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Author(s)	Xiao, Yingqing; Zhang, Chaohui
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A CLASSIFICATION PROBLEM ON MAPPING CLASSES ON FIBER SPACES OVER TEICHMÜLLER SPACES

YINGQING XIAO and CHAOHUI ZHANG

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Abstract

Let \tilde{S} be an analytically finite Riemann surface which is equipped with a hyperbolic metric. Let $S = \tilde{S} \setminus \{\text{one point } x\}$. There exists a natural projection Π of the x -pointed mapping class group Mod_S^x onto the mapping class group $\text{Mod}(\tilde{S})$. In this paper, we classify elements in the fiber $\Pi^{-1}(\chi)$ for an elliptic element $\chi \in \text{Mod}(\tilde{S})$, and give a geometric interpretation for each element in $\Pi^{-1}(\chi)$. We also prove that $\Pi^{-1}(t_a^n \circ \chi)$ or $\Pi^{-1}(t_a^n \circ \chi^{-1})$ consists of hyperbolic mapping classes provided that $t_a^n \circ \chi$ and $t_a^n \circ \chi^{-1}$ are hyperbolic, where a is a simple closed geodesic on \tilde{S} and t_a is the positive Dehn twist along a .

1. Introduction

Let S be an analytically finite Riemann surface of genus p which contains $n + 1$ punctures $\{x, x_1, \dots, x_n\}$. Assume that $2p + n > 4$. Let Mod_S^x denote the subgroup of the mapping class group $\text{Mod}(S)$ that consists of isotopy classes of self-maps of S that fix x , which implies that Mod_S^x is a subgroup of $\text{Mod}(S)$ with a finite index and each element of Mod_S^x can be projected, under the natural projection

$$\Pi : \text{Mod}_S^x \rightarrow \text{Mod}(\tilde{S}),$$

to an element of $\text{Mod}(\tilde{S})$, where $\tilde{S} = S \cup \{x\}$.

Fix an element $\chi \in \text{Mod}(\tilde{S})$. Consider

$$\Pi^{-1}(\chi) = \{\theta \in \text{Mod}_S^x \mid \Pi(\theta) = \chi\}.$$

In the case where $\chi = \text{id}$, $\Pi^{-1}(\chi)$ is a normal subgroup of Mod_S^x which can be identified with the fundamental group $\pi_1(\tilde{S}, x)$, and thus it is isomorphic to a covering group of \tilde{S} . In [8] Kra classified all elements in $\Pi^{-1}(\text{id})$ by using the terms introduced in [3] and [11]. He also investigated on elements in $\Pi^{-1}(\chi)$ for a non-trivial element χ and showed that if $\chi \in \text{Mod}(\tilde{S})$ is hyperbolic, then elements in $\Pi^{-1}(\chi)$ are either hyperbolic or pseudo-hyperbolic (see the definitions in Section 2). The problem of whether or not $\Pi^{-1}(\chi)$ contains hyperbolic mapping classes were also studied in [13, 15].

In this paper, we study the problem of classifying elements in $\Pi^{-1}(\chi)$ when $\chi \in \text{Mod}(\tilde{S})$ is elliptic with prime order $m \geq 3$. We also study compositions of elliptic mapping classes and mapping classes induced by simple Dehn twists.

This paper is organized as follows. Section 2 is an overview of Teichmüller spaces and Bers fiber spaces. Main results of this paper are stated in the section also. In Section 3 we

prove some lemmas. Section 4 is devoted to the proof of Theorem 2.6 and Theorem 2.7. In Section 5 we prove Theorem 2.8.

2. Main Results

We begin with an overview of Teichmüller spaces and some basic properties. More details can be found in Bers [3, 4].

§2.1. Let S be an analytically finite Riemann surface which is endowed with a hyperbolic metric. The *Teichmüller space* $T(S)$ is the space of equivalence classes $[\mu]$ of conformal structures $(w : S \rightarrow \mu(S))$, where $(w : S \rightarrow \mu(S))$ and $(w' : S \rightarrow \mu'(S))$ are in the same equivalence class $[\mu]$ if there is a conformal map $h : \mu(S) \rightarrow \mu'(S)$ such that $h \circ w$ is isotopic to w' .

Every self-map f of S induces an element in $\text{Mod}(S)$. It is well known that $\text{Mod}(S)$ acts effectively and discontinuously on $T(S)$ as a group of biholomorphisms when $2p + n > 4$. Following Thurston [11], a non-periodic self-map f of S is called reducible if there is a non-contractible homotopically independent curve system $\{\gamma_1, \dots, \gamma_s\}$, $s \geq 1$, on S so that γ_i is not homotopic to γ_j whenever $i \neq j$ and $1 \leq i, j \leq s$, and for each γ_i , $f(\gamma_i)$ is homotopic to a γ_j . A self-map f of S is called irreducible if no such system can be found. By [11], f is irreducible if and only if it is isotopic to a pseudo-Anosov map (see [11] for the definition of a pseudo-Anosov map).

The mapping class group $\text{Mod}(S)$ can be viewed as a group of biholomorphisms of $T(S)$. Let $\langle \cdot, \cdot \rangle$ denote the Teichmüller distance. By introducing the index

$$a(\sigma) = \inf\{\langle y, \sigma(y) \rangle \text{ for } y \in T(S)\},$$

Bers [4] classified elements σ of $\text{Mod}(S)$ as follows. A mapping class σ is elliptic if $a(\sigma) = 0$ and the value is achievable; parabolic if $a(\sigma) = 0$ and the value is not achievable; hyperbolic if $a(\sigma) > 0$ and the value is achievable; or pseudo-hyperbolic if $a(\sigma) > 0$ and the value is not achievable. Remarkable theorems in [4] state that (i) σ is elliptic if and only if it is induced by a periodic map, (ii) σ is hyperbolic if and only if σ is induced by an irreducible (or pseudo-Anosov) map, and (iii) σ is parabolic or pseudo-hyperbolic if and only if σ is induced by a reducible map.

§2.2. Let \tilde{S}, S be as in the introduction. Associated to each $[\mu] \in T(\tilde{S})$ is a Jordan domain \mathcal{D}_μ depending holomorphically on the equivalence class $[\mu]$. The *Bers fiber space* $F(\tilde{S})$ is the set of pairs

$$\{([\mu], z) \mid [\mu] \in T(\tilde{S}), z \in \mathcal{D}_\mu\}.$$

There is a holomorphic projection $\pi : F(\tilde{S}) \rightarrow T(\tilde{S})$.

By Theorem 10 of [3], there is a holomorphic bijection (Bers isomorphism, see [3]) φ of $F(\tilde{S})$ onto $T(S)$ that makes the diagram

$$\begin{array}{ccc} F(\tilde{S}) & \xrightarrow{\varphi} & T(S) \\ \pi \downarrow & & \downarrow \iota \\ T(\tilde{S}) & \xrightarrow{\text{id}} & T(\tilde{S}) \end{array}$$

commutative, where $\iota : T(S) \rightarrow T(\tilde{S})$ is the forgetful map. Notice that S is of type $(p, n+1)$.

§2.3. The mapping class group $\text{Mod}(\tilde{S})$ can be lifted to a group $\text{mod}(\tilde{S})$ that acts biholomorphically and effectively on $F(\tilde{S})$. To do this, we let \mathbf{D} be a unit disk endowed with a hyperbolic metric. Let $\chi \in \text{Mod}(\tilde{S})$ be represented by f . Fix a lift $\hat{f} : \mathbf{D} \rightarrow \mathbf{D}$ under a universal covering map

$$(2.1) \quad \varrho : \mathbf{D} \rightarrow \tilde{S}.$$

Let G be the covering group. Then every lift of f is of the form $g_1 \circ \hat{f} \circ g_2$, where g_1 and $g_2 \in G$. Let \hat{f} and \hat{f}' be two lifts of self-maps of \tilde{S} . We say \hat{f} and \hat{f}' are equivalent if $\hat{f}|_{S^1} = \hat{f}'|_{S^1}$. The equivalence class of \hat{f} is denoted by $[\hat{f}]$. The group $\text{mod}(\tilde{S})$ consists of all such $[\hat{f}]$. It is known that the group G is regarded as a normal subgroup of $\text{mod}(\tilde{S})$ by conjugation, which keeps each fiber of $F(\tilde{S})$ invariant.

