Xiao, Y. and Zhang, C.
Osaka J. Math.
56 (2019), 213–227

A CLASSIFICATION PROBLEM ON MAPPING CLASSES ON FIBER SPACES OVER TEICHMÜLLER SPACES

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(Received October 27, 2016, revised October 12, 2017)

Abstract

Let $\tilde{S}$ be an analytically finite Riemann surface which is equipped with a hyperbolic metric. Let $S = \tilde{S}\backslash\{\text{one point } x\}$. There exists a natural projection Π of the x-pointed mapping class group Mod_S^x onto the mapping class group $\mathrm{Mod}(\tilde{S})$. In this paper, we classify elements in the fiber $\Pi^{-1}(\chi)$ for an elliptic element $\chi \in \mathrm{Mod}(\tilde{S})$, and give a geometric interpretation for each element in $\Pi^{-1}(\chi)$. We also prove that $\Pi^{-1}(t_a^n \circ \chi)$ or $\Pi^{-1}(t_a^n \circ \chi^{-1})$ consists of hyperbolic mapping classes provided that $t_a^n \circ \chi$ and $t_a^n \circ \chi^{-1}$ are hyperbolic, where a is a simple closed geodesic on $\tilde{S}$ and t_a is the positive Dehn twist along a.

1. Introduction

Let S be an analytically finite Riemann surface of genus p which contains $n+1$ punctures $\{x, x_1, \cdots, x_n\}$. Assume that $2p + n > 4$. Let Mod_S^x denote the subgroup of the mapping class group $\mathrm{Mod}(S)$ that consists of isotopy classes of self-maps of S that fix x, which implies that Mod_S^x is a subgroup of $\mathrm{Mod}(S)$ with a finite index and each element of Mod_S^x can be projected, under the natural projection

$$\Pi : \mathrm{Mod}_S^x \to \mathrm{Mod}(\tilde{S}),$$

to an element of $\mathrm{Mod}(\tilde{S})$, where $\tilde{S} = S \cup \{x\}$.

Fix an element $\chi \in \mathrm{Mod}(\tilde{S})$. Consider

$$\Pi^{-1}(\chi) = \{\theta \in \mathrm{Mod}_S^x \mid \Pi(\theta) = \chi\}.$$

In the case where $\chi = \mathrm{id}$, $\Pi^{-1}(\chi)$ is a normal subgroup of Mod_S^x which can be identified with the fundamental group $\pi_1(\tilde{S}, x)$, and thus it is isomorphic to a covering group of $\tilde{S}$. In [8] Kra classified all elements in $\Pi^{-1}(\mathrm{id})$ by using the terms introduced in [3] and [11]. He also investigated on elements in $\Pi^{-1}(\chi)$ for a non-trivial element χ and showed that if $\chi \in \mathrm{Mod}(\tilde{S})$ is hyperbolic, then elements in $\Pi^{-1}(\chi)$ are either hyperbolic or pseudo-hyperbolic (see the definitions in Section 2). The problem of whether or not $\Pi^{-1}(\chi)$ contains hyperbolic mapping classes were also studied in [13, 15].

In this paper, we study the problem of classifying elements in $\Pi^{-1}(\chi)$ when $\chi \in \mathrm{Mod}(\tilde{S})$ is elliptic with prime order $m \geq 3$. We also study compositions of elliptic mapping classes and mapping classes induced by simple Dehn twists.

This paper is organized as follows. Section 2 is an overview of Teichmüller spaces and Bers fiber spaces. Main results of this paper are stated in the section also. In Section 3 we

2010 Mathematics Subject Classification. Primary 30F40; Secondary 32G05.

prove some lemmas. Section 4 is devoted to the proof of Theorem 2.6 and Theorem 2.7. In Section 5 we prove Theorem 2.8.

2. Main Results

We begin with an overview of Teichmüller spaces and some basic properties. More details can be found in Bers [3, 4].

§2.1. Let S be an analytically finite Riemann surface which is endowed with a hyperbolic metric. The *Teichmüller space* $T(S)$ is the space of equivalence classes $[\mu]$ of conformal structures $(w : S \to \mu(S))$, where $(w : S \to \mu(S))$ and $(w' : S \to \mu'(S))$ are in the same equivalence class $[\mu]$ if there is a conformal map $h : \mu(S) \to \mu'(S)$ such that $h \circ w$ is isotopic to w'.

Every self-map f of S induces an element in Mod(S). It is well known that Mod(S) acts effectively and discontinuously on $T(S)$ as a group of biholomorphisms when $2p + n > 4$. Following Thurston [11], a non-periodic self-map f of S is called reducible if there is a non-contractible homotopically independent curve system $\{\gamma_1, \ldots, \gamma_s\}$, $s \geq 1$, on S so that γ_i is not homotopic to γ_j whenever $i \neq j$ and $1 \leq i, j \leq s$, and for each γ_i, $f(\gamma_i)$ is homotopic to a γ_j. A self-map f of S is called irreducible if no such system can be found. By [11], f is irreducible if and only if it is isotopic to a pseudo-Anosov map (see [11] for the definition of a pseudo-Anosov map).

The mapping class group Mod(S) can be viewed as a group of biholomorphisms of $T(S)$. Let $\langle\cdot,\cdot\rangle$ denote the Teichmüller distance. By introducing the index

$$a(\sigma) = \inf\{\langle y, \sigma(y)\rangle \text{ for } y \in T(S)\},$$

Bers [4] classified elements σ of Mod(S) as follows. A mapping class σ is elliptic if $a(\sigma) = 0$ and the value is achievable; parabolic if $a(\sigma) = 0$ and the value is not achievable; hyperbolic if $a(\sigma) > 0$ and the value is achievable; or pseudo-hyperbolic if $a(\sigma) > 0$ and the value is not achievable. Remarkable theorems in [4] state that (i) σ is elliptic if and only if it is induced by a periodic map, (ii) σ is hyperbolic if and only if σ is induced by a irreducible (or pseudo-Anosov) map, and (iii) σ is parabolic or pseudo-hyperbolic if and only if σ is induced by a reducible map.

§2.2. Let $\tilde{S}, S$ be as in the introduction. Associated to each $[\mu] \in T(\tilde{S})$ is a Jordan domain $\mathscr{D}_\mu$ depending holomorphically on the equivalence class $[\mu]$. The *Bers fiber space* $F(\tilde{S})$ is the set of pairs

$$\{([\mu], z) \mid [\mu] \in T(\tilde{S}),\ z \in \mathscr{D}_\mu\}.$$

There is a holomorphic projection $\pi : F(\tilde{S}) \to T(\tilde{S})$.

By Theorem 10 of [3], there is a holomorphic bijection (Bers isomorphism, see [3]) φ of $F(\tilde{S})$ onto $T(S)$ that makes the diagram

$$\begin{array}{ccc} F(\tilde{S}) & \xrightarrow{\ \varphi\ } & T(S) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\ T(\tilde{S}) & \xrightarrow{\ \text{id}\ } & T(\tilde{S}) \end{array}$$

commutative, where $\iota : T(S) \to T(\tilde{S})$ is the forgetful map. Notice that S is of type $(p, n+1)$.

§2.3. The mapping class group $\mathrm{Mod}(\tilde{S})$ can be lifted to a group $\mathrm{mod}(\tilde{S})$ that acts biholomorphically and effectively on $F(\tilde{S})$. To do this, we let $\mathbf{D}$ be a unit disk endowed with a hyperbolic metric. Let $\chi \in \mathrm{Mod}(\tilde{S})$ be represented by f. Fix a lift $\hat{f} : \mathbf{D} \to \mathbf{D}$ under a universal covering map

$$\varrho : \mathbf{D} \to \tilde{S}. \tag{2.1}$$

Let G be the covering group. Then every lift of f is of the form $g_1 \circ \hat{f} \circ g_2$, where g_1 and $g_2 \in G$. Let $\hat{f}$ and $\hat{f}'$ be two lifts of self-maps of $\tilde{S}$. We say $\hat{f}$ and $\hat{f}'$ are equivalent if $\hat{f}|_{\mathbf{S}^1} = \hat{f}'|_{\mathbf{S}^1}$. The equivalence class of $\hat{f}$ is denoted by $[\hat{f}]$. The group $\mathrm{mod}(\tilde{S})$ consists of all such $[\hat{f}]$. It is known that the group G is regarded as a normal subgroup of $\mathrm{mod}(\tilde{S})$ by conjugation, which keeps each fiber of $F(\tilde{S})$ invariant.

