



Title	UNIFORM WELL-POSEDNESS FOR A TIME-DEPENDENT GINZBURG-LANDAU MODEL IN SUPERCONDUCTIVITY
Author(s)	Fan, Jishan; Samet, Bessem; Zhou, Yong
Citation	Osaka Journal of Mathematics. 2019, 56(2), p. 269-276
Version Type	VoR
URL	https://doi.org/10.18910/72318
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

UNIFORM WELL-POSEDNESS FOR A TIME-DEPENDENT GINZBURG-LANDAU MODEL IN SUPERCONDUCTIVITY

JISHAN FAN, BESSEM SAMET and YONG ZHOU*

(Received November 16, 2017)

Abstract

We study the initial boundary value problem for a time-dependent Ginzburg-Landau model in superconductivity. First, we prove the uniform boundedness of strong solutions with respect to diffusion coefficient $0 < \epsilon < 1$ in the case of Coulomb gauge. Our second result is the global existence and uniqueness of the weak solutions to the limit problem when $\epsilon = 0$.

1. Introduction

This paper is concerned with the following Ginzburg-Landau model in superconductivity:

$$(1.1) \quad \eta \partial_t \psi + i\eta k \phi \psi + \left(i \frac{\epsilon}{k} \nabla + A \right)^2 \psi + (|\psi|^2 - 1)\psi = 0,$$

$$(1.2) \quad \partial_t A + \nabla \phi + \operatorname{curl}^2 A + \operatorname{Re} \left\{ \left(i \frac{\epsilon}{k} \nabla \psi + \psi A \right) \bar{\psi} \right\} = 0$$

in $Q_T := (0, T) \times \Omega$, with boundary and initial conditions

$$(1.3) \quad \epsilon \nabla \psi \cdot \nu = 0, \quad A \cdot \nu = 0, \quad \operatorname{curl} A \times \nu = 0 \quad \text{on } (0, T) \times \partial\Omega,$$

$$(1.4) \quad (\psi, A)(x, 0) = (\psi_0, A_0)(x) \quad \text{in } \Omega.$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward normal to $\partial\Omega$, and T is any given positive constant. The unknowns ψ, A , and ϕ are \mathbb{C} -valued, \mathbb{R}^d -valued, and \mathbb{R} -valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. η and k are Ginzburg-Landau positive constants. $\bar{\psi}$ denotes the complex conjugate of ψ , $\operatorname{Re}\psi := (\psi + \bar{\psi})/2$, $|\psi|^2 := \psi \bar{\psi}$ is the density of superconducting carriers, and $i := \sqrt{-1}$. ϵ is a positive constant.

It is well known that the Ginzburg-Landau equations are gauge invariant, namely if (ψ, A, ϕ) is a solution of (1.1)-(1.4), then for any real-valued smooth function χ , $(\psi e^{ik\chi}, A + \nabla \chi, \phi - \partial_t \chi)$ is also a solution of (1.1)-(1.4). So, in order to obtain the well-posedness of the problem, we need to impose suitable gauge condition. From the physical point of view, one usually has four types of the gauge conditions:

- Coulomb gauge: $\operatorname{div} A = 0$ in Ω and $\int_{\Omega} \phi dx = 0$.
- Lorentz gauge: $\phi = -\operatorname{div} A$ in Ω .
- Lorenz gauge: $\partial_t \phi = -\operatorname{div} A$ in Ω .
- Temporal gauge(Weyl gauge): $\phi = 0$ in Ω .

2010 Mathematics Subject Classification. Primary 35Q35, 35K55.

*Corresponding author.

