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Osaka University
CONVERGENCE TOWARDS AN ELASTICA
IN A RIEMANNIAN MANIFOLD

NORIHITO KOISO

(Received May 27, 1998)

0. Introduction

Consider a springy circle wire in a riemannian manifold $M$. We describe it as a closed curve $\gamma$ with unit line element and fixed length. For such a curve, its elastic energy is given by

$$E(\gamma) = \int |D_x \gamma'|^2 \, dx.$$ 

Solutions of the corresponding Euler-Lagrange equation are called elastic curves. We discuss a corresponding parabolic equation in this paper. We will see that the equation becomes an initial value problem:

$$\begin{align*}
\partial_t \gamma &= -D_t^3 \gamma' + R(\gamma', D_t \gamma')\gamma' + D_t(w \gamma'), \\
-w'' + |D_x \gamma'|^2 w &= 2(|D_x \gamma'|^2)' - |D_x^2 \gamma'|^2 - (R(\gamma', D_x \gamma')\gamma', D_x \gamma'), \\
\gamma(x, 0) &= \gamma_0(x),
\end{align*}$$

where $w = w(x, t)$ is an unknown real valued function.

In [3], we treated the case of euclidean spaces and saw that the above equation has a unique long time solution and that the solution converges to an elastica. In this paper, we treat general riemannian manifolds, and get the following

**Theorem 5.6.** Let $M$ be a compact real analytic riemannian manifold, and let $\gamma_0(x)$ be a closed curve with unit line element and length $L$. Suppose that there are no closed geodesics of length $L$ in $M$. Then (EP) has a unique solution $\gamma(x, t)$ for all time, and the solution $\gamma(\ast, t)$ converges to an elastica when $t \to \infty$.

**Remark 0.1.** Even if the metric of $M$ is not real analytic, there is a solution of (EP) which has a subsequence converging to an elastica (Theorem 4.1, 5.5). This proves the existence of an elastica. Existence of an elastica has been originally shown in [4] by using Palais-Smale's condition (C). Another proof has been given in [1] by using a direct method.
Remark 0.2. The equation (EP) is not the so-called curve shortening equation. The principal part of (EP) is \((\partial/\partial t + \partial^4/\partial x^4)\gamma + (\partial^2/\partial x^2)w\). Main difficulty of our equation comes from being coupled.

1. Preliminaries

By scaling, we may assume that the length of the initial curve \(\gamma_0\) is 1. From now on, a closed curve means a map from \(S^1 \equiv \mathbb{R}/\mathbb{Z}\) into a Riemannian manifold \(M\). The variable in \(S^1\) is denoted by \(x\), and differentiation with respect to \(x\) is denoted by \(*'\) or \(*^{(n)}\). The covariant differentiation on \(M\) is denoted by \(D\).

For tensors on \(M\), we use pointwise inner product \((*, *)\) and norm \(|*|\). For functions on \(S^1\) and vector fields along a closed curve \(\gamma\), we use \(L_2\) inner product \((*, *)\) and \(L_2\) norm \(|*|\). Sobolev \(H^s\) norm is denoted by \(|*|_s\). For a tensor field \(\xi\) along a closed curve \(\gamma\), \(H^s\) norm \(|\xi|_s\) is defined by \(|\xi|_s^2 = \sum_{i=0}^s |D^i\xi|^2|/\).

We recall basic lemmas from [3]. Some of them are extended to the case of tensor fields. We frequently use them to get estimation, but always makes no mention of them.

Lemma 1.1 ([3, Lemma 3.1]). For a tensor field \(\xi\) along a closed curve \(\gamma\),
\[
\max |\xi|^2 \leq 2|\xi 2 \cdot (|\xi| + |D_x\xi|).
\]

Lemma 1.2 ([3, Lemma 3.2]). For integers \(0 < p < q < r\),
\[
|D^p\xi|^q \leq |D^p\xi|^q \cdot (r-p) \cdot |D^q\xi|^{(r-p)/(r-p)}.
\]

Lemma 1.3 ([3, Lemma 4.1]). Let \(a\) and \(b\) be \(L_1\) functions on \(S^1\) such that \(a \geq 0\) and \(|a|_{L_1} > 0\). Then, the ODE for a function \(v\) on \(S^1\):
\[
-v'' + av = b
\]
has a unique solution, and the solution is estimated as
\[
\max |v| \leq 2\left[1 + |a|^{-1}_{L_1}\right] \cdot |b|_{L_1},
\]
\[
\max |v'| \leq 2\left[1 + |a|_{L_1}\right] \cdot |b|_{L_1}.
\]

We need also Hölder norms. The usual Hölder space for functions on \(S^1\) is denoted by \(C^{n+4}\). The weighted Hölder space (time derivative is counted 4 times) for functions on \(S^1 \times [0, T]\) is denoted by \(C^{n+4}\). See [3] for the detailed definition.

Lemma 1.4 ([3, Proposition 5.6]). Set \(D = S^1 \times [0, T]\). Let \(a : D \to \mathbb{R}; b_i, d_i, f : D \to \mathbb{R}^N; c_i : D \to \mathbb{R}^{N \times N}\) be \(C^4\) functions and \(\phi : S^1 \to \mathbb{R}^N\) a \(C^{4+4}\) function. Suppose that \(a\) is non-negative and \(|a|_{L_1} \geq C > 0\). Then, the linear PDE for a \(\mathbb{R}^N\)
valued function \( u \) and a function \( v \):

\[
\begin{align*}
\partial_t u + u^{(4)} + \sum_{i=0}^{3} c_i u^{(i)} + \sum_{i=0}^{1} d_i v^{(i)} &= f, \\
-\nu'' + av &= \sum_{i=0}^{3} b_i u^{(i)}, \\
u(x, 0) &= \phi(x)
\end{align*}
\]

has a unique \( C^{4+4\mu} \) solution on \( D \), and the \( C^{4+4\mu} \) norm of the solution is bounded by a constant depending on the \( C^4 \) norms of \( f, a, b, c, d \), the \( C^{4+4\mu} \) norm of \( \phi \), and \( C^{-1} \).

