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UNIFORM WELL-POSEDNESS FOR A TIME-DEPENDENT GINZBURG-LANDAU MODEL IN SUPERCONDUCTIVITY

JISHAN FAN, BESSEM SAMET and YONG ZHOU*

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Abstract
We study the initial boundary value problem for a time-dependent Ginzburg-Landau model in superconductivity. First, we prove the uniform boundedness of strong solutions with respect to diffusion coefficient $0 < \epsilon < 1$ in the case of Coulomb gauge. Our second result is the global existence and uniqueness of the weak solutions to the limit problem when $\epsilon = 0$.

1. Introduction

This paper is concerned with the following Ginzburg-Landau model in superconductivity:

\begin{align*}
\eta \partial_t \psi + i \eta k \phi \psi + \left( i \frac{\epsilon}{k} \nabla + A \right)^2 \psi + (|\psi|^2 - 1) \psi &= 0, \\
\partial_t A + \nabla \phi + \text{curl} A + \text{Re} \left\{ \left( i \frac{\epsilon}{k} \nabla \psi + \psi A \right) \bar{\psi} \right\} &= 0
\end{align*}

in $Q_T := (0, T) \times \Omega$, with boundary and initial conditions

\begin{align*}
\epsilon \nabla \psi \cdot \nu &= 0, \quad A \cdot \nu = 0, \quad \text{curl} A \times \nu = 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \\
(\psi, A)(x, 0) &= (\psi_0, A_0)(x) \quad \text{in} \quad \Omega.
\end{align*}

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial \Omega$, $\nu$ is the outward normal to $\partial \Omega$, and $T$ is any given positive constant. The unknowns $\psi, A$, and $\phi$ are $\mathbb{C}$-valued, $\mathbb{R}^d$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. $\eta$ and $k$ are Ginzburg-Landau positive constants. $\bar{\psi}$ denotes the complex conjugate of $\psi$, $\text{Re} \psi := (\psi + \bar{\psi})/2$, $|\psi|^2 := \psi \bar{\psi}$ is the density of superconducting carriers, and $i := \sqrt{-1}$. $\epsilon$ is a positive constant.

It is well known that the Ginzburg-Landau equations are gauge invariant, namely if $(\psi, A, \phi)$ is a solution of (1.1)-(1.4), then for any real-valued smooth function $\chi$, $(\psi e^{i\chi}, A + \nabla \chi, \phi - \partial_t \chi)$ is also a solution of (1.1)-(1.4). So, in order to obtain the well-posedness of the problem, we need to impose suitable gauge condition. From the physical point of view, one usually has four types of the gauge conditions:

- **Coulomb gauge**: $\text{div} A = 0$ in $\Omega$ and $\int_\Omega \phi d\chi = 0$.
- **Lorentz gauge**: $\phi = -\text{div} A$ in $\Omega$.
- **Lorenz gauge**: $\partial_t \phi = -\text{div} A$ in $\Omega$.
- **Temporal gauge (Weyl gauge)**: $\phi = 0$ in $\Omega$.

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For the initial data $\psi_0 \in H^1(\Omega), |\psi_0| \leq 1, A_0 \in H^1(\Omega)$, Chen, Elliott and Tang [1], Chen, Hoffmann and Liang [2], Du [3] and Tang [4] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb and Lorentz as well as temporal gauges. For the initial data $\psi_0 \in H^1(\Omega), A_0 \in H^1(\Omega)$, Tang and Wang [5] obtained the existence and uniqueness of global strong solutions, while Fan and Jiang [6] showed the existence of global weak solutions when $\psi_0, A_0 \in L^2$. Fan and Ozawa [7] (2-D) and Fan, Gao and Guo [8, 9] (3-D) prove the uniqueness of weak solutions for $\psi_0, A_0 \in L^d$ with $d = 2, 3$, which is critical. This comes from a scaling argument for (1.1) and (1.2). Move precisely, if $(\psi(t, x), A(t, x), \phi(t, x))$ is a solution of (1.1) and (1.2) associated with the initial data $(\psi_0(x), A_0(x))$ without linear lower order term $\psi$, then

\begin{equation}
(\lambda \psi(\lambda^2 t, \lambda x), \lambda A(\lambda^2 t, \lambda x), \lambda^2 \phi(\lambda^2 t, \lambda x)) =: (\psi_\lambda, A_\lambda, \phi_\lambda)
\end{equation}

is also a solution for any $\lambda > 0$. A Banach space $B$ of distributions on $\mathbb{R} \times \mathbb{R}^d$ is a critical space if its norm verifies for any $\lambda$ and any $u \in B$,

$$||u||_B = ||u(\lambda^2, \lambda .)||_B.$$ 

If we choose $B$ as $L'(0, \infty; L^p(\mathbb{R}^d))$, then $(r, p)$ should satisfy

$$\frac{2}{r} + \frac{d}{p} = 1.$$

In this paper, we will choose the Coulomb gauge.

