{equation}

According to the Theorem of Kambe ([4] Lemma 4.8),

\begin{equation}
\gamma'(q_n L^m) = (1 + 3(t - i^2))^q
= \sum_{\alpha = 0}^q \left( \sum_{j=0}^{[\alpha/2]} A_j \right) \sigma^\alpha t^\alpha
\end{equation}

where $A_j = (-1)^j (-3)^{s-j-1} \binom{q}{\alpha-j} \binom{t-j}{j}$.

Then we have $\gamma'(q_n L^m) \sum_{j=0}^{[\alpha/2]} A_j \sigma^\alpha = 0 \in \tilde{KO}(L^n(3)) = \mathbb{Z}_{t \equiv m \equiv m/3}$. Therefore

\begin{equation}
(24) \quad \nu_i \left( \sum_{j=0}^{[\alpha/2]} A_j \right) = \left[ \frac{m}{2} \right] \quad \text{for all } i > t.
\end{equation}

Now, according to Lemma 4, we have

\begin{equation}
\nu_i \left( \sum_{j=0}^{[\alpha/2]} A_j \right) = \nu_i (A_{i \equiv 0 \mod 2} \land i \equiv 1 \mod 6)
\end{equation}

And so we have

\begin{equation}
(25) \quad \left( \begin{array}{c}
1 - \left[ \frac{i}{2} \right] - 1
\end{array} \right) + \nu_i \left( \begin{array}{c}
\binom{q}{i/2}
\end{array} \right) \nu_i \left( \begin{array}{c}
\left( i - \left[ i/2 \right] \right)
\end{array} \right) \equiv \left[ \frac{m}{2} \right] \quad \text{for } i > t, i \equiv 0 \mod 2
\end{equation}

and $i \equiv 1 \mod 6$.

Now the total Pontrjagin class mod 3 of $q_n L^m$ is given by the equation $p(q_n L^m) = (1 + x^2)^q$. Since $m > 2t - 1$, Theorem 1 implies that there exists an integer $s$ such that

$p(\zeta') = (1 + x^2)^s, \quad 0 \leq 2s \leq t$.

Hence we have
\[ (1+x^2)^q = (1+x^2)^s \mod 3, \text{ i.e.,} \]
\[ 1 + \left( \frac{q-s}{1} \right)x^2 + \cdots + \left( \frac{q-s}{m/2} \right)x^{(m/2)} \equiv 1 \mod 3. \]

This implies that there exists an integer \( u \) such that
\[ q-s = 3^u r, \ (r, 3) = 1 \text{ and } 3^u > [m/2]. \]

Then we obtain the following
\[ \nu_s \left( \left( \frac{q}{i-[i/2]} \right) \right) = \nu_s \left( \left( \frac{r3^u+s}{i-[i/2]} \right) \right) \]
\[ \leq \nu_s \left( \left( \frac{3^u}{i-[i/2]} \right) \right) \text{ for } t < i < m \text{ (by Lemma 2)} \]
\[ = u - \nu_s (i-[i/2]). \]

Hence from (25) \( u + (i-[i/2]-1) - \nu_s (i-[i/2]) + \nu_s \left( \left( \frac{i-[i/2]}{i-[i/2]} \right) \right) \geq \left[ \frac{m}{2} \right] \) for \( t < i < m \) and \( i \equiv 0 \mod 2, i \equiv 1 \mod 6 \). By the assumption (5) of Theorem 3, we have
\[ u \geq [m/2] - \min \left[ (i-[i/2]-1) - \nu_s (i-[i/2]) + \nu_s \left( \left( \frac{i-[i/2]}{i-[i/2]} \right) \right) \right] \]
\[ = [m/2] - \beta_s (t, m) \geq [n/2]. \]

According to (23), (26) and (27), there exists an integer \( s \) such that
\[ 0 \leq 2s \leq t, \]
\[ \xi - t = (r3^u+s)s_n = sa_n. \]

This completes the proof of Theorem 3.

Proof of Theorem 4. By the contraposition of Theorem 3 and the main theorem of Kambe ([4] Th. 2), it is clear.

Proof of Theorem 5. Since \( \tau(L^n(3)) \oplus 1 = (n+1)\eta_{L^n} \) and \( n+1 > n = \left[ \frac{2n+1}{2} \right] \)
\[ = [1/2 \dim \tau(L^n(3))], \text{ Theorem 4 implies that the tangent bundle } \tau \text{ is not exten-} \]
dible to \( L^{a\cdot(m+1,n)}(3) \). For every \( m \geq 2n+1, \beta_s (2n+1, m) \leq n \) whenever \( n \equiv 0 \)
mod 3, \( n \equiv 1 \mod 3 ) \beta_s (2n+1, m) < n \) whenever \( n \equiv 2 \mod 3 \). Then \( \phi(2n+1, n) \)
\[ = 2 (2n+1). \]
This completes the proof of Theorem 5.
REMARK. The following table shows the value of $\phi(t, n)$ where $1 \leq t \leq 10$ and $1 \leq n \leq 16$.

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References