2. The equation

To derive the equation of motion governed by an energy, we perturb the curve \( \gamma = \gamma(x) \) with a time parameter \( t \): \( \gamma = \gamma(x, t) \). Then the elastic energy changes at \( t = 0 \) as

\[
\frac{d}{dt} E(\gamma) = 2\langle D_x \gamma', D_t D_x \gamma' \rangle \\
= 2\langle D_x \gamma', R(\partial_t \gamma', \gamma')\gamma' + D_x^2 \partial_t \gamma \rangle \\
= -2\langle \partial_t \gamma, R(\gamma', D_x \gamma')\gamma' \rangle + 2\langle \partial_t \gamma, D_x^3 \gamma' \rangle \\
= 2\langle \partial_t \gamma, D_x^3 \gamma' - R(\gamma', D_x \gamma')\gamma' \rangle,
\]

where \( \gamma(x, 0) = \gamma(x) \). Therefore, \(-D_x^3 \gamma' + R(\gamma', D_x \gamma')\gamma' \) would be the most efficient direction to minimize the elastic energy. However, this direction does not preserve the condition \(|\gamma'| = 1\). To force to preserve the condition we have to add certain term. Let \( V \) be the space of all directions satisfying the condition in the sense of first derivative. Namely,

\[
V = \{ \alpha | (\gamma', D_x \alpha) = 0 \}.
\]

We can check that a direction is \( L_2 \) orthogonal to \( V \) if and only if it has a form \( D_x (w \gamma') \) with some function \( w(x) \). Therefore, the “true direction” should be

\[
\partial_t \gamma = -D_x^3 \gamma' + R(\gamma', D_x \gamma')\gamma' + D_x (w \gamma'),
\]

where the function \( w \) has to satisfy the condition

\[
(\gamma', D_x \partial_t \gamma) = 0.
\]

To simplify this relation, we use the following
Lemma 2.1. For a curve $\gamma$ with $|\gamma'| \equiv 1$, we have the following identities.

\[
\begin{align*}
(y', D_x y') &= 0, \\
(y', D_x^2 y') &= -|D_x y'|^2, \\
(y', D_x^3 y') &= -\frac{3}{2}(|D_x y'|^2)', \\
(y', D_x^4 y') &= -2(|D_x y'|^2)'' + |D_x^2 y'|^2.
\end{align*}
\]

Proof. We can get these by a simple calculation. \hfill \Box


Therefore we have

\[
\begin{align*}
0 &= (D_x(-D_x^3 y' + R(y', D_x y') y' + D_x(w y'))), y') \\
&= -(y', D_x^4 y') + (y', D_x(R(y', D_x y') y')) + (y', D_x^2(w y'))) \\
&= 2(|D_x y'|^2)'' - |D_x^2 y'|^2 + (R(y', D_x y') D_x y', y') \\
&\quad + (y', w'' y' + 2w' D_x y' + w D_x^2 y') \\
&= 2(|D_x y'|^2)'' - |D_x^2 y'|^2 - (R(y', D_x y') y', D_x y') + w'' - |D_x y'|^2 w.
\end{align*}
\]

Thus the equation for the function $w(x)$ becomes

\[
-w'' + |D_x y'|^2 w = 2(|D_x y'|^2)'' - |D_x^2 y'|^2 - (R(y', D_x y') y', D_x y').
\]

If we put

\[
v = w + 2|D_x y'|^2,
\]

then we have

\[
-v'' + |D_x y'|^2 v = -|D_x^2 y'|^2 + 2|D_x y'|^4 - (R(y', D_x y') y', D_x y').
\]

Therefore our equation becomes

\[
\begin{align*}
\partial_t y &= -D_x^3 y' + R(y', D_x y') y' + D_x(w y'), \\
-w'' + |D_x y'|^2 w &= 2(|D_x y'|^2)'' - |D_x^2 y'|^2 - (R(y', D_x y') y', D_x y'), \\
\gamma(x, 0) &= \gamma_0(x).
\end{align*}
\]

Or, equivalently,

\[
\begin{align*}
\partial_t y &= -D_x^3 y' + R(y', D_x y') y' + D_x((v - 2|D_x y'|^2) y'), \\
-v'' + |D_x y'|^2 v &= -|D_x^2 y'|^2 + 2|D_x y'|^4 - (R(y', D_x y') y', D_x y'), \\
\gamma(x, 0) &= \gamma_0(x).
\end{align*}
\]

Note that both $\gamma$ and $w$ (or $v$) are unknown functions on $S^1 \times \mathbb{R}_+$. 
3. Short time existence

In this section, we consider a modified equation for an $\mathbb{R}^N$ valued function $\gamma$ and a function $v$:

$$\begin{cases}
\partial_t \gamma = -\gamma^{(4)} + F(x, \gamma, \gamma', \gamma'', \gamma^{(3)}, v, v'), \\
-\nu'' + G(x, \gamma, \gamma', \gamma'', \gamma^{(3)}) \cdot v = H(x, \gamma, \gamma', \gamma'', \gamma^{(3)}),
\end{cases}$$

(3.1) where $F$, $G$ and $H$ are given $C^\infty$ functions on $S^1 \times (\mathbb{R}^N)^3$, $S^1 \times (\mathbb{R}^N)^4$ and $S^1 \times (\mathbb{R}^N)^4$, respectively, and the function $G$ is non-negative. For functions $\gamma$ and $v$, we take their jets and use abbreviated notations such as $F(x, j_3 \gamma, j_1 v)$, $G(x, j_3 \gamma)$ and $H(x, j_3 \gamma)$.

**Theorem 3.1.** For any $C^\infty$ initial data $\gamma_0$ with $G(x, j_3 \gamma_0) > 0$ at some point $x \in S^1$, there is a positive time $T$ so that (3.1) has a unique $C^\infty$ solution on the time interval $[0, T)$.