It is important to note that the Bers isomorphism $\varphi : F(\tilde{S}) \to T(S)$ induces an isomorphism φ^* of $\mathrm{mod}(\tilde{S})$ onto Mod_S^x by carrying each element $[\hat{f}]$ to $\varphi^*([\hat{f}]) = \varphi \circ [\hat{f}] \circ \varphi^{-1}$, where we recall that Mod_S^x is a subgroup of $\mathrm{Mod}(S)$ with index $n+1$. Observe also that every element Mod_S^x can be described as a mapping class that fixes x. So an element $\theta \in \mathrm{Mod}_S^x$ naturally projects to an element χ in $\mathrm{Mod}(\tilde{S})$.

§2.4. Interestingly, there are highly non-trivial mapping classes in Mod_S^x that project to the trivial mapping class. To describe them, let $[\alpha] \in \pi_1(\tilde{S}, x)$. Then α is considered a trace of an isotopy of x. Extend the isotopy to an isotopy $\{f_t : \tilde{S} \to \tilde{S}\}$ so that $f_0 = \mathrm{id}$ and $f_1(x) = x$. Thus $f_1|_S$ defines a mapping class in Mod_S^x. Define

$$j : \pi_1(\tilde{S}, x) \to \mathrm{Mod}_S^x. \tag{2.2}$$

by sending $[\alpha]$ to the mapping class of $f_1|_S$. A closed curve $c \subset \tilde{S}$ is called to fill $\tilde{S}$ if its complement in $\tilde{S}$ is a disjoint union of topological disks and (possibly) punctured disks. Likewise, c is called semi-filling with respect to a curve system $\{\gamma_1, \cdots, \gamma_u\}$, $u \geq 1$, if c fills the component of $\tilde{S}\backslash\{\gamma_1, \cdots, \gamma_u\}$ where c resides. It was shown in [8] that $j([c])$ is hyperbolic if and only if c fills $\tilde{S}$, and $j([c])$ is pseudo-hyperbolic if and only if c is a semi-filling geodesic.

§2.5. To state our main result, we return to an elliptic mapping class $\chi \in \mathrm{Mod}(\tilde{S})$ with prime order $m \geq 3$. Let $\theta \in \Pi^{-1}(\chi)$ be induced by a self-map f_θ of S. In the case where f_θ is non-periodic reducible self-map of S, θ can be further classified as a type (I) or type (II) mapping class, where θ is called a type (I) mapping class if f_θ is reduced by the boundary $\partial\Delta$ of a twice punctured disk $\Delta \subset S$ enclosing x and another puncture of $\tilde{S}$ such that the restriction $f_\theta|_{S\backslash\Delta}$ is isotopic to a periodic self-map with prime order m. θ is called to be of type (II) if there is a curve system $\mathscr{A} = \{\alpha_1, \ldots, \alpha_v\}$, $v \geq 1$, where each α_i is non-contractible on $\tilde{S}$, such that the following three conditions hold:

- f_θ leaves invariant the component $\mathscr{R}$ of $S\backslash\mathscr{A}$ that contains x,
- $f_\theta|_{\mathscr{R}}$ is irreducible, and permutes other components of $S\backslash\mathscr{A}$, and
- f_θ^m can be expressed as $j([c])$ for a semi-filling loop c with respect to $\mathscr{A}$.

It is obvious that each type (I) or type (II) mapping class projects to a periodic mapping class of order m. Hence f has some fixed points on the compactification of $\tilde{S}$ some of which may be punctures of $\tilde{S}$. As mentioned before, for each $\chi \in \mathrm{Mod}(\tilde{S})$, $\Pi^{-1}(\chi)$ consists of elliptic, hyperbolic, and non-hyperbolic elements, where non-hyperbolic elements are either parabolic or pseudo-hyperbolic. From the definition, we know that any type (I) mapping class is parabolic, while any type (II) mapping class is pseudo-hyperbolic.

§2.6. Under our circumstances, χ is elliptic with prime order $m \geq 3$. Our main theorems below state that for any element $\theta \in \Pi^{-1}(\chi)$, if θ is not elliptic, then θ is either of type (I), or of type (II), or hyperbolic. More precisely, from Nielsen's theorem (see Ivanov [7] for example), χ has a fixed point in $T(\tilde{S})$. We assume without loss of generality that the fixed point is represented by $\tilde{S}$. Thus there is a representative f of χ that can be realized as a conformal automorphism of $\tilde{S}$ with order m.

Theorem. *For each non-trivial elliptic mapping class* $\chi \in \mathrm{Mod}(\tilde{S})$ *with prime order* $m \geq 3$*, we have:*

(1) $\Pi^{-1}(\chi)$ *contains (infinitely many) elliptic elements if and only if* f *fixes at least one point of* $\tilde{S}$*,*

(2) $\Pi^{-1}(\chi)$ *contains (infinitely many)* (I) *parabolic elements if and only if* f *fixes at least one puncture of* $\tilde{S}$*,*

(3) $\Pi^{-1}(\chi)$ *always contains (infinitely many) type* (II) *or hyperbolic mapping classes, and*

(4) *if in addition* $\tilde{S}/\langle f\rangle$ *has genus* $\tilde{p} = p/m > 1$*, then* $\Pi^{-1}(\chi)$ *contains (infinitely many) hyperbolic elements.*

§2.7. Let $\hat{f} : \mathbf{D} \to \mathbf{D}$ be a lift of f. Then $\hat{f}$ is a conformal automorphism of $\mathbf{D}$. Thus $\hat{f} \in \mathbf{PSL}(2, \mathbf{R})$ but $\hat{f}$ is not an element of G. Note that any element in $\Pi^{-1}(\chi)$ can be written in the form $\varphi^*([\hat{f}])$ for a conformal automorphism $\hat{f}$ of $\mathbf{D}$.

More information on $\Pi^{-1}(\chi)$ is contained in the following result.

Theorem. *Let* $\chi \in \mathrm{Mod}(\tilde{S})$ *be a non-trivial elliptic element with prime order* $m \geq 3$*, and let* $\theta \in \Pi^{-1}(\chi)$ *be non-elliptic which can be expressed as* $\theta = \varphi^*([\hat{f}])$ *for some conformal automorphism* $\hat{f}$ *of* $\mathbf{D}$*. Then* θ *is either hyperbolic, or of type* (I) *or of type* (II)*. More precisely, we have*

(1) θ *is of type* (I) *if and only if* $\hat{f}$ *fixes a fixed point of a parabolic element of* G*,*

(2) θ *is of type* (II) *if and only if* $\hat{f}$ *fixes a geodesic* λ_c *that can be projected to a semi-filling closed geodesic* $c \subset \tilde{S}$ *that is invariant under* f*,*

(3) θ *is hyperbolic if and only if* $\hat{f}$ *keeps invariant a geodesic* $\lambda_c \in \mathbf{D}$ *that can be projected to a filling closed geodesic* $c \subset \tilde{S}$ *that is invariant under* f*.*

§2.8. Finally, we consider some compositions of elliptic mapping classes χ and Dehn twists t_α along a simple closed curve $\alpha \subset \tilde{S}$, and study the corresponding fiber in Mod_S^x. Our last result states:

Theorem. *Let* $\chi \in \mathrm{Mod}(\tilde{S})$ *be elliptic with prime order* $m \geq 3$*. There exist simple closed geodesics* $\alpha \subset \tilde{S}$ *such that* $t_\alpha^n \circ \chi$ *and* $t_\alpha^n \circ \chi^{-1}$ *are both hyperbolic for all integers n with a finite number of exclusions. In the case where both* $t_\alpha^n \circ \chi$ *and* $t_\alpha^n \circ \chi^{-1}$ *are hyperbolic, either* $\Pi^{-1}(t_\alpha^n \circ \chi)$ *or* $\Pi^{-1}\left(t_\alpha^n \circ \chi^{-1}\right)$ *consists of hyperbolic mapping classes.*

§2.9. Remark: When $n = 0$, that is, $\tilde{S}$ is closed, it was shown in [13] that for any hyperbolic mapping class $\chi \in \mathrm{Mod}(\tilde{S})$, $\Pi^{-1}(\chi)$ consists of hyperbolic mapping classes. It is not known, however, whether $\Pi^{-1}(\chi')$ contains hyperbolic mapping class for a general mapping class χ' of $\tilde{S}$ if $n > 0$. Theorem 2.8 above provides an example that the single fiber $\Pi^{-1}(\chi')$ may contain infinitely many hyperbolic mapping classes. Another example is given in Theorem 2 of [13].

