Title: A BLOCK REFINEMENT OF THE GREEN–PUIG PARAMETERIZATION OF THE ISOMORPHISM TYPES OF INDECOMPOSABLE MODULES

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Osaka University
A BLOCK REFINEMENT OF THE GREEN-PUIG
PARAMETERIZATION OF THE ISOMORPHISM TYPES OF
INDECOMPOSABLE MODULES

Dedicated to the Memory of J.A. Green

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Abstract

Let $p$ be a prime integer, let $\mathcal{O}$ be a commutative complete local Noetherian ring with an algebraically closed residue field $k$ of characteristic $p$ and let $G$ be a finite group. Let $P$ be a $p$-subgroup of $G$ and let $X$ be an indecomposable $\mathcal{O}P$-module with vertex $P$. Let $\Lambda(G, P, X)$ denote a set of representatives for the isomorphism classes of indecomposable $\mathcal{O}G$-modules with vertex-source pair $(P, X)$ (so that $\Lambda(G, P, X)$ is a finite set by the Green correspondence). As mentioned in [5, Notes on Section 26], L. Puig asserted that a defect multiplicity module determined by $(P, X)$ can be used to obtain an extended parameterization of $\Lambda(G, P, X)$. In [5, Proposition 26.3], J. Thévenaz completed this program under the hypotheses that $X$ is $\mathcal{O}$-free. Here we use the methods of proof of [5, Theorem 26.3] to show that the $\mathcal{O}$-free hypothesis on $X$ is superfluous. (M. Linckelmann has also proved this, cf. [3]). Let $B$ be a block of $\mathcal{O}G$. Then we obtain a corresponding parameterization of the $(\mathcal{O}G)B$-modules in $\Lambda(G, P, X)$.

1. Introduction and Statements

Our notation and terminology are standard and tend to follow [5]. All rings have identities and are Noetherian and all modules over a ring are unitary and finitely generated left modules.

Let $R$ be a ring. Then $R$-mod will denote the abelian category of finitely generated (left) $R$-modules. Let $U, V$ be $R$-modules. Then $U|V$ in $R$-mod signifies that $U$ is isomorphic in $R$-mod to a direct summand of $V$. Also if $R$ has the unique decomposition property (cf.[1, p. 37]), then $U$ is a component of $V$ if $U|V$ and $U$ is indecomposable in $R$-mod.

Throughout this paper, $G$ is a finite group, $p$ is a prime integer and $\mathcal{O}$ is a commutative complete local Noetherian ring with an algebraically closed residue field of characteristic $p$ ([5, Assumption 2.1]).

The statements of the main results of this paper are given in this section. The required proofs are presented in Section 2.

Let $P$ be a $p$-subgroup of $G$ and let $X$ be an indecomposable $\mathcal{O}P$-module with vertex $P$. Let $E$ be an idempotent of $Z(\mathcal{O}G)$. Then, as in [2], $IT(G, P, X, E)$ will denote a set of representatives for the isomorphism types of indecomposable $(\mathcal{O}G)E$-modules with vertex-source pair $(P, X)$. Clearly we may assume that the modules in $IT(G, P, X, E)$ are components of $E\text{Ind}_P^G(X)$ in $\mathcal{O}G$-mod.

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Set $H = N_G(P, X)$. By [5, Proposition 20.8]:

(1.1) There is a natural bijection $P^H_{u}$, from $IT(G, P, X, 1)$ to $IT(H, P, X, 1)$ with inverse $P^H_{u} : IT(H, P, X, 1) \rightarrow IT(G, P, X, 1)$.

Here $P \leq H$ and $C_G(P) \leq H$.

Clearly we may assume:

(1.2) The modules in $IT(H, P, X, 1)$ are components of $\text{Ind}^H_P(X)$.

Set $A = \text{End}_\sigma(\text{Ind}^H_P(X))$ so that $A \cong \text{Ind}^H_P(\text{End}_\sigma(X))$ as interior $H$-algebras, $A^H = \text{End}_{\sigma H}(\text{Ind}^H_P(X))$ and $A^P = \text{End}_{\sigma P}(\text{Ind}^H_P(X))$. Also the points of $A^H$, $P(A^H)$, biject with the isomorphism types of components of $\text{Ind}^H_P(X)$ in $\mathcal{O}H$-mod and similarly for the points of $A^P$, $P(A^P)$.