First, we will prove

**Theorem 1.1.** Let $d = 3$ and $0 < \epsilon < 1$. Let $\psi_0 \in H^1, |\psi_0| \leq 1$ and $A_0 \in H^1$. Then the solution $(\psi, A, \phi)$ satisfies

\begin{equation}
||\psi|| \leq 1, ||\psi||_{L^\infty(0, T; H^1)} \leq C, ||\partial_t \psi||_{L^2(0, T; L^2)} \leq C,
\end{equation}

\begin{equation}
||A||_{L^\infty(0, T; H^1)} + ||A||_{L^2(0, T; H^1)} + ||\partial_t A||_{L^2(0, T; L^2)} \leq C,
\end{equation}

\begin{equation}
||\phi||_{L^2(0, T; H^1)} \leq C
\end{equation}

for any $0 < T < \infty$. Here and later $C$ will denote a constant independent of $\epsilon$.

When $\epsilon = 0$, we will prove

**Theorem 1.2.** Let $d = 3, \epsilon = 0$, and $\psi_0, A_0 \in L^2$. If $\psi, A \in L^2(0, T; H^1) \cap W$ with $W := \{(\psi, A); \psi \in L^\infty(0, T; L^2) \cap L^2(0, T; L^3), A \in L^\infty(0, T; L^3) \cap L^\infty(0, T; L^p) \text{ with some } 3 < p \leq \infty\}$, then the problem (1.1)-(1.4) has at most a unique weak solution.

**Remark 1.1.** The space $W$ is scaling invariant due to (1.5).

**Theorem 1.3.** Let $d = 3, \epsilon = 0, \psi_0 \in H^1, |\psi_0| \leq 1$ and $A_0 \in L^4$. Then the problem (1.1)-(1.4) has a unique weak solution.

**Remark 1.2.** Our results also hold true with the choice of Lorentz gauge.

In our proofs, we will use the following lemmas.

**Lemma 1.1** ([10, 11]). Let $\Omega$ be a smooth and bounded open set in $\mathbb{R}^3$. Then there exists $C > 0$ such that
for any $1 < p < \infty$ and $f : \Omega \to \mathbb{R}^3$ be in $W^{1,p}(\Omega)$.

**Lemma 1.2** ([12]). Let $\Omega$ be a regular bounded domain in $\mathbb{R}^3$, let $f : \Omega \to \mathbb{R}^3$ be a smooth enough vector field, and let $1 < p < \infty$. Then, the following identity holds true:

\[
(1.8) \quad - \int_{\Omega} \Delta f \cdot f |f|^{p-2} \, dx = \int_{\Omega} |f|^p |\nabla f|^2 \, dx + \frac{4(p-2)}{p^2} \int_{\Omega} |\nabla |f|^2|^2 \, dx - \int_{\partial \Omega} |f|^{p-2}(\nu \cdot \nabla) f \cdot f \, dS.
\]

2. **Proof of Theorem 1.1**

This section is devoted to the proof of Theorem 1.1, we only need to show a priori estimates (1.6).

To begin with, it is easy to show that [1, 2, 3, 4]:

\[
(2.1) \quad |\psi| \leq 1 \text{ in } \Omega \times (0, T).
\]

Testing (1.1) by $\psi$ and taking the real parts, we see that

\[
\eta \frac{d}{dt} \int |\psi|^2 \, dx + \int \left| \frac{\varepsilon}{k} \nabla \psi + \psi A \right|^2 \, dx + \int |\psi|^4 \, dx = \int |\psi|^2 \, dx,
\]

which gives

\[
(2.2) \quad \int_0^T \int \left| \frac{\varepsilon}{k} \nabla \psi + \psi A \right|^2 \, dtdx \leq C.
\]

In [6], we have proved that

\[
(2.3) \quad \nabla \phi \cdot \nu = 0 \text{ on } (0, T) \times \partial \Omega.
\]