3. Some preliminary results

§3.1. Let $\theta, \theta' \in \mathrm{Mod}(S)$ be non-trivial. We call θ and θ' commuting mapping classes of S if $\theta \circ \theta'(\tau) = \theta' \circ \theta(\tau)$ for every $\tau \in T(S)$. We have

Lemma. *Suppose that θ and θ' are infinite order commuting mapping classes of S. Then θ is hyperbolic if and only if θ' is hyperbolic.*

Remark. The authors are grateful to the referee for pointing out that this result is essentially known, whose proof was given in Ivanov [7]. Here we provide with an alternate approach.

Proof. Let f_θ and $f_{\theta'}$ denote a self-maps of S that induce θ and θ', respectively. Obviously, the condition is equivalent to that $f_\theta \circ f_{\theta'}$ is isotopic to $f_{\theta'} \circ f_\theta$ on S. Suppose that $f_{\theta'}$ is reduced by a loop system $E = \{e_1, \ldots, e_k\}$. By taking a suitable power we may assume that $f_{\theta'}$ is a component map. In particular, $f_{\theta'}(e_i) = e_i$, $i = 1, \ldots, k$. Let $\mathscr{P} = \{P_1, \ldots, P_{s_0}\}$ denote all components of $S \backslash E$ on which $f_{\theta'}$ is isotopic to a pseudo-Anosov map. Let $\{Q_1, \ldots, Q_s\}$ denote all components of $S \backslash E$ on which $f_{\theta'}$ is the identity. Let $E_0 = \{e_1, \ldots, e_t\}$ be the subset of E consisting of boundary components of P_i, $i = 1, \ldots, s_0$. $E \backslash E_0$ consists of loops on each of which $f_{\theta'}$ is either the identity or a power of a non-trivial Dehn twist.

Consider the self-map $\xi = f_\theta \circ f_{\theta'} \circ f_\theta^{-1}$. Then $\xi(f_\theta(\alpha)) = f_\theta(\alpha)$, which says that ξ restricts to the identity or non-trivial Dehn twist on the loop $f_\theta(\alpha)$ for each $\alpha \in E \backslash E_0$. By hypothesis, ξ is isotopic to $f_{\theta'}$. It turns out that $f_{\theta'}$ restricts to the identity, or a non-trivial Dehn twist on $f_\theta(\alpha)$. As such, $f_\theta(\alpha)$ is also in $E \backslash E_0$. It follows that f_θ leaves invariant the set $E \backslash E_0$. If this set is not empty, we are done. Otherwise, $\mathscr{P}$ is not empty. Let $P_1 \in \mathscr{P}$. The map ξ is isotopic to pseudo-Anosov on $f_\theta(P_1)$. Since ξ is isotopic to $f_{\theta'}$, $f_\theta(P_1)$ is in $\mathscr{P}$ as well. This means that f_θ permutes P_i in $\mathscr{P}$. Thus f_θ is reduced by the boundary loops of P_i, $P_i \in \mathscr{P}$. So f_θ is reducible. Since θ and θ' are symmetric, the lemma is proved. □

§3.2. By assumption, $\tilde{S}$ is of type (p, n) with $2p + n > 4$. In particular, $\tilde{S}$ is not of type $(0,3)$, $(0,4)$, $(1,1)$, $(1,2)$, or $(2,0)$. The following lemma is merely a special case (torsion free) of Lemma 3.8 of [12].

Lemma. *Let θ be a holomorphic automorphism of the Bers fiber space $F(\tilde{S})$ that leaves each fiber invariant. Then θ coincides with an element of G.*

§3.3. Let c be a primitive, closed filling geodesic on $\tilde{S}$, which means that $[c]$ is not a power of any element of $\pi_1(\tilde{S}, x)$. Let $j : \pi_1(\tilde{S}, x) \to \mathrm{Mod}_S^x$ be defined in (2.2). By Theorem 2 of [8], $j([c])$ is hyperbolic. Hence by Lemma 3.1, any infinite order mapping class commuting with $j([c])$ must also be hyperbolic. More precisely, we have

Lemma. *Let $\theta \in \mathrm{Mod}_S^x$ be of infinite order and $\theta \neq (j([c]))^p$ for any $p \in \mathbf{Z}$. Then θ commutes with $j([c])$ if and only if $\Pi(\theta)$ is a non-trivial elliptic element and is induced by a conformal automorphism $f : \tilde{S} \to \tilde{S}$ that keeps the filling closed geodesic c invariant (as a set).*

Proof. Suppose that an infinite order element $\theta \in \mathrm{Mod}_S^x$ satisfies the condition

$$\theta \circ j([c]) = j([c]) \circ \theta. \tag{3.1}$$

Since c is a filling geodesic, from Theorem 2 of [8], $j([c])$ is hyperbolic, and thus it is induced by a pseudo-Anosov map of S. By Lemma 3.1, θ is also hyperbolic. Hence by Theorem 15.7 of [7], both θ and $j([c])$ are powers of the same hyperbolic mapping class δ of of Mod_S^x. Write

$$\theta = \delta^s \text{ and } j([c]) = \delta^r.$$

If $\delta = j([c_0])$ for a $[c_0] \in \pi_1(\tilde{S}, x)$, then $\theta = j([c_0])^s = j([c_0]^s)$. In this case, $\Pi(\theta)$ is trivial. Thus $\theta \in G$. Since G is centerless, either $\theta = (j([c]))^p$ or $\theta^p = j([c])$ for some $p \in \mathbf{Z}$. By assumption, the first case does not occur. The second case says that c is not primitive. This again contradicts the hypothesis. We conclude that $\delta \neq j([c_0])$.

Let $G_0 = \langle \delta, \varphi^*(G) \rangle$. As a subgroup of Mod_S^x, G_0 acts on $T(S)$ discontinuously. Note also that G is a normal subgroup of $\mathrm{mod}(\tilde{S})$ as biholomorphisms on $F(\tilde{S})$. The group $\mathrm{mod}(\tilde{S})$ is isomorphic to Mod_S^x under φ^*. So $\varphi^*(G)$ is a normal subgroup of Mod_S^x and thus $\varphi^*(G)$ is a normal subgroup of G_0.

From Nielsen's theorem, there is a point $\sigma \in T(\tilde{S})$ such that $\Pi(\delta)(\sigma) = \sigma$. We assume that $\sigma = [0]$ is represented by $\tilde{S}$. Consider the fiber $\mathbf{D} = \pi^{-1}([0]) \subset F(\tilde{S})$. Note that $\varphi^{-1}(\delta)|_{\mathbf{D}}$ acts as a conformal automorphism. $\varphi^{-1}(\delta)|_{\mathbf{D}}$ is an element of $\mathbf{PSL}(2, \mathbf{R})$. Denote $\hat{f} = \varphi^{-1}(\delta)|_{\mathbf{D}}$, and let $G_0' = (\varphi^*)^{-1}(G_0)|_{\mathbf{D}}$. We see that $G_0' = \left\langle \hat{f}, G \right\rangle$ acts on $\mathbf{D}$ discontinuously, $\hat{f}$ does not belong to G, and G is a normal subgroup of G_0'. In particular, we have

$$\hat{f} \circ G \circ \hat{f}^{-1} = G.$$

It follows that $\hat{f}$ can be projected to a conformal automorphism f of $\tilde{S}$ under the projection $\varrho : \mathbf{D} \to \tilde{S}$. It is also easy to see that $\mathbf{D}/G_0' = \tilde{S} / \langle f \rangle$.

