Aihara, Y.
Osaka J. Math.
35 (1998), 593–616

FINITENESS THEOREMS FOR MEROMORPHIC MAPPINGS

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(Received April 8, 1997)

Introduction

The main purpose of this paper is to prove finiteness theorems for some families of meromorphic mappings that are transcendental in general. The finiteness problem for meromorphic mappings under the condition on the preimages of divisors was first studied by H. Cartan and R. Nevanlinna and they obtained a finiteness theorem for meromorphic functions on the complex plane $\mathbf{C}$ ([2] and [19]). The finiteness theorem of Cartan-Nevanlinna states that there exist at most two meromorphic functions on $\mathbf{C}$ that have the same inverse images with multi-plicities for distinct three values in $\mathbf{P}_1(\mathbf{C})$. In 1981, H. Fujimoto generalized the theorem of Cartan-Nevanlinna to the case of meromorphic mappings of $\mathbf{C}^m$ into complex projective spaces $\mathbf{P}_n(\mathbf{C})$ by making use of Borel's identity ([9], IV and [10]). He proved the finiteness of families of linearly nondegenerate meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}_n(\mathbf{C})$ with the same inverse images for some hyperplanes. In his results, the number of hyperplanes in general position is essential and must be larger than a certain number depending on the dimension of the projective spaces. Furthermore, the finiteness theorem of Fujimoto has been extended to the case of meromorphic mappings into a projective algebraic manifold ([10] and [12]). In this paper, we mainly deal with the finiteness problem for meromorphic mappings f of $\mathbf{C}^m$ into a compact complex manifold M and for a divisor D on M.

Let $L \to M$ be a fixed line bundle over M, and let $\sigma_1, \cdots, \sigma_s$ be linearly independent holomorphic sections of $L \to M$ with $s \geq 2$. Throughout this paper, we assume that $(\sigma_j) = dD_j (1 \leq j \leq s)$ for some positive integer d, where D_j are effective divisors on M. Set

$$\varpi = c_1\sigma_1 + \cdots + c_s\sigma_s,$$

where $c_j \in \mathbf{C}^*$. Let D be a divisor defined by $\varpi = 0$. We define a meromorphic mapping $\Psi : M \to \mathbf{P}_{s-1}(\mathbf{C})$ by

$$\Psi = (\sigma_1, \cdots, \sigma_s).$$

DEFINITION 0.1. Let p be a nonnegative integer. For divisors E_1 and E_2 on $\mathbf{C}^m$, we write

$$E_1 \equiv E_2 \pmod{p}$$

if there exists a divisor E' on $\mathbf{C}^m$ such that $E_1 - E_2 = pE'$; in the special case of $p = 0$, $E_1 \equiv E_2 \pmod{0}$ if and only if $E_1 = E_2$.

Let E be a nonzero effective divisor on $\mathbf{C}^m$. We denote by

$$\mathcal{F}(p; (\mathbf{C}^m, E), (M, D))$$

the set of all meromorphic mappings $f : \mathbf{C}^m \to M$ such that

$$f^*D \equiv E \pmod{p}.$$

DEFINITION 0.2. A meromorphic mapping $f : \mathbf{C}^m \to M$ is said to be *analytically nondegenerate* if $f(\mathbf{C}^m)$ is not included in any proper analytic subset of M.

Let

$$\mathcal{F}^*(p; (\mathbf{C}^m, E), (M, D))$$

denote the subset of all $f \in \mathcal{F}(p; (\mathbf{C}^m, E), (M, D))$ that are *analytically nondegenerate*. The main result of the present paper is as follows:

Main Theorem 0.3. *If* $\operatorname{rank} \Psi = \dim M$ *and* $d > (s+1)!\{(s+1)! - 2\}$, *then the number of mappings in* $\mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$ *is bounded by a constant depending only on* D.

We prove Main Theorem 0.3 in §5. For the proof of this theorem, we need to generalize Fujimoto's finiteness theorem as follows.

DEFINITION 0.4. A meromorphic mapping $f : \mathbf{C}^m \to \mathbf{P}_n(\mathbf{C})$ is said to be *linearly nondegenerate* if $f(\mathbf{C}^m)$ is not included in any proper linear subspace of $\mathbf{P}_n(\mathbf{C})$.

Let $E_1, \cdots, E_{n+2}$ be effective divisors on $\mathbf{C}^m$ and let $H_1, \cdots, H_{n+2}$ be hyperplanes in general position in $\mathbf{P}_n(\mathbf{C})$. Let

$$\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$$

be the set of all *linearly nondegenerate* meromorphic mappings f of $\mathbf{C}^m$ into $\mathbf{P}_n(\mathbf{C})$ such that

$$f^*H_j \equiv E_j \pmod{p}$$

for $1 \leq j \leq n+2$. Then we have the following theorem that is a generalization of the finiteness theorem of Fujimoto (see §4):

Theorem 0.5. *Suppose that $p > (n+2)!\{(n+2)! - 2\}$. Then the number of mappings in $\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$ is bounded by by a constant depending only on n.*

The unicity problem for meromorphic mappings may be considered as a special case where the finite set of a family of meromorphic mappings reduces to one point set. The classical unicity theorem for meromorphic functions due to G. Pólya and R. Nevanlinna is well known ([18] and [22]). There have been many researches about the unicity of meromorphic functions on $\mathbf{C}$ as well in the multidimensional case (cf. [1], [4], [5], [7], [8], [9], [15], [23], [25] and [27]). In §6, we prove the following unicity theorem:

Theorem 0.6. *Assume that there exist big line bundles $L_j \to M (1 \leq j \leq s)$ such that $L = L_j^{\otimes d}, 1 \leq j \leq s$, and $\sigma_j = \tau_j^{\otimes d}$ for some holomorphic sections τ_j of $L_j \to M$. Let $f, g : \mathbf{C}^m \to M$ be analytically nondegenerate meromorphic mappings whose ranks are not less than μ. Suppose that the following conditions are satisfied*:

(1) $\operatorname{rank} \Psi = \dim M$.

(2) $\bigcap_{j=1}^{s} \operatorname{Supp}(\sigma_j) = \emptyset$.

(3) $f^{-1}(D) = g^{-1}(D)$ *as point sets (say Z).*

(4) $f = g$ *on* $Z - (I(f) \cup I(g))$.

Then there exists a positive integer d_0 depending only on $L_j (1 \leq j \leq s)$ such that if $d > (s - \mu)(s + d_0)$, $f = g$ on $\mathbf{C}^m$.

In §1 we explain some known facts in Nevanlinna theory and in §2 we prove two lemmas. In §3 we give some remarks on analytic dependence of meromorphic mappings. In §§4-6, we give the proofs of the above theorems. In the proofs, we use the second main theorem for meromorphic mappings into complex projective spaces and the generalized Borel identity.

ACKNOWLEDGEMENT. The author is grateful to Professor Junjiro Noguchi for his valuable advice. He gave useful comments and remarks to the first draft of this paper.

1. Preliminaries

Let $z = (z_1, \cdots, z_m)$ be the natural coordinate system in $\mathbf{C}^m$, and set

$$\|z\|^2 = \sum_{\nu=1}^{m} z_\nu \bar{z}_\nu, \quad B(r) = \{z \in \mathbf{C}^m; \|z\| < r\},$$

$$\partial B(r) = \{z \in \mathbf{C}^m; \|z\| = r\}, \quad d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial),$$
$$\alpha = dd^c\|z\|^2, \qquad \eta = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1}.$$

For a (1,1)-current φ of order zero on $\mathbf{C}^m$ we set

$$n(r,\varphi) = r^{2-2m}\langle \varphi \wedge \alpha^{m-1}, \chi_{B(r)}\rangle$$

and

$$N(r,\varphi) = \int_1^r n(t,\varphi)\frac{dt}{t},$$

where $\chi_{B(r)}$ denotes the characteristic function of $B(r)$.