Testing (1.2) by $\partial_t A + \text{curl}^2 A$, using (2.1), (2.2) and (2.3), we find that

\[
\frac{d}{dt} \int |\text{curl} A|^2 \, dx + \int (|\partial_t A|^2 + |\text{curl}^2 A|^2) \, dx \leq \int \left| \frac{\varepsilon}{k} \nabla \psi + \psi A \right| \partial_t A + \text{curl}^2 \, dx \\
\leq \frac{1}{2} \int (|\partial_t A|^2 + |\text{curl}^2 A|^2) \, dx + C \int \left| \frac{\varepsilon}{k} \nabla \psi + \psi A \right|^2 \, dx,
\]

which leads to

\[
(2.4) \quad \|A\|_{L^2(0,T; H^1)} + \|A\|_{L^2(0,T; H^2)} + \|\partial_t A\|_{L^2(0,T; L^2)} \leq C,
\]

whence

\[
(2.5) \quad \|\phi\|_{L^2(0,T; H^1)} \leq C.
\]

Multiplying (1.1) by $-\Delta \overline{\psi}$, integrating by parts and taking the real part, using (2.1), (2.4) and (2.5), we obtain
which yields

\[ \| \psi \|_{L^2(0,T;H')} + \epsilon \| \psi \|_{L^2(0,T,H')} \leq C, \]

whence

\[ \| \partial_t \psi \|_{L^2(0,T,L^2)} \leq C. \]

This completes the proof. \( \square \)

3. Proof of Theorem 1.2

In this section, we will prove the uniqueness. To this end, let \((\psi_i, A_i, \phi_i) (i = 1, 2)\) be the two weak solutions and let

\[ \psi := \psi_1 - \psi_2, A := A_1 - A_2, \phi := \phi_1 - \phi_2. \]

Then it is easy to verify that

\[ \eta \partial_t \psi + i \eta k \phi \psi_1 + i \eta k \phi \psi_2 + A_1^2 \psi_1 - A_2^2 \psi_2 + |\psi_1|^2 \psi_1 - |\psi_2|^2 \psi_2 - \psi = 0, \]

\[ \partial_t A + \nabla \phi + \text{curl} A = |\psi_1|^2 A_1 - |\psi_2|^2 A_2 = 0, \]

\[ -\Delta \phi = \text{div} (|\psi_1|^2 A_1 - |\psi_2|^2 A_2). \]

Testing (3.1) by \( \psi \) and taking the real part, we get

\[ \frac{\eta}{2} \frac{d}{dt} \int |\psi|^2 \, dx \]

\[ \leq \eta k \left( \int \psi \partial_t \psi \, dx \right) + \int (A_1^2 - A_2^2) \psi_2 \, dx + \int |\psi_1|^2 |\psi|^2 \, dx + \int |\psi|^2 \, dx \]

\[ \leq C \| \phi \|_{L^2} \| \psi_1 \|_{L^2} \| \psi_2 \|_{L^2} + C \| A_1 \| + A_2 \| |A| \|_{L^2} + \| \psi_1 \|_{L^2} \| \psi_2 \|_{L^2} + \int |\psi|^2 |\phi|^2 \, dx + \int |\psi|^2 \, dx \]

\[ \leq \eta k \| \psi \|_{L^2} + C(\| \psi_1 \|_{L^2}^2 + \| \psi_2 \|_{L^2}^2 + 1) \| \psi_1 \|_{L^2}^2 + C \| A_1 \| + A_2 \| |A| \|_{L^2} \]

for any \( 0 < \delta < 1 \).

On the other hand, we have

\[ \| \phi \|_{L^2} \leq C \| \nabla \phi \|_{L^2} + C \| \psi_1 \|^2 A_1 - |\psi_2|^2 A_2 \]

\[ \leq C \| \psi_1 \|^2 A_1 + C(\| \psi_1 \| - |\psi_2|)\| \psi_1 \| + |\psi_2|)A_2 \|_{L^2} \]
Using the Gagliardo-Nirenberg inequality

\[(3.6) \quad \|A\|_{L^{2p/2}} \leq C \|\phi\|_{L^2}^{1-\frac{2}{p}} \|A\|_{L^2}^{\frac{2}{p}},\]

we have

\[(3.7) \quad C\|A_1 + A_2\|_{L^2}^2 \leq \delta\|A\|_{H_p}^2 + C\|A_1 + A_2\|_{L^2}^{2\delta} \|A\|_{L^2}^2\]

for any \(0 < \delta < 1\).