By construction, $\hat{f}^r = j([c])$. As elements of $\mathbf{PSL}(2, \mathbf{R})$, both $\hat{f}$ and $j([c])$ keep a geodesic λ_c invariant. Since $\varrho(\lambda_c) = c$, $f(c) = c$, as desired.

Conversely, we assume that c is a filling geodesic on $\tilde{S}$, and $f : \tilde{S} \to \tilde{S}$ satisfies $f(c) = c$. We lift the map f to a conformal automorphism $\hat{f}$ of $\mathbf{D}$ so that $\hat{f}(\lambda_c) = \lambda_c$, where λ_c is a geodesic in $\mathbf{D}$ such that $\varrho(\lambda_c) = c$. $\hat{f}$ is a hyperbolic element of $\mathbf{PSL}(2, \mathbf{R})$. Since $\chi^m = \mathrm{id}$, by Lemma 3.2, $\hat{f}^m$ is an element g_c of G, where g_c corresponds to c under the isomorphism $\pi_1(\tilde{S}, x) \xrightarrow{\cong} G$. $[\hat{f}]$ commutes with g_c if both are considered elements of $\mathrm{mod}(\tilde{S})$. It follows that θ commutes with $j([c])$. □

§3.4. Now let $\chi \in \mathrm{Mod}(\tilde{S})$ be elliptic with prime order $m \geq 3$. Let $f : \tilde{S} \to \tilde{S}$ be a representative of χ and let $\theta \in \mathrm{Mod}_S^x$ be such that $\Pi(\theta) = \chi$. Let $f_\theta : S \to S$ be a representative of θ. Suppose that there is a subsurface $\mathscr{R}$ of S satisfying the properties:

- $x \in \mathscr{R}$, f_θ keeps $\mathscr{R}$ invariant, and
- $\partial\mathscr{R} = \{d_1, \cdots, d_u\}$, where $u \geq 1$ and d_i are also non-contractible loops on $\tilde{S}$.

Under these circumstances, we have:

Lemma. *$f_\theta|_{\mathscr{R}}$ is periodic if and only if f_θ is periodic.*

Proof. Suppose that $f_\theta|_{\mathscr{R}}$ is periodic. Then the restriction $f_\theta^n|_{\mathscr{R}}$ is the identity for some $n \in \mathbf{Z}$. On the other hand, by assumption, we know that $\Pi(\theta^m) = \chi^m$ is the identity, by

Lemma 3.2, $f_\theta^m = \varphi \circ \gamma \circ \varphi^{-1}$ for some element $\gamma \in G$. This tells us that f_θ^m leaves the identity on any component of $S \backslash \{d_1, \ldots, d_u\}$ other than $\mathscr{R}$.

If $f_\theta^m|_{\mathscr{R}}$ is not the identity, then $\{f_\theta^m|_{\mathscr{R}}\}$ is infinitely cyclic, which says that $\left(f_\theta^m|_{\mathscr{R}}\right)^q \neq \mathrm{id}$ for any integer $q \neq 0$. In particular, $\left(f_\theta^m|_{\mathscr{R}}\right)^n \neq \mathrm{id}$. But $\left(f_\theta^m|_{\mathscr{R}}\right)^n = \left(f_\theta^n|_{\mathscr{R}}\right)^m = \mathrm{id}$. This is a contradiction, showing that $f_\theta^m|_{\mathscr{R}}$ is the identity. This also implies that $m = n$.

We conclude that f_θ^m restricts to the identity on any components of $S \backslash \{d_1, \ldots, d_u\}$. It remains to exclude the case where f_θ^m is a multi-twists along some loops d_i.

Notice that d_i is non-contractible on $\tilde{S}$. Assume that $f_\theta^m|_{N_1}$ is non-trivial, where N_1 is a thin annular neighborhood of d_1. N_1 avoids the puncture x and is disjoint from any other loops d_j for $j \neq 1$. Since d_1 is non-contractible on $\tilde{S}$ and since $m \geq 3$ is a prime integer, $\mathscr{R}$ is not an x-punctured cylinder, which means that $\Pi(\theta^m)$ is non-trivial on the image of N_1 under the forgetful map. Thus χ^m is not the identity. But this contradicts that χ is periodic with order m. It follows that θ^m is the identity and hence θ is an elliptic mapping class of order m.

The converse is trivial. □

§3.5. A similar argument yields the following result:

Lemma. *Under the same notation and hypothesis of* Lemma 3.4, *if $f_\theta^m|_{\mathscr{R}}$ is a non-trivial Dehn twist along $\partial\Delta$ where $\Delta \subset \mathscr{R} \subset S$ is a twice punctured disk enclosing x and another puncture, then f_θ represents a type* (I) *mapping class.*

Proof. By definition, $f_\theta|_{\mathscr{R}}$ is a type (I) reducible on $\mathscr{R}$. By the same argument of Lemma 3.4, f_θ is itself a type (I) reducible map on S. □

§3.6. Let $\chi \in \mathrm{Mod}(\tilde{S})$ be elliptic which is represented by a conformal automorphism f of $\tilde{S}$. Suppose that χ has a prime order $m \geq 3$.

Lemma. *Let c be a simple non-contractible loop on $\tilde{S}$ such that c is not homotopic to $f(c)$. If $f^q(c)$ is homotopic to c for an integer q with* $1 < q \leq m$, *then* $q = m$.

Proof. Without loss of generality we may assume that c is a simple geodesic. Since f is conformal, $f(c)$ is also a geodesic and is not homotopic to c.

Suppose that $f^q(c) = c$. Then either $f^q|_c$ is the identity or cyclic. If $f^q|_c$ is the identity, f^q is the identity on $\tilde{S}$. So $q = m$. If $f^q|_c$ is cyclic, since m is a prime number, $f^q|_c$ must be of order m, which means that f is of order qm. But $q > 1$. This is impossible. □

§3.7. Recall that for any $[c] \in \pi_1(\tilde{S}, x)$, $j([c]) \in \mathrm{Mod}_S^x$ is the mapping class defined in (2.2). Let $f_c : S \to S$ be a suitable representative of $j([c])$ and c a representative of $[c]$.

If c is a loop around a puncture of $\tilde{S}$, f_c is a Dehn twist along a twice punctured disc Δ enclosing x. So f_c restricts to the identity on any component of $S \backslash \partial\Delta$.

If c is a simple non-contracting loop of $\tilde{S}$, f_c is a spin map which is reducible. Denote by $\mathscr{C}$ the corresponding cylinder containing x. Then f_c restricts to the identity on any component of $S \backslash \mathscr{C}$.

Consider the universal covering map (2.1). Fix a point $\hat{x}$. Let $g_c \in G$ be the element corresponding to c. Assume that $\hat{x}$ is in the axis λ_c of g_c. Construct a quasiconformal automorphism w of $\mathbf{D}$ that is supported on a thin neighborhood of λ_c and has the properties that

- $w(\hat{x}) = g_c(\hat{x})$ and
- w commutes with each element of G.