Let M be a compact complex manifold and $L \to M$ a line bundle over M. We denote by $\Gamma(M,L)$ the space of all holomorphic sections of $L \to M$. Let $|L| = \mathbf{P}(\Gamma(M,L))$ be the complete linear system defined by L. When $|L| \neq \emptyset$, we define the base locus of $|L|$ by

$$\mathrm{Bs}|L| = \bigcap_{D \in |L|} \mathrm{Supp}D.$$

Let $\{\varphi_0, \cdots, \varphi_n\}$ be a basis for $\Gamma(M,L)$. We define a meromorphic mapping $\Phi_L : M \to \mathbf{P}_n(\mathbf{C})$ by

$$\Phi_L(z) = (\varphi_0(z), \cdots, \varphi_n(z)), \; z \in M - \mathrm{Bs}|L|.$$

Let $|\cdot|$ be a hermitian fiber metric in L and let ω be its Chern form. Let $f : \mathbf{C}^m \to M$ be a meromorphic mapping. We set

$$T_f(r,L) = N(r, f^*\omega)$$

and call it the characteristic function of f with respect to L. In the case where $M = \mathbf{P}_n(\mathbf{C})$ and $L = [H]$ is the hyperplane bundle, we simply write $T_f(r)$ for $T_f(r,[H])$. Let $D = (\sigma) \in |L|$ with $|\sigma| \leq 1$ on M. Assume that $f(\mathbf{C}^m) \not\subseteq \mathrm{Supp}D$. We define the proximity function of D by

$$m_f(r,D) = \int_{\partial B(r)} \log\left(\frac{1}{|f^*\sigma|}\right)\eta.$$

Then we have the following first main theorem for meromorphic mappings (cf. [27]):

Theorem 1.1. *Let $L \to M$ be a line bundle over M and let $f : \mathbf{C}^m \to M$ be a meromorphic mapping. Then*

$$N(r, f^*D) + m_f(r,D) = T_f(r,L) + O(1)$$

for $D \in |L|$ *with* $f(\mathbf{C}^m) \not\subseteq \mathrm{Supp}D$, *where* $O(1)$ *stands for a bounded term as* $r \to +\infty$.

Let E be an effective divisor on $\mathbf{C}^m$ and let l be a positive integer. If

$$E = \sum_j \nu_j E_j$$

for irreducible hypersurfaces E_j in $\mathbf{C}^m$ and for nonnegative integers ν_j, then we set

$$N_l(r, E) = \sum_j \min\{l, \nu_j\} N(r, E_j).$$

For a meromorphic mapping $f : \mathbf{C}^m \to M$, we denote by $I(f)$ the indeterminacy locus of f. Set

$$\mathrm{rank}\, f = \max\{\mathrm{rank}\, df(z); z \in \mathbf{C}^m - I(f)\}.$$

By making use of "Lemma on the logarithmic derivative" on $\mathbf{C}^m$, which was first proved by A. Vitter ([28]) and was refined by B. Shiffman ([24, Lemma 3.11]), we have the following second main theorem for meromorphic mappings $f : \mathbf{C}^m \to \mathbf{P}_n(\mathbf{C})$ that plays an essential role in this paper (cf. [11] and [21]):

Theorem 1.2. *Let* $f : \mathbf{C}^m \to \mathbf{P}_n(\mathbf{C})$ *be a meromorphic mapping with rank* μ, *and let* l *be the dimension of the smallest linear subspace of* $\mathbf{P}_n(\mathbf{C})$ *including* $f(\mathbf{C}^m)$. *Let* $H_1, \cdots, H_q$ *be hyperplanes in* $\mathbf{P}_n(\mathbf{C})$ *located in general position. Then*

$$(q - 2n + l - 1)T_f(r) \leq \sum_{j=1}^{q} N_{l-\mu+1}(r, f^* H_j) + S_f(r),$$

where

$$S_f(r) = O(\log T_f(r)) + o(\log r)$$

except on a Borel subset $E \subseteq [1, +\infty)$ *with finite measure.*

2. Two lemmas

Let $f : \mathbf{C}^m \to \mathbf{P}_n(\mathbf{C})$ be a nonconstant meromorphic mapping and let H be a hyperplane in $\mathbf{P}_n(\mathbf{C})$ with $f(\mathbf{C}^m) \not\subseteq H$.

DEFINITION 2.1. We say that f is *ramified to order at least* $d(>0)$ *over* H if

$$f^* H \geq d\mathrm{Supp} f^* H.$$

In the case $\mathrm{Supp} f^*H = \emptyset$, we may say that f is ramified to order $+\infty$.

We first show the following (cf. H. Cartan [3]):

Lemma 2.2. *Let $f : \mathbf{C}^m \to \mathbf{P}_n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping with rank μ. Let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbf{P}_n(\mathbf{C})$. Suppose that f is ramified to order at least d_j over H_j $(1 \leq j \leq q)$. Then*

$$\sum_{j=1}^{q}\left(1 - \frac{n-\mu+1}{d_j}\right) \leq n+1.$$

Proof. Let $k = n - \mu + 1$. By Theorem 1.1 and Theorem 1.2, it follows that

$$\begin{aligned} n+1 &\geq \sum_{j=1}^{q}\left(1 - \limsup_{r\to+\infty} \frac{N_k(r, f^*H_j)}{T_f(r)}\right) \\ &\geq \sum_{j=1}^{q}\left(1 - \limsup_{r\to+\infty} \frac{k}{d_j}\frac{d_j N_1(r, f^*H_j)}{T_f(r)}\right) \\ &\geq \sum_{j=1}^{q}\left(1 - \limsup_{r\to+\infty} \frac{k}{d_j}\frac{N(r, f^*H_j)}{T_f(r)}\right) \\ &\geq \sum_{j=1}^{q}\left(1 - \frac{k}{d_j}\right). \end{aligned}$$

□

The following lemma is a generalization of the classical theorem of E. Borel due to H. Fujimoto and M. Green (cf. [6, Corollary 6.4] and [13, p. 70]):

Lemma 2.3. *Let $\varphi_1, \cdots, \varphi_t$ be nonzero meromorphic functions on $\mathbf{C}^m$ satisfying the functional equation*

$$\varphi_1 + \cdots + \varphi_t \equiv 0. \tag{2.4}$$

Suppose that $(\varphi_j) \equiv 0 \pmod d$ for all $1 \leq j \leq t$ and $d > t(t-2)$. Then there exists a decomposition of indices, $\{1, \cdots, t\} = \cup I_\nu$, such that

(1) *every I_ν contains at least two indices*

(2) *the ratio of φ_i and φ_j is nonzero constant if and only if $i, j \in I_\nu$*

(3) *$\sum_{j \in I_\nu} \varphi_j \equiv 0$ for every ν.*

Proof. We prove Lemma 2.3 by induction on t. The case of $t = 2$ is trivial. Suppose that our assertion holds up to $t - 1$. We introduce an equivalence relation in $\{1, \cdots, t\}$ as follows: $\varphi_i \sim \varphi_j$ if φ_i/φ_j is constant. Let $\{I_1, \cdots, I_{\nu_0}\} =$

$\{1, \cdots, t\}/\sim$. By definition, we have (2). For the proof of (1), we assume that there exists I_ν that contains only one index, say t. We show that $\varphi_1, \cdots, \varphi_{t-1}$ are linearly dependent over $\mathbf{C}$. We define a meromorphic mapping $F : \mathbf{C}^m \to \mathbf{P}_{t-2}(\mathbf{C})$ by

$$F = (\varphi_1, \cdots, \varphi_{t-1}).$$

Suppose that F is linearly nondegenerate. Take the following hyperplanes in general position:

$$H_1 = \{\zeta_1 = 0\}, \cdots, \{\zeta_{t-1} = 0\},$$
$$H_t = \{\zeta_1 + \cdots + \zeta_{t-1} = 0\}.$$

Then F is ramified to order at least d over H_j for all $1 \le j \le t$. By Lemma 2.2, it follows that

$$\sum_{j=1}^{t} \left(1 - \frac{t-2}{d}\right) \le t - 1.$$

Hence $d \le t(t-2)$. This contradicts the assumption. Thus there exists a nontrivial linear relation

(2.5) $$a_1\varphi_1 + \cdots + a_{t-1}\varphi_{t-1} \equiv 0.$$

We may assume that $a_1 = 1$. By (2.4) and (2.5), we have

$$(1 - a_2)\varphi_2 + \cdots + (1 - a_{t-1})\varphi_{t-1} + \varphi_t \equiv 0.$$

By the induction hypothesis, there exists an index $i (2 \le i \le t-1)$ such that $1 - a_i \ne 0$ and $(1 - a_i)\varphi_i/\varphi_t$ is constant. Thus $i, t \in I_\nu$. This is absurd. Hence we have (1). Finally we show (3). We choose an index $i_\nu \in I_\nu$ and set

$$\sum_{i \in I_\nu} \varphi_i = b_\nu \varphi_{i_\nu},$$

where $b_\nu \in \mathbf{C}$. Then (2.4) can be written as

$$\sum_{\nu=1}^{\nu_0} b_\nu \varphi_{i_\nu} \equiv 0.$$

By (1), we infer that all $b_\nu = 0$. This shows (3). □

3. Analytic dependence of meromorphic mappings

In this section we deal with analytic dependence of meromorphic mappings belonging to a certain class. Let

$$\mathcal{F}^*_\mu(d; (\mathbf{C}^m, E), (M, D)) = \{f \in \mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D)); \operatorname{rank} f \ge \mu\}.$$

Set $M^2 = M \times M$. For meromorphic mappings $f,\ g : \mathbf{C}^m \to M$, we define a meromorphic mapping $f \times g : \mathbf{C}^m \to M^2$ by

$$(f \times g)(z) = (f(z), g(z))$$

for $z \in \mathbf{C}^m - (I(f) \cup I(g))$.