As $X$ is an indecomposable $\mathcal{O}P$-module with vertex $P$, $\text{End}_\sigma(X)$ is a primitive interior $P$-algebra with defect group $P$. Since $\text{End}_\sigma(X)^P = \text{End}_{\sigma P}(X)$ is a local algebra, $P_{[Id_X]}$ is a defect of $\text{End}_\sigma(X)$.

Let

$$D^H_P : \text{End}_\sigma(\text{Ind}^H_P(X)) \rightarrow \text{Res}^H_P(A) = \text{Res}^H_P(\text{Ind}_\sigma(\text{Ind}^H_P(X)))$$

$$\cong \text{Res}^H_P(\text{Ind}^H_P(\text{End}^H_P(X)))$$

denote the canonical embedding. Here $\{\text{Id}_X\}$ is the unique point of $\text{End}_\sigma(X)^P = \text{End}_{\sigma P}(X)$ and $(1 \otimes \text{Id}_X \otimes 1)\text{Ind}^H_P(X) = 1 \otimes X$ so that $D^H_P(\{\text{Id}_X\}) = \gamma \in P(A^P)$ by [5, Proposition 15.1] and $P_\gamma$ is a local pointed group of $A$ by [5, Proposition 15.1(d)].

Thus $N_G(P_\gamma) = N_G(P, X)$ since $\text{Ind}^H_P(X) \cong X$ in $\mathcal{O}P$-mod for any $j \in \gamma$ as on [5, p. 106].

Also $A = \text{End}_\sigma(\text{Ind}^H_P(X)) \cong \text{Ind}^H_P(\text{End}_\sigma(X))$ as interior $H$-algebras and $A^P = \text{End}_{\sigma P}(\text{Res}^H_P(\text{Ind}^H_P(X)))$. Note that:

(1.3) $\text{Res}^H_P(\text{Ind}^H_P(X)) \cong [H/P]X$ in $\mathcal{O}P$-mod.

Thus:

(1.4) $A^H \cong \text{Mat}_{H/H}(\text{End}_{\sigma P}(X))$ as $\mathcal{O}$-algebras; $A^P/J(A^P) \cong \text{Mat}_{H/P}(k)$ as $k$-algebras and $P(A^P) = \gamma$.

**Lemma 1.1.** Let $W$ be a component of $\text{Ind}^H_P(X)$ in $\mathcal{O}H$-mod and let $e \in B(\mathcal{O}H)$ be such that $eW = W$. Then: (a) $(P, X)$ is the “unique” vertex-source pair of $W$; (b) Let $j \in \tau \in P(A^H)$ be such that $j\text{Ind}^H_P(X) \cong W$ in $\mathcal{O}H$-mod so that $jA_j \cong \text{End}_d(j\text{Ind}^H_P(X))$ as interior $H$-algebras. Then $P$ is a defect group of $jA_j$ and if $P_\gamma$ is a defect of $jA_j$ and if $I : jA_j \rightarrow A$ is the inclusion embedding of interior $H$-algebras, then $I(P_\gamma) = P_\gamma$; and (c) $e^G$ is a defined block of $G$ and $e^G_P\text{Ind}^G_P(W) = \text{Ind}^G_P(W)$.

**Corollary 1.2.** $IT(H, P, X, 1)$ bijects with $P(A^H)$.

**Remark 1.3.** By [4, Theorem 5.5.15], the map that associates each block $e$ of $\mathcal{O}H$ to the $H$-conjugacy class $C$ of blocks of $\mathcal{O}C_G(P)$ covered by $e$ is a bijection. Moreover $C = \{b \in B(\mathcal{O}C_G(P)) | b^{H} = e\}$. Also $IT(H, P, X, e) \neq \phi$ for each $e \in B(\mathcal{O}H)$ hence $P$ is contained in a defect group of $e$ by [2, Corollary 1.8]. Moreover $e^G$ is defined for each $e \in B(\mathcal{O}H)$ by [4, Theorem 5.3.5].
Also \( A^H = \text{End}_\mathcal{H}(\text{Ind}_P^H(X)) \) is an interior \( C_G(H) \)-algebra where \( C_G(H) \leq C_G(P) \leq H \) so that \( C_G(H) = Z(H) \).