Let $W : \tilde{S} \to \tilde{S}$ be the projection of w. There is a homotopy ω_t (which is called the Ahlfors homotopy in the literature) between w and the identity so that for any $t \in [0, 1]$, ω_t commutes with each element of G. Hence ω_t can be projected to a homotopy Ω_t on $\tilde{S}$ so that Ω_0 =id and $\Omega_1 = W$. Since W fixes x, $j([c])$ is the mapping class of $\Omega_1|_S$. From this construction, we see that W is the identity outside a neighborhood of c. Let $\{e_1, \ldots, e_{k_0}\}$ be the curve system on S so that one component of $S\backslash\{e_1, \ldots, e_{k_0}\}$ is a minimum surface containing c. This means that f_c restricts to the identity on each component of $S\backslash\{e_1, \ldots, e_{k_0}\}$ that avoids c.

Putting all the information together, we summarize:

Lemma. *For any $[c] \in \pi_1(\tilde{S}, x)$, $j([c])$ is represented by a map f_c which restricts to the identity on any subsurface of S that avoids c.*

§3.8. We continue to assume that $\chi \in \text{Mod}(\tilde{S})$ is elliptic which is represented by a conformal automorphism f of $\tilde{S}$ fixing a point or a puncture of $\tilde{S}$. Let $f^* : \mathbf{D} \to \mathbf{D}$ be a (conformal) lift of f that fixes a pre-image y^* of a fixed point y of f. The point $y^* \in \partial\mathbf{D}$ if and only if y is a puncture of $\tilde{S}$.

Lemma. *With the above notation and terminology, there is a hyperbolic element $g \in G$ and an integer N such that for all $k \geq N$, $g^k \circ f^*$ are hyperbolic Möbius transformations.*

Proof. There are two cases to discuss.

Case 1. f fixes a point $y \in \tilde{S}$. In this case, for any two non-antipodal points $\alpha,\beta \in \mathbf{S}^1$, we use $[\alpha,\beta]$ (resp. (α,β)) to denote the minor closed (resp. open) arc on $\mathbf{S}^1$. Also, the Euclidean length of the segment is denoted by $\text{len}(\alpha,\beta)$. The lift f^* of f that fixes y^*, where y^* is a point in the orbit $\{\varrho^{-1}(y)\} \subset \mathbf{D}$. In this setting f^* is a Möbius transformation keeping $\mathbf{S}^1$ invariant. One may assume that $y^* = 0$ and f^* is of the form

$$z \to [\exp(2\pi i/m)]\, z, \quad z \in \mathbf{D}.$$

That is, f^* is a rotation. It sends any point $\alpha \in \mathbf{S}^1$ to a point $\beta = f^*(\alpha)$. The length $\lambda := \text{len}(\alpha,\beta)$ does not depend on α.

Let $g \in G$ be a hyperbolic element so that its axis A_g has a relatively large Euclidean length, in the sense that A_g and $f^*(A_g)$ intersect. This is achievable by Theorem 5.3.8 of Beardon [1]. Let $A_g \cap \mathbf{S}^1 = \{\text{Fix}^+(g), \text{Fix}^-(g)\}$, where $\text{Fix}^+(g)$ and $\text{Fix}^-(g)$) are repelling and attracting fixed point of g. Orient A_g so that it points from $\text{Fix}^+(g)$ to $\text{Fix}^-(g)$. It is clear that A_g divides $\mathbf{D}$ into two half-plane U and U', where $U$and U' are the half planes lying in the right and left side of A_g, respectively. Assume that U contains some diameter of $\mathbf{D}$.

We see that $\text{len}(z, g(z))$, $z \in U \cap \mathbf{S}^1$, attains its maximum value at some point $z_0 \in \mathbf{S}^1$ that is away from $\text{Fix}^+(g)$ and $\text{Fix}^-(g)$. As a point $z \in U \cap \mathbf{S}^1$ tends to either $\text{Fix}^+(g)$ or $\text{Fix}^-(g)$, $\text{len}(z,\ g(z))$ tends to zero.

Choose a sufficiently large integer N so that when $k \geq N$, the maximum value of $\text{len}(z, g^k(z))$, $z \in U \cap \mathbf{S}^1$, is $\lambda_0 > \lambda$. Let $z_0 \in U \cap \mathbf{S}^1$ be the point so that $\text{len}(z_0, g^N(z_0)) = \lambda_0$. Then it still holds that for a fixed $k > N$, $\text{len}(z,\ g^k(z))$ tends to zero, whenever $z \in U \cap \mathbf{S}^1$ tends to either $\text{Fix}^+(g^k)$ or $\text{Fix}^-(g^k)$, Now from the intermediate-value theorem in Calculus, there is a point z' in the arc $(\mathrm{Fix}^+(g), z_0)$ and point z'' in $(z_0, \mathrm{Fix}^-(g))$ so that $\mathrm{len}(z'\ g^k(z')) = \lambda$ and $\mathrm{len}(z''\ g^k(z'')) = \lambda$. Since $\lambda = \mathrm{len}(z', f^*(z')) = \mathrm{len}(z'', f^*(z''))$, we conclude that

$$g^k(z') = f^*(z') \text{ and } g^k(y'') = f^*(z''),\ \ z', z'' \in U \cap \mathbf{S}^1.$$

It follows that $(g^k)^{-1} \circ f^*(z') = g^{-k} \circ f^*(z') = z'$ and $(g^k)^{-1} \circ f^*(z'') = g^{-k} \circ f^*(z'') = z''$, which says that $g^{-k} \circ f^*$ has two distinct fixed points in $U \cap \mathbf{S}^1$. Notice that g^{-k} and f^* are Möbius transformations. We assert that $g^{-k} \circ f^*$ must be a hyperbolic Möbius transformation. Since $G_0 = \langle G,\ f^* \rangle$ is discrete, as a hyperbolic element of G_0, $g^{-k} \circ f^*$ cannot fix any fixed points of parabolic elements of G_0. Similarly, $g^k \circ f^*$ has two distinct fixed points in $U' \cap \mathbf{S}^1$. So $g^k \circ f^*$ is also a hyperbolic element.

Case 2. f fixes a puncture y of $\tilde{S}$. In this case, we take the upper half space model for the hyperbolic space $\mathbf{H}$. The boundary is $\partial \mathbf{H} = \mathbf{R}$. We assume that ∞ lies above y under (2.1), and the parabolic element γ that fixes ∞ is $z \to z + 1$. The lift f^* fixes ∞. Under the assumption, since f^m is the identity, f^* sends each point z in $\mathbf{R} \cup \mathbf{H}$ to $z + k/m$ for some integer k that is prime to m.

We then use the same argument in Case 1. Let $g \in G$ be a hyperbolic element so that $\mathrm{len}(\mathrm{Fix}^+(g), \mathrm{Fix}^-(g)) > 1$ (by Theorem 5.3.8 of Beardon [1]). Its axis A_g divides $\mathbf{R} \cup \mathbf{H}$ into two regions. Let U be the bounded region. For a large but fixed k, there are two distinguished points $z', z'' \in U \cap \mathbf{R}$ so that $\mathrm{len}(z',\ g^k(z')) = 1$ and $\mathrm{len}(z'',\ g^k(z'')) = 1$. It follows that $g^{-k} \circ f^* \in G_0$ is hyperbolic. Similarly, $g^k \circ f^* \in G_0$ is also hyperbolic with two distinct fixed points in $U' \cap \mathbf{R}$. Once again, both $g^{-k} \circ f^*$ and $g^k \circ f^*$ cannot fix any fixed points of parabolic elements of G; otherwise, G_0 would not be discrete. Lemma 3.8 is proved. □

4. Proof of Theorem 2.6 and Theorem 2.7

§4.1. We first prove Theorem 2.7. Theorem 2.6 will be derived easily. As usual, we let θ be induced by a self-map f_θ of S. First we assume that f_θ is completely reduced by

$$\mathscr{L} = \{c_1, \cdots, c_u\},\ \ u \geq 1, \tag{4.1}$$

(see Bers [3] for the definition of completely reducible map. From Lemma 5.9 of [3], each reducible mapping class is completely reducible). Let σ_t be a smooth flow in $T(S)$ that is obtained from pinching all the loops in (4.1) to cusps. Let $\partial T(S)$ denote the Bers boundary of $T(S)$ (see [2]). Let $\{x_i\} \in \sigma_t \subset T(S)$ be any discrete instances represented by S_i so that $x_i \to \partial T(S)$. Let $\tilde{x}_i = \pi \circ \varphi^{-1}(x_i) \in T(\tilde{S})$ be represented by $\tilde{S}_i$. Then $\tilde{S}_i$ is obtained from S_i by filling in the puncture x. Let

$$\mathcal{R}(\tilde{S}) = T(\tilde{S})\ /\ \mathrm{Mod}(\tilde{S})$$

denote the Riemann moduli space of $\tilde{S}$, and let $\varpi : T(\tilde{S}) \to \mathcal{R}(\tilde{S})$ be the natural projection.