DEFINITION 3.1. Nonconstant meromorphic mappings $f,\ g : \mathbf{C}^m \to M$ are said to be *analytically dependent* if there exists a proper analytic subset S of M^2 such that S is not of the type $S_1 \times S_2$ with analytic subsets $S_j \subseteq M$ $(j = 1, 2)$ and $(f \times g)(\mathbf{C}^m) \subseteq S$.

For a positive integer p, we denote by $\mathcal{M}_p^*$ the set of all meromorphic functions f on $\mathbf{C}^m$ such that $f = g^p$ for some nonzero meromorphic functions g on $\mathbf{C}^m$. For $p = 0$, let $\mathcal{M}_0^*$ denote the set of all nonvanishing holomorphic functions on $\mathbf{C}^m$. We first give the following proposition:

Proposition 3.2. *Let $\widetilde{L} \to M^2$ be a holomorphic line bundle over M^2 such that*

$$\widetilde{L} = \pi_1^* L \otimes \pi_2^* L,$$

where $\pi_j : M^2 \to M (j = 1, 2)$ are the natural projections. Let f and g be arbitrary mappings in $\mathcal{F}(d; (\mathbf{C}^m, E), (M, D))$. If $d > 4s(s-1)$, then there exists $\widetilde{D} \in |\widetilde{L}|$ such that $\widetilde{D}$ is not of the type $D_1 \times M + M \times D_2$ with $D_1,\ D_2 \in |L|$ and

$$(f \times g)(\mathbf{C}^m) \subseteq \mathrm{Supp}\widetilde{D}.$$

Proof. Let $f,\ g \in \mathcal{F}(d; (\mathbf{C}^m, E), (M, D))$. By the definition of $\mathcal{F}(d; (\mathbf{C}^m, E), (M, D))$, there exists a meromorphic function α in $\mathcal{M}_d^*$ such that

$$\varpi(f) = \alpha \varpi(g). \tag{3.3}$$

We note that $\varpi(f) \not\equiv 0$ and $\varpi(g) \not\equiv 0$. There exist subsets $\{j_1, \cdots, j_a\}$, $\{j_{a+1}, \cdots, j_{a+b}\}$ of $\{1, \cdots, s\}$ and nonzero constants c_j' such that the relation (3.3) can be written as

$$\sum_{\mu=1}^{a} c_{j_\mu}' \sigma_{j_\mu} \circ f - \alpha \sum_{\mu=a+1}^{a+b} c_{j_\mu}' \sigma_{j_\mu} \circ g \equiv 0, \tag{3.4}$$

where $\{\sigma_{j_1} \circ f, \cdots, \sigma_{j_a} \circ f\}$ and $\{\sigma_{j_{a+1}} \circ g, \cdots, \sigma_{j_{a+b}} \circ g\}$ are linearly independent over $\mathbf{C}$ respectively. Since $d > 4s(s-1)$, applying Lemma (2.3) to (3.4), we have a decomposition $\{1,\cdots,a+b\}=\cup I_\nu$ of indices such that each I_ν contains two indices and

$$\sigma_{j_k}\circ f=\alpha a_{kl}\sigma_{j_l}\circ g, \tag{3.5}$$

where $a_{kl}\in\mathbf{C}^*$ and $I_\nu=\{k,l\}$. Thus, by (3.3) and (3.5), we can eliminate α and obtain at least one relation $\tau(f;g)=0$, where $\tau(\xi;\zeta)$ is a holomorphic section of $\widetilde{L}\to M^2$. Since $\{\sigma_1,\cdots,\sigma_s\}$ is linearly independent over $\mathbf{C}$, it follows that $\tau\neq 0$. Let $\widetilde{D}=(\tau)$. Then we have $(f\times g)(\mathbf{C}^m)\subseteq \mathrm{Supp}\widetilde{D}$. It is clear that $\widetilde{D}\neq D_1\times M+M\times D_2$ for any D_1, $D_2\in|L|$. □

For the family $\mathcal{F}^*_\mu(d;(\mathbf{C}^m,E),(M,D))$, we have the following:

Proposition 3.6. *Let $\widetilde{L}$ be as in Proposition* 3.2. *Suppose that* $\mathrm{rank}\,\Psi=\dim M$. *Let f and g be arbitrary mappings in $\mathcal{F}^*_\mu(d;(\mathbf{C}^m,E),(M,D))$. If $d>2s(2s-\mu-1)$, then there exists $\widetilde{D}\in|\widetilde{L}|$ such that $\widetilde{D}$ is not of the type $D_1\times M+M\times D_2$ with D_1, $D_2\in|L|$ and*

$$(f\times g)(\mathbf{C}^m)\subseteq \mathrm{Supp}\widetilde{D}.$$

Proof. Let f, $g\in\mathcal{F}(d;(\mathbf{C}^m,E),(M,D))$. As in the proof of Proposition 3.2, we have a relation

$$\varpi(f)=\alpha\varpi(g), \tag{3.7}$$

where $\alpha\in\mathcal{M}^*_d$. We define a meromorphic mapping $F:\mathbf{C}^m\to\mathbf{P}_{2s-2}(\mathbf{C})$ by

$$F=(\sigma_1\circ f:\cdots:\sigma_s\circ f:\alpha\sigma_1\circ g:\cdots:\alpha\sigma_{s-1}\circ g).$$

Since $\mathrm{rank}\,\Psi=\dim M$, it follows that $\mathrm{rank}\,F\geq\mu$. Assume that F is linearly nondegenerate. Take the following hyperplanes in general position:

$$H_1=\{\zeta_1=0\},\cdots,H_{2s-1}=\{\zeta_{2s-1}=0\},$$
$$H_{2s}=\{c_1\zeta_1+\cdots+c_{2s-1}\zeta_{2s-1}=0\},$$

where $\{\zeta_1,\cdots,\zeta_{2s-1}\}$ is a homogeneous coordinate system in $\mathbf{P}_{2s-2}(\mathbf{C})$. Then, by Lemma 2.1, we have

$$2s\left(1-\frac{2s-\mu-1}{d}\right)\leq 2s-1$$

and hence $d\leq 2s(2s-\mu-1)$. This is absurd. Thus there exists a nontrivial linear relation

$$a_1\sigma_1\circ f+\cdots+a_s\sigma_s\circ f+\alpha\{b_1\sigma_1\circ g+\cdots+b_{s-1}\sigma_{s-1}\circ g\}\equiv 0, \tag{3.8}$$

where $(a_1, \cdots, a_s) \neq (0, \cdots, 0)$ and $(b_1, \cdots, b_{s-1}) \neq (0, \cdots, 0)$. By (3.7) and (3.8), we eliminate α and obtain a relation $\tau(f; g) = 0$, where τ is a holomorphic section of $\widetilde{L} \to M$ defined by

$$\tau(\xi; \zeta) = \left\{ \sum_{j=1}^{s} a_j \sigma_j(\xi) \right\} \left\{ \sum_{j=1}^{s} c_j \sigma_j(\zeta) \right\} + \left\{ \sum_{j=1}^{s} c_j \sigma_j(\xi) \right\} \left\{ \sum_{j=1}^{s-1} b_j \sigma_j(\zeta) \right\}.$$