Set \( \overline{H} = H/P \). Thus \( A^P \) is an \( \overline{H} \)-algebra and an interior \( C_G(P) \)-algebra. Since \( \gamma \) is the unique point of \( A^P \), the maximal ideal \( m_\gamma \) of \( A^P \) such that \( m_\gamma \cap \gamma = \phi \) is just \( J(A^P) \). Hence the multiplicity algebra of \( P_\gamma \), \( S(\gamma) = A^P/J(A^P) \cong \text{Mat}_\mu(k) \) where \( \mu \) is the multiple of \( \gamma \) as \( k \)-algebras and \( S(\gamma) = \text{End}_S(\mathcal{A}(\gamma)) \) where \( \mathcal{A}(\gamma) \) is a multiplicity module of \( P_\gamma \) on \( A \). Clearly \( S(\gamma) \) is an \( \overline{H} \)-algebra and \cite[Theorem 19.1]{5} yields:

**Proposition 1.4.** There is a bijection between \( \mathcal{P}(A^H) \) and \( \{ \delta \in \mathcal{P}(S(\gamma)^\overline{H}) \mid \overline{H}_\delta \text{ is a projective pointed group on } S(\gamma) \} \).

Let \( \alpha : H \to A^x = \text{End}_\mathcal{O}(\text{Ind}_P^H(X))^x \) denote the group homomorphism that expresses the fact that \( A \) is an interior \( H \)-algebra. \( \text{Thus, if } h \in H \text{ and } v \in \text{Ind}_P^H(X), \alpha(h)v = hv \). Consequently \( \alpha \) induces an \( H \)-algebra homomorphism \( \alpha : \mathcal{O}H \to A \), an \( \overline{H} \)-group homomorphism \( \alpha : C_G(P) \to (A^P)^x \) and an \( \overline{H} \)-algebra homomorphism \( \alpha : \mathcal{O}C_G(P) \to A^P \).

Let \( T \) be a transversal of \( Z(P) \) in \( C_G(P) \) with \( 1 \in T \) so that \( C_G(P) = \bigcup_{t \in T}\{tZ(P)\} \) and \( PC_G(P) = \bigcup_{t \in T}\{tP\} \) are disjoint. Also let \( \pi : A^P \to S(\gamma) = A^P/J(A^P) \) denote the canonical \( \overline{H} \)-algebra epimorphism. \( \text{Thus } \pi \circ \alpha : C_G(P) \to S(\gamma)^x \text{ is an } \overline{H} \)-group homomorphism with } \( Z(P) \leq \text{Ker}(\pi \circ \alpha) \) by \cite[p. 104]{5} and \( \pi \circ \alpha \) induces an \( \overline{H} \)-algebra homomorphism \( \pi \circ \alpha : \mathcal{O}C_G(P) \to S(\gamma) \).

As above \( S(\gamma) = \text{End}_S(\mathcal{A}(\gamma)) \) where \( \mathcal{A}(\gamma) \) is a multiplicity module of \( P_\gamma \) on \( A \).

Let \( \mathcal{H} = \{ (\overline{\sigma}, \overline{h}) \in (S(\gamma)^x \times \overline{H}) \mid \overline{\sigma} \overline{\sigma}^{-1} = h \sigma \text{ for all } \sigma \in S(\gamma) \} \) and set \( \mathcal{N} = \{ ((\pi \circ \alpha)(t), \overline{t}) \mid t \in T \} \). Let \( t \in T \) and \( a \in A^P \). Then, \( \overline{\tau} = \alpha(t) a \overline{\tau}^{-1} \), and so \( \overline{\tau} \pi(a) = (\pi \circ \alpha)(t) \pi(a) (\pi \circ \alpha)(t)^{-1} \). Thus \( \mathcal{N} \subseteq \mathcal{H} \subseteq S(\gamma)^x \times \overline{H} \).

**Proposition 1.5.** (a) \( \mathcal{H} \leq S(\gamma)^x \times \overline{H} \) and with \( \overline{H} \) acting by conjugation on \( \overline{H}, \overline{H} \) acts diagonally on \( \mathcal{H} \); \( b \mathcal{N} \leq \mathcal{H} \) and \( \mathcal{N} \) is \( \overline{H} \)-invariant; \( \text{and (c) If } (\overline{\sigma}, \overline{h}) \text{ and } (\overline{u}, \overline{x}) \in \mathcal{H}, \text{ then } \overline{\tau}(\overline{\sigma}, \overline{h}) = (\overline{u}, \overline{x})(\overline{\sigma}, \overline{h})(\overline{u}, \overline{x})^{-1} \).