To prove this, we need “cut off” functions for $F$, $G$ and $H$. Let $\rho_a(y)$ be a $C^\infty$ function of $y$ such that $\rho_a(y) = 1$ for $|y| \leq a$, $\rho_a(y) = 0$ for $|y| \geq 2a$, and $0 \leq \rho_a(y) \leq 1$ for all $y$. Let $v_0$ be the solution of the ODE: $-\nu'' + G(x, j_3 \gamma_0) \cdot v = H(x, j_3 \gamma_0)$ and put $A = \max(|j_3 \gamma_0|^2 + |j_1 v_0|^2)$. Set

$$\tilde{F}(x, j_3 \gamma, j_1 v) = \rho_{2A}(|j_3 \gamma|^2 + |j_1 v|^2) \cdot F(x, j_3 \gamma, j_1 v),$$

$$\tilde{H}(x, j_3 \gamma) = \rho_{2A}(|j_3 \gamma|^2) \cdot H(x, j_3 \gamma).$$

For the function $G$, we take a point $x_0 \in S^1$ and positive numbers $B \leq 1$ and $C$ so that $G(x, j_3 \gamma) \geq C$ for all 3-jets $\{x, \gamma\}$ with $|x - x_0|, |j_3(\gamma - \gamma_0)|^2 \leq B$. Set

$$\tilde{G}(x, j_3 \gamma) = \rho_{B/2}(|j_3 (\gamma - \gamma_0)|^2) \cdot G(x, j_3 \gamma) + 1 - \rho_{B/2}(|j_3 (\gamma - \gamma_0)|^2).$$

Take any point $x$ with $|x - x_0| \leq B$. If $|j_3(\gamma - \gamma_0)|^2 \leq B$, then $\tilde{G}(x, j_3 \gamma) \geq \min\{G(x, j_3 \gamma), 1\} \geq C$. If $|j_3(\gamma - \gamma_0)|^2 \geq B$, then $\tilde{G}(x, j_3 \gamma) = 1$. In particular, for any function $\gamma$, we have

$$\int \tilde{G}(x, j_3 \gamma) \, dx \geq BC.$$

Note that if $\gamma$ is sufficiently close to $\gamma_0$ in $C^3$ topology, then $\tilde{G}(x, j_3 \gamma) = G(x, j_3 \gamma)$ and $\tilde{H}(x, j_3 \gamma) = H(x, j_3 \gamma)$. It also implies that the solution $\tilde{v}$ of the ODE: $-\tilde{v}'' + \tilde{G}(x, j_3 \gamma) \cdot \tilde{v} = \tilde{H}(x, j_3 \gamma)$ coincides with $v$. Therefore, if we have a solution for the equation

$$\begin{cases}
\partial_t \gamma = -\gamma^{(4)} + \tilde{F}(x, j_3 \gamma, j_1 v), \\
-\nu'' + \tilde{G}(x, j_3 \gamma) \cdot v = \tilde{H}(x, j_3 \gamma),
\end{cases}$$

(3.2)
then it is a solution for the original equation for some short time.

Now we consider the equation

\begin{equation}
\tag{\bar{S}T_\lambda}
\begin{cases}
\partial_t \gamma = -\gamma^{(4)} + \lambda F(x, j_3 \gamma, j_1 v), \\
-v'' + \bar{G}(x, j_3 \gamma) \cdot v = \bar{H}(x, j_3 \gamma),
\end{cases}
\end{equation}

where \(\lambda\) is a constant in \([0, 1]\).

**Lemma 3.2.** Let \(\gamma = \gamma(t, x)\) be a \(C^{4+4\mu}\) solution of \((\bar{S}T_\lambda)\) with a \(C^\infty\) initial data \(\gamma_0(x)\). Then \(\gamma\) is \(C^\infty\).

**Proof.** If \(\gamma\) belongs in the class \(C^{n+4+4\mu}\), then the functions \(\bar{G}(x, j_3 \gamma)\) and \(\bar{H}(x, j_3 \gamma)\) belong to the class \(C^{n+1+4\mu}\). Hence Lemma 1.4 implies that \(v\) and \(v'\) belong to \(C^{n+1+4\mu}\), therefore also \(F(x, j_3 \gamma, j_1 v)\) belongs to \(C^{n+1+4\mu}\). Thus we see that \(\gamma\) belongs to \(C^{n+5+4\mu}\). By induction, we see the smoothness of the solution \(\gamma\). \(\square\)

**Lemma 3.3.** Consider the ODE: \(-v'' + \bar{G}(x, j_3 \gamma) \cdot v = \bar{H}(x, j_3 \gamma)\). For any non-negative integer \(n\) and a positive number \(C\), there is a positive number \(K\) with the following property:

If \(\|\gamma\|_n \leq C\), then \(\|v\|_n \leq K \cdot \{1 + \|\gamma^{(n+1)}\|\}\).

**Proof.** Since \(|v|\) and \(|v'|\) are bounded by Lemma 1.3, the claim holds for \(n = 0, 1\). Suppose that the claim holds for an integer \(n \geq 1\) and that \(\|\gamma\|_{n+1} \leq C\). Then, by Lemmas 1.1 and 1.2, we have

\[
\|v\|_{n+1} \leq \|v\|_n + \|v^{(n+1)}\| + C + \|\bar{G}(x, j_3 \gamma) \cdot v\|_{n-1} + \|\bar{H}(x, j_3 \gamma)\|_{n-1} \\
\leq C + C_1 \cdot \|\bar{G}(x, j_3 \gamma)\|_{n-1} + \|\bar{H}(x, j_3 \gamma)\|_{n-1}.
\]

The last expression involves the derivatives of \(\gamma\) up to \(\gamma^{(n+2)}\). Counting the fact that \(\|\gamma^{(n)}\|\) is bounded, we see

\[
\|v\|_{n+1} \leq C_2 \cdot [1 + \|\gamma^{(n+2)}\| + \|\gamma^{(n+1)}\| \cdot \|\gamma^{(4)}\|_{\#3}] \\
\leq C_2 \cdot [1 + \|\gamma^{(n+2)}\| + \|\gamma^{(n+1)}\| \cdot \max \|\gamma^{(4)}\|_{\#3}] \\
\leq C_3 \cdot [1 + \|\gamma^{(n+2)}\| + \max \|\gamma^{(n+1)}\|_{\#3}] \\
\leq C_4 \cdot [1 + \|\gamma^{(n+2)}\|],
\]

where \((\#3)\) means that the indicated term appears only if \(n \geq 3\). \(\square\)