For the initial data $\psi_0 \in H^1(\Omega)$, $|\psi_0| \leq 1$, $A_0 \in H^1(\Omega)$, Chen, Elliott and Tang [1], Chen, Hoffmann and Liang [2], Du [3] and Tang [4] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb and Lorentz as well as temporal gauges. For the initial data $\psi_0 \in H^1(\Omega)$, $A_0 \in H^1(\Omega)$, Tang and Wang [5] obtained the existence and uniqueness of global strong solutions, while Fan and Jiang [6] showed the existence of global weak solutions when $\psi_0, A_0 \in L^2$. Fan and Ozawa [7] (2-D) and Fan, Gao and Guo [8, 9] (3-D) prove the uniqueness of weak solutions for $\psi_0, A_0 \in L^d$ with $d = 2, 3$, which is critical. This comes from a scaling argument for (1.1) and (1.2). More precisely, if $(\psi(t, x), A(t, x), \phi(t, x))$ is a solution of (1.1) and (1.2) associated with the initial data $(\psi_0(x), A_0(x))$ without linear lower order term ψ , then

$$(1.5) \quad (\lambda\psi(\lambda^2 t, \lambda x), \lambda A(\lambda^2 t, \lambda x), \lambda^2 \phi(\lambda^2 t, \lambda x)) =: (\psi_\lambda, A_\lambda, \phi_\lambda)$$

is also a solution for any $\lambda > 0$. A Banach space \mathbf{B} of distributions on $\mathbb{R} \times \mathbb{R}^d$ is a critical space if its norm verifies for any λ and any $u \in \mathbf{B}$,

$$\|u\|_{\mathbf{B}} = \|\lambda u(\lambda^2 \cdot, \lambda \cdot)\|_{\mathbf{B}}.$$

If we choose \mathbf{B} as $L^r(0, \infty; L^p(\mathbb{R}^d))$, then (r, p) should satisfy

$$\frac{2}{r} + \frac{d}{p} = 1.$$

In this paper, we will choose the Coulomb gauge.

First, we will prove

Theorem 1.1. *Let $d = 3$ and $0 < \epsilon < 1$. Let $\psi_0 \in H^1$, $|\psi_0| \leq 1$ and $A_0 \in H^1$. Then the solution (ψ, A, ϕ) satisfies*

$$(1.6) \quad \begin{aligned} |\psi| &\leq 1, \|\psi\|_{L^\infty(0, T; H^1)} \leq C, \|\partial_t \psi\|_{L^2(0, T; L^2)} \leq C, \\ \|A\|_{L^\infty(0, T; H^1)} + \|A\|_{L^2(0, T; H^2)} + \|\partial_t A\|_{L^2(0, T; L^2)} &\leq C, \\ \|\phi\|_{L^2(0, T; H^1)} &\leq C \end{aligned}$$

for any $0 < T < \infty$. Here and later C will denote a constant independent of ϵ .

When $\epsilon = 0$, we will prove

Theorem 1.2. *Let $d = 3, \epsilon = 0$, and $\psi_0, A_0 \in L^2$. If $\psi, A \in L^2(0, T; H^1) \cap W$ with $W := \{(\psi, A); \psi \in L^\infty(0, T; L^3) \cap L^2(0, T; L^\infty), A \in L^\infty(0, T; L^3) \cap L^{\frac{2p}{p-3}}(0, T; L^p)$ with some $3 < p \leq \infty\}$, then the problem (1.1)-(1.4) has at most a unique weak solution.*

REMARK 1.1. *The space W is scaling invariant due to (1.5).*

Theorem 1.3. *Let $d = 3, \epsilon = 0, \psi_0 \in H^1$, $|\psi_0| \leq 1$ and $A_0 \in L^4$. Then the problem (1.1)-(1.4) has a unique weak solution.*

REMARK 1.2. *Our results also hold true with the choice of Lorentz gauge.*

In our proofs, we will use the following lemmas.

Lemma 1.1 ([10, 11]). *Let Ω be a smooth and bounded open set in \mathbb{R}^3 . Then there exists $C > 0$ such that*

$$(1.7) \quad \|f\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\Omega)}^{1-\frac{1}{p}} \|f\|_{W^{1,p}(\Omega)}^{\frac{1}{p}}$$

for any $1 < p < \infty$ and $f : \Omega \rightarrow \mathbb{R}^3$ be in $W^{1,p}(\Omega)$.