Moreover, we have the following diagram in the category of groups:

\[
\begin{array}{cccc}
1 & \rightarrow & k^x & \xrightarrow{\rho} & 1 \\
\downarrow & & \downarrow & & \\
S(\gamma)^x & \xrightarrow{\eta} & \overline{H} \end{array}
\]

(1.5)

where \( \phi(u) = (u \text{ Id}_{S(\gamma)}, 1_{\overline{H}}) \), for all \( u \in k^x \), \( \eta \) is the projection on the second component and \( \rho \) is the projection on the first component.

Clearly:

(1.6) \( \text{The row in (1.5) is exact and letting } \overline{H} \text{ act trivially on } k^x, \text{ all of the group homomorphisms in (1.5) are } \overline{H}\text{-homomorphisms.} \)

Clearly \( k\mathcal{H} \) is an \( \overline{H} \)-algebra.

**Lemma 1.6.** \( Z(k\mathcal{H}) = (k\mathcal{H})^\overline{H} \).

Let \( \omega : C_G(P) \to N \) be such that \( tz \mapsto ((\pi \circ \alpha)(t), \overline{t}) \) for all \( t \in T \) and \( z \in Z(P) \). Clearly \( \omega \) is an \( \overline{H} \)-group epimorphism with \( Z(P) = \text{Ker}(\omega) \) and \( \omega \) induces an \( \overline{H} \)-group isomorphism \( \overline{\omega} : C_G(P) \to N \) and an \( \overline{H} \)-algebra homomorphism \( \overline{\omega} : \mathcal{O}C_G(P) \to kN \) and an \( \overline{H} \)-algebra isomorphism \( \overline{\omega} : kC_G(P) \to kN \).
Clearly $\rho : H \to S(\gamma)^X$ induces an $\overline{H}$-algebra homorphism $\rho : kH \to S(\gamma)$.

Since $T$ is a transversal of $P$ in $PC_G(P)$, we may extend $T$ to a transversal $S$ of $P$ in $H$.

For $t \in T$, set $\Sigma(t) = (\pi(t, \alpha(t, \gamma)), \overline{\gamma})$ and for $v \in S - T$, set $\Sigma(v) = (\overline{v}, \overline{\gamma})$ for any element $\overline{v}$ of $\overline{\gamma}^{-1}(v)$.

Clearly the ideal $\mathcal{I}$ of $kH$ generated by $\{\phi(u) - u(1_{S(\gamma)}, 1_{\overline{\gamma}}) \mid u \in k^X\}$ is contained in the kernel of $\rho : kH \to S(\gamma)$. Then $kH/\mathcal{I} \cong \bigoplus_{s \in S} k\Sigma(s)$ as $k$-vector spaces and $kH/\mathcal{I}$ is a twisted group algebra over $\overline{H}$ of dimension $|\overline{H}|$ denoted by $k_\mathcal{I}H$ and $\rho$ induces an $\overline{H}$-algebra homorphism $\overline{\rho} : k_\mathcal{I}H \to S(\gamma)$. Consequently $V_\mathcal{I}(\gamma)$ is via $\overline{\rho}$ a $k_\mathcal{I}H$-module.

In our situation, [5, Lemma 26.1] directly yields:

(1.7) The multiplicity module $V_\mathcal{I}(\gamma)$ is isomorphic to the regular (left) module $k_\mathcal{I}H$ in $k_\mathcal{I}H$-mod.

At this point, (1.7) and the discussion on [5, p. 157] (applying [5, Example 13.5, Lemma 12.4, and Corollary 17.8]) yields our generalization of [5, Proposition 26.3] which has been independently obtained by M. Linckelmann in [3, Proposition 5.7.6]:

**Theorem 1.7.** There are bijections between:

(a) $IT(G, P, X, 1)$;
(b) $IT(H, P, X, 1)$;
(c) $\mathcal{P}(A^H)$;
(d) $\{\delta \in \mathcal{P}(S(\gamma)^H) \mid \overline{\gamma}_\delta \text{ is projective}\}$; and
(e) The isomorphism classes of components of the $k_H$-module $k_\mathcal{I}H$.

**Remark 1.8.** As remarked in [5, Notes to Section 26], the idea of using the defect multiplicity module as a third invariant for the parameterization of indecomposable modules is an idea of L. Puig. Theorem 1.7 generalizes [5, Theorem 26.3] where $X$ is an $\mathcal{O}$-lattice to arbitrary pairs $(P, X)$ of $G$ where $X$ is any indecomposable $\mathcal{O}P$-module with vertex $P$. Our proof of Theorem 1.7 essentially follows the proof of [5, Theorem 26.3]. (If $X$ is also an $\mathcal{O}$-lattice, then so is any module in $IT(G, P, X, 1)$).