Since f and g are analytically nondegenerate, $\sum_{j=1}^{s} a_j \sigma_j \circ f \not\equiv 0$ and $\sum_{j=1}^{s-1} b_j \sigma_j \circ g \not\equiv 0$. Since $c_s \neq 0$ and $\{\sigma_1, \cdots, \sigma_s\}$ is linearly independent over $\mathbf{C}$, we infer that $\tau \not\equiv 0$. Let $\widetilde{D} = (\tau)$. Then we have $(f \times g)(\mathbf{C}^m) \subseteq \mathrm{Supp}\widetilde{D}$ and $\widetilde{D} \neq D_1 \times M + M \times D_2$ for any $D_1, D_2 \in |L|$. □

4. A finiteness theorem for meromorphic mappings into $\mathbf{P}_n(\mathbf{C})$

In this section we give a finiteness theorem for the family of meromorphic mappings $\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$ as follows:

Theorem 4.1. *Suppose that $p > (n+2)!\{(n+2)! - 2\}$. Then the number of mappings in $\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$ is bounded by a constant depending only on n.*

This theorem is a generalization of the following finiteness theorem for the family

$$\mathcal{E}((\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\})) := \mathcal{E}(0; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$$

proved by H. Fujimoto ([10, Theorem 2.1]):

Theorem 4.2 (Fujimoto). *The number of mappings in $\mathcal{E}((\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$ is bounded by a constant depending only on n.*

REMARK 4.3. In general, the number of mappings in $\mathcal{E}((\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$ is not less than $(n+1)!$ for all n (cf. [9, IV, p. 153]).

The proof of Theorem 4.1 is obtained by a modification of the proof of the finiteness theorem of Fujimoto. We give the proof of Theorem 4.1 step by step in what follows.

Suppose that $\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$ contains mutually distinct meromorphic mappings $f_1, \cdots, f_q$. We have to show that there exists a positive integer q_n depending only on n such that $q \leq q_n$ if $p > (n+2)!\{(n+2)! - 2\}$. We show this by induction on n. In the case $n = 1$, we can take $q_1 = 2$ by the following lemma:

Lemma 4.4. *Let a_1, a_2, a_3 be distinct three points in $\mathbf{P}_1(\mathbf{C})$ and let f_1, f_2,*

f_3 be nonconstant meromorphic functions on $\mathbf{C}^m$ such that $f_1^* a_j \equiv f_2^* a_j \equiv f_3^* a_j \pmod{p}$ for $j = 1, 2, 3$. If $p > 24$, then $f_1 = f_2$, $f_2 = f_3$ or $f_3 = f_1$.

This lemma is a generalization of the finiteness theorem of Cartan-Nevanlinna. We give here a proof based on the idea of [20, p. 126] and suggested by J. Noguchi.

Proof. Without loss of generality, we may assume that $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$. Put

$$\psi_1 = \frac{f_1}{f_2}, \psi_2 = \frac{f_1 - 1}{f_2 - 1}, \psi_3 = \frac{f_1}{f_3}, \psi_4 = \frac{f_1 - 1}{f_3 - 1}. \tag{4.5}$$

Then it is clear that $(\psi_j)_0 \equiv (\psi_j)_\infty \equiv (\psi_j - 1)_0 \equiv (\psi_j - 1)_\infty \pmod{p}$. Eliminating f_1, f_2, f_3 in (4.5), we have a relation

$$\psi_1\psi_2\psi_3 + \psi_2\psi_3\psi_4 - \psi_3\psi_4\psi_1 - \psi_4\psi_1\psi_2 + \psi_1\psi_4 - \psi_2\psi_3 \equiv 0. \tag{4.6}$$

We define meromorphic functions φ_i $(1 \leq i \leq 6)$ on $\mathbf{C}^m$ by

$$\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\} = \{\psi_1\psi_2\psi_3, \psi_2\psi_3\psi_4, \psi_3\psi_4\psi_1, \psi_4\psi_1\psi_2, \psi_1\psi_4, \psi_2\psi_3\}.$$

Since $p > 24$, applying Lemma (2.3) to (4.6), we have a decomposition $\{1, \cdots, 6\} = \bigcup_\nu I_\nu$ of indices with $\sharp I_\nu \geq 2$. We note that $i, j \in I_\nu$ if and only if φ_i/φ_j is nonzero constant. From this, we can easily verify our assertion. □

Assume that Theorem 4.1 is true for $1, \cdots, n-1$. Hence we have constants $q_1, \cdots, q_{n-1}$ that have the above property. We identify $\mathbf{P}_n(\mathbf{C})$ with the hyperplane

$$H := \{\zeta_1 + \cdots + \zeta_{n+2} = 0\}$$

in $\mathbf{P}_{n+1}(\mathbf{C})$, where $\{\zeta_1, \cdots, \zeta_{n+2}\}$ is a homogeneous coordinate system in $\mathbf{P}_{n+1}(\mathbf{C})$. Then we may assume that $H_i = \{\zeta_i = 0\} \cap H$ for $1 \leq i \leq n+2$. For $j = 1, \cdots, q$, let $(f_1^j, \cdots, f_{n+2}^j)$ be a reduced representation of f_j. We take holomorphic functions k_i on $\mathbf{C}^m$ such that $(k_i) = E_i$ for $1 \leq i \leq n+2$. Put $h_{ij} = f_i^j/k_i$. Then we have $h_{ij} \in \mathcal{M}_p^*$ and

$$h_{1j}k_1 + \cdots + h_{n+2j}k_{n+2} \equiv 0$$

for $1 \leq j \leq q$. Thus we have

$$\det(h_{ij_l}; 1 \leq i, l \leq p_0) \equiv 0 \tag{4.7}$$

for every $1 \leq j_1 < \cdots < j_{p_0} \leq q$, where $p_0 = n + 2$.

We note here that we can perform the following operation without loss of generality: multiplying a row or column of the matrix $\Lambda := (h_{ij}; 1 \le i \le p_0, 1 \le j \le q)$ by a common elements in $\mathcal{M}_p^*$.

Lemma 4.8 ([10, Lemma 2.2]). *Suppose that there exists r with $2 \le r \le p_0$ such that h_{ij} are constant for $1 \le i \le r$, $1 \le j \le q$ and* $\operatorname{rank}(h_{ij}; 1 \le i \le r, 1 \le j \le q) < r$. *Then $q \le q_{n-1}$.*

In [10], the proof of Lemma 4.8 is given in the case where h_{ij} are holomorphic functions without zero. We give here a proof after [10] for a convenience.

Proof. After changing indices, we may assume that

$$h_{1j} = \sum_{i=2}^{r} \lambda_i h_{ij} \tag{4.9}$$

for $1 \le j \le q$, where $\lambda_2, \cdots, \lambda_r \in \mathbf{C}^*$. Put $g_i = k_i + \lambda_i k_1$ for $i = 2, \cdots, r$ and $g_i = k_i$ for $i = r+1, \cdots, p_0$. We may assume that all $g_j \not\equiv 0$. We identify $\mathbf{P}_{n-1}(\mathbf{C})$ with the subspace

$$H' := \{(\xi_2 : \cdots : \xi_{n+2}) \in \mathbf{P}_n(\mathbf{C}); \xi_2 + \cdots + \xi_{n+2} = 0\}$$

in $\mathbf{P}_n(\mathbf{C})$. We define meromorphic mappings $\widetilde{f}_j : \mathbf{C}^m \to \mathbf{P}_{n-1}(\mathbf{C})$ by

$$\widetilde{f}_j = (h_{2j} g_2 : \cdots : h_{rj} g_r : h_{r+1j} g_{r+1} : \cdots : h_{n+2j} g_{n+2})$$

for $j = 1, \cdots, q$. It is easy to see that $\widetilde{f}_j$ are linearly nondegenerate. Moreover, $\widetilde{f}_j \ne \widetilde{f}_{j'}$ for $j \ne j'$. Indeed, if $\widetilde{f}_j = \widetilde{f}_{j'}$, then there exists a nonzero meromorphic function α on $\mathbf{C}^m$ such that

$$h_{ij} g_i = \alpha h_{ij'} g_i \tag{4.10}$$

for $i = 2, \cdots, n+2$. Multiplying (4.10) by k_i/g_i, we have

$$f_i^j = \alpha f_i^{j'} \tag{4.11}$$

for $i = 2, \cdots, n+2$. By (4.9) and (4.11), it is easy to see that $f_1^j = \alpha f_1^{j'}$. This shows $f_j = f_{j'}$. For $2 \le i \le n+2$, let $H_i' = \{\xi_i = 0\} \cap H'$ and $E_i' = (g_i)$. Then it is clear that $\widetilde{f}_j \in \mathcal{E}' := \mathcal{E}(p; (\mathbf{C}^m, \{E_j'\}), (\mathbf{P}_{n-1}(\mathbf{C}), \{H_j'\}))$. By the hypothesis of induction, we have $\sharp \mathcal{E}' \le q_{n-1}$. Thus $q \le q_{n-1}$. $\square$

Next we show the following lemma (cf. [7, Proposition 4.5]):

Lemma 4.12. *Let $h_1, \cdots, h_t \in \mathcal{M}_p^*$ and let*

$$P(w_1, \cdots, w_t) = \sum_{j=1}^{l} c_j M_j(w_1, \cdots, w_t)$$

be a nonzero Laurent polynomial in $w_1, \cdots, w_t$, where $c_j \in \mathbf{C}^*$ and M_j are distinct monomials. Suppose that $h_1^{l_1} \cdots h_t^{l_t} \notin \mathbf{C}^*$ for any $(l_1, \cdots, l_t) \in \mathbf{Z}^t - \{0\}$ and $l \leq (n+2)!$. Then $P(h_1, \cdots, h_t) \not\equiv 0$.