Let $\Lambda \subset \mathcal{R}(\tilde{S})$ and $\Lambda' \subset T(\tilde{S})$ denote the sequences $\{\varpi(\tilde{x}_i)\}$ and $\{\tilde{x}_i\}$, respectively. There are two cases to consider.

Case 1. Λ lies in a compact subset of $\mathcal{R}(\tilde{S})$. In this case, there is a sub-sequence $\{\tilde{x}_i\} \subset \Lambda'$, which may tend to the boundary $\partial T(\tilde{S})$, yet $\tilde{S}_i$ does not possess short closed geodesics. By Lemma 2 of [13], all loops in (4.1) bound disks $D_j \subset S$ that encloses x and another puncture

x_j. But all loops in (4.1) are disjoint and homotopically independent. It follows that $u = 1$, which means that (4.1) consists of only one single loop, say c_1, that is the boundary of a twice punctured disk Δ enclosing x and a puncture x_1 of $\tilde{S}$.

Since $\Pi(\theta) = \chi \in \mathrm{Mod}(\tilde{S})$ is elliptic with prime order m, $f_\theta|_{S\setminus\Delta}$ is isotopic to a periodic self-map of prime order m. Hence $\theta = \varphi^*([\hat{f}])$ is a type (I) mapping class.

Case 2. Λ is not compact in $\mathcal{R}(\tilde{S})$. In this case, we claim that θ is a type (II) mapping class as defined in §2.5. Indeed, we pick an arbitrary loop, say c_i, in (4.1) that is non-contractible on $\tilde{S}$ (the image of c_i on $\tilde{S}_i$ is denoted by $\tilde{c}_i$). By assumption, $m \geq 3$ is a prime number. Consider the image loops $\mathscr{L}^i = \{f_\theta^j(c_i),\ j \geq 0\}$. There is a possibility that $\mathscr{L}^i$ consists of a simple loop c_i only. That is, f_θ keeps c_i invariant. But if c_i is not homotopic to $f_\theta(c_i)$, by Lemma 3.6, c_i is not homotopic to $f_\theta^q(c_i)$ for any q with $1 < q < m$.

We claim that c_i is homotopic to $f_\theta^m(c_i)$. Indeed, $\tilde{c}_i$ is homotopic to $f^m(\tilde{c}_i)$. So if c_i is not homotopic to $f_\theta^m(c_i)$, then since c_i and $f_\theta^m(c_i)$ lie in $\mathscr{L}$, they are disjoint. Hence c_i and $f_\theta^m(c_i)$ must bound a cylinder $\mathscr{P}_i$ that encloses the puncture x. It follows that $f_\theta(\mathscr{P}_i)$ is also a cylinder that encloses the puncture x as well.

This means that either $\mathscr{P}_i = f_\theta(\mathscr{P}_i)$, or $\mathscr{P}_i$ and $f_\theta(\mathscr{P}_i)$ intersect. In the later case, the boundary $\partial\mathscr{P}_i$ of $\mathscr{P}_i$ is $\{c_i,\ f_\theta^m(c_i)\}$ and the boundary $\partial f_\theta(\mathscr{P}_i)$ of $f_\theta(\mathscr{P}_i)$ is $\{f_\theta(c_i),\ f_\theta(f_\theta^m(c_i))\}$. Since c_i and $f_\theta(c_i)$ is disjoint, c_i must intersect with $f_\theta(f_\theta^m(c_i))$. This is impossible. In the former case, if c_i is homotopic to $f_\theta(c_i)$, then this contradicts Lemma 3.6; if c_i is homotopic to $f_\theta(f_\theta(c_i))$, then $f_\theta(c_i)$ is homotopic to $f_\theta^m(c_i)$, which says that c_i is homotopic to $f_\theta^{m-1}(c_i)$, which again contradicts Lemma 3.6.

We conclude that c_i is homotopic to $f_\theta^m(c_i)$, and that (4.1) is a disjoint union

$$\bigcup_{i=0}^{N} \mathscr{L}^i,\ N < \infty, \tag{4.2}$$

where $\mathscr{L}^i$ is either an m-cycle of loops, or consists of a simple loop only.

Note that every $\mathscr{L}^i$ in (4.2) consists of either dividing loops or non-dividing loops. Let $\mathscr{R}^*$ be the component of $S - \bigcup \mathscr{L}^i$ containing the puncture x. We claim that $\mathscr{R}^*$ is invariant under f_θ. To see this, we construct $\mathscr{R}^*$ in the following steps:

(A) If $\mathscr{L}^1$ consists of non-dividing loops and $S\backslash\{\mathscr{L}^1\}$ is a single component, we define $\mathscr{R}_1$ to be $S\backslash\{\mathscr{L}^1\}$. If $\mathscr{L}^1$ consists of dividing loops, or non-dividing loops but $S\backslash\{\mathscr{L}^1\}$ has more than one components, we let $\mathscr{R}_1$ be the component of $S\backslash\{\mathscr{L}^1\}$ that the puncture x resides. It is easy to see that $\mathscr{R}_1$ is invariant under f_θ.

(B) If $\mathscr{R}_1$ contains a loop in $\mathscr{L}^2$ then since $\mathscr{R}_1$ is invariant under f_θ, $\mathscr{R}_1$ contains every element in $\mathscr{L}^2$ and we follow the same procedure as in (A) to obtain $\mathscr{R}_2$. If $\mathscr{R}_1$ does not contain any loop in $\mathscr{L}^2$, then we simply ignore $\mathscr{L}^2$ and examine $\mathscr{L}^3$, and so on.

Since N is finite, the process terminates after finite many steps. The resulting subsurface is $\mathscr{R}^*$. From the construction we see that $\mathscr{R}^*$ is invariant under f_θ and encloses the puncture x.

Our next claim is that $f_\theta|_{\mathscr{R}^*}$ is irreducible (or pseudo-Anosov) self-map of $\mathscr{R}^*$. Suppose that $f_\theta|_{\mathscr{R}^*}$ is periodic, by Lemma 3.4, f_θ is periodic. Thus θ is elliptic. This is a contradiction. Next we assume that $f_\theta|_{\mathscr{R}^*}$ is reducible. The only possibility for this to occur is that $f_\theta|_{\mathscr{R}^*}$ is reduced by the single loop c_1, where c_1 is the boundary of a twice punctured disk Δ enclosing x and another puncture (otherwise, we would continue the above procedure (A)). This leads to three possibilities:

- $f_\theta^m|_{\mathscr{R}^*}$ is the Dehn twist along c_1,
- $f_\theta^m|_{\mathscr{R}^*}$ is irreducible (or pseudo-Anosov) on $S\backslash\Delta$, or
- $f_\theta^m|_{\mathscr{R}^*}$ is irreducible (or pseudo-Anosov) on Δ but the identity on $S\backslash\Delta$.

If the first situation occurs, by Lemma 3.5, f_θ is of type (I). We claim that the second situation does not occur. Suppose for the contrary. By filling in the puncture x, c_1 shrinks to a puncture, and all other loops in (4.1) is non-contractible. So $f_\theta^m|_{\mathscr{R}^*}$ descends to a pseudo-Anosov self-map on a component of $\tilde{S}\backslash\{\tilde{c_2},\dots,\tilde{c_u}\}$. This contradicts that χ^m is the identity.