**Lemma 3.4.** Let \(\gamma\) be a solution of \((\bar{S}T_\lambda)\) on a finite time interval \([0, T)\). For any non-negative integer \(n\), the norm \(\|\gamma^{(n)}\|\) is uniformly bounded with respect to \(\lambda \in [0, 1]\).
Proof. First of all, for \( n \leq 2 \), we have
\[
\frac{d}{dt} \| \gamma^{(n)} \|^2 = 2 \langle \gamma^{(n)}, \partial_t \gamma^{(n)} \rangle
\]
\[
= 2 \langle \gamma^{(n)}, -\gamma^{(n+4)} + \lambda \bar{F}(x, j_3 \gamma, j_1 \nu)^{(n)} \rangle
\]
\[
= -2 \| \gamma^{(n+2)} \|^2 + 2 \lambda \langle \gamma^{(2n)}, \bar{F}(x, j_3 \gamma, j_1 \nu) \rangle
\]
\[
\leq -2 \| \gamma^{(n+2)} \|^2 + 2 \| \bar{F}(x, j_3 \gamma, j_1 \nu) \| \cdot \| \gamma^{(2n)} \|.
\]

Thus for \( n = 0 \), we have
\[
\frac{d}{dt} \| \gamma \|^2 \leq 2C_1 \cdot \| \gamma \|,
\]
hence \((d/dt)\|\gamma\|\) is bounded. Also, for \( n = 2 \), we have
\[
\frac{d}{dt} \| \gamma^{(4)} \|^2 \leq -2 \| \gamma^{(4)} \|^2 + 2C_2 \| \gamma^{(4)} \| \leq C_3.
\]
Therefore, the norm \( \| \gamma \|_2 \) increases at most linear order.

Suppose that we know estimation of \( \| \gamma \|_{n+1} \) for an integer \( n \geq 1 \). By Lemma 1.2, we can estimate each factor as:

\[
\| \gamma^{(n+3)} \|, \quad \| \gamma^{(n+2)} \| \cdot \| \gamma^{(4)} \|, \quad \| \gamma^{(n+2)} \| \cdot \| \nu'' \|,
\]
\[
\| \gamma^{(n+1)} \| \cdot \| \gamma^{(5)} \|, \quad \| \gamma^{(n+1)} \| \cdot \| \gamma^{(4)} \|, \quad \| \gamma^{(4)} \|, \quad \| \gamma^{(n+1)} \| \cdot \| \nu^{(3)} \|,
\]
\[
\| \gamma^{(n+1)} \| \cdot \| \nu'' \|, \quad \| \gamma^{(n+1)} \| \cdot \| \nu'' \|, \quad \| \gamma^{(n+1)} \| \cdot \| \nu'' \|, \quad \| \gamma^{(n+1)} \| \cdot \| \nu'' \|, \quad \| \gamma^{(n+1)} \| \cdot \| \nu'' \|, \quad \| \gamma^{(n+1)} \| \cdot \| \nu'' \|, \quad \| \gamma^{(n+1)} \| \cdot \| \nu'' \|.
\]

Note that terms with multiple factors appear only if \( n \geq (\text{their number of factors}) \).

By Lemma 1.2, we can estimate each factor as:
\[
\| \gamma^{(n+3)} \| \leq C_6 \cdot \| \gamma^{(n+4)} \|^{2/3},
\]
When $n \geq 2$, we have

\[ \|\gamma^{(4)}\| \leq C_{16} \cdot \{1 + \|\gamma^{(n+2)}\|\} \leq C_{17} \cdot \{1 + \|\gamma^{(n+4)}\|^{1/3}\}, \]
\[ \|v''\| \leq C_{18} \cdot \{1 + \|v^{(n)}\|\} \leq C_{19}, \]
\[ \|\gamma^{(5)}\| \leq C_{20} \cdot \{1 + \|\gamma^{(n+3)}\|\} \leq C_{21} \cdot \{1 + \|\gamma^{(n+4)}\|^{2/3}\}, \]
\[ \|v^{(3)}\| \leq C_{22} \cdot \{1 + \|v^{(n+1)}\|\} \leq C_{23} \cdot \{1 + \|\gamma^{(n+4)}\|^{1/3}\}. \]

When $n \geq 3$, we have

\[ \max |v''| \leq C_{24} \cdot \{1 + \max |v^{(n-1)}|\} \leq C_{25}, \]
\[ \max |\gamma^{(4)}| \leq C_{26} \cdot \{1 + \max |\gamma^{(n+1)}|\} \leq C_{27} \cdot \{1 + \|\gamma^{(n+2)}\|^{1/2}\} \]
\[ \leq C_{28} \cdot \{1 + \|\gamma^{(n+4)}\|^{1/6}\}. \]

Combining all, we conclude

\[ \|\tilde{F}(x, j_3\gamma, j_1v)^{(n)}\| \leq C_{29} \cdot \{1 + \|\gamma^{(n+4)}\|^{5/6}\}, \]

and

\[ \frac{d}{dt} \|\gamma^{(n+2)}\|^2 \leq C_{30}. \]

Proof (of Theorem 3.1). We use the so-called open closed method. Take any positive time $T$. By the implicit function theorem with Lemma 1.4, the set $\Lambda$ of $\lambda$ which has a solution $\gamma$ of \((ST_\lambda)\) on $[0, T)$ is open in the interval $[0, 1]$. On the other hand, by Lemma 3.4, $\Lambda$ is closed in $[0, 1]$. Since $\Lambda$ contains $0$, it should coincide with $[0, 1]$. By definition, the solution of \((ST_\lambda)\) with $\lambda = 1$ is a solution of \((ST)\), which gives a short time solution of \((ST)\). For detailed discussion, see [3, Proof of Theorem 6.5].