Lemma 1.2 ([12]). *Let Ω be a regular bounded domain in \mathbb{R}^3 , let $f : \Omega \rightarrow \mathbb{R}^3$ be a smooth enough vector field, and let $1 < p < \infty$. Then, the following identity holds true:*

$$(1.8) \quad \begin{aligned} & - \int_{\Omega} \Delta f \cdot f |f|^{p-2} dx \\ &= \int_{\Omega} |f|^{p-2} |\nabla f|^2 dx + \frac{4(p-2)}{p^2} \int_{\Omega} |\nabla |f|^{\frac{p}{2}}|^2 dx - \int_{\partial\Omega} |f|^{p-2} (\nu \cdot \nabla) f \cdot f dS. \end{aligned}$$

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, we only need to show a priori estimates (1.6).

To begin with, it is easy to show that [1, 2, 3, 4]:

$$(2.1) \quad |\psi| \leq 1 \text{ in } \Omega \times (0, T).$$

Testing (1.1) by $\bar{\psi}$ and taking the real parts, we see that

$$\frac{\eta}{2} \frac{d}{dt} \int |\psi|^2 dx + \int \left| i \frac{\epsilon}{k} \nabla \psi + \psi A \right|^2 dx + \int |\psi|^4 dx = \int |\psi|^2 dx,$$

which gives

$$(2.2) \quad \int_0^T \int \left| i \frac{\epsilon}{k} \nabla \psi + \psi A \right|^2 dx dt \leq C.$$

In [6], we have proved that

$$(2.3) \quad \nabla \phi \cdot \nu = 0 \text{ on } (0, T) \times \partial\Omega.$$

Testing (1.2) by $\partial_t A + \operatorname{curl}^2 A$, using (2.1), (2.2) and (2.3), we find that

$$\begin{aligned} & \frac{d}{dt} \int |\operatorname{curl} A|^2 dx + \int (|\partial_t A|^2 + |\operatorname{curl}^2 A|^2) dx \\ & \leq \int \left| i \frac{\epsilon}{k} \nabla \psi + \psi A \right| |\partial_t A + \operatorname{curl}^2 A| dx \\ & \leq \frac{1}{2} \int (|\partial_t A|^2 + |\operatorname{curl}^2 A|^2) dx + C \int \left| i \frac{\epsilon}{k} \nabla \psi + \psi A \right|^2 dx, \end{aligned}$$

which leads to

$$(2.4) \quad \|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} + \|\partial_t A\|_{L^2(0,T;L^2)} \leq C,$$

whence

$$(2.5) \quad \|\phi\|_{L^2(0,T;H^1)} \leq C.$$

Multiplying (1.1) by $-\Delta \bar{\psi}$, integrating by parts and taking the real part, using (2.1), (2.4) and (2.5), we obtain

$$\begin{aligned}
& \frac{\eta}{2} \frac{d}{dt} \int |\nabla \psi|^2 dx + \frac{\epsilon^2}{k^2} \int |\Delta \psi|^2 dx \\
& \leq \left| \operatorname{Re} \int i \eta k \phi \psi \cdot \Delta \bar{\psi} dx \right| + 2 \left| \operatorname{Re} \frac{\epsilon}{k} \int i A \nabla \psi \cdot \Delta \bar{\psi} dx \right| \\
& \quad + \operatorname{Re} \int A^2 \psi \Delta \bar{\psi} dx + \operatorname{Re} \int (|\psi|^2 - 1) \psi \cdot \Delta \bar{\psi} dx \\
& \leq \frac{1}{2} \frac{\epsilon^2}{k^2} \int |\Delta \psi|^2 dx + C \int |\nabla \phi| |\nabla \psi| dx \\
& \quad + C \|A\|_{L^\infty}^2 \|\nabla \psi\|_{L^2}^2 + C \|A\|_{L^\infty} \|\nabla A\|_{L^2} \|\nabla \psi\|_{L^2} + C \|\nabla \psi\|_{L^2}^2,
\end{aligned}$$

which yields

$$(2.6) \quad \|\psi\|_{L^\infty(0,T;H^1)} + \epsilon \|\psi\|_{L^2(0,T;H^2)} \leq C,$$

whence

$$(2.7) \quad \|\partial_t \psi\|_{L^2(0,T;L^2)} \leq C.$$

This completes the proof. \square

3. Proof of Theorem 1.2

In this section, we will prove the uniqueness. To this end, let (ψ_i, A_i, ϕ_i) ($i = 1, 2$) be the two weak solutions and let