It is important to note that the Bers isomorphism $\varphi : F(\tilde{S}) \rightarrow T(S)$ induces an isomorphism φ^* of $\text{mod}(\tilde{S})$ onto Mod_S^x by carrying each element $[\hat{f}]$ to $\varphi^*([\hat{f}]) = \varphi \circ [\hat{f}] \circ \varphi^{-1}$, where we recall that Mod_S^x is a subgroup of $\text{Mod}(S)$ with index $n+1$. Observe also that every element Mod_S^x can be described as a mapping class that fixes x . So an element $\theta \in \text{Mod}_S^x$ naturally projects to an element χ in $\text{Mod}(\tilde{S})$.

§2.4. Interestingly, there are highly non-trivial mapping classes in Mod_S^x that project to the trivial mapping class. To describe them, let $[\alpha] \in \pi_1(\tilde{S}, x)$. Then α is considered a trace of an isotopy of x . Extend the isotopy to an isotopy $\{f_t : \tilde{S} \rightarrow \tilde{S}\}$ so that $f_0 = \text{id}$ and $f_1(x) = x$. Thus $f_1|_S$ defines a mapping class in Mod_S^x . Define

$$(2.2) \quad j : \pi_1(\tilde{S}, x) \rightarrow \text{Mod}_S^x.$$

by sending $[\alpha]$ to the mapping class of $f_1|_S$. A closed curve $c \subset \tilde{S}$ is called to fill \tilde{S} if its complement in \tilde{S} is a disjoint union of topological disks and (possibly) punctured disks. Likewise, c is called semi-filling with respect to a curve system $\{\gamma_1, \dots, \gamma_u\}$, $u \geq 1$, if c fills the component of $\tilde{S} \setminus \{\gamma_1, \dots, \gamma_u\}$ where c resides. It was shown in [8] that $j([c])$ is hyperbolic if and only if c fills \tilde{S} , and $j([c])$ is pseudo-hyperbolic if and only if c is a semi-filling geodesic.

§2.5. To state our main result, we return to an elliptic mapping class $\chi \in \text{Mod}(\tilde{S})$ with prime order $m \geq 3$. Let $\theta \in \Pi^{-1}(\chi)$ be induced by a self-map f_θ of S . In the case where f_θ is non-periodic reducible self-map of S , θ can be further classified as a type (I) or type (II) mapping class, where θ is called a type (I) mapping class if f_θ is reduced by the boundary $\partial\Delta$ of a twice punctured disk $\Delta \subset S$ enclosing x and another puncture of \tilde{S} such that the restriction $f_\theta|_{S \setminus \Delta}$ is isotopic to a periodic self-map with prime order m . θ is called to be of type (II) if there is a curve system $\mathcal{A} = \{\alpha_1, \dots, \alpha_v\}$, $v \geq 1$, where each α_i is non-contractible on \tilde{S} , such that the following three conditions hold:

- f_θ leaves invariant the component \mathcal{R} of $S \setminus \mathcal{A}$ that contains x ,
- $f_\theta|_{\mathcal{R}}$ is irreducible, and permutes other components of $S \setminus \mathcal{A}$, and
- f_θ^m can be expressed as $j([c])$ for a semi-filling loop c with respect to \mathcal{A} .

It is obvious that each type (I) or type (II) mapping class projects to a periodic mapping class of order m . Hence f has some fixed points on the compactification of \tilde{S} some of which may be punctures of \tilde{S} . As mentioned before, for each $\chi \in \text{Mod}(\tilde{S})$, $\Pi^{-1}(\chi)$ consists of elliptic, hyperbolic, and non-hyperbolic elements, where non-hyperbolic elements are either

parabolic or pseudo-hyperbolic. From the definition, we know that any type (I) mapping class is parabolic, while any type (II) mapping class is pseudo-hyperbolic.

§2.6. Under our circumstances, χ is elliptic with prime order $m \geq 3$. Our main theorems below state that for any element $\theta \in \Pi^{-1}(\chi)$, if θ is not elliptic, then θ is either of type (I), or of type (II), or hyperbolic. More precisely, from Nielsen's theorem (see Ivanov [7] for example), χ has a fixed point in $T(\tilde{S})$. We assume without loss of generality that the fixed point is represented by \tilde{S} . Thus there is a representative f of χ that can be realized as a conformal automorphism of \tilde{S} with order m .

Theorem. *For each non-trivial elliptic mapping class $\chi \in \text{Mod}(\tilde{S})$ with prime order $m \geq 3$, we have:*

- (1) $\Pi^{-1}(\chi)$ contains (infinitely many) elliptic elements if and only if f fixes at least one point of \tilde{S} ,
- (2) $\Pi^{-1}(\chi)$ contains (infinitely many) (I) parabolic elements if and only if f fixes at least one puncture of \tilde{S} ,
- (3) $\Pi^{-1}(\chi)$ always contains (infinitely many) type (II) or hyperbolic mapping classes, and
- (4) if in addition $\tilde{S}/\langle f \rangle$ has genus $\tilde{p} = p/m > 1$, then $\Pi^{-1}(\chi)$ contains (infinitely many) hyperbolic elements.

§2.7. Let $\hat{f} : \mathbf{D} \rightarrow \mathbf{D}$ be a lift of f . Then \hat{f} is a conformal automorphism of \mathbf{D} . Thus $\hat{f} \in \text{PSL}(2, \mathbf{R})$ but \hat{f} is not an element of G . Note that any element in $\Pi^{-1}(\chi)$ can be written in the form $\varphi^*([\hat{f}])$ for a conformal automorphism \hat{f} of \mathbf{D} .

More information on $\Pi^{-1}(\chi)$ is contained in the following result.

Theorem. *Let $\chi \in \text{Mod}(\tilde{S})$ be a non-trivial elliptic element with prime order $m \geq 3$, and let $\theta \in \Pi^{-1}(\chi)$ be non-elliptic which can be expressed as $\theta = \varphi^*([\hat{f}])$ for some conformal automorphism \hat{f} of \mathbf{D} . Then θ is either hyperbolic, or of type (I) or of type (II). More precisely, we have*

- (1) θ is of type (I) if and only if \hat{f} fixes a fixed point of a parabolic element of G ,
- (2) θ is of type (II) if and only if \hat{f} fixes a geodesic λ_c that can be projected to a semi-filling closed geodesic $c \subset \tilde{S}$ that is invariant under f ,
- (3) θ is hyperbolic if and only if \hat{f} keeps invariant a geodesic $\lambda_c \in \mathbf{D}$ that can be projected to a filling closed geodesic $c \subset \tilde{S}$ that is invariant under f .