**Theorem 3.5.** The equation \((EP)\) with non-geodesic initial data of unit line element has a unique short time solution $\gamma(x, t)$. Moreover, every closed curve $\gamma(\ast, t)$ has unit line element.
Proof. We may assume that the induced tangent bundle of the initial data $\gamma_0$ is orientable, taking a double cover if necessary. Then, using a tubular neighbourhood of $\gamma_0$, $(EP_\gamma)$ is expressed as $(ST)$, hence has a short time solution. Let $\{\gamma, v\}$ be a solution. Since $\partial_t |\gamma'|^2 = 2(\gamma', D_t \gamma') = 2(\gamma', D_\gamma \partial_t \gamma) = 0$, we have $|\gamma'|^2 \equiv 1$. Let $\{\gamma + \xi, v + u\}$ be another solution of $(ST)$ in the tubular neighbourhood of $\gamma_0$. Then $\{\xi, u\}$ satisfies the equation:

$$
\begin{cases}
\partial_t \xi = -\xi^{(4)} + f(x, t, \xi, \xi', \xi'', \xi^{(3)}, u, u'), \\
-u'' + G(x, \gamma, \gamma', \gamma'', \gamma^{(3)}) \cdot u = h(x, t, \xi, \xi', \xi'', \xi^{(3)}, u).
\end{cases}
$$

Here, $|f|$ and $|h|$ are bounded by $C(\sum_{i=0}^{3} |\xi^{(i)}| + |u| + |u'|)$, because $\{\gamma + \xi, v + u\}$ is bounded. Therefore, we have $\|u\|_1 \leq C_1 \|\xi\|_3$, and

$$
\frac{1}{2} \frac{d}{dt} \|\xi\|_1^2 = \langle \xi, \partial_t \xi \rangle + \langle \xi', \partial_t \xi' \rangle = \langle \xi - \xi'', -\xi^{(4)} + f \rangle
$$

$$
= -\|\xi''\|^2 - \|\xi^{(3)}\|^2 + \langle \xi, f \rangle
$$

$$
\leq -\|\xi''\|^2 - \|\xi^{(3)}\|^2 + C_2 \cdot \|\xi\| \cdot (\|\xi\|_3 + \|u\|_1)
$$

$$
\leq C_3 \cdot \|\xi\|^2.
$$

Since $\xi = 0$ at $t = 0$, we have $\xi \equiv 0$. Replacing $t = 0$ to arbitrary $t = t_0$, we see that the set of all $t$ such that two solutions coincide is open. Hence the solutions coincide for all time. 

4. Long time existence

In this section, we consider the original equation:

$$
\begin{cases}
\partial_t \gamma = -D_x^3 \gamma' + R(\gamma', D_x \gamma') \gamma' + D_x (w \gamma'), \\
-w'' + |D_x \gamma'|^2 w = 2(|D_x \gamma'|^2)'' - |D_x^2 \gamma'|^2 - (R(\gamma', D_x \gamma') \gamma' + D_x \gamma'), \\
\gamma(x, 0) = \gamma_0(x),
\end{cases}
$$

where $\gamma_0$ is a closed curve of unit line element.

**Theorem 4.1.** Let $M$ be a compact riemannian manifold and $\gamma_0$ a closed curve of unit line element. Then $(EP)$ has a unique solution for a time interval $[0, T)$ and one of the followings holds.

1) There is a sequence of times $t_i \to T$ such that $\gamma(*, t_i)$ converges to a closed geodesic in $C^1$ topology.

2) $T = \infty$.

To prove this, we need some preparation. For a closed curve $\gamma$, let $v$ and $w$ be solutions of the ODE:
\[-v'' + |D_x \gamma'|^2 v = -|D_x^2 \gamma'|^2 + 2|D_x \gamma'|^4 - (R(\gamma', D_x \gamma') \gamma', D_x \gamma'), \]
\[-w'' + |D_x \gamma'|^2 w = 2(|D_x \gamma'|^2)'' - |D_x^2 \gamma'|^2 - (R(\gamma', D_x \gamma') \gamma', D_x \gamma'), \]

and put

\[\delta = -D_x^3 \gamma' + R(\gamma', D_x \gamma') \gamma' + D_x (w \gamma').\]

In Lemmas 4.2–4.6, we consider this ODE and estimate \(v, w\) and \(\delta\) by \(\gamma'\). They will be applied to the PDE (EP) later.

**Lemma 4.2.** For any non-negative integer \(n\) and any positive real number \(C\), there is a positive number \(K\) with the following property:

If \(\|D_x \gamma'\| \geq C^{-1}\), \(\|\gamma'|_1 \leq C\) and \(\|\gamma'|_n \leq C\), then

\[\|w\|_{n+1} \leq K \cdot \{1 + \|D_x^2 \gamma'| \cdot \|D_x^{n+2} \gamma'|\}.\]

**Proof.** The assumption and Lemma 1.3 imply that

\[\|v\|_{C^1} \leq \|D_x^2 \gamma'\|^2 + 2\|D_x \gamma'|^2 + \|D_x \gamma'\|^2.\]

But we know that \(\max |D_x \gamma'| \leq C_1 \cdot \{1 + \|D_x^2 \gamma'|^{1/2}\}\). Therefore,

\[\|v\|_{C^1} \leq C_2 \cdot \{1 + \|D_x^2 \gamma'|^2\}.\]

Moreover,

\[\|D_x \gamma'|^2 \leq C_3 \cdot \{1 + \|D_x^2 \gamma'|^{1/2}\},\]
\[\|\{|D_x \gamma'|^2\}' \| \leq 2\|D_x \gamma'| \cdot \|D_x^2 \gamma'| \| \leq C_4 \cdot \{1 + \|D_x^2 \gamma'|^{3/2}\}.\]

Thus we proved the claim for \(n = 0\):

\[\|w\|_1 \leq C_5 \cdot \{1 + \|D_x^2 \gamma'|^2\}.\]

Suppose that the claim holds for a non-negative integer \(n\) and that \(\|\gamma'|_{n+1} \leq C\). Then, we know \(\|w\|_{n+1} \leq C_6 \cdot \{1 + \|D_x^2 \gamma'| \cdot \|D_x^{n+2} \gamma'|\}\). Therefore,

\[\|w^{(n+2)}\| \leq \{\|D_x \gamma'|^2 \cdot w\}'^{(n)}\| + 2\|\|D_x \gamma'|^2\|^{(n+2)}\|
+\|\|D_x^2 \gamma'|^2\|^{(n)}\| + \|R(\gamma', D_x \gamma') \gamma', D_x \gamma')^{(n)}\|
\leq C_7 \cdot \{1 + \|D_x^{n+1} \gamma'| \cdot |D_x \gamma'| \cdot |w|| + \|w_n\|
+C_8 \cdot \{1 + \|D_x^{n+2} \gamma'| \cdot |D_x \gamma'| \| + \|D_x^{n+2} \gamma'| \cdot |D_x^2 \gamma'|\|
+ \|\|D_x^{n+1} \gamma'| \cdot |D_x^3 \gamma'|\|_{(n)}\}
+C_9 \cdot \{1 + \|D_x^{n+1} \gamma'| \cdot |D_x \gamma'| \|\}.\]
where (\#2) means that the indicated term appears only when \( n \geq 2 \).