$$\psi := \psi_1 - \psi_2, A := A_1 - A_2, \phi := \phi_1 - \phi_2.$$

Then it is easy to verify that

$$(3.1) \quad \eta \partial_t \psi + i \eta k \phi \psi_1 + i \eta k \phi_2 \psi + A_1^2 \psi_1 - A_2^2 \psi_2 + |\psi_1|^2 \psi_1 - |\psi_2|^2 \psi_2 - \psi = 0,$$

$$(3.2) \quad \partial_t A + \nabla \phi + \operatorname{curl}^2 A + |\psi_1|^2 A_1 - |\psi_2|^2 A_2 = 0,$$

$$(3.3) \quad -\Delta \phi = \operatorname{div} (|\psi_1|^2 A_1 - |\psi_2|^2 A_2).$$

Testing (3.1) by $\bar{\psi}$ and taking the real part, we get

$$\begin{aligned}
(3.4) \quad & \frac{\eta}{2} \frac{d}{dt} \int |\psi|^2 dx \\
& \leq \eta k \left| \int \phi \psi_1 \psi dx \right| + \left| \int (A_1^2 - A_2^2) \psi_2 \bar{\psi} dx \right| + \int |\psi_2|^2 |\psi|^2 dx + \int |\psi|^2 dx \\
& \leq C \|\phi\|_{L^2} \|\psi_1\|_{L^\infty} \|\psi\|_{L^2} + C \|A_1 + A_2\|_{L^p} \|A\|_{L^{\frac{2p}{p-2}}} \|\psi_2\|_{L^\infty} \|\psi\|_{L^2} + \int |\psi_2|^2 |\psi|^2 dx + \int |\psi|^2 dx \\
& \leq \delta \|\phi\|_{L^2}^2 + C (\|\psi_1\|_{L^\infty}^2 + \|\psi_2\|_{L^\infty}^2 + 1) \|\psi\|_{L^2}^2 + C \|A_1 + A_2\|_{L^p}^2 \|A\|_{L^{\frac{2p}{p-2}}}^2
\end{aligned}$$

for any $0 < \delta < 1$.

On the other hand, we have

$$\begin{aligned}
(3.5) \quad \|\phi\|_{L^2} & \leq C \|\nabla \phi\|_{L^{\frac{6}{5}}} \leq C \|\psi_1|^2 A_1 - |\psi_2|^2 A_2\|_{L^{\frac{6}{5}}} \\
& \leq C \|\psi_1|^2 A\|_{L^{\frac{6}{5}}} + C \|(|\psi_1| - |\psi_2|)(|\psi_1| + |\psi_2|) A_2\|_{L^{\frac{6}{5}}}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|\psi_1\|_{L^3}^3\|A\|_{L^6} + C\|\psi_1| + |\psi_2\|_{L^\infty}\|\psi\|_{L^2}\|A_2\|_{L^3} \\
&\leq C\|A\|_{L^6} + C(\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty})\|\psi\|_{L^2} \\
&\leq C\|\operatorname{curl} A\|_{L^2} + C(\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty})\|\psi\|_{L^2}.
\end{aligned}$$

Using the Gagliardo-Nirenberg inequality

$$(3.6) \quad \|A\|_{L^{\frac{2p}{p-2}}} \leq C\|A\|_{L^2}^{1-\frac{3}{p}}\|A\|_{H^1}^{\frac{3}{p}},$$

we have

$$(3.7) \quad C\|A_1 + A_2\|_{L^p}^2\|A\|_{L^{\frac{2p}{p-2}}}^2 \leq \delta\|A\|_{H^1}^2 + C\|A_1 + A_2\|_{L^p}^{\frac{2p}{p-3}}\|A\|_{L^2}^2$$

for any $0 < \delta < 1$.

Inserting (3.5) and (3.7) into (3.4), we have

$$\begin{aligned}
(3.8) \quad &\frac{\eta}{2}\frac{d}{dt}\int|\psi|^2dx \\
&\leq C\delta\|A\|_{H^1}^2 + C(1 + \|\psi_1\|_{L^\infty}^2 + \|\psi_2\|_{L^\infty}^2)\|\psi\|_{L^2}^2 + C(\|A_1\|_{L^p}^{\frac{2p}{p-3}} + \|A_2\|_{L^p}^{\frac{2p}{p-3}})\|A\|_{L^2}^2
\end{aligned}$$

for any $0 < \delta < 1$.