Otherwise, we conclude that the third case must occur. That is, $f_\theta|_{\mathscr{R}^*}$ is irreducible (or pseudo-Anosov). Hence ${f_\theta}^m|_{\mathscr{R}^*}$ is also irreducible (or pseudo-Anosov) and f_θ^m restricts to the identity on any other component of $S\backslash\mathscr{L}$. Since f_θ^m projects to the identity, by Theorem 2 of [8], $(f_\theta)^m = j([c])$, where c is a semi-filling loop on $\tilde{S}$ but a filling loop on $\mathscr{R}^*$. This proves that θ must be of type (II).

§4.2. To prove (1) of Theorem 2.7, suppose that $\hat{f}$ fixes a point $y^* \in \partial\mathbf{D} = \mathbf{S}^1$, where y^* is the fixed point of a parabolic element T of G. There is an integer n such that

$$[\hat{f}] \circ T \circ [\hat{f}]^{-1} = T^n. \tag{4.3}$$

From Theorem 2 of [8, 9], $\varphi^*(T^n) = (\varphi^*(T))^n$ is a power of the Dehn twist along the boundary $\partial\Delta$ of a twice punctured disc Δ enclosing x and another puncture x_1. Thus $\varphi^*([\hat{f}]) \circ \varphi^*(T) \circ \varphi^*([\hat{f}])^{-1}$ keeps $\varphi^*([\hat{f}])(\partial\Delta)$ invariant. From (4.3), $\varphi^*([\hat{f}])(\partial\Delta) = \partial\Delta$. Therefore, $n = 1$. Since $\Pi \circ \varphi^*([\hat{f}]) = \chi$ is of order m, $f_\theta|_{S\backslash\Delta}$ is isotopic to a periodic self-map of order m. It follows that $\varphi^*([\hat{f}])$ is of type (I) and is reduced by $\partial\Delta$.

Conversely, if θ is of type (I), we are in Case 1 of §4.1. In this case, from Lemma 1 of [13], there is a parabolic element $T \in G$ such that $\varphi^*(T)$ is the Dehn twist along $\partial\Delta$. In particular, $\varphi^*(T)$ commutes with θ. So T commutes with $[\hat{f}]$. But $[\hat{f}]|_{\mathbf{D}} = \hat{f}$. From the same argument of Corollary 1 of [13], $\hat{f}$ and T share a common fixed point if both are viewed as elements of real transformations.

§4.3. To prove (2) of Theorem 2.7, we suppose that θ is a type (II) mapping class, we are in Case 2 of §4.1. Then f_θ keeps invariant some subsurface $\mathscr{R}^*$ of S, where $\mathscr{R}^*$ is a subsurface of S described in steps (A) and (B) right after (4.2). By hypothesis, $(f_\theta|_{\mathscr{R}^*})^m$ is irreducible, which means that there is a filling closed geodesic c on $\mathscr{R}^*$ such that $(f_\theta|_{\mathscr{R}^*})^m$ is isotopic to $j([c])|_{\mathscr{R}^*}$.

By Lemma 3.7, $j([c])$ is the identity outside $\mathscr{R}^*$. We see that f_θ commutes with $j([c])$ as self-maps of S, which implies that the element $g_c \in G$ that corresponds to c under (2.1) commutes with $[\hat{f}]$. The curve c, as a loop on $\tilde{S}$, is semi-filling. So g_c is semi-essential hyperbolic. Since $[\hat{f}]|_{\mathbf{D}} = \hat{f}$, g_c and $\hat{f}$ share a common axis λ_c if both are considered real Möbius transformations. λ_c projects to a semi-filling geodesic c on $\tilde{S}$. It is immediate that f keeps the geodesic $\varrho(\lambda_c)$ invariant.

Finally, we assume that $\hat{f}$ fixes a geodesic $\lambda \subset \mathbf{D}$ with $\varrho(\lambda) \subset \tilde{S}$ being a semi-filling geodesic. Then $\hat{f}$ commutes with an element g_c of G, where g_c corresponds to the semi-filling geodesic $c = \varrho(\lambda_c)$. This implies that $\theta = \varphi^*([\hat{f}])$ commutes with $\varphi^*(g_c) = j([c])$. Since $j([c])$ is a reducible mapping class. By Lemma 3.1, θ is reducible as well. From the same discussion as above, θ is not of type (I); otherwise, there is a parabolic element T of G such that $[\hat{f}]$ commutes with T. As real Möbius transformations, T and $\hat{f}$ share a common fixed point in $\mathbf{S}^1$. By assumption, $\hat{f}$ and g_c share the two fixed points. It follows that the fixed point of T is also a fixed point of g_c. As a consequence, the group generated by T and g_c, which is a subgroup of G, is not discrete. This is absurd. It follows from §4.1 that θ must be a type (II) mapping class.

§4.4. To prove (3) of Theorem 2.7, we note that $\theta^m = \theta'$ is also a hyperbolic mapping class. Since $\Pi(\theta') = \mathrm{id}$, by Lemma 3.2, $\varphi^{*-1}(\theta') \in G$. From Theorem 2 of [8], $\theta' = \varphi^*(g_c) = j([c])$ for an essential hyperbolic element g_c of G that corresponds to a closed filling geodesic c on $\tilde{S}$. Obviously, θ commutes with θ'. From the proof of Lemma 3.3, $\hat{f}$ and g_c share the same axis $\lambda_c \subset \mathbf{D}$. It is clear that $\varrho(\lambda_c)$ is a filling geodesic that is invariant under f.

Conversely, suppose that $\hat{f}(\lambda_c) = \lambda_c$, where λ_c is the axis of an essential hyperbolic element g_c of G. As real Möbius transformations, $\hat{f}$ and g_c share the same fixed points. So $\hat{f}$ commutes with g_c. It follows that $\theta = \varphi([\hat{f}])$ commutes with $\varphi^*(g_c)$. From Theorem 2 of [8] again, $\varphi^*(g_c)$ is hyperbolic. Hence from Lemma 3.1, θ is hyperbolic as well. This completes the proof of Theorem 2.7.

Now we proceed to prove Theorem 2.6.

§4.5. If θ is elliptic, θ has a fixed point $\tau \in T(S)$. So $[\hat{f}](\varphi^{-1}(\tau)) = \varphi^{-1}(\tau)$. This means that $[\hat{f}]$ keeps the fiber determined by $\varphi^{-1}(\tau)$ invariant. We may assume that the fiber is $\mathbf{D}$. In this case $\varphi^{-1}(\tau) \in \mathbf{D}$. Hence f fixes the point $\varrho(\varphi^{-1}(\tau)) \in \tilde{S}$.

Conversely, if f fixes a point $\tilde{y} \in \tilde{S}$, then we may choose a lift $\hat{f} \in \mathbf{PSL}(2, \mathbf{R})$ so that $\hat{f}$ fixes a point $y \in \{\varrho^{-1}(\tilde{y})\}$ in $\mathbf{D}$. Notice that in our case, $\hat{f} = [\hat{f}]|_{\mathbf{D}}$. We see that as an element of $\mathrm{mod}(\tilde{S})$, $[\hat{f}]$ has a fixed point in $F(\tilde{S})$. Hence $\varphi^*([\hat{f}])$ has a fixed point in $T(S)$, which says $\varphi^*([\hat{f}])$ is elliptic. This proves (1) of Theorem 2.6. (2) of Theorem 2.6 can be similarly handled.