Proof. Assume that $P(h_1, \cdots, h_t) \equiv 0$. By Lemma 2.3, it follows that the ratio

$$M_j(h_1, \cdots h_t)/M_{j'}(h_1, \cdots, h_t)$$

is constant for some distinct j, j' and hence

$$h_1^{\nu_1{}^0} \cdots h_t^{\nu_t{}^0} / h_1^{\nu_1{}^1} \cdots h_t^{\nu_t{}^1}$$

is constant for some $(\nu_1^0, \cdots, \nu_t^0) \neq (\nu_1^1, \cdots, \nu_t^1)$. Thus we have $h_1^{\mu_1} \cdots h_t^{\mu_t} \in \mathbf{C}^*$, where $\mu_j = \nu_j^0 - \nu_j^1$ for $1 \leq j \leq t$. This contradicts the assumption. $\square$

Now we consider a multiplicative group $G = \mathcal{M}_p^*/\mathbf{C}^*$. It is clear that G is a torsion free abelian group. For $h \in \mathcal{M}_p^*$, we denote by $[h]$ the equivalence class containing h. Let G_0 be a finitely generated subgroup of G with all $[h_{ij}] \in G_0$ and $[\eta_1], \cdots, [\eta_t]$ a basis for G_0 over $\mathbf{Z}$. Then h_{ij} can be written as

$$h_{ij} = c_{ij} \eta_1^{l_{ij}^1} \cdots \eta_t^{l_{ij}^t}, \tag{4.13}$$

where $c_{ij} \in \mathbf{C}^*$ and $l_{ij}^1, \cdots, l_{ij}^t \in \mathbf{Z}$. For integers $m_1, \cdots, m_t$, we set

$$l_{ij} = l_{ij}^1 m_1 + \cdots + l_{ij}^t m_t.$$

We choose $m_1, \cdots, m_t$ such that $l_{ij} \neq l_{i'j'}$ whenever $(l_{ij}^1, \cdots, l_{ij}^t) \neq (l_{i'j'}^1, \cdots, l_{i'j'}^t)$. We show the existence of such integers $m_1, \cdots, m_t$ by induction on t (cf. [7, p. 2]). The case $t = 1$ is trivial. Assume that our assertion holds up to $t-1$ and we can take integers $m_1, \cdots, m_{t-1}$ with the property that $l_{ij}^* \neq l_{i'j'}^*$ if $(l_{ij}^1, \cdots, l_{ij}^{t-1}) \neq (l_{i'j'}^1, \cdots, l_{i'j'}^{t-1})$, where

$$l_{ij}^* = l_{ij}^1 m_1 + \cdots + m_{t-1} l_{ij}^{t-1}.$$

Then it is easy to see that there exist only finitely many integers m_t such that $m_1, \cdots, m_t$ do not satisfy the above property. Thus we have the desired conclusion. Without loss of generality, after multiplying a row or column of the matrix Λ by a common elements in $\mathcal{M}_p^*$, we may assume that $l_{ij} \geq 0$ for all i, j. By (4.7) and (4.13), we have

$$\det(c_{ij}\eta_1^{l_{ij}^1} \cdots \eta_t^{l_{ij}^t}; 1 \leq i, j \leq p_0) \equiv 0. \tag{4.14}$$

The relation (4.14) can be written as

$$\sum_{j=1}^{l} c_j M_j(\eta_1, \cdots, \eta_t) \equiv 0,$$

where

$$P(w_1, \cdots, w_t) := \sum_{j=1}^{l} c_j M_j(w_1, \cdots, w_t)$$

is a Laurent polynomial in $w_1, \cdots, w_t$ with $l := (n+2)!$. By Lemma 4.12, it follows that $P = 0$ as polynomial. Hence (4.14) remains valid if we substitute monomials $\eta_i = w_i^{m_i}$ in one variable $w(1 \leq i \leq p_0)$. Thus we have

$$\det(P_{ij}(w); i = 1, \cdots p_0, j = j_1, \cdots, j_{p_0}) \equiv 0$$

for all $j_1, \cdots, j_{p_0}$, where $P_{ij}(w) := c_{ij}w^{l_{ij}}$.

We now state a lemma on monomials due to H. Fujimoto. We consider $p_0 \times q$ matrices $\Pi = (P_{ij}(w))$ with monomials $P_{ij}(w) = c_{ij}w^{l_{ij}}$ as entries. By rank Π we mean the rank of matrix in the field $\mathbf{C}(w)$ of rational functions.

Lemma 4.15 ([10, Main Lemma]). *For each q_0 (≥ 1), there exists a positive integer $Q(p_0; q_0)$ depending only on p_0 and q_0 with the following property:*

Suppose that $q > Q(p_0; q_0)$ and $\operatorname{rank}\Pi < p_0$. *Then there exists a positive integer r depending on Π with $2 \leq r \leq p_0$ such that, after changing indices,*

$$l_{i1} - l_{i'1} = l_{i2} - l_{i'2} = \cdots = l_{iq_0} - l_{i'q_0} \tag{4.16}$$

for all i, i' with $1 \leq i \leq i' \leq r$ and

$$\operatorname{rank}(P_{ij}(w); 1 \leq i \leq r, 1 \leq j \leq q_0) < r. \tag{4.17}$$

For the proof, see §3 in [10].

Let $q_0 = q_{n-1} + 1$ and set $q_n = Q(p_0; q_0)$. Suppose that $q > q_n$. By Lemma 4.15, we have (4.16) and (4.17). Hence $h_{ij}(1 \leq i \leq p_0, 1 \leq j \leq q_0)$ satisfy the assumption of Lemma 4.8. Thus we have $q_0 \leq q_{n-1}$. This contradicts the choice of $q_0(= q_{n-1} + 1)$. Therefore we have $q \leq q_n$. This completes the proof of Theorem 4.1.

REMARK 4.18. The proof of Main Lemma in [10] and the above argument give us a way of actually finding an upper bound for $\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$. Indeed, we first note that the constant $Q(p_0; q_0)$ is defined as follows. For each $q_0(\geq 1)$, we set $Q(2; q_0) = q_0 - 1$. Assume that there exist $Q(2; q_0), \cdots, Q(p_0 - 1; q_0)$ with the conclusion of Lemma 4.15. Set

$$q^* = \max\{q_0, Q(2; q_0), \cdots, Q(p_0 - 1; q_0)\}.$$

Moreover we set $q'_s (1 \leq s \leq p_0)$ by

$$q'_1 = q^* \qquad \text{and} \qquad q'_s = q^* + p_0 + q'_{s-1}(p_0!)^2$$

inductively. Then we define $Q(p_0; q_0) = q'_s$ (see [10, p. 534]). On the other hand, by the above proof of Theorem 4.1, we have the constants

$$q_1 = 2 \qquad \text{and} \qquad q_n = Q(n+2; q_{n-1} + 1)$$

such that $\sharp\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\})) \leq q_n$ for each n. For instance, we have

$$\sharp\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_2(\mathbf{C}), \{H_j\})) \leq q_2 = 786800531602.$$

We do not know whether the upper bound q_n is sharp or not in the case of $n \geq 2$. It is an interesting problem to determine the least upper bound for the numbers of mappings in $\mathcal{E}(p; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_n(\mathbf{C}), \{H_j\}))$.