§2.8. Finally, we consider some compositions of elliptic mapping classes χ and Dehn twists t_α along a simple closed curve $\alpha \subset \tilde{S}$, and study the corresponding fiber in Mod_S^x . Our last result states:

Theorem. *Let $\chi \in \text{Mod}(\tilde{S})$ be elliptic with prime order $m \geq 3$. There exist simple closed geodesics $\alpha \subset \tilde{S}$ such that $t_\alpha^n \circ \chi$ and $t_\alpha^n \circ \chi^{-1}$ are both hyperbolic for all integers n with a finite number of exclusions. In the case where both $t_\alpha^n \circ \chi$ and $t_\alpha^n \circ \chi^{-1}$ are hyperbolic, either $\Pi^{-1}(t_\alpha^n \circ \chi)$ or $\Pi^{-1}(t_\alpha^n \circ \chi^{-1})$ consists of hyperbolic mapping classes.*

§2.9. Remark: When $n = 0$, that is, \tilde{S} is closed, it was shown in [13] that for any hyperbolic mapping class $\chi \in \text{Mod}(\tilde{S})$, $\Pi^{-1}(\chi)$ consists of hyperbolic mapping classes. It is not known, however, whether $\Pi^{-1}(\chi')$ contains hyperbolic mapping class for a general mapping class χ' of \tilde{S} if $n > 0$. Theorem 2.8 above provides an example that the single fiber $\Pi^{-1}(\chi')$ may contain infinitely many hyperbolic mapping classes. Another example is given in Theorem 2

of [13].

3. Some preliminary results

§3.1. Let $\theta, \theta' \in \text{Mod}(S)$ be non-trivial. We call θ and θ' commuting mapping classes of S if $\theta \circ \theta'(\tau) = \theta' \circ \theta(\tau)$ for every $\tau \in T(S)$. We have

Lemma. *Suppose that θ and θ' are infinite order commuting mapping classes of S . Then θ is hyperbolic if and only if θ' is hyperbolic.*

REMARK. The authors are grateful to the referee for pointing out that this result is essentially known, whose proof was given in Ivanov [7]. Here we provide with an alternate approach.

Proof. Let f_θ and $f_{\theta'}$ denote a self-maps of S that induce θ and θ' , respectively. Obviously, the condition is equivalent to that $f_\theta \circ f_{\theta'}$ is isotopic to $f_{\theta'} \circ f_\theta$ on S . Suppose that $f_{\theta'}$ is reduced by a loop system $E = \{e_1, \dots, e_k\}$. By taking a suitable power we may assume that $f_{\theta'}$ is a component map. In particular, $f_{\theta'}(e_i) = e_i$, $i = 1, \dots, k$. Let $\mathcal{P} = \{P_1, \dots, P_{s_0}\}$ denote all components of $S \setminus E$ on which $f_{\theta'}$ is isotopic to a pseudo-Anosov map. Let $\{Q_1, \dots, Q_s\}$ denote all components of $S \setminus E$ on which $f_{\theta'}$ is the identity. Let $E_0 = \{e_1, \dots, e_t\}$ be the subset of E consisting of boundary components of P_i , $i = 1, \dots, s_0$. $E \setminus E_0$ consists of loops on each of which $f_{\theta'}$ is either the identity or a power of a non-trivial Dehn twist.

Consider the self-map $\xi = f_\theta \circ f_{\theta'} \circ f_\theta^{-1}$. Then $\xi(f_\theta(\alpha)) = f_\theta(\alpha)$, which says that ξ restricts to the identity or non-trivial Dehn twist on the loop $f_\theta(\alpha)$ for each $\alpha \in E \setminus E_0$. By hypothesis, ξ is isotopic to $f_{\theta'}$. It turns out that $f_{\theta'}$ restricts to the identity, or a non-trivial Dehn twist on $f_\theta(\alpha)$. As such, $f_\theta(\alpha)$ is also in $E \setminus E_0$. It follows that f_θ leaves invariant the set $E \setminus E_0$. If this set is not empty, we are done. Otherwise, \mathcal{P} is not empty. Let $P_1 \in \mathcal{P}$. The map ξ is isotopic to pseudo-Anosov on $f_\theta(P_1)$. Since ξ is isotopic to $f_{\theta'}$, $f_\theta(P_1)$ is in \mathcal{P} as well. This means that f_θ permutes P_i in \mathcal{P} . Thus f_θ is reduced by the boundary loops of P_i , $P_i \in \mathcal{P}$. So f_θ is reducible. Since θ and θ' are symmetric, the lemma is proved. \square

§3.2. By assumption, \tilde{S} is of type (p, n) with $2p + n > 4$. In particular, \tilde{S} is not of type $(0, 3)$, $(0, 4)$, $(1, 1)$, $(1, 2)$, or $(2, 0)$. The following lemma is merely a special case (torsion free) of Lemma 3.8 of [12].

Lemma. *Let θ be a holomorphic automorphism of the Bers fiber space $F(\tilde{S})$ that leaves each fiber invariant. Then θ coincides with an element of G .*

§3.3. Let c be a primitive, closed filling geodesic on \tilde{S} , which means that $[c]$ is not a power of any element of $\pi_1(\tilde{S}, x)$. Let $j : \pi_1(\tilde{S}, x) \rightarrow \text{Mod}_S^x$ be defined in (2.2). By Theorem 2 of [8], $j([c])$ is hyperbolic. Hence by Lemma 3.1, any infinite order mapping class commuting with $j([c])$ must also be hyperbolic. More precisely, we have

Lemma. *Let $\theta \in \text{Mod}_S^x$ be of infinite order and $\theta \neq (j([c]))^p$ for any $p \in \mathbb{Z}$. Then θ commutes with $j([c])$ if and only if $\Pi(\theta)$ is a non-trivial elliptic element and is induced by a conformal automorphism $f : \tilde{S} \rightarrow \tilde{S}$ that keeps the filling closed geodesic c invariant (as a set).*

Proof. Suppose that an infinite order element $\theta \in \text{Mod}_S^x$ satisfies the condition

$$(3.1) \quad \theta \circ j([c]) = j([c]) \circ \theta.$$

Since c is a filling geodesic, from Theorem 2 of [8], $j([c])$ is hyperbolic, and thus it is induced by a pseudo-Anosov map of S . By Lemma 3.1, θ is also hyperbolic. Hence by Theorem 15.7 of [7], both θ and $j([c])$ are powers of the same hyperbolic mapping class δ of Mod_S^x . Write

$$\theta = \delta^s \text{ and } j([c]) = \delta^r.$$

If $\delta = j([c_0])$ for a $[c_0] \in \pi_1(\tilde{S}, x)$, then $\theta = j([c_0])^s = j([c_0]^s)$. In this case, $\Pi(\theta)$ is trivial. Thus $\theta \in G$. Since G is centerless, either $\theta = (j([c]))^p$ or $\theta^p = j([c])$ for some $p \in \mathbf{Z}$. By assumption, the first case does not occur. The second case says that c is not primitive. This again contradicts the hypothesis. We conclude that $\delta \neq j([c_0])$.

Let $G_0 = \langle \delta, \varphi^*(G) \rangle$. As a subgroup of Mod_S^x , G_0 acts on $T(S)$ discontinuously. Note also that G is a normal subgroup of $\text{mod}(\tilde{S})$ as biholomorphisms on $F(\tilde{S})$. The group $\text{mod}(\tilde{S})$ is isomorphic to Mod_S^x under φ^* . So $\varphi^*(G)$ is a normal subgroup of Mod_S^x and thus $\varphi^*(G)$ is a normal subgroup of G_0 .

From Nielsen's theorem, there is a point $\sigma \in T(\tilde{S})$ such that $\Pi(\delta)(\sigma) = \sigma$. We assume that $\sigma = [0]$ is represented by \tilde{S} . Consider the fiber $\mathbf{D} = \pi^{-1}([0]) \subset F(\tilde{S})$. Note that $\varphi^{-1}(\delta)|_{\mathbf{D}}$ acts as a conformal automorphism. $\varphi^{-1}(\delta)|_{\mathbf{D}}$ is an element of $\mathbf{PSL}(2, \mathbf{R})$. Denote $\hat{f} = \varphi^{-1}(\delta)|_{\mathbf{D}}$, and let $G'_0 = (\varphi^*)^{-1}(G_0)|_{\mathbf{D}}$. We see that $G'_0 = \langle \hat{f}, G \rangle$ acts on \mathbf{D} discontinuously, \hat{f} does not belong to G , and G is a normal subgroup of G'_0 . In particular, we have

$$\hat{f} \circ G \circ \hat{f}^{-1} = G.$$

It follows that \hat{f} can be projected to a conformal automorphism f of \tilde{S} under the projection $\varrho : \mathbf{D} \rightarrow \tilde{S}$. It is also easy to see that $\mathbf{D}/G'_0 = \tilde{S}/\langle f \rangle$.