§4.6. Since χ is elliptic, f has fixed points on the compactification of $\tilde{S}$. In the case where $\tilde{S}$ is closed, f only fixes some points of $\tilde{S}$. However, if $\tilde{S}$ is not closed, f could fix some punctures as well as some points of $\tilde{S}$. In particular, if f does not fix any punctures of $\tilde{S}$, f must fix some points of $\tilde{S}$. In this situation, let $\xi_1, \ldots, \xi_s$ denote all the points of $\tilde{S}$ fixed by f. From §4.5, we know that $i^{-1}(\chi)$ contains infinitely many elliptic elements. By Lemma 3.8, among the lifts of f there exist some lifts $\hat{f}$ (that can be written as $g^k \circ f^*$) that are hyperbolic Möbius transformations. Hence $\hat{f}$ does not fix any pre-images of $\xi_1, \ldots, \xi_s$. Those $\hat{f}$ induce $[\hat{f}] \in \mathrm{mod}(\tilde{S})$. It is easy to see that $\varphi^*([\hat{f}])$ is not elliptic. Since f does not fix any puncture, $i^{-1}(\chi)$ does not contain any type (I) mapping classes, from §4.1 it follows that either $\varphi^*([\hat{f}])$ is of type (II), or $\varphi^*([\hat{f}])$ is hyperbolic.

Now we assume that f does fix some punctures, which are denoted by $y_1, \ldots, y_{s_0}$. If f fixes a point $x_1 \in \tilde{S}$, by Case 1 of Lemma 3.8, there is a lift $\hat{f}$ that is a hyperbolic element. So it cannot fix any fixed points of parabolic elements of G. From §4.1, either $\varphi^*([\hat{f}])$ is of type II, or $\varphi^*([\hat{f}])$is hyperbolic.

If f fixes no points of $\tilde{S}$, f must fix a puncture y of $\tilde{S}$. By Lemma 3.8 again, there is a lift $\hat{f}$ that is a hyperbolic element. From §4.1, either $\varphi^*([\hat{f}])$ is of type II, or it is hyperbolic. This proves (3) of Theorem 2.6.

§4.7. We proceed to prove (4) of Theorem 2.6. Let $\eta : \tilde{S} \to \tilde{S}/\langle f \rangle$ denote the branched covering. If the orbifold $\tilde{S}/\langle f \rangle$ has genus $\tilde{p} = p/m \geq 1$, we fix a branch point $\tilde{\xi}$ on $\tilde{S}/\langle f \rangle$, let $\xi = \eta^{-1}(\tilde{\xi}) \in \tilde{S}$. Also we take a filling loop $\tilde{c}$ on $\tilde{S}^*$ that passes through $\tilde{\xi}$ and avoids any other branch points, where $\tilde{S}^*$ is the Riemann surface obtained from forgetting the branch points of $\tilde{S}/\langle f\rangle$. Let $c^* = \eta^{-1}(\tilde{c})$, and c the loop obtained from reparameterizing c^*. Note that c is a (self-intersecting) closed loop on $\tilde{S}$ that passes through ξ and is invariant under f.

We claim that c is a filling loop of $\tilde{S}$. Otherwise, there is a component Δ of $\tilde{S}\backslash c$ that is neither a disk nor a punctured disk. Then either $f(\Delta) = \Delta$ or that $f(\Delta) \cap \Delta$ is empty. In the second case, Δ is homeomorphic to a component of $(\tilde{S}/\langle f\rangle)\backslash\tilde{c}$, contradicting that $\tilde{c}$ is a filling loop of $\tilde{S}^*$. In the first case, $\eta|_\Delta : \Delta \to \Delta/\langle f\rangle$ is a finite branched covering. Since Δ projects to a disk or punctured disk, $\eta(\Delta)$ is a disk or punctured disk. This implies that Δ must be a disk or punctured disk. This is again a contradiction.

We conclude that c is a filling loop of $\tilde{S}$ that is invariant under f. From (3) of Theorem 2.6, there is a lift θ of χ, so that θ is irreducible. This proves (4) of Theorem 2.6 and hence this completes the proof of Theorem 2.6.

5. Proof of Theorem 2.8

§5.1. Let χ be represented by a conformal automorphism f on $\tilde{S}$ which has some fixed points. Let $\tilde{z}_0$ be one of the fixed points. Let $z_0 \in \mathbf{D}$ be a point such that $\pi(z_0) = \tilde{z}_0$. We may assume without loss of generality that $z_0 = 0$.

Let $\hat{f} : \mathbf{D} \to \mathbf{D}$ be the lift of f so that $\hat{f}(0) = 0$. Then $\hat{f}$ is a conformal automorphism on $\mathbf{D}$, and hence it is a Möbius transformation. It follows that $\hat{f}$ is a rotation on $\mathbf{D}$ with rotation angle no larger than $2\pi/3$.

§5.2. A power of a Dehn twist t_α^n about a simple closed geodesic $\alpha \subset \tilde{S}$ can also be lifted to map $\tau^n : \mathbf{D} \to \mathbf{D}$ so that $\varrho \circ \tau^n = t_\alpha^n \circ \varrho$. It is known that τ determines a disjoint union of half planes Δ_j so that τ keeps each Δ_j invariant. As such, the complement $\Omega = \mathbf{D}\backslash \cup \Delta_j$ is also an invariant set by τ (see [15] for more details). By post composing a suitable element of G, one may assume that $0 \in \Omega$. This implies that for any $z \in \mathbf{S}^1$ and any n, the Euclidean distance between $\tau^n(z)$ and z is no greater than $2\pi/2 = \pi$.

§5.3. If $n > 0$, the motion direction of $\tau|_{\mathbf{S}^1}$ is in the clockwise direction. Now either $\hat{f}$ or $\hat{f}^{-1}$ is in the clockwise motion direction. Thus we may assume that $\tau|_{\mathbf{S}^1}$ and $\hat{f}|_{\mathbf{S}^1}$ are in the same clockwise motion direction. We assert that for any $z \in \mathbf{S}^1$, the distance between z and $\tau^n\hat{f}$ is no greater than $2\pi/3 + \pi < 2\pi$. It follows that $(\tau^n\hat{f})|_{\mathbf{S}^1}$ has no fixed points. In particular, $(\tau^n\hat{f})|_{\mathbf{S}^1}$ does not fix any parabolic fixed point of G. From Lemma 5.1 and Lemma 5.2 of [14], $\varphi^*([\tau^n\hat{f}])$ cannot fix the boundary of any twice punctured disk enclosing x. By the same argument of [13, 15], $\{\varphi^*([\tau^n\hat{f}])\} \subset \Pi^{-1}(t_\alpha^n \circ \chi)$ consists of hyperbolic mapping classes if $t_\alpha^n \circ \chi$ itself is a hyperbolic mapping class.

§5.4. We need to prove that there exists a simple closed geodesic $\alpha \subset \tilde{S}$ such that $t_\alpha^n \circ f$ and $f \circ t_\alpha^n$ are pseudo-Anosov for almost all integers n. To see this, we invoke Theorem III.3 of FLP [5] (see also Ivanov [7]) which asserts that if

$$\{f^k(\alpha) \text{ for } k = 0, \ldots, m-1\} \tag{5.1}$$

fills $\tilde{S}$ in the sense that $\tilde{S}\backslash\{f^k(\alpha),\ k = 0, \ldots, m-1\}$ is a union of disks or punctured disks, then for almost all integers n (with possibly seven exceptional cases) $t_\alpha^n \circ f$ and $f \circ t_\alpha^n$ represent hyperbolic mapping classes.

It is now easy to obtain a simple curve α on S satisfying condition (5.1). Figure 1 demonstrates a conformal automorphism of order 4 on a compact Riemann surface of genus 4. The curve α in the figure together with all the images $f^i(\alpha)$ fill the surface S.

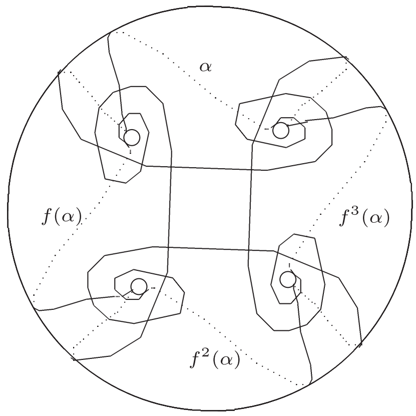

Fig. 1

ACKNOWLEDGEMENTS. This work has been partially supported by NSFC under grants 11371126 and 11571099. The authors thank the referee for carefully reading the manuscript and for his/her valuable suggestions and comments, which much improve the quality of the paper.

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