Here, we know that

\[
\max |D^{n+1}_x \gamma'| \leq C_{10} \cdot \{1 + \|D^{n+2}_x \gamma\|^1/2\} \leq C_{11} \cdot \{1 + \|D^{n+3}_x \gamma\|^1/4\},
\]
\[
\max |\omega| \leq \|\omega\|_1 \leq C_{12} \cdot \{1 + \|D^2_x \gamma\|\} \leq C_{13} \cdot \{1 + \|D^2_x \gamma\| \cdot \|D^{n+3}_x \gamma\|^1/2\},
\]
\[
\max |D_x \gamma'| \leq C_{14} \cdot \{1 + \|D^2_x \gamma\|\},
\]
\[
\max |D^{n+2}_x \gamma'| \leq C_{15} \cdot \{1 + \|D^{n+2}_x \gamma\|^1/2 \cdot \|D^{n+3}_x \gamma\|^1/2\} \leq C_{16} \cdot \{1 + \|D^{n+3}_x \gamma\|^{3/4}\},
\]
\[
\|D^3_x \gamma\|_{(\#2)} \leq C_{17}.
\]

Thus we have

\[
\|\omega\|_{n+2} \leq C_{18} \cdot \{1 + \|D^2_x \gamma\| \cdot \|D^{n+3}_x \gamma\|\},
\]

and the induction completes the proof. \( \square \)

**Lemma 4.3.** Set

\[
\phi = R(\gamma', D_x \gamma') \gamma' + D_x (w \gamma').
\]

For any non-negative integer \( n \) and any positive real number \( C \), there is a positive number \( K \) with the following property:

If \( \|D_x \gamma\| \geq C^{-1} \), \( \|\gamma\|_1 \leq C \) and \( \|\gamma\|_n \leq C \), then

\[
\|\phi\|_n \leq K \cdot \{1 + \|D^2_x \gamma\| \cdot \|D^{n+2}_x \gamma\|\}.
\]

**Proof.** The assumption and Lemma 4.2 imply that

\[
\|\phi\| \leq C_1 \cdot \{1 + \|w\| + \max |w|\} \leq C_2 \cdot \{1 + \|w\|_1\} \leq C_3 \cdot \{1 + \|D^2_x \gamma\|^2\}.
\]

Thus the claim holds for \( n = 0 \).

Suppose that the claim holds for a non-negative integer \( n \) and that \( \|\gamma\|_{n+1} \leq C \). Then, we know \( \|\phi\|_n \leq C_4 \cdot \{1 + \|D^2_x \gamma\| \cdot \|D^{n+2}_x \gamma\|\}. \) Therefore,

\[
\|\phi^{(n+1)}\| \leq \|D^{n+1}_x (R(\gamma', D_x \gamma') \gamma')\| + \|D^{n+2}_x (w \gamma')\| \leq C_5 \cdot \{1 + \|D^{n+2}_x \gamma\|^2\} + \|D^{n+2}_x \gamma\| \cdot \|D_x \gamma\| \leq C_6 \cdot \{1 + \|D^{n+2}_x \gamma\|^2\} \cdot \|D_x \gamma\| \leq C_7 \cdot \{1 + \|D^{n+2}_x \gamma\|^2\} \cdot \|D_x \gamma\| \leq C_8 \cdot \{1 + \|D^{n+2}_x \gamma\|^2\} \cdot \|D_x \gamma\| + \|w\|_1 \cdot \|D^{n+1}_x \gamma\|_{(\#2)} + \|w\|_{n+2}.
\]

Here, by Lemma 4.2, the terms except 4th and 5th are estimated linearly by \( \|D^2_x \gamma\| \cdot \|D^{n+3}_x \gamma\| \). For the excepted terms, Lemma 4.2 also implies that
Lemma 4.4. For any non-negative integer \( n \) and any positive real number \( C \), there is a positive number \( K \) with the following property:

If \( \| D_x y' \| \geq C^{-1} \) and \( \| y' \|_{n+1} \leq C \), then

\[
C_1 \cdot \{ 1 + \| D_x^n y' \| + \| D_x^{n+2} y' \| \}.
\]

where \( \delta \) is defined below Theorem 4.1.

Proof. Lemma 4.3 implies that

\[
\| D_x^n \delta \| \leq \| D_x^n \phi \| + \| D_x^n \psi \|.
\]

Here, we know

\[
\| D_x^2 y' \| \leq C_2 \cdot \| D_x^3 y' \|^{1/2} \leq C_3 \cdot \{ 1 + \| D_x^{n+3} y' \|^{1/2} \},
\]

\[
\| D_x^{n+2} y' \| \leq C_4 \cdot \| D_x^{n+3} y' \|^{1/2},
\]

which completes the proof.

Lemma 4.5. Let \( \gamma \) be the solution of (EP). For any non-negative integer \( n \) and any positive real number \( C \), there is a positive number \( K \) with the following property:

If \( \| D_x y' \| \geq C^{-1} \) and \( \| y' \|_{n+1} \leq C \), then

\[
\frac{d}{dt} \| D_x^{n+2} y' \|^{2} \leq K \cdot \{ 1 + \| D_x^2 y' \|^{2} \cdot \| D_x^{n+3} y' \|^{2} \} - \| D_x^{n+4} y' \|^{2}.
\]

Proof.

\[
\frac{d}{dt} \| D_x^{n+2} y' \|^{2} = 2\langle D_x^{n+2} y', D_x^{n+2} y' \rangle
\]

\[
= 2 \left( D_x^{n+2} y', \sum_{i=0}^{n+1} D_x^{i} (R(\delta, y') D_x^{n+1-i} y') + D_x^{n+3} \delta \right)
\]

\[
= 2 \langle D_x^{n+2} y', R(\delta, y') D_x^{n+1} y' \rangle.
\]
\[-2(D_x^{n+3} \gamma', R(\delta, \gamma') D_x^2 \gamma')
+ \sum_{i=2}^{n+1} (D_x^{n+4} \gamma', D_x^{i-2} (R(\delta, \gamma') D_x^{n+1-i} \gamma'))
+ 2(D_x^{n+4} \gamma', D_x^{n+1} \{-D_x^3 \gamma' + \phi\})\]
\[\leq C_1 \cdot \{C_2 - \frac{\|D_x^{n+3} \gamma'\|}{1/3}, \quad \frac{\|D_x^{n+3} \gamma'\|}{1/3}\}
+ \|D_x^{n+3} \gamma'\| \cdot \{\|\delta\| + \|D_x^{n-1} \delta\|\}\}
- 2\|D_x^{n+4} \gamma'\|^2 + 2\|D_x^{n+4} \gamma'\| \cdot \|D_x^{n+1} \phi\|,\]

where (#1) means that the indicated term appears only when \(n \geq 1\).