Next we proceed to present our block version of Theorem 1.7

**Lemma 1.9.** (a) Let $x \in C_G(P)$. Then $(\pi \circ \alpha)(x) = (\rho \circ \omega)(x)$; and (b) Let $e \in Blt(\mathcal{O}H)$ so that $e = \sum_{\delta \in \mathcal{C}} \delta$ where $\mathcal{C}$ is an orbit of $H$ on $Blt(O\mathcal{C}_G(P))$. Then $0 \neq \omega(e) \in (k\overline{\gamma})N \leq (kH)^\mathcal{I} = Z(kH)$. Thus $\omega(e)$ is an idempotent of $Z(kH)$.

For the remainder of this article, fix $B \in Blt(\mathcal{O}G)$ and set $N = N_G(P)$.

**Lemma 1.10.** The following two conditions are equivalent:

(a) $P$ is contained in a defect group of $B$; and
(b) $IT(G, P, X, B) \neq \emptyset$.

Let $\mathcal{B}(B, H) = \{e \in Blt(\mathcal{O}H) \mid Br_{r_B}(B; \overline{e}) = \overline{e}\}$ so that $\mathcal{B}(B, H) = \{e \in Blt(\mathcal{O}H) \mid e^G = B\}$ by [4, Theorem 5.3.5]. Set $E = \sum_{e \in \mathcal{B}(B, H)} e$. Thus we may assume:

(1.8) $IT(H, P, X, E) = \bigcup_{e \in \mathcal{B}(B, H)} IT(H, P, X, e) \text{ is disjoint and } IT(H, P, X, e) \neq \emptyset$ for each $e \in \mathcal{B}(B, H)$. (since $P \not\leq H$).

From [2, Theorem 1.6(c)] we have:
(1.9) \(Pu^H_G\) induces a bijection from \(IT(G, P, X, B)\) to \(IT(H, P, X, E)\).

Fix \(e \in \mathcal{B}(B, H)\).

Here \(\alpha: \mathcal{O}H \to A\) such that \(\alpha(u)w = uw\) for all \(u \in \mathcal{O}H\) and \(w \in \mathcal{I}nd^H_P(X)\) induces the \(\mathcal{O}\)-algebra homomorphism \(\alpha: \mathcal{Z}(\mathcal{O}H) \to A^H\). Also \(IT(H, P, X, e) \neq \emptyset\) and we may assume that each module in \(IT(H, P, X, e)\) is a component of \(e\text{Ind}^H_P(X)\). Thus \(\alpha(e)\) is an idempotent of \(A^H\).

Set \(C = \alpha(e)A\alpha(e)\) so that \(C\) is an interior \(H\)-subalgebra of \(A, C \cong \text{End}_\mathcal{O}(e\text{Ind}^H_P(X))\) as interior \(H\)-algebras and the inclusion map \(I: C \to A\) is an embedding of interior \(H\)-algebras.

Lemma 1.1.1 implies that each component of \(e\text{Ind}^H_P(X)\) has a unique vertex-source pair and \(IT(H, P, X, e)\) bijects with \(\mathcal{P}\text{End}_{\mathcal{O}H}(e\text{Ind}^H_P(X))\) where \(\text{End}_{\mathcal{O}H}(e\text{Ind}^H_P(X)) \cong \text{End}_{\mathcal{O}}(e\text{Ind}^H_P(X)) \cong C^H\) as \(\mathcal{O}\)-algebras.

Thus \([5, \text{Proposition 15.1 (a)}]\) implies that \(C^P\) has a unique point \(y'\) such that \(I_\gamma(P_{y'}) = P_y\) and \(P_{y'}\) is a local pointed group on \(C\). Thus \(H = N_H(P_y) = N_H(P_{y'})\) and if \(M_{y'}\) denotes the maximal ideal of \(C^P\) such that \(M_{y'} \cap y' = \emptyset\) and if \(\pi'\): \(C^P \to S(y') = C^P/M_{y'}\) denotes the canonical \(\overline{H}\)-algebra epimorphism, then \(I\) induces an \(\overline{H}\)-algebra embedding \(\overline{I}(y')\): \(S(y') \to S(\gamma)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
C^P & \xrightarrow{\pi'} & A^P \\
\downarrow{\pi} & & \downarrow{\pi} \\
S(y') & \xrightarrow{\overline{I}(y')} & S(\gamma).
\end{array}
\]

Since \(e \in (\mathcal{O}C_G(P))^H, \alpha(e) \in A^H \leq A^P\) so that \(0 \neq (\pi \circ \alpha)(e) \in S(\gamma)\overline{H}\) and \((\pi \circ \alpha)(e)\) is an idempotent of \(S(\gamma)^H\) as \(\text{Ker}(\pi) = J(A^P)\).