By construction, $\hat{f}^r = j([c])$. As elements of $\mathbf{PSL}(2, \mathbf{R})$, both \hat{f} and $j([c])$ keep a geodesic λ_c invariant. Since $\varrho(\lambda_c) = c$, $f(c) = c$, as desired.

Conversely, we assume that c is a filling geodesic on \tilde{S} , and $f : \tilde{S} \rightarrow \tilde{S}$ satisfies $f(c) = c$. We lift the map f to a conformal automorphism \hat{f} of \mathbf{D} so that $\hat{f}(\lambda_c) = \lambda_c$, where λ_c is a geodesic in \mathbf{D} such that $\varrho(\lambda_c) = c$. \hat{f} is a hyperbolic element of $\mathbf{PSL}(2, \mathbf{R})$. Since $\chi^m = \text{id}$, by Lemma 3.2, \hat{f}^m is an element g_c of G , where g_c corresponds to c under the isomorphism $\pi_1(\tilde{S}, x) \xrightarrow{\cong} G$. $[\hat{f}]$ commutes with g_c if both are considered elements of $\text{mod}(\tilde{S})$. It follows that θ commutes with $j([c])$. \square

§3.4. Now let $\chi \in \text{Mod}(\tilde{S})$ be elliptic with prime order $m \geq 3$. Let $f : \tilde{S} \rightarrow \tilde{S}$ be a representative of χ and let $\theta \in \text{Mod}_S^x$ be such that $\Pi(\theta) = \chi$. Let $f_\theta : S \rightarrow S$ be a representative of θ . Suppose that there is a subsurface \mathcal{R} of S satisfying the properties:

- $x \in \mathcal{R}$, f_θ keeps \mathcal{R} invariant, and
- $\partial\mathcal{R} = \{d_1, \dots, d_u\}$, where $u \geq 1$ and d_i are also non-contractible loops on \tilde{S} .

Under these circumstances, we have:

Lemma. $f_\theta|_{\mathcal{R}}$ is periodic if and only if f_θ is periodic.

Proof. Suppose that $f_\theta|_{\mathcal{R}}$ is periodic. Then the restriction $f_\theta^n|_{\mathcal{R}}$ is the identity for some $n \in \mathbf{Z}$. On the other hand, by assumption, we know that $\Pi(\theta^m) = \chi^m$ is the identity, by

Lemma 3.2, $f_\theta^m = \varphi \circ \gamma \circ \varphi^{-1}$ for some element $\gamma \in G$. This tells us that f_θ^m leaves the identity on any component of $S \setminus \{d_1, \dots, d_u\}$ other than \mathcal{R} .

If $f_\theta^m|_{\mathcal{R}}$ is not the identity, then $\{f_\theta^m|_{\mathcal{R}}\}$ is infinitely cyclic, which says that $(f_\theta^m|_{\mathcal{R}})^q \neq \text{id}$ for any integer $q \neq 0$. In particular, $(f_\theta^m|_{\mathcal{R}})^n \neq \text{id}$. But $(f_\theta^m|_{\mathcal{R}})^n = (f_\theta^n|_{\mathcal{R}})^m = \text{id}$. This is a contradiction, showing that $f_\theta^m|_{\mathcal{R}}$ is the identity. This also implies that $m = n$.

We conclude that f_θ^m restricts to the identity on any components of $S \setminus \{d_1, \dots, d_u\}$. It remains to exclude the case where f_θ^m is a multi-twists along some loops d_i .

Notice that d_i is non-contractible on \tilde{S} . Assume that $f_\theta^m|_{N_1}$ is non-trivial, where N_1 is a thin annular neighborhood of d_1 . N_1 avoids the puncture x and is disjoint from any other loops d_j for $j \neq 1$. Since d_1 is non-contractible on \tilde{S} and since $m \geq 3$ is a prime integer, \mathcal{R} is not an x -punctured cylinder, which means that $\Pi(\theta^m)$ is non-trivial on the image of N_1 under the forgetful map. Thus χ^m is not the identity. But this contradicts that χ is periodic with order m . It follows that θ^m is the identity and hence θ is an elliptic mapping class of order m .

The converse is trivial. □

§3.5. A similar argument yields the following result:

Lemma. *Under the same notation and hypothesis of Lemma 3.4, if $f_\theta^m|_{\mathcal{R}}$ is a non-trivial Dehn twist along $\partial\Delta$ where $\Delta \subset \mathcal{R} \subset S$ is a twice punctured disk enclosing x and another puncture, then f_θ represents a type (I) mapping class.*

Proof. By definition, $f_\theta|_{\mathcal{R}}$ is a type (I) reducible on \mathcal{R} . By the same argument of Lemma 3.4, f_θ is itself a type (I) reducible map on S . □

§3.6. Let $\chi \in \text{Mod}(\tilde{S})$ be elliptic which is represented by a conformal automorphism f of \tilde{S} . Suppose that χ has a prime order $m \geq 3$.

Lemma. *Let c be a simple non-contractible loop on \tilde{S} such that c is not homotopic to $f(c)$. If $f^q(c)$ is homotopic to c for an integer q with $1 < q \leq m$, then $q = m$.*

Proof. Without loss of generality we may assume that c is a simple geodesic. Since f is conformal, $f(c)$ is also a geodesic and is not homotopic to c .

Suppose that $f^q(c) = c$. Then either $f^q|_c$ is the identity or cyclic. If $f^q|_c$ is the identity, f^q is the identity on \tilde{S} . So $q = m$. If $f^q|_c$ is cyclic, since m is a prime number, $f^q|_c$ must be of order m , which means that f is of order qm . But $q > 1$. This is impossible. □

§3.7. Recall that for any $[c] \in \pi_1(\tilde{S}, x)$, $j([c]) \in \text{Mod}_S^x$ is the mapping class defined in (2.2). Let $f_c : S \rightarrow S$ be a suitable representative of $j([c])$ and c a representative of $[c]$.

If c is a loop around a puncture of \tilde{S} , f_c is a Dehn twist along a twice punctured disc Δ enclosing x . So f_c restricts to the identity on any component of $S \setminus \partial\Delta$.

If c is a simple non-contracting loop of \tilde{S} , f_c is a spin map which is reducible. Denote by \mathcal{C} the corresponding cylinder containing x . Then f_c restricts to the identity on any component of $S \setminus \mathcal{C}$.

Consider the universal covering map (2.1). Fix a point \hat{x} . Let $g_c \in G$ be the element corresponding to c . Assume that \hat{x} is in the axis λ_c of g_c . Construct a quasiconformal automorphism w of \mathbf{D} that is supported on a thin neighborhood of λ_c and has the properties

that

- $w(\hat{x}) = g_c(\hat{x})$ and
- w commutes with each element of G .

Let $W : \tilde{S} \rightarrow \tilde{S}$ be the projection of w . There is a homotopy ω_t (which is called the Ahlfors homotopy in the literature) between w and the identity so that for any $t \in [0, 1]$, ω_t commutes with each element of G . Hence ω_t can be projected to a homotopy Ω_t on \tilde{S} so that $\Omega_0 = \text{id}$ and $\Omega_1 = W$. Since W fixes x , $j([c])$ is the mapping class of $\Omega_1|_S$. From this construction, we see that W is the identity outside a neighborhood of c . Let $\{e_1, \dots, e_{k_0}\}$ be the curve system on S so that one component of $S \setminus \{e_1, \dots, e_{k_0}\}$ is a minimum surface containing c . This means that f_c restricts to the identity on each component of $S \setminus \{e_1, \dots, e_{k_0}\}$ that avoids c .