5. A finiteness theorem for meromorphic mappings into M

In this section we give the proof of our main theorem. Namely, we prove the following finiteness theorem:

Theorem 5.1. *If* $\operatorname{rank}\Psi = \dim M$ *and* $d > (s+1)!\{(s+1)! - 2\}$, *then the number of mappings in* $\mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$ *is bounded by a constant depending only on* D.

Proof. Let $f,\ g \in \mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$. Then we have a relation

$$\sum_{j=1}^{s} c_j\sigma_j \circ f - \alpha \sum_{j=s+1}^{2s} c_j\sigma_j \circ g \equiv 0, \tag{5.2}$$

where $\alpha \in \mathcal{M}_d^*$ and $c_{s+j} := c_j, \sigma_{s+j} := \sigma_j$ for $j = 1, \cdots, s$. Applying Lemma 2.3 to the relation (5.2), we have a decomposition $\{1, \cdots, 2s\} = \cup I_\nu$ of indices. Since f and g are analytically nondegenerate, we have that $I_\nu = \{j, s+k\}$ $(1 \leq j, k \leq s)$ for all ν. Thus we obtain

$$\sigma_j \circ f = \alpha a_j \sigma_{\tau(j)} \circ g \quad (a_j \in \mathbf{C}^*) \tag{5.3}$$

for $1 \leq j \leq s$, where τ is a permutation of $\{1, \cdots, s\}$. Let Γ be the subgroup of $\mathrm{Aut}(\mathbf{P}_{s-1}(\mathbf{C}))$ generated by all diagonal matrices, that is, an automorphism T of $\mathbf{P}_{s-1}(\mathbf{C})$ belongs to Γ if and only if

$$T((\zeta_1 : \cdots : \zeta_s)) = (\lambda_1 \zeta_1 : \cdots : \lambda_s \zeta_s)$$

for some $\lambda_1, \cdots, \lambda_s \in \mathbf{C}^*$, where $\{\zeta_1, \cdots, \zeta_s\}$ is a homogeneous coordinate system in $\mathbf{P}_{s-1}(\mathbf{C})$. We set

$$(\Psi \circ g)^\tau = (\sigma_{\tau(1)} \circ g, \cdots, \sigma_{\tau(s)} \circ g).$$

Then the relation (5.3) yields that

$$(\Psi \circ f)(z) = T((\Psi \circ g)^\tau(z))$$

for some $T \in \Gamma$ and for $z \in \mathbf{C}^m - (I(f) \cup I(g))$ with $f(z)$, $g(z) \notin I(\Psi)$. We note that

$$E \equiv (\Psi \circ f)^* H \pmod{d},$$

where $H := \{c_1 \zeta_1 + \cdots + c_s \zeta_s = 0\}$.

The above argument shows that there exist finitely many linearly nondegenerate meromorphic mappings $\varphi_1, \cdots, \varphi_t : \mathbf{C}^m \to \mathbf{P}_{s-1}(\mathbf{C})$ with $t \leq s!$ that satisfy the following property: For arbitrary $f \in \mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$, there exists j ($1 \leq j \leq t$) and $T \in \Gamma$ such that $\Psi \circ f = T \circ \varphi_j$. For $1 \leq j \leq t$, we define

$$\mathcal{F}_j = \{f \in \mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D)); \Psi \circ f = T \circ \varphi_j \text{ for some } T \in \Gamma\}.$$

To prove that $\mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$ is finite, it suffices to show that $\mathcal{F}_j$ is finite for each j. Let j ($1 \leq j \leq t$) be fixed. We define effective divisors $E_1, \cdots, E_{s+1}$ on $\mathbf{C}^m$ as follows. Let E_k be the zero divisor of φ_k^j for $1 \leq k \leq s$ and $E_{s+1} = E$, where $(\varphi_1^j, \cdots, \varphi_s^j)$ is a reduced representation of φ_j. Take the following hyperplanes in general position in $\mathbf{P}_{s-1}(\mathbf{C})$:

$$H_1 = \{\zeta_1 = 0\}, \cdots, H_s = \{\zeta_s = 0\},$$
$$H_{s+1} = H.$$

Let $\mathcal{E} := \mathcal{E}(d; (\mathbf{C}^m, \{E_j\}), (\mathbf{P}_{s-1}(\mathbf{C}), \{H_j\}))$ be the set of all linearly nondegenerate meromorphic mappings $\varphi : \mathbf{C}^m \to \mathbf{P}_{s-1}(\mathbf{C})$ such that $\varphi^* H_k \equiv E_k \pmod{d}$ for $1 \leq k \leq s+1$. Thanks to Theorem 4.1, $\sharp\mathcal{E}$ is bounded by a constant depending only on s. For $f \in \mathcal{F}_j$, by the definition of $\mathcal{F}_j$, we have $\Psi \circ f \in \mathcal{E}$. Let f_0 be an arbitrary meromorphic mapping in $\mathcal{F}_j$. Then it is easy to see that

$$\mathcal{F}_j(f_0) := \{f \in \mathcal{F}_j; \Psi \circ f = \Psi \circ f_0\}$$

is finite. Indeed, since the restriction of Ψ to an open dense subset W of M is a locally biholomorphic mapping, there exists a positive integer e_0 such that $\sharp\Psi^{-1}\Psi(w) \leq e_0$ for each point w in W. Suppose that there exist mutually distinct meromorphic mappings $f_0, \cdots, f_p \in \mathcal{F}_j(f_0)$. Set

$$W' = \{z \in \mathbf{C}^m; f_j(z) \in W \text{ for all } j \text{ and } f_j(z) \neq f_{j'}(z) \text{ for } 0 \leq j < j' \leq p\}.$$

Then W' is an open dense subset of $\mathbf{C}^m$. For $z_0 \in W'$, we have $f_0(z_0) \in W$ and

$$\{f_0(z_0), \cdots, f_p(z_0)\} \subseteq \Psi^{-1}\Psi(f_0(z_0)).$$

Hence $p+1 \leq e_0$ and $\sharp\mathcal{F}_j(f_0) \leq e_0$. Therefore $\sharp\mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$ is bounded by a constant depending only on D. □

By Theorem 5.1, we have the following corollary (cf. [9, IV, Theorem 4.3]):

Corollary 5.4. *Let γ be an automorphism of $\mathbf{C}^m$ and let $f : \mathbf{C}^m \to M$ be an analytically nondegenerate meromorphic mapping. Suppose that* $\operatorname{rank}\Psi = \dim M$ *and* $\gamma^* f^* D \equiv f^* D \pmod{d}$. *If $d > (s+1)!\{(s+1)! - 2\}$, then there exists a positive integer j_0 depending only on D such that $f \circ \gamma^{j_0} = f$, where $\gamma^j = \gamma \circ \cdots \circ \gamma$ (j-times) for a positive integer j.*

Proof. Take $E = f^*D$. By the assumption, we have $f \circ \gamma^j \in \mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$ for any positive integers j. Since $\sharp\mathcal{F}^*(d; (\mathbf{C}^m, E), (M, D))$ is bounded by a constant depending only on D, it follows that $f \circ \gamma^{j_1} = f \circ \gamma^{j_2}$ for some j_1, j_2 with $j_1 < j_2$. We take the smallest j_1, j_2 with the above property. Hence we see $f \circ \gamma^{j_0} = f$ for $j_0 = j_2 - j_2$. □

For the family

$$\mathcal{F}^*((\mathbf{C}^m, E), (M, D)) := \mathcal{F}^*(0; (\mathbf{C}^m, E), (M, D)),$$

we have the following theorem by making use of Theorem 4.2:

Theorem 5.5. *Suppose that* $\operatorname{rank}\Psi = \dim M$. *If $d > 4s(s-1)$, then the number of mappings in $\mathcal{F}^*((\mathbf{C}^m, E), (M, D))$ is bounded by a constant depending only on D.*

We give here examples of pairs (M, D) satisfying the assumption of the above theorems. We first consider in the case where $M = \mathbf{P}_n(\mathbf{C})$. In the following two examples, let $M = \mathbf{P}_n(\mathbf{C})$ and $\{w_1, \cdots, w_{n+1}\}$ a homogeneous coordinate system in $\mathbf{P}_n(\mathbf{C})$. As usual, we denote by $[H]$ the hyperplane bundle over $\mathbf{P}_n(\mathbf{C})$.