Note that \(I(\text{Id}_C) = \alpha(e) \in A^H\).

Set \(\mathcal{H}' = \{(\mathcal{S}', h) \in S(\gamma)^\times \times \overline{H} | \mathcal{S}' \mathcal{S}'(\mathcal{S}')^{-1} = h\mathcal{S}' \text{ for all } \mathcal{S}' \in S(\gamma)\}\).

From \([5, \text{Theorem 19.1}]\), we have:

**Proposition 1.11.** There are bijections between
(a) \(IT(H, P, X, e)\);
(b) \(\mathcal{P}(C^H)\); and
(c) \(\{\delta \in \mathcal{P}(S(\gamma)\overline{H}) | \overline{H}_\delta \text{ is projective}\}\).

Moreover we have the following diagram of \(\overline{H}\)-group homomorphisms:

\[
\begin{array}{ccc}
1 & \xrightarrow{\phi^\times} & \mathcal{H}' \\
\downarrow{\rho^\times} & & \downarrow{\rho^\times} \\
S(\gamma)^\times
\end{array}
\]

where \(\overline{H}\) acts trivially on \(k^\times\) and diagonally on \(\mathcal{H}'\), \(\phi'(\mathcal{A}) = (\mathcal{A}1_{S(\gamma)}, \overline{1})\) for all \(\mathcal{A} \in k^\times\). \(\eta^\times, \rho^\times\) are the projections on the second and first component, respectively, and the row is exact.

Since \(\overline{I}(\gamma')\): \(S(\gamma') \to S(\gamma)\) is an \(\overline{H}\)-algebra embedding, \(\overline{I}(\gamma')\) induces the \(\overline{H}\)-algebra isomorphism \(\overline{I}(\gamma')\): \(S(\gamma') \to ((\pi \circ \alpha)(e))S(\gamma)((\pi \circ \alpha)(e))\). Note that \(S(\gamma') = \text{End}_C(V_C(\gamma'))\) where \(V_C(\gamma')\) is the multiplicity module of \(P_y\) on \(C\).

From \([5, \text{Proposition 15.4}]\) and the fact that \(\overline{I}(\gamma')\text{Id}_{S(\gamma')}) = (\pi \circ \alpha)(e)\) we have:
There is an isomorphism of the short exact sequence of groups:

$$1 \longrightarrow k^\times \overset{\varphi}{\longrightarrow} H \overset{\eta}{\longrightarrow} \overline{H} \longrightarrow 1$$

$$1 \longrightarrow k^\times \overset{\varphi'}{\longrightarrow} H' \overset{\eta'}{\longrightarrow} \overline{H} \longrightarrow 1$$

where \( i \) and \( j \) are identity maps and \( f((\overline{s}, h)) = (\overline{J}(y'))^{-1}((\pi \circ \alpha)(e))\overline{s}((\pi \circ \alpha(e)), h) \) for all \((\overline{s}, h) \in \overline{H}\).

Thus \( f \) induces \( k \)-algebra isomorphisms \( f: kH \rightarrow kH' \) and \( f_\#: k_\#H \rightarrow k_\#H' \). Also \( \rho: H \rightarrow S(\gamma)^{\times} \) and \( \rho': H' \rightarrow S(\gamma')^{\times} \) induce \( k \)-algebra homomorphisms \( \rho: kH \rightarrow S(\gamma) = \text{End}_k(V_A(\gamma)), \rho_\#: k_\#H \rightarrow S(\gamma), \rho': kH' \rightarrow S(\gamma') = \text{End}_k(V_C(\gamma')) \) and \( \rho'_\#: k_\#H' \rightarrow S(\gamma') \).

Thus:

(1.13) \( \rho'_\#: k_\#H' \rightarrow S(\gamma') = \text{End}_k(V_C(\gamma')) \) describes \( V_C(\gamma') \) as a \( k_\#H' \)-module; and

(1.14) \( \rho'_\# \circ f_\#: k_\#H \rightarrow S(\gamma') \) describes \( V_C(\gamma') \) as a \( k_\#H \)-module.