Testing (3.2) by A , we deduce that

$$\begin{aligned}
(3.9) \quad &\frac{1}{2}\frac{d}{dt}\int A^2dx + \int |\operatorname{curl} A|^2dx + \int |\psi_1|^2Adx \\
&= -\int(|\psi_1|^2 - |\psi_2|^2)A_2Adx \\
&\leq (\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty})\|\psi\|_{L^2}\|A_2\|_{L^p}\|A\|_{L^{\frac{2p}{p-2}}} \\
&\leq (\|\psi_1\|_{L^\infty}^2 + \|\psi_2\|_{L^\infty}^2)\|\psi\|_{L^2}^2 + C\|A_2\|_{L^p}^2\|A\|_{L^{\frac{2p}{p-2}}}^2 \\
&\leq \delta\|A\|_{H^1}^2 + (\|\psi_1\|_{L^\infty}^2 + \|\psi_2\|_{L^\infty}^2)\|\psi\|_{L^2}^2 + C\|A_2\|_{L^p}^{\frac{2p}{p-3}}\|A\|_{L^2}^2
\end{aligned}$$

for any $0 < \delta < 1$.

Using the well-known Poincaré inequality

$$(3.10) \quad \|A\|_{H^1} \leq C\|\operatorname{curl} A\|_{L^2},$$

summing up (3.8) and (3.9), taking δ small enough, using the Gronwall inequality, we arrive at

$$\psi = 0, A = 0$$

and thus $\phi = 0$, whence $\psi_1 = \psi_2, A_1 = A_2$ and $\phi_1 = \phi_2$.

This completes the proof. \square

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3, we only need to show a priori estimates.

We still have (2.1).

Testing (1.2) by A , we see that

$$(4.1) \quad \|A\|_{L^2(0,T;H^1)} \leq C.$$

Testing (1.2) by $|A|^2 A$ and using (1.8), we have

$$(4.2) \quad \begin{aligned} & \frac{1}{4} \frac{d}{dt} \int |A|^4 dx + \int |A|^2 |\nabla A|^2 dx + \frac{1}{2} \int |\nabla|A|^2|^2 dx + \int |\psi|^2 |A|^4 dx \\ &= \int \nabla \phi \cdot |A|^2 A dx + \int_{\partial\Omega} |A|^2 (\nu \cdot \nabla) A \cdot A dS =: I_1 + I_2. \end{aligned}$$

Using the formula

$$\begin{aligned} (\nu \cdot \nabla) A \cdot A &= (A \cdot \nabla) A \cdot \nu + (\operatorname{curl} A \times \nu) \cdot A \\ &= (A \cdot \nabla) A \cdot \nu \\ &= -(A \cdot \nabla) \nu \cdot A, \end{aligned}$$

we observe that

$$\begin{aligned} I_2 &= - \int_{\partial\Omega} |A|^2 (A \cdot \nabla) \nu \cdot A dS \leq C \int_{\partial\Omega} |A|^4 dS \\ &= C \int_{\partial\Omega} f^2 dS \leq C \|f\|_{L^2(\Omega)} \|f\|_{H^1(\Omega)} (f := |A|^2) \\ &\leq \frac{1}{8} \int |\nabla f|^2 dx + C \|f\|_{L^2}^2. \end{aligned}$$

Using (2.1), we bound I_1 as follows

$$\begin{aligned} I_1 &\leq \|\nabla \phi\|_{L^4} \|A\|_{L^4}^3 \\ &\leq C \|\psi\|^2 A \|A\|_{L^4}^3 \leq C \|A\|_{L^4}^4. \end{aligned}$$