Here, we know that
\[\|D_x^{n+2} \gamma'\| \leq C_2 \cdot \|D_x^{n+4} \gamma'\|^{1/3},\]
\[\max\|D_x^{n+1} \gamma'\| \leq C_3 \cdot (1 + \|D_x^{n+2} \gamma'\|^{1/2}) \leq C_4 \cdot (1 + \|D_x^{n+4} \gamma'\|^{1/6}),\]
\[\|D_x^{n+3} \gamma'\| \leq C_5 \cdot \|D_x^{n+4} \gamma'\|^{2/3}.\]

Moreover, by Lemma 4.4,
\[\|\delta\| \leq C_6 \cdot (1 + \|D_x^{3} \gamma'\|),\]
\[\leq C_7 \cdot (1 + \|D_x^{n+3} \gamma'\|) \leq C_8 \cdot (1 + \|D_x^{n+4} \gamma'\|^{2/3}),\]
\[\|D_x^{n-1} \delta\| \leq C_9 \cdot (1 + \|D_x^{n+2} \gamma'\|) \quad \text{(when} \quad n \geq 1),\]

and by Lemma 4.3,
\[\|D_x^{n+1} \phi\| \leq C_{10} \cdot (1 + \|D_x^{2} \gamma'\| \cdot \|D_x^{n+3} \gamma'\|).\]

Combining all gives the result. \(\square\)

**Lemma 4.6.** For any positive real number \(C\) and a \(C^1\) neighbourhood \(U\) of the set of all closed geodesics of unit line element, there is a positive number \(K\) with the following property:

If \(\gamma\) is a closed curve of unit line element not in the set \(U\) and if \(\|D_x \gamma'\| \leq C\), then
\[\|D_x^3 \gamma'\| \leq K \cdot \|\delta\|.\]

**Proof.** Since
\[(\gamma', \delta) = -(\gamma', D_x^2 \gamma') + (\gamma', w' \gamma' + w D_x \gamma') = \frac{3}{2} \|D_x \gamma' \|^2 + w',\]
we see
\[\|w'\| \leq \|\delta\| + 3\|D_x \gamma', D_x^2 \gamma'\|,\]
\[ \varphi = -D_x^2 \gamma' + w \gamma'. \]

Then we have
\[ (\gamma', \varphi) = -(\gamma', D_x^2 \gamma') + w = |D_x \gamma'|^2 + w. \]

Therefore,
\[ \int w \, dx = (\gamma', \varphi) - |D_x \gamma'|^2. \]

Let \( \alpha \) be a vector field along \( \gamma \) such that \( D_x \alpha = \gamma' \) on \( 0 \leq x \leq 1 \) and \( \alpha(0) = 0. \)

Then,
\[ (\gamma', \varphi) = (D_x \alpha, \varphi) = \int_0^1 (D_x \alpha, \varphi) \, dx \]
\[ = [(\alpha, \varphi)]^1_0 - \int_0^1 (\alpha, D_x \varphi) \, dx \]
\[ = (\alpha(1), \varphi(0)) - \int_0^1 (\alpha, \delta - R(\gamma', D_x \gamma') \gamma') \, dx \]
\[ = -(\alpha(1), D_x^2 \gamma'(0)) + w(0) \cdot (\alpha(1), \gamma'(0)) \]
\[ - \int_0^1 (\alpha, \delta) \, dx + \int_0^1 (\alpha, R(\gamma', D_x \gamma') \gamma') \, dx. \]

Therefore,
\[ \int w \, dx - (\alpha(1), \gamma'(0)) \cdot w(0) \]
\[ = -(\alpha(1), D_x^2 \gamma'(0)) - \int_0^1 (\alpha, \delta) \, dx \]
\[ + \int_0^1 (\alpha, R(\gamma', D_x \gamma') \gamma') \, dx - |D_x \gamma'|^2. \]

Here,
\[ (|\alpha|^2)' = 2(\alpha, D_x \alpha) = 2(\alpha, \gamma') \leq 2|\alpha|. \]
and so
\[ |α'| \leq 1 \quad \text{and} \quad |α| \leq 1 \quad \text{on} \quad 0 \leq x \leq 1. \]

Thus,
\[
\left| \int w \, dx - (α(1), γ'(0)) \cdot w(0) \right|
\]
\[
\leq C_4 \cdot \{1 + \max |D_2^2 γ'| + \|δ\| + \|D_γ γ'\| + \|D_γ γ'\|^2\}
\]
\[
\leq C_5 \cdot \{1 + \|δ\| + \|D_γ^2 γ'\|^{1/2} \cdot \|D_γ^3 γ'\|^{1/2}\}
\]
\[
\leq C_6 \cdot \{1 + \|δ\| + \|D_γ^3 γ'\|^{3/4}\}.
\]

We know that \((α(1), γ'(0)) \leq 1\) and the equality holds if and only if the curve \(γ\) is a closed geodesic. If there is a sequence \(γ_i\) of closed curves such that \((α_i(1), γ_i'(0)) \rightarrow 1\) for the corresponding vector field \(α_i\), then the sequence has a \(C^1\) convergent subsequence, because the curves are \(H^2\) bounded. Since the limiting curve is a closed geodesic, this contradicts the assumption. Therefore we have a positive number \(C_0 < 1\) such that
\[ (α(1), γ'(0)) \leq 1 - C_0 \]
for all closed curves satisfying the condition.