Since \((\pi \circ \alpha)(e) \in S(\gamma)\overline{H}\) and if \((\overline{s}, h) \in \overline{H}\), then \(\overline{s}((\pi \circ \alpha)(e))\overline{s}^{-1} = h((\pi \circ \alpha)(e)) = (\pi \circ \alpha)(e)\).

Thus

(1.15) \( (\pi \circ \alpha)(e)V_A(\gamma) \) is a \( k_\#H \)-submodule of \( V_A(\gamma) \).

By [5, Lemma 12.4], there is an interior \( H \)-algebra isomorphism \( A: ((\pi \circ \alpha)(e))S(\gamma)((\pi \circ \alpha)(e)) \rightarrow \text{End}_k(((\pi \circ \alpha)(e))V_A(\gamma)) \). Consequently \( A \circ \overline{J}(\gamma')(\rho' \circ f): H \rightarrow \text{End}_k(((\pi \circ \alpha)(e))V_A(\gamma))^\times \) is a group homomorphism that induces the \( k \)-algebra homomorphism \( A \circ \overline{J}(\gamma') \circ \rho'_\# \circ f_\#: k_\#H \rightarrow \text{End}_k(((\pi \circ \alpha)(e))V_A(\gamma)) \).

From the above, \( \overline{J}(\gamma '): S(\gamma ') \rightarrow ((\pi \circ \alpha)(e))S(\gamma)((\pi \circ \alpha)(e)) \) is an \( \overline{H} \)-algebra isomorphism of simple \( k \)-algebras and \(((\pi \circ \alpha)(e))V_A(\gamma)) \) is an irreducible module for \(((\pi \circ \alpha)(e))S(\gamma)((\pi \circ \alpha)(e)) \).

Thus there is a \( k \)-module isomorphism \( j': V_C(\gamma') \rightarrow (\pi \circ \alpha)(e)V_A(\gamma) \) such that if \( \overline{v'} \in V_C(\gamma') \) and \( \sigma' \in S(\gamma') \), then \( j'(\sigma' \overline{v'}) = ((A \circ \overline{J}(\gamma'))(\sigma'))j'(\overline{v'}) \).

Here \( A \circ \overline{J}(\gamma') \circ \rho' \circ f: H \rightarrow \text{End}_k(((\pi \circ \alpha)(e))V_A(\gamma))^\times \) is a group homomorphism that induces the \( k \)-algebra homomorphism:

\[ A \circ \overline{J}(\gamma') \circ \rho'_\# \circ f_\#: k_\#H \rightarrow \text{End}_k(((\pi \circ \alpha)(e))V_A(\gamma)). \]

Thus if \((\overline{s}, h) \in \overline{H}\) and \( \sigma' \in V_C(\gamma') \), then

\[ j'(\rho' \circ f(\overline{s}, h))(\sigma') = (A \circ \overline{J}(\gamma'))((\rho' \circ f)(\overline{s}, h))j'(\sigma') = A \circ \overline{J}(\gamma')(\overline{J}(\gamma')^{-1}((\pi \circ \alpha)(e))\overline{s}((\pi \circ \alpha)(e)))j'(\sigma') = A((\pi \circ \alpha)(e)\overline{s}((\pi \circ \alpha)(e)))j'(\sigma') = \overline{s}(\overline{J}(\gamma'))j'(\sigma') = \overline{s}(\overline{J}(\gamma'))(\sigma')j'(\sigma') = \overline{s}(\overline{J}(\gamma'))j'(\sigma'). \]

We have proved:

**Theorem 1.12.** There are bijections between:

(a) \( IT(H, P, X, e) \);
(b) \( P(C^H) \);
(c) \( \{ \delta \in P(S(\gamma)\overline{H}) \mid \overline{H}_S \text{ is a projective pointed group on } S(\gamma) \} \);
(d) The isomorphism classes of projective components of the \( k_\#H' \)-module \( V_C(\gamma') \); and
Corollary 1.13. There is a bijection between
(a) $IT(G,P,X,B)$; and
(b) The isomorphism classes of components of the $kH$-module $\omega(x)kH$.