Inserting (3.5) and (3.7) into (3.4), we have

\[(3.8) \quad \frac{\eta}{2} \frac{d}{dt} \int |\phi|^2 dx \leq C\delta\|A\|_{H_p}^2 + C(1 + \|\phi_1\|_{L^2}^2 + \|\phi_2\|_{L^2}^2)\|A\|_{L^2}^2 + C(\|A_1\|_{L^2}^{\frac{2\delta}{p}} + \|A_2\|_{L^2}^{\frac{2\delta}{p}}) \|A\|_{L^2}^2\]

for any \(0 < \delta < 1\).

Testing (3.2) by \(A\), we deduce that

\[(3.9) \quad \frac{1}{2} \frac{d}{dt} \int A^2 dx + \int |\text{curl} A|^2 dx + \int |\phi_1|^2 A dx = -\int (|\phi_1|^2 - |\phi_2|^2) A_2 A dx \leq (\|\phi_1\|_{L^2} + \|\phi_2\|_{L^2})\|\phi\|_{L^2}\|A_2\|_{L^2} \leq (\|\phi_1\|_{L^2}^2 + \|\phi_2\|_{L^2}^2)\|\phi\|_{L^2}^2 + C\|A_2\|_{L^2}^{\frac{2\delta}{p}} \|A\|_{L^2}^2 \leq \delta\|A\|_{H_p}^2 + (\|\phi_1\|_{L^2}^2 + \|\phi_2\|_{L^2}^2)\|\phi\|_{L^2}^2 + C\|A_2\|_{L^2}^{\frac{2\delta}{p}} \|A\|_{L^2}^2\]

for any \(0 < \delta < 1\).

Using the well-known Poincaré inequality

\[(3.10) \quad \|A\|_{H^1} \leq C |\text{curl} A|_{L^2},\]

summing up (3.8) and (3.9), taking \(\delta\) small enough, using the Gronwall inequality, we arrive at

\[\psi = 0, A = 0\]

and thus \(\phi = 0\), whence \(\phi_1 = \phi_2, A_1 = A_2\) and \(\phi_1 = \phi_2\).

This completes the proof.

\[\square\]

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3, we only need to show a priori estimates.

We still have (2.1).
Testing (1.2) by $A$, we see that
\[
\|A\|_{L^2(0,T;H^1)} \leq C.
\]

Testing (1.2) by $|A|^2 A$ and using (1.8), we have
\[
\int \nabla \cdot |A|^2 A \, dx + \int |A|^2 (\nabla A)^2 \, dx + \int |\psi|^2 |A|^4 \, dx
\]
\[
= \int \nabla \phi \cdot |A|^2 A \, dx + \int_{\partial \Omega} |A|^2 (\nu \cdot \nabla) A \cdot A \, dS =: I_1 + I_2.
\]

Using the formula
\[
(\nu \cdot \nabla) A \cdot A = (A \cdot \nabla) A \cdot \nu + (\text{curl} \, A \times \nu) \cdot A
\]
\[
= (A \cdot \nabla) A \cdot \nu
\]
\[
= -(A \cdot \nabla) \nu \cdot A,
\]
we observe that
\[
I_2 = -\int_{\partial \Omega} |A|^2 (A \cdot \nabla) \nu \cdot A \, dS \leq C \int_{\partial \Omega} |A|^4 \, dS
\]
\[
= C \int_{\partial \Omega} f^2 \, dS \leq C\|f\|_{L^2(\Omega)}\|f\|_{H^1(\Omega)} (f := |A|^2)
\]
\[
\leq \frac{1}{8} \int |\nabla f|^2 \, dx + C\|f\|_{L^2}^2.
\]

Using (2.1), we bound $I_1$ as follows
\[
I_1 \leq \|\nabla \phi\|_{L^2} \|A\|_{L^2}^2
\]
\[
\leq C \|\psi\|_{L^2} |A|_{L^2} \|A\|_{L^2}^4 \leq C\|A\|_{L^4}^4.
\]

Inserting the above estimates into (4.2), we have
\[
\int \nabla \psi \cdot |A|^2 \, dx \leq \frac{\eta}{2} \int \frac{d}{dt} |A|^2 \, dx + C \int \|\nabla A\|^2 \, dx + C \int |\psi|^2 |A|^4 \, dx
\]
\[
\leq C\|\nabla \phi\|_{L^2} \|\nabla \psi\|_{L^2} + C \|\nabla \psi\|_{L^2}^2 + C \int |A|^2 |\nabla A|^2 \, dx,
\]
which implies
\[
\|\psi\|_{L^\infty(0,T;H^1)} \leq C.
\]

This completes the proof. \(\square\)
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