Putting all the information together, we summarize:

Lemma. *For any $[c] \in \pi_1(\tilde{S}, x)$, $j([c])$ is represented by a map f_c which restricts to the identity on any subsurface of S that avoids c .*

§3.8. We continue to assume that $\chi \in \text{Mod}(\tilde{S})$ is elliptic which is represented by a conformal automorphism f of \tilde{S} fixing a point or a puncture of \tilde{S} . Let $f^* : \mathbf{D} \rightarrow \mathbf{D}$ be a (conformal) lift of f that fixes a pre-image y^* of a fixed point y of f . The point $y^* \in \partial\mathbf{D}$ if and only if y is a puncture of \tilde{S} .

Lemma. *With the above notation and terminology, there is a hyperbolic element $g \in G$ and an integer N such that for all $k \geq N$, $g^k \circ f^*$ are hyperbolic Möbius transformations.*

Proof. There are two cases to discuss.

Case 1. f fixes a point $y \in \tilde{S}$. In this case, for any two non-antipodal points $\alpha, \beta \in \mathbf{S}^1$, we use $[\alpha, \beta]$ (resp. (α, β)) to denote the minor closed (resp. open) arc on \mathbf{S}^1 . Also, the Euclidean length of the segment is denoted by $\text{len}(\alpha, \beta)$. The lift f^* of f that fixes y^* , where y^* is a point in the orbit $\{\varrho^{-1}(y)\} \subset \mathbf{D}$. In this setting f^* is a Möbius transformation keeping \mathbf{S}^1 invariant. One may assume that $y^* = 0$ and f^* is of the form

$$z \rightarrow [\exp(2\pi i/m)]z, \quad z \in \mathbf{D}.$$

That is, f^* is a rotation. It sends any point $\alpha \in \mathbf{S}^1$ to a point $\beta = f^*(\alpha)$. The length $\lambda := \text{len}(\alpha, \beta)$ does not depend on α .

Let $g \in G$ be a hyperbolic element so that its axis A_g has a relatively large Euclidean length, in the sense that A_g and $f^*(A_g)$ intersect. This is achievable by Theorem 5.3.8 of Beardon [1]. Let $A_g \cap \mathbf{S}^1 = \{\text{Fix}^+(g), \text{Fix}^-(g)\}$, where $\text{Fix}^+(g)$ and $\text{Fix}^-(g)$ are repelling and attracting fixed point of g . Orient A_g so that it points from $\text{Fix}^+(g)$ to $\text{Fix}^-(g)$. It is clear that A_g divides \mathbf{D} into two half-plane U and U' , where U and U' are the half planes lying in the right and left side of A_g , respectively. Assume that U contains some diameter of \mathbf{D} .

We see that $\text{len}(z, g(z))$, $z \in U \cap \mathbf{S}^1$, attains its maximum value at some point $z_0 \in \mathbf{S}^1$ that is away from $\text{Fix}^+(g)$ and $\text{Fix}^-(g)$. As a point $z \in U \cap \mathbf{S}^1$ tends to either $\text{Fix}^+(g)$ or $\text{Fix}^-(g)$, $\text{len}(z, g(z))$ tends to zero.

Choose a sufficiently large integer N so that when $k \geq N$, the maximum value of $\text{len}(z, g^k(z))$, $z \in U \cap \mathbf{S}^1$, is $\lambda_0 > \lambda$. Let $z_0 \in U \cap \mathbf{S}^1$ be the point so that $\text{len}(z_0, g^N(z_0)) = \lambda_0$. Then it still holds that for a fixed $k > N$, $\text{len}(z, g^k(z))$ tends to zero, whenever $z \in U \cap \mathbf{S}^1$ tends to either $\text{Fix}^+(g^k)$ or $\text{Fix}^-(g^k)$,

Now from the intermediate-value theorem in Calculus, there is a point z' in the arc $(\text{Fix}^+(g), z_0)$ and point z'' in $(z_0, \text{Fix}^-(g))$ so that $\text{len}(z' g^k(z')) = \lambda$ and $\text{len}(z'' g^k(z'')) = \lambda$. Since $\lambda = \text{len}(z', f^*(z')) = \text{len}(z'', f^*(z''))$, we conclude that

$$g^k(z') = f^*(z') \text{ and } g^k(z'') = f^*(z''), \quad z', z'' \in U \cap \mathbf{S}^1.$$

It follows that $(g^k)^{-1} \circ f^*(z') = g^{-k} \circ f^*(z') = z'$ and $(g^k)^{-1} \circ f^*(z'') = g^{-k} \circ f^*(z'') = z''$, which says that $g^{-k} \circ f^*$ has two distinct fixed points in $U \cap \mathbf{S}^1$. Notice that g^{-k} and f^* are Möbius transformations. We assert that $g^{-k} \circ f^*$ must be a hyperbolic Möbius transformation. Since $G_0 = \langle G, f^* \rangle$ is discrete, as a hyperbolic element of G_0 , $g^{-k} \circ f^*$ cannot fix any fixed points of parabolic elements of G_0 . Similarly, $g^k \circ f^*$ has two distinct fixed points in $U' \cap \mathbf{S}^1$. So $g^k \circ f^*$ is also a hyperbolic element.

Case 2. f fixes a puncture y of \tilde{S} . In this case, we take the upper half space model for the hyperbolic space \mathbf{H} . The boundary is $\partial\mathbf{H} = \mathbf{R}$. We assume that ∞ lies above y under (2.1), and the parabolic element γ that fixes ∞ is $z \rightarrow z + 1$. The lift f^* fixes ∞ . Under the assumption, since f^m is the identity, f^* sends each point z in $\mathbf{R} \cup \mathbf{H}$ to $z + k/m$ for some integer k that is prime to m .

We then use the same argument in Case 1. Let $g \in G$ be a hyperbolic element so that $\text{len}(\text{Fix}^+(g), \text{Fix}^-(g)) > 1$ (by Theorem 5.3.8 of Beardon [1]). Its axis A_g divides $\mathbf{R} \cup \mathbf{H}$ into two regions. Let U be the bounded region. For a large but fixed k , there are two distinguished points $z', z'' \in U \cap \mathbf{R}$ so that $\text{len}(z', g^k(z')) = 1$ and $\text{len}(z'', g^k(z'')) = 1$. It follows that $g^{-k} \circ f^* \in G_0$ is hyperbolic. Similarly, $g^k \circ f^* \in G_0$ is also hyperbolic with two distinct fixed points in $U' \cap \mathbf{R}$. Once again, both $g^{-k} \circ f^*$ and $g^k \circ f^*$ cannot fix any fixed points of parabolic elements of G ; otherwise, G_0 would not be discrete. Lemma 3.8 is proved. \square

4. Proof of Theorem 2.6 and Theorem 2.7

§4.1. We first prove Theorem 2.7. Theorem 2.6 will be derived easily. As usual, we let θ be induced by a self-map f_θ of S . First we assume that f_θ is completely reduced by

$$(4.1) \quad \mathcal{L} = \{c_1, \dots, c_u\}, \quad u \geq 1,$$

(see Bers [3] for the definition of completely reducible map. From Lemma 5.9 of [3], each reducible mapping class is completely reducible). Let σ_t be a smooth flow in $T(S)$ that is obtained from pinching all the loops in (4.1) to cusps. Let $\partial T(S)$ denote the Bers boundary of $T(S)$ (see [2]). Let $\{x_i\} \in \sigma_t \subset T(S)$ be any discrete instances represented by S_i so that $x_i \rightarrow \partial T(S)$. Let $\tilde{x}_i = \pi \circ \varphi^{-1}(x_i) \in T(\tilde{S})$ be represented by \tilde{S}_i . Then \tilde{S}_i is obtained from S_i by filling in the puncture x . Let

$$\mathcal{R}(\tilde{S}) = T(\tilde{S}) / \text{Mod}(\tilde{S})$$

denote the Riemann moduli space of \tilde{S} , and let $\varpi : T(\tilde{S}) \rightarrow \mathcal{R}(\tilde{S})$ be the natural projection.