Inserting the above estimates into (4.2), we have

$$(4.3) \quad \|A\|_{L^\infty(0,T;L^4)} + \int_0^T \int |A|^2 |\nabla A|^2 dx dt \leq C,$$

whence

$$(4.4) \quad \|A\|_{L^5(0,T;L^5)} \leq C,$$

$$(4.5) \quad \|\nabla \phi\|_{L^\infty(0,T;L^4)} \leq C.$$

Taking ∇ to (1.1), testing by $\nabla \bar{\psi}$ and taking the real part, using (2.1), (4.3) and (4.5), we have

$$\begin{aligned} \frac{\eta}{2} \frac{d}{dt} \int |\nabla \psi|^2 dx &\leq \eta k \int |\nabla \phi| |\nabla \psi| dx + \int |\nabla|A|^2| |\nabla \psi| dx + C \int |\nabla \psi|^2 dx \\ &\leq C \|\nabla \phi\|_{L^2} \|\nabla \psi\|_{L^2} + C \|\nabla \psi\|_{L^2}^2 + C \int |A|^2 |\nabla A|^2 dx, \end{aligned}$$

which implies

$$\|\psi\|_{L^\infty(0,T;H^1)} \leq C.$$

This completes the proof. \square

ACKNOWLEDGEMENTS. This paper is partially supported by NSFC (No. 11171154). The second author extends his appreciation to Distinguished Scientist Fellowship Program (DSFP) at King Saud University (Saudi Arabia).

References

- [1] Z.M. Chen, C. Elliott and Q. Tang: *Justification of a two-dimensional evolutionary Ginzburg-Landau superconductivity model*, RAIRO Model Math. Anal. Numer. **32** (1998), 25–50.
- [2] Z.M. Chen, K.H. Hoffmann and J.Liang: *On a nonstationary Ginzburg-Landau superconductivity model*, Math. Meth. Appl. Sci. **16** (1993), 855–875.
- [3] Q. Du: *Global existence and uniqueness of solutions of the time dependent Ginzburg-Landau model for superconductivity*, Appl. Anal. **53** (1994), 1–17.
- [4] Q. Tang: *On an evolutionary system of Ginzburg-Landau equations with fixed total magnetic flux*, Comm. PDE, **20** (1995), 1–36.
- [5] Q. Tang and S. Wang: *Time dependent Ginzburg-Landau equations of superconductivity*, Physica D, **88** (1995), 139–166.
- [6] J. Fan and S. Jiang: *Global existence of weak solutions of a time-dependent 3-D Ginzburg-Landau model for superconductivity*, Appl. Math. Lett. **16** (2003), 435–440.
- [7] J. Fan and T. Ozawa: *Uniqueness of weak solutions to the Ginzburg-Landau model for superconductivity*, Z. Angew. Math. Phys. **63** (2012), 453–459.
- [8] J. Fan, H. Gao and B. Guo: *Uniqueness of weak solutions to the 3D Ginzburg-Landau superconductivity model*, Int. Math. Res. Not. IMRN (2015), 1239–1246.
- [9] J. Fan and H. Gao: *Uniqueness of weak solutions in critical spaces of the 3-D time-dependent Ginzburg-Landau equations for superconductivity*, Math. Nachr. **283** (2010), 1134–1143.
- [10] R.A. Adams and J.J.F. Fournier: Sobolev Spaces, 2nd ed., Pure and Applied Mathematics (Amsterdam) **140**. Elsevier/Academic Press, Amsterdam, 2003.
- [11] A. Lunardi: Interpolation Theory, 2nd ed., Lecture Notes, Scuola Normale Superiore di Pisa (New Series), Edizioni della Normale, Pisa, 2009.
- [12] H. Beirão da Veiga and F. Crispo: *Sharp inviscid limit results under Navier type boundary conditions: An L^p theory*, J. Math. Fluid Mech. **12**, (2010), 307–411.

Jishan Fan
Department of Applied Mathematics
Nanjing Forestry University
Nanjing 210037
China
e-mail: fanjishan@njfu.edu.cn

Bessem Samet
Department of Mathematics
College of Science
King Saud University, P.O. Box 2455, Riyadh 11451
Saudi Arabia

Yong Zhou
School of Mathematics (Zhuhai)
Sun Yat-Sen University, Zhuhai, Guangdong 519082
China
email: zhouyong3@mail.sysu.edu.cn

and
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, Zhejiang
China