EXAMPLE 5.6. Let D be a Fermat hypersurface defined by

$$w_1^d + \cdots + w_{n+1}^d = 0.$$

Put $\sigma_j = w_j^d$ and $H_j = \{w_j = 0\}$ for $1 \le j \le n+1$. Then $\sigma_j \in \Gamma(\mathbf{P}_n(\mathbf{C}), [H]^{\otimes d})$ and $(\sigma_j) = dH_j$. It is easy to see that $\operatorname{rank}\Psi = n$.

EXAMPLE 5.7. Let $\{M_j(w_1, \cdots, w_{n+1})\}_{j=1}^s$ be a set of monomials with nonnegative rational exponents that is $(n+1)$-admissible (for the definition, see [17]). Let l be the smallest positive integer such that all exponents of $M_1^l, \cdots, M_s^l$ are integers. Put $\sigma_j = M_j^{ld}$ and $D_j = \{M_j^l = 0\}$ for $1 \le j \le s$. Then $\sigma_j \in \Gamma(\mathbf{P}_n(\mathbf{C}), [H]^{\otimes ld})$ and $(\sigma_j) = dD_j$. Since $\{M_j(w_1, \cdots, w_{n+1})\}_{j=1}^s$ is $(n+1)$-admissible, we may assume that $M_j = w_j$ for $1 \le j \le n+1$. Hence $\operatorname{rank}\Psi = n$.

Next we give an example of (M, D) such that M is other than $\mathbf{P}_n(\mathbf{C})$. The following example is due to J. Noguchi:

EXAMPLE 5.8. Let E_1 and E_2 be smooth elliptic curves. We denote by e_1 (resp. e_2) the identity of an abelian group E_1 (resp. E_2). Let p_1 (resp. p_2) be a d-torsion point in E_1 (resp. E_2). Let $L_i = [p_i]^{\otimes d}$ be the line bundle over E_i determined by a divisor de_i for $i = 1, 2$. By Abel's theorem, de_i and dp_i are linearly equivalent. Hence there exist holomorphic sections φ_0, $\varphi_1 \in \Gamma(E_1, L_1)$ and ψ_0, $\psi_1 \in \Gamma(E_2, L_2)$ such that $(\varphi_0) = de_1, (\varphi_1) = dp_1, (\psi_0) = de_2$ and $(\psi_1) = dp_2$. Set $M = E_1 \times E_2$. We define a line bundle $L \to M$ by $L = \pi_1^* L_1 \otimes \pi_2^* L_2$, where $\pi_i : M \to E_i$ are the natural projections. Put $\sigma_1 = \pi_1^*\varphi_0 \otimes \pi_2^*\psi_0, \sigma_2 = \pi_1^*\varphi_0 \otimes \pi_2^*\psi_1, \sigma_3 = \pi_1^*\varphi_1 \otimes \pi_2^*\psi_0$ and $\sigma_4 = \pi_1^*\varphi_1 \otimes \pi_2^*\psi_1$. Then $\sigma_j \in \Gamma(M, L)$ for all j. By the construction of σ_j, it is easy to see that $(\sigma_j) = dD_j$ for some effective divisors D_j on M and rank $\Psi = 2$.

6. A unicity theorem

In this section we give a unicity theorem for meromorphic mappings. We first recall the definition of big line bundle.

DEFINITION 6.1. A line bundle $L \to M$ is said to be *big* provided that

$$\dim_{\mathbf{C}} \Gamma(M, L^{\otimes \nu}) \ge C\nu^{\dim\ M}$$

for some positive constant C and for all sufficiently large positive integers ν.

We now show the following unicity theorem (cf. [1] and [4]):

Theorem 6.2. *Assume that there exist big line bundles $L_j \to M (1 \le j \le s)$ such that $L = L_j^{\otimes d}, 1 \le j \le s$, and $\sigma_j = \tau_j^{\otimes d}$ for some holomorphic sections τ_j of $L_j \to M$. Let f, $g : \mathbf{C}^m \to M$ be analytically nondegenerate meromorphic mappings whose ranks are not less than μ. Suppose that the following conditions are satisfied:*

(1) $\operatorname{rank}\Psi = \dim M$.

(2) $\bigcap_{j=1}^s \operatorname{Supp}(\sigma_j) = \emptyset$.

(3) *$f^{-1}(D) = g^{-1}(D) \neq \emptyset$ as point sets (say Z).*
(4) *$f = g$ on $Z - (I(f) \cup I(g))$.*
Then there exists a positive integer d_0 depending only on L_j $(1 \leq j \leq s)$ such that if $d > (s-\mu)(s+d_0)$, $f = g$ on $\mathbf{C}^m$.

For the proof of Theorem 6.2, we need two lemmas. The following lemma is well known (cf. [16, p. 42]):

Lemma 6.3 (Kodaira's Lemma). *Let L_1 and L_2 be line bundles over M. Suppose that L_1 is big. Then there exists a positive integer ν such that*

$$\Gamma(M, L_1^{\otimes \nu} \otimes L_2) \neq \{0\}.$$

We next prove an inequality of second main theorem type as follows:

Lemma 6.4. *Let L_j be as in Theorem 6.2. Let $f : \mathbf{C}^m \to M$ be an analytically nondegenerate meromorphic mapping with $\operatorname{rank} f \geq \mu$. Put $\widetilde{T}_f(r) = \max_{1 \leq j \leq s} T_f(r, L_j)$. Suppose that $\operatorname{rank} \Psi = \dim M$ and $\bigcap_{j=1}^{s} \operatorname{Supp}(\sigma_j) = \emptyset$. Then*

$$\{d - s(s-\mu)\}\widetilde{T}_f(r) \leq N_{s-\mu}(r, f^*D) + \widetilde{S}_f(r),$$

where

$$\widetilde{S}_f(r) = O(\log \widetilde{T}_f(r)) + o(\log r)$$

except on a Borel subset $E \subseteq [1, +\infty)$ with finite measure.

Proof. We first note that $I(\Psi) = \emptyset$. We define a meromorphic mapping $F : \mathbf{C}^m \to \mathbf{P}_{s-1}(\mathbf{C})$ by $F = \Psi \circ f$. Since f is analytically nondegenerate, it is clear that F is linearly nondegenerate. Since $\operatorname{rank} \Psi = \dim M$, it is easy to see that rank F=rank f. Let

$$H_1 = \{\zeta_1 = 0\}, \cdots, H_s = \{\zeta_s = 0\},$$
$$H_{s+1} = \{c_1\zeta_1 + \cdots + c_s\zeta_s = 0\}$$

be hyperplanes in general position in $\mathbf{P}_{s-1}(\mathbf{C})$, where $\{\zeta_1, \cdots, \zeta_s\}$ is a homogeneous coordinate system of $\mathbf{P}_{s-1}(\mathbf{C})$. By Theorem (1.2), we obtain

$$T_F(r) \leq \sum_{j=1}^{s+1} N_{s-\mu}(r, F^*H_j) + S_F(r)$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{s} N_{s-\mu}(r,(\sigma_j \circ f)_0) + N_{s-\mu}(r, f^*D) + S_F(r) \\
&\leq \sum_{j=1}^{s} (s-\mu) N_1(r,(\tau_j^{\otimes d} \circ f)_0) + N_{s-\mu}(r, f^*D) + S_F(r) \\
&\leq (s-\mu) \sum_{j=1}^{s} N_1(r,(\tau_j \circ f)_0) + N_{s-\mu}(r, f^*D) + S_F(r) \\
&\leq (s-\mu) \sum_{j=1}^{s} T_f(r, L_j) + N_{s-\mu}(r, f^*D) + S_F(r) \\
&\leq s(s-\mu)\widetilde{T}_f(r) + N_{s-\mu}(r, f^*D) + S_F(r).
\end{aligned}
$$

We set $h = \sum_{i=1}^{s} |\sigma_i|^2$. Since $\bigcap_{i=1}^{s} \operatorname{Supp}(\sigma_i) = \emptyset$, it follows that h gives a metric in the line bundle $L \to M$. Then we have

$$T_f(r, L) = \int_{\partial B(r)} \log \left(\frac{f^* h}{|f^* \sigma_j|^2} \right)^{1/2} \eta + N(r, f^*(\sigma_j)) + O(1).$$

On the other hand, taking the Fubini-Study metric in $[H]$, we see

$$T_F(r) = \int_{\partial B(r)} \log \left(\frac{\sum_{i=1}^{s} |f^* \sigma_i|^2}{|f^* \sigma_j|^2} \right)^{1/2} \eta + N(r, f^*(\sigma_j)) + O(1).$$