Let $\Lambda \subset \mathcal{R}(\tilde{S})$ and $\Lambda' \subset T(\tilde{S})$ denote the sequences $\{\varpi(\tilde{x}_i)\}$ and $\{\tilde{x}_i\}$, respectively. There are two cases to consider.

Case 1. Λ lies in a compact subset of $\mathcal{R}(\tilde{S})$. In this case, there is a sub-sequence $\{\tilde{x}_i\} \subset \Lambda'$, which may tend to the boundary $\partial T(\tilde{S})$, yet \tilde{S}_i does not possess short closed geodesics. By Lemma 2 of [13], all loops in (4.1) bound disks $D_j \subset S$ that encloses x and another puncture

x_j . But all loops in (4.1) are disjoint and homotopically independent. It follows that $u = 1$, which means that (4.1) consists of only one single loop, say c_1 , that is the boundary of a twice punctured disk Δ enclosing x and a puncture x_1 of \tilde{S} .

Since $\Pi(\theta) = \chi \in \text{Mod}(\tilde{S})$ is elliptic with prime order m , $f_\theta|_{S \setminus \Delta}$ is isotopic to a periodic self-map of prime order m . Hence $\theta = \varphi^*([\hat{f}])$ is a type (I) mapping class.

Case 2. Λ is not compact in $\mathcal{R}(\tilde{S})$. In this case, we claim that θ is a type (II) mapping class as defined in §2.5. Indeed, we pick an arbitrary loop, say c_i , in (4.1) that is non-contractible on \tilde{S} (the image of c_i on \tilde{S}_i is denoted by \tilde{c}_i). By assumption, $m \geq 3$ is a prime number. Consider the image loops $\mathcal{L}^i = \{f_\theta^j(c_i), j \geq 0\}$. There is a possibility that \mathcal{L}^i consists of a simple loop c_i only. That is, f_θ keeps c_i invariant. But if c_i is not homotopic to $f_\theta(c_i)$, by Lemma 3.6, c_i is not homotopic to $f_\theta^q(c_i)$ for any q with $1 < q < m$.

We claim that c_i is homotopic to $f_\theta^m(c_i)$. Indeed, \tilde{c}_i is homotopic to $f^m(\tilde{c}_i)$. So if c_i is not homotopic to $f_\theta^m(c_i)$, then since c_i and $f_\theta^m(c_i)$ lie in \mathcal{L} , they are disjoint. Hence c_i and $f_\theta^m(c_i)$ must bound a cylinder \mathcal{P}_i that encloses the puncture x . It follows that $f_\theta(\mathcal{P}_i)$ is also a cylinder that encloses the puncture x as well.

This means that either $\mathcal{P}_i = f_\theta(\mathcal{P}_i)$, or \mathcal{P}_i and $f_\theta(\mathcal{P}_i)$ intersect. In the later case, the boundary $\partial\mathcal{P}_i$ of \mathcal{P}_i is $\{c_i, f_\theta^m(c_i)\}$ and the boundary $\partial f_\theta(\mathcal{P}_i)$ of $f_\theta(\mathcal{P}_i)$ is $\{f_\theta(c_i), f_\theta(f_\theta^m(c_i))\}$. Since c_i and $f_\theta(c_i)$ is disjoint, c_i must intersect with $f_\theta(f_\theta^m(c_i))$. This is impossible. In the former case, if c_i is homotopic to $f_\theta(c_i)$, then this contradicts Lemma 3.6; if c_i is homotopic to $f_\theta(f_\theta(c_i))$, then $f_\theta(c_i)$ is homotopic to $f_\theta^m(c_i)$, which says that c_i is homotopic to $f_\theta^{m-1}(c_i)$, which again contradicts Lemma 3.6.

We conclude that c_i is homotopic to $f_\theta^m(c_i)$, and that (4.1) is a disjoint union

$$(4.2) \quad \bigcup_{i=0}^N \mathcal{L}^i, \quad N < \infty,$$

where \mathcal{L}^i is either an m -cycle of loops, or consists of a simple loop only.

Note that every \mathcal{L}^i in (4.2) consists of either dividing loops or non-dividing loops. Let \mathcal{R}^* be the component of $S - \bigcup \mathcal{L}^i$ containing the puncture x . We claim that \mathcal{R}^* is invariant under f_θ . To see this, we construct \mathcal{R}^* in the following steps:

(A) If \mathcal{L}^1 consists of non-dividing loops and $S \setminus \{\mathcal{L}^1\}$ is a single component, we define \mathcal{R}_1 to be $S \setminus \{\mathcal{L}^1\}$. If \mathcal{L}^1 consists of dividing loops, or non-dividing loops but $S \setminus \{\mathcal{L}^1\}$ has more than one components, we let \mathcal{R}_1 be the component of $S \setminus \{\mathcal{L}^1\}$ that the puncture x resides. It is easy to see that \mathcal{R}_1 is invariant under f_θ .

(B) If \mathcal{R}_1 contains a loop in \mathcal{L}^2 then since \mathcal{R}_1 is invariant under f_θ , \mathcal{R}_1 contains every element in \mathcal{L}^2 and we follow the same procedure as in (A) to obtain \mathcal{R}_2 . If \mathcal{R}_1 does not contain any loop in \mathcal{L}^2 , then we simply ignore \mathcal{L}^2 and examine \mathcal{L}^3 , and so on.

Since N is finite, the process terminates after finite many steps. The resulting subsurface is \mathcal{R}^* . From the construction we see that \mathcal{R}^* is invariant under f_θ and encloses the puncture x .

Our next claim is that $f_\theta|_{\mathcal{R}^*}$ is irreducible (or pseudo-Anosov) self-map of \mathcal{R}^* . Suppose that $f_\theta|_{\mathcal{R}^*}$ is periodic, by Lemma 3.4, f_θ is periodic. Thus θ is elliptic. This is a contradiction. Next we assume that $f_\theta|_{\mathcal{R}^*}$ is reducible. The only possibility for this to occur is that $f_\theta|_{\mathcal{R}^*}$ is reduced by the single loop c_1 , where c_1 is the boundary of a twice punctured disk Δ enclosing x and another puncture (otherwise, we would continue the above procedure (A)). This leads

to three possibilities:

- $f_\theta^m|_{\mathcal{R}^*}$ is the Dehn twist along c_1 ,
- $f_\theta^m|_{\mathcal{R}^*}$ is irreducible (or pseudo-Anosov) on $S \setminus \Delta$, or
- $f_\theta^m|_{\mathcal{R}^*}$ is irreducible (or pseudo-Anosov) on Δ but the identity on $S \setminus \Delta$.

If the first situation occurs, by Lemma 3.5, f_θ is of type (I). We claim that the second situation does not occur. Suppose for the contrary. By filling in the puncture x , c_1 shrinks to a puncture, and all other loops in (4.1) is non-contractible. So $f_\theta^m|_{\mathcal{R}^*}$ descends to a pseudo-Anosov self-map on a component of $\tilde{S} \setminus \{\tilde{c}_2, \dots, \tilde{c}_u\}$. This contradicts that χ^m is the identity.