2. Required Proofs

2.1. A Proof of Lemma 1.1. Assume the situation of Lemma 1.1. Here $\text{Res}_P^H(W) \cong sX$ in $\mathcal{O}P^\text{-mod}$ for some positive integer $s$. Let $(R,U)$ be a vertex-source pair of $W$ so that $W \mid P$ and $\text{Ind}_P^R(U)$ in $\mathcal{O}P^\text{-mod}$. Let $T$ be the set of representatives for the $(P,R)$-double cosets in $H$ with $1 \in T$. Then $\text{Res}_P^H(\text{Ind}_P^R(U)) \cong \bigoplus_{t \in T}(\text{Ind}_P^R(\text{Res}_{P,R}(U)))$ in $\mathcal{O}P^\text{-mod}$. Thus there is a $t \in T$ such that $X \mid P \text{Ind}_P^R(\text{Res}_{P,R}(U))$ in $\mathcal{O}P^\text{-mod}$. Whence $P \leq R$, $P \leq R$ and since $W \mid P$ and $\text{Ind}_P^R(X)$ in $\mathcal{O}H$-mod, $R = P$ and (a) holds. Now [2, Corollary 1.7] implies (c): Since $P(A^t) = \gamma$ by (1.4), (b) holds and we are done.

2.2. A Proof of Proposition 1.5. Assume the situation of Proposition 1.5. Clearly $H \leq S(\gamma) \times \overline{H}$ and $N \leq H$. Let $\overline{x} \in \overline{H}$ and $(\overline{x},h) \in S(\gamma) \times \overline{H}$. Then $\overline{\gamma}(\overline{x},h) = (\overline{\gamma},\overline{\gamma}h\overline{x}^{-1})$. Since $\overline{x} = x\sigma^{-1} = h\sigma$ for all $\sigma \in S(\gamma)$, $(\overline{x},h) = (\overline{x},\overline{\gamma}h\overline{x}^{-1})$ for all $\sigma \in S(\gamma)$. Thus (a) is proved. Let $t \in T$. Then $\overline{\gamma}(\pi \circ \alpha)(t),\overline{\gamma} = ((\pi \circ \alpha)(t),\overline{\gamma}h\overline{x}^{-1})$ and $N$ is $\overline{H}$ invariant. Also let $(\overline{x},h) \in H$. Then

$$
(\overline{x},h)((\pi \circ \alpha)(t),\overline{\gamma})(\overline{x}^{-1},h^{-1}) = (\overline{\gamma}(\pi \circ \alpha)(t)\overline{x}^{-1},h\overline{h}^{-1})
$$

and so $N \leq H$. Let $(\overline{x},h)$ and $(\overline{u},\overline{x}) \in H$ as in (c). Then $\overline{\gamma}(\overline{x},h) = (\overline{u\overline{w}^{-1}},\overline{xh}\overline{x}^{-1})$ and we are done.

A Proof of Lemma 1.6. Let $\Gamma = \sum_{i=1}^{n} k_i(\overline{x}_i,h_i)$ where $k_i \in k^\times$ and $(\overline{x}_i,h_i) \in H$ for all $1 \leq i \leq n$. Let $(\overline{t},\overline{x}) \in H$. Then

$$
(\overline{t},\overline{x})\Gamma(\overline{t}^{-1},\overline{x}^{-1}) = \sum_{i=1}^{n} k_i(\overline{t}\overline{x}^{-1},\overline{x}h_i\overline{x}^{-1})
$$

Thus $\Gamma \in Z(kH)$ if and only if $\Gamma \in (kH)\overline{\Gamma}$ and we are done.

A Proof of Lemma 1.9. Let $x \in C_G(P)$ so that $x = tz$ for a unique $t$ in $T$ and $z \in Z(P)$. Then

$$
(\rho \circ \omega)(x) = \rho((\pi \circ \alpha)(t),\overline{\gamma}) = (\pi \circ \alpha)(t)h = (\pi \circ \alpha)(t)z
$$

since $Z(P) \leq \text{Ker}(\pi \circ \alpha)$ and (a) follows. As in (b), clearly $\omega(x) \in (kN)\overline{\pi} \leq (kH)\overline{\pi} = Z(kH)$ by Lemma 1.6. Let $\pi : \mathcal{O} \rightarrow k$ denote the canoncal $\mathcal{O}$-algebra epimorphism and let $- : \mathcal{O}C_G(P) \rightarrow kC_G(P)$ denote the canoncal $H$-algebra epimorphism induced by $-$. Here
\[ \omega(\overline{e}) = \omega(e). \] As \( \overline{e} \) is an idempotent of \( kCG(P) \) and \( \overline{\omega} : kCG(G) \to kN \) is an \( \overline{H} \)-algebra isomorphism (b) also follows.

A Proof of Lemma 1.10. Clearly [1, III, Corollary 6.8] yields (b) implies (a). Since [2, Corollary 1.8] implies the converse, we are done.

References