Hence $T_f(r, L) = T_F(r) + O(1)$. Since $L = L_j^{\otimes d}$ for $1 \leq j \leq s$, it follows that

$$dT_f(r, L_j) \leq s(s-\mu)\widetilde{T}(r) + N_{s-\mu}(r, f^*D) + S(r)$$

for $1 \leq j \leq s$. Thus we have the desired conclusion. □

Proof of Theorem 6.2. Put $\Phi_{k,\nu} = \Phi_{L_k^{\otimes \nu}}$ for $1 \leq k \leq s$ and $\nu \in \mathbf{Z}^+$. Since L_k $(1 \leq k \leq s)$ are big,

$$\Phi_{k,\nu} : M \to W_{k,\nu}$$

are bimeromorphic mappings for all sufficiently large integers ν, where $W_{k,\nu} = \Phi_{k,\nu}(M)$ for $1 \leq k \leq s$ (see [14, Theorem 5]). For each $1 \leq k \leq s$, let $\nu(k)$ be the smallest positive integer such that

$$\iota_k := \Phi_{k,\nu(k)} : M \to W_{k,\nu(k)}$$

is bimeromorphic. Assume that $W_{k,\nu(k)} \subseteq \mathbf{P}_{n_k}(\mathbf{C})$. We denote by $[H]_k$ the hyperplane bundle over $\mathbf{P}_{n_k}(\mathbf{C})$. By Lemma 6.3, for each pair (j,k) with $1 \leq j,\ k \leq s$, there exists a positive integer l such that

$$\Gamma(M, L_j^{\otimes l} \otimes \iota_k^*[H]_k^{-1}) \neq \{0\}.$$

Let l_{jk} be the smallest positive integer that has the above property for each pair (j,k) and let

$$l_0 = \min_{1 \le j,k \le s} l_{jk}.$$

Without loss of generality, we may assume that $l_0 = l_{11}$. Set $n = n_1, \iota = \iota_1$ and $[H] = [H]_1$. Then it is easy to see that

$$(6.5) \qquad T_f(r, \iota^*[H]) \le l_0 \widetilde{T}_f(r) + O(1) \qquad \text{and} \qquad T_g(r, \iota^*[H]) \le l_0 \widetilde{T}_g(r) + O(1).$$

Indeed, there exists a holomorphic section $\tau \in \Gamma(M, L_1^{\otimes l_0} \otimes \iota^*[H]^{-1})$ with $f^*\tau \not\equiv 0$. By Theorem 1.1, it follows that

$$\begin{aligned} 0 &\le N(r, f^*(\tau)) \\ &\le T_f(r, L_1^{\otimes l_0} \otimes \iota^*[H]^{-1}) + O(1) \\ &= l_0 T_f(r, L_1) - T_f(r, \iota^*[H]) + O(1) \\ &\le l_0 \widetilde{T}_f(r) - T_f(r, \iota^*[H]) + O(1). \end{aligned}$$

Thus we have (6.5).

Let Δ be the diagonal of $\mathbf{P}_n(\mathbf{C})^2 = \mathbf{P}_n(\mathbf{C}) \times \mathbf{P}_n(\mathbf{C})$. We define a meromorphic mapping $\varphi : \mathbf{C}^m \to \mathbf{P}_n(\mathbf{C})^2$ by $\varphi = \iota \circ f \times \iota \circ g$. For the proof of Theorem 6.2, it suffices to show $\varphi(\mathbf{C}^m) \subseteq \Delta$. Assume the contrary. Let $\pi_j : \mathbf{P}_n(\mathbf{C})^2 \to \mathbf{P}_n(\mathbf{C})$ be the projections on the j-th factor. Set

$$\widetilde{L} = \pi_1^*[H] \otimes \pi_2^*[H].$$

Then there exists a holomorphic section $\widetilde{\sigma}$ of $\widetilde{L} \to \mathbf{P}_n(\mathbf{C})^2$ such that $\varphi^*\widetilde{\sigma} \not\equiv 0$ and $\Delta \subseteq \mathrm{Supp}(\widetilde{\sigma})$ (cf. [4, p. 354]). It is easy to see that

$$(6.6) \qquad N(r, \varphi^*(\widetilde{\sigma})) \le T_f(r, \iota^*[H]) + T_g(r, \iota^*[H]) + O(1).$$

By the assumption (4), we have

$$(6.7) \qquad N_1(r, f^*D) \le N(r, \varphi^*(\widetilde{\sigma})) \qquad \text{and} \qquad N_1(r, g^*D) \le N(r, \varphi^*(\widetilde{\sigma})).$$

By (6.6), (6.7) and Lemma 6.4, we obtain

$$(6.8) \quad \{d - s(s-\mu)\}T(r) \le 2(s-\mu)\{T_f(r, \iota^*[H]) + T_g(r, \iota^*[H])\} + S(r),$$

where $T(r) = \widetilde{T}_f(r) + \widetilde{T}_g(r)$ and $S(r) = \widetilde{S}_f(r) + \widetilde{S}_g(r)$. Thus, by (6.5) and (6.8), we have

$$(6.9) \qquad \{d - s(s-\mu)\}T(r) \le d_0(s-\mu)T(r) + S(r),$$

where $d_0 = 2l_0$. Since L_j $(1 \leq j \leq s)$ are big, we have

$$C \log r \leq \widetilde{T}_f(r) + O(1) \qquad \text{and} \qquad C \log r \leq \widetilde{T}_g(r) + O(1)$$

for some positive constant C (cf. [1, Proposition 1.2]). Thus, by (6.9), it follows that $d \leq (s - \mu)(s + d_0)$. This is absurd. Therefore $\varphi(\mathbf{C}^m) \subseteq \Delta$. $\square$

REMARK 6.10. We note that the condition (4) in Theorem 6.2 can not be removed. Let $M = \mathbf{P}_2(\mathbf{C})$ and $L = [H]^{\otimes d}$, where $[H]$ is the hyperplane bundle over $\mathbf{P}_2(\mathbf{C})$. Let D be a curve defined by

$$w_1^d + w_2^d + w_3^d = 0,$$

where $\{w_1, w_2, w_3\}$ is a homogeneous coordinate system in $\mathbf{P}_2(\mathbf{C})$. Then D satisfies the assumptions in Theorem 6.2. Let $f,\ g : \mathbf{C}^m \to \mathbf{P}_2(\mathbf{C})$ be meromorphic mappings defined by $f = (\varphi : \psi : 1)$ and $g = (\psi : \varphi : 1)$, where φ and ψ are holomorphic functions on $\mathbf{C}^m$ with $\varphi \not\equiv \psi$. Then it is clear that $f^{-1}(D) = g^{-1}(D)$ as point sets and $f \not\equiv g$. We also note that $f^*D = g^*D$ as divisors. It is an interesting problem to find more natural condition other than the condition (4).

We can also prove some theorems on propagation of analytic dependence as in [5] and [27] by making use of Lemma 6.4. For example, we have the following theorem by an argument similar to the proof of Theorem 6.2.

Theorem 6.11. *Let L and L_j be as in Theorem 6.2. Let S be a hypersurface in M^2 such that the line bundle F over M^2 defined by S is of the type $\pi_1^* F_1 \otimes \pi_2^* F_2$, where $\pi_k : M^2 \to M (k = 1, 2)$ are the natural projections and $F_k (k = 1, 2)$ are line bundles over M. For each k, let l_k be the smallest positive integer l such that $\Gamma(M, L_j^{\otimes l} \otimes F_k^{-1}) \neq \{0\}$ for some j. Set $d_0 = l_1 + l_2$. Let $f,\ g : \mathbf{C}^m \to M$ be analytically nondegenerate meromorphic mappings whose ranks are not less than μ. Suppose that the following conditions are satisfied*:

(1) $\operatorname{rank} \Psi = \dim M$.

(2) $\bigcap_{j=1}^s \operatorname{Supp}(\sigma_j) = \emptyset$.

(3) *there exists a hypersurface Z of $\mathbf{C}^m$ such that $f^{-1}(D) = g^{-1}(D) = Z$.*

(4) $(f \times g)(Z) \subseteq S$.

If $d > (s - \mu)(s + d_0)$, then $(f \times g)(\mathbf{C}^m) \subseteq S$.

We note that Theorem 6.2 is also deduced from Theorem 6.11. We omit here the details in this direction.

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