Otherwise, we conclude that the third case must occur. That is, $f_\theta|_{\mathcal{R}^*}$ is irreducible (or pseudo-Anosov). Hence $f_\theta^m|_{\mathcal{R}^*}$ is also irreducible (or pseudo-Anosov) and f_θ^m restricts to the identity on any other component of $S \setminus \mathcal{L}$. Since f_θ^m projects to the identity, by Theorem 2 of [8], $(f_\theta)^m = j([c])$, where c is a semi-filling loop on \tilde{S} but a filling loop on \mathcal{R}^* . This proves that θ must be of type (II).

§4.2. To prove (1) of Theorem 2.7, suppose that \hat{f} fixes a point $y^* \in \partial \mathbf{D} = \mathbf{S}^1$, where y^* is the fixed point of a parabolic element T of G . There is an integer n such that

$$(4.3) \quad [\hat{f}] \circ T \circ [\hat{f}]^{-1} = T^n.$$

From Theorem 2 of [8, 9], $\varphi^*(T^n) = (\varphi^*(T))^n$ is a power of the Dehn twist along the boundary $\partial \Delta$ of a twice punctured disc Δ enclosing x and another puncture x_1 . Thus $\varphi^*([\hat{f}]) \circ \varphi^*(T) \circ \varphi^*([\hat{f}])^{-1}$ keeps $\varphi^*([\hat{f}])(\partial \Delta)$ invariant. From (4.3), $\varphi^*([\hat{f}])(\partial \Delta) = \partial \Delta$. Therefore, $n = 1$. Since $\Pi \circ \varphi^*([\hat{f}]) = \chi$ is of order m , $f_\theta|_{S \setminus \Delta}$ is isotopic to a periodic self-map of order m . It follows that $\varphi^*([\hat{f}])$ is of type (I) and is reduced by $\partial \Delta$.

Conversely, if θ is of type (I), we are in Case 1 of §4.1. In this case, from Lemma 1 of [13], there is a parabolic element $T \in G$ such that $\varphi^*(T)$ is the Dehn twist along $\partial \Delta$. In particular, $\varphi^*(T)$ commutes with θ . So T commutes with $[\hat{f}]$. But $[\hat{f}]|_{\mathbf{D}} = \hat{f}$. From the same argument of Corollary 1 of [13], \hat{f} and T share a common fixed point if both are viewed as elements of real transformations.

§4.3. To prove (2) of Theorem 2.7, we suppose that θ is a type (II) mapping class, we are in Case 2 of §4.1. Then f_θ keeps invariant some subsurface \mathcal{R}^* of S , where \mathcal{R}^* is a subsurface of S described in steps (A) and (B) right after (4.2). By hypothesis, $(f_\theta|_{\mathcal{R}^*})^m$ is irreducible, which means that there is a filling closed geodesic c on \mathcal{R}^* such that $(f_\theta|_{\mathcal{R}^*})^m$ is isotopic to $j([c])|_{\mathcal{R}^*}$.

By Lemma 3.7, $j([c])$ is the identity outside \mathcal{R}^* . We see that f_θ commutes with $j([c])$ as self-maps of S , which implies that the element $g_c \in G$ that corresponds to c under (2.1) commutes with $[\hat{f}]$. The curve c , as a loop on \tilde{S} , is semi-filling. So g_c is semi-essential hyperbolic. Since $[\hat{f}]|_{\mathbf{D}} = \hat{f}$, g_c and \hat{f} share a common axis λ_c if both are considered real Möbius transformations. λ_c projects to a semi-filling geodesic c on \tilde{S} . It is immediate that \hat{f} keeps the geodesic $\varrho(\lambda_c)$ invariant.

Finally, we assume that \hat{f} fixes a geodesic $\lambda \subset \mathbf{D}$ with $\varrho(\lambda) \subset \tilde{S}$ being a semi-filling geodesic. Then \hat{f} commutes with an element g_c of G , where g_c corresponds to the semi-filling geodesic $c = \varrho(\lambda_c)$. This implies that $\theta = \varphi^*([\hat{f}])$ commutes with $\varphi^*(g_c) = j([c])$. Since $j([c])$ is a reducible mapping class. By Lemma 3.1, θ is reducible as well. From the same discussion as above, θ is not of type (I); otherwise, there is a parabolic element T of G such that $[\hat{f}]$ commutes with T . As real Möbius transformations, T and \hat{f} share a common

fixed point in \mathbf{S}^1 . By assumption, \hat{f} and g_c share the two fixed points. It follows that the fixed point of T is also a fixed point of g_c . As a consequence, the group generated by T and g_c , which is a subgroup of G , is not discrete. This is absurd. It follows from §4.1 that θ must be a type (II) mapping class.

§4.4. To prove (3) of Theorem 2.7, we note that $\theta^m = \theta'$ is also a hyperbolic mapping class. Since $\Pi(\theta') = \text{id}$, by Lemma 3.2, $\varphi^{*-1}(\theta') \in G$. From Theorem 2 of [8], $\theta' = \varphi^*(g_c) = j([c])$ for an essential hyperbolic element g_c of G that corresponds to a closed filling geodesic c on \tilde{S} . Obviously, θ commutes with θ' . From the proof of Lemma 3.3, \hat{f} and g_c share the same axis $\lambda_c \subset \mathbf{D}$. It is clear that $\varrho(\lambda_c)$ is a filling geodesic that is invariant under f .

Conversely, suppose that $\hat{f}(\lambda_c) = \lambda_c$, where λ_c is the axis of an essential hyperbolic element g_c of G . As real Möbius transformations, \hat{f} and g_c share the same fixed points. So \hat{f} commutes with g_c . It follows that $\theta = \varphi([\hat{f}])$ commutes with $\varphi^*(g_c)$. From Theorem 2 of [8] again, $\varphi^*(g_c)$ is hyperbolic. Hence from Lemma 3.1, θ is hyperbolic as well. This completes the proof of Theorem 2.7.

Now we proceed to prove Theorem 2.6.

§4.5. If θ is elliptic, θ has a fixed point $\tau \in T(S)$. So $[\hat{f}](\varphi^{-1}(\tau)) = \varphi^{-1}(\tau)$. This means that $[\hat{f}]$ keeps the fiber determined by $\varphi^{-1}(\tau)$ invariant. We may assume that the fiber is \mathbf{D} . In this case $\varphi^{-1}(\tau) \in \mathbf{D}$. Hence f fixes the point $\varrho(\varphi^{-1}(\tau)) \in \tilde{S}$.

Conversely, if f fixes a point $\tilde{y} \in \tilde{S}$, then we may choose a lift $\hat{f} \in \mathbf{PSL}(2, \mathbf{R})$ so that \hat{f} fixes a point $y \in \{\varrho^{-1}(\tilde{y})\}$ in \mathbf{D} . Notice that in our case, $\hat{f} = [\hat{f}]|_{\mathbf{D}}$. We see that as an element of $\text{mod}(\tilde{S})$, $[\hat{f}]$ has a fixed point in $F(\tilde{S})$. Hence $\varphi^*([\hat{f}])$ has a fixed point in $T(S)$, which says $\varphi^*([\hat{f}])$ is elliptic. This proves (1) of Theorem 2.6. (2) of Theorem 2.6 can be similarly handled.

§4.6. Since χ is elliptic, f has fixed points on the compactification of \tilde{S} . In the case where \tilde{S} is closed, f only fixes some points of \tilde{S} . However, if \tilde{S} is not closed, f could fix some punctures as well as some points of \tilde{S} . In particular, if f does not fix any punctures of \tilde{S} , f must fix some points of \tilde{S} . In this situation, let ξ_1, \dots, ξ_s denote all the points of \tilde{S} fixed by f . From §4.5, we know that $i^{-1}(\chi)$ contains infinitely many elliptic elements. By Lemma 3.8, among the lifts of f there exist some lifts \hat{f} (that can be written as $g^k \circ f^*$) that are hyperbolic Möbius transformations. Hence \hat{f} does not fix any pre-images of ξ_1, \dots, ξ_s . Those \hat{f} induce $[\hat{f}] \in \text{mod}(\tilde{S})$. It is easy to see that $\varphi^*([\hat{f}])$ is not elliptic. Since f does not fix any puncture, $i^{-1}(\chi)$ does not contain any type (I) mapping classes, from §4.1 it follows that either $\varphi^*([\hat{f}])$ is of type (II), or $\varphi^*([\hat{f}])$ is hyperbolic.

Now we assume that f does fix some punctures, which are denoted by y_1, \dots, y_{s_0} . If f fixes a point $x_1 \in \tilde{S}$, by Case 1 of Lemma 3.8, there is a lift \hat{f} that is a hyperbolic element. So it cannot fix any fixed points of parabolic elements of G . From §4.1, either $\varphi^*([\hat{f}])$ is of type II, or $\varphi^*([\hat{f}])$ is hyperbolic.

If f fixes no points of \tilde{S} , f must fix a puncture y of \tilde{S} . By Lemma 3.8 again, there is a lift \hat{f} that is a hyperbolic element. From §4.1, either $\varphi^*([\hat{f}])$ is of type II, or it is hyperbolic. This proves (3) of Theorem 2.6.

§4.7. We proceed to prove (4) of Theorem 2.6. Let $\eta : \tilde{S} \rightarrow \tilde{S}/\langle f \rangle$ denote the branched covering. If the orbifold $\tilde{S}/\langle f \rangle$ has genus $\tilde{p} = p/m \geq 1$, we fix a branch point $\tilde{\xi}$ on $\tilde{S}/\langle f \rangle$, let $\xi = \eta^{-1}(\tilde{\xi}) \in \tilde{S}$. Also we take a filling loop \tilde{c} on \tilde{S}^* that passes through $\tilde{\xi}$ and avoids any

other branch points, where \tilde{S}^* is the Riemann surface obtained from forgetting the branch points of $\tilde{S}/\langle f \rangle$. Let $c^* = \eta^{-1}(\tilde{c})$, and c the loop obtained from reparameterizing c^* . Note that c is a (self-intersecting) closed loop on \tilde{S} that passes through ξ and is invariant under f .

We claim that c is a filling loop of \tilde{S} . Otherwise, there is a component Δ of $\tilde{S} \setminus c$ that is neither a disk nor a punctured disk. Then either $f(\Delta) = \Delta$ or that $f(\Delta) \cap \Delta$ is empty. In the second case, Δ is homeomorphic to a component of $(\tilde{S}/\langle f \rangle) \setminus \tilde{c}$, contradicting that \tilde{c} is a filling loop of \tilde{S}^* . In the first case, $\eta|_{\Delta} : \Delta \rightarrow \Delta/\langle f \rangle$ is a finite branched covering. Since Δ projects to a disk or punctured disk, $\eta(\Delta)$ is a disk or punctured disk. This implies that Δ must be a disk or punctured disk. This is again a contradiction.

We conclude that c is a filling loop of \tilde{S} that is invariant under f . From (3) of Theorem 2.6, there is a lift θ of χ , so that θ is irreducible. This proves (4) of Theorem 2.6 and hence this completes the proof of Theorem 2.6.

5. Proof of Theorem 2.8

§5.1. Let χ be represented by a conformal automorphism f on \tilde{S} which has some fixed points. Let \tilde{z}_0 be one of the fixed points. Let $z_0 \in \mathbf{D}$ be a point such that $\pi(z_0) = \tilde{z}_0$. We may assume without loss of generality that $z_0 = 0$.

Let $\hat{f} : \mathbf{D} \rightarrow \mathbf{D}$ be the lift of f so that $\hat{f}(0) = 0$. Then \hat{f} is a conformal automorphism on \mathbf{D} , and hence it is a Möbius transformation. It follows that \hat{f} is a rotation on \mathbf{D} with rotation angle no larger than $2\pi/3$.

§5.2. A power of a Dehn twist t_{α}^n about a simple closed geodesic $\alpha \subset \tilde{S}$ can also be lifted to map $\tau^n : \mathbf{D} \rightarrow \mathbf{D}$ so that $\varrho \circ \tau^n = t_{\alpha}^n \circ \varrho$. It is known that τ determines a disjoint union of half planes Δ_j so that τ keeps each Δ_j invariant. As such, the complement $\Omega = \mathbf{D} \setminus \bigcup \Delta_j$ is also an invariant set by τ (see [15] for more details). By post composing a suitable element of G , one may assume that $0 \in \Omega$. This implies that for any $z \in \mathbf{S}^1$ and any n , the Euclidean distance between $\tau^n(z)$ and z is no greater than $2\pi/2 = \pi$.

§5.3. If $n > 0$, the motion direction of $\tau|_{\mathbf{S}^1}$ is in the clockwise direction. Now either \hat{f} or \hat{f}^{-1} is in the clockwise motion direction. Thus we may assume that $\tau|_{\mathbf{S}^1}$ and $\hat{f}|_{\mathbf{S}^1}$ are in the same clockwise motion direction. We assert that for any $z \in \mathbf{S}^1$, the distance between z and $\tau^n \hat{f}$ is no greater than $2\pi/3 + \pi < 2\pi$. It follows that $(\tau^n \hat{f})|_{\mathbf{S}^1}$ has no fixed points. In particular, $(\tau^n \hat{f})|_{\mathbf{S}^1}$ does not fix any parabolic fixed point of G . From Lemma 5.1 and Lemma 5.2 of [14], $\varphi^*([\tau^n \hat{f}])$ cannot fix the boundary of any twice punctured disk enclosing x . By the same argument of [13, 15], $\{\varphi^*([\tau^n \hat{f}])\} \subset \Pi^{-1}(t_{\alpha}^n \circ \chi)$ consists of hyperbolic mapping classes if $t_{\alpha}^n \circ \chi$ itself is a hyperbolic mapping class.

§5.4. We need to prove that there exists a simple closed geodesic $\alpha \subset \tilde{S}$ such that $t_{\alpha}^n \circ f$ and $f \circ t_{\alpha}^n$ are pseudo-Anosov for almost all integers n . To see this, we invoke Theorem III.3 of FLP [5] (see also Ivanov [7]) which asserts that if

$$(5.1) \quad \{f^k(\alpha) \text{ for } k = 0, \dots, m-1\}$$

fills \tilde{S} in the sense that $\tilde{S} \setminus \{f^k(\alpha), k = 0, \dots, m-1\}$ is a union of disks or punctured disks, then for almost all integers n (with possibly seven exceptional cases) $t_{\alpha}^n \circ f$ and $f \circ t_{\alpha}^n$ represent hyperbolic mapping classes.

It is now easy to obtain a simple curve α on S satisfying condition (5.1). Figure 1 demon-

strates a conformal automorphism of order 4 on a compact Riemann surface of genus 4. The curve α in the figure together with all the images $f^i(\alpha)$ fill the surface S .

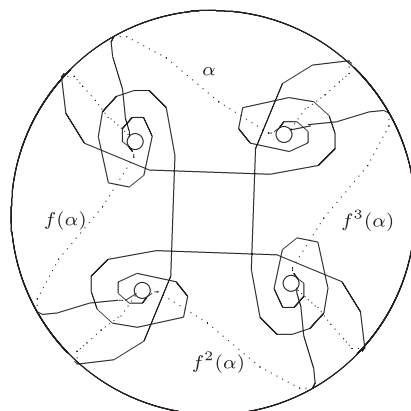


Fig. 1

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Yingqing Xiao
College of Mathematics and Econometrics
Hunan University
Changsha, 410082
China
e-mail: ouxyq@hnu.edu.cn

Chaohui Zhang
Zhang, Department of Mathematics
Morehouse College
Atlanta, GA 30314
USA
e-mail: czhang@morehouse.edu