We choose the origin 0 so that \(\int w \, dx = w(0)\). Then
\[
\left| \int w \, dx - (α(1), γ'(0)) \cdot w(0) \right|
\]
\[
= \left| \{1 - (α(1), γ'(0))\} \cdot w(0) \right|
\]
\[
\geq C_0 |w(0)|.
\]

Thus, we see
\[
|w(0)| \leq C_8 \cdot \{1 + \|δ\| + \|D_γ^3 γ'\|^{3/4}\},
\]
hence
\[
\max |w| \leq |w(0)| + \|w'\| \leq C_9 \cdot \{1 + \|δ\| + \|D_γ^3 γ'\|^{3/4}\}.
\]

Therefore, we have
\[
\|D_γ^2 γ'\| = \|δ - R(γ', D_γ γ') γ' - D_γ (wg')\|
\]
\[
\leq C_{10} \cdot \{\|δ\| + \|D_γ γ'\| + \max |w| \cdot \|D_γ γ'\| + \|w'\|\}
\]
\[
\leq C_{11} \cdot \{1 + \|δ\| + \|D_γ^3 γ'\|^{3/4}\},
\]
Let $\gamma$ be a solution of (EP). Since $|\gamma'| = 1$, we have
\[
\frac{d}{dt} \|D_t \gamma'\|^2 = 2\langle \delta, -\delta + D_t (w \gamma') \rangle = -2\|\delta\|^2 - 2\langle D_t \delta, w \gamma' \rangle = -2\|\delta\|^2 - 2\langle D_t \gamma', w \gamma' \rangle = -2\|\delta\|^2.
\]
Thus we have the following

**Lemma 4.7.** For a solution $\gamma$ of (EP), $\|D_t \gamma'\|^2$ is non-increasing.

**Lemma 4.8.** For any positive real numbers $C$, $T$ and any non-negative integer $n$, there is a positive number $K$ with the following property:
If $\gamma$ is a solution of (EP) on $[0, T)$ and if $\|D^3_t \gamma'\| \leq C \cdot \{1 + \|\delta\|\}$, then $\|\gamma\|_n \leq K$.

Proof. We know that $\|D_t \gamma'\| \leq C_1$. From Lemma 4.5, we have
\[
\frac{d}{dt} \|D^2_t \gamma'\|^2 \leq C_2 \cdot \{1 + \|D^2_t \gamma'\|^2 \cdot \|D^3_t \gamma'\| \} - \|D^4_t \gamma'\|^2.
\]
It implies that
\[
\frac{d}{dt} \log \|D^2_t \gamma'\|^2 \leq C_3 \cdot \{1 + \|D^3_t \gamma'\|^2 \}.
\]
Combining it with inequality
\[
\frac{d}{dt} \|D_t \gamma'\|^2 = -2\|\delta\|^2 \leq -C_4 \|D^3_t \gamma'\|^2 + C_5
\]
which follows from the assumption, we have
\[
\frac{d}{dt} (\log \|D^2_t \gamma'\|^2 + C_6 \cdot \|D_t \gamma'\|_2^2) \leq C_7.
\]
Hence,
\[
\|D^2_t \gamma'\| \leq C_8.
\]
Suppose that $\|\gamma'\|_{n+1} \leq C$ for an integer $n \geq 1$. Then, Lemma 4.5 implies that
\[
\frac{d}{dt} \|D^{n+2}_t \gamma'\|^2 \leq C_9 \cdot \{1 + \|D^{n+3}_t \gamma'\|_2^2 \} - \|D^{n+4}_t \gamma'\|^2 \leq C_{10}.
\]
Thus the induction completes the proof. \qed
Proof (of Theorem 4.1). Suppose that no sequences $\gamma(*, t_*)$ converge to closed geodesics. By Lemmas 4.7 and 4.6, the assumption of Lemma 4.8 is satisfied. Therefore, for any finite time interval $[0, T)$, the solution $\gamma$ is bounded in $C^\infty$ norm. Thus the solution in Theorem 3.1 can be continued onto $[0, \infty)$.

5. Convergence

In this section, we assume that the solution $\gamma$ of (EP) does not have the property (1) of Theorem 4.1. In particular, $\|D_x \gamma'^2 \| \geq C^{-1}$ and the solution is defined for all time interval $[0, \infty)$. To show the convergence of the solution $\gamma$, we need some preparation.

Lemma 5.1. For any non-negative integer $n$ and a positive real number $C$, there is a positive number $K$ with the following property:
If $\|\delta\|_n \leq C$, then $\|\gamma\|_{n+3} \leq K$.

Proof. For $n = 0$, the claim holds by Lemma 4.6. Suppose that the claim holds for $n$ and that $\|\delta\|_{n+1} \leq C$. Then we know that $\|\gamma\|_{n+3} \leq C_1$. Thus, from Lemma 4.3, we have

$$
\|D^{n+4}_x \gamma'\| \leq C_2 \cdot \{\|D^{n+1}_x \delta\| + \|D^{n+1}_x \phi\| \}
\leq C_3 \cdot \{1 + \|D^2_x \gamma'\| \cdot \|D^{n+3}_x \gamma'\| \}.
$$

Proposition 5.2. For any non-negative integer $n$ and any positive number $C$, there is a positive number $K$ with the following property:
If $\gamma$ is a solution of (EP) and if $\|\delta\|_n \leq C$, then
$$
\|\partial_t w\|_{n+1} \leq K \cdot \{\|\delta\| + \|D^{n+3}_x \delta\| \}.
$$

Proof. From the defining equation of $v$:

$$
-v'' + |D_x \gamma'|^2 \cdot v = 2|D_x \gamma'|^4 - |D^2_x \gamma'|^2 - (R(\gamma', D_x \gamma') \gamma', D_x \gamma'),
$$

we have

$$
-\partial_t v'' + |D_x \gamma'|^2 \cdot \partial_t v
= -\partial_t \{ |D_x \gamma'|^2 \} \cdot v + \partial_t \{2|D_x \gamma'|^4 - |D^2_x \gamma'|^2 - (R(\gamma', D_x \gamma') \gamma', D_x \gamma') \}
= -2(D_x \gamma', R(\delta, \gamma') \gamma' + D^2_x \delta) \cdot v
+ 8(D_x \gamma', R(\delta, \gamma') \gamma' + D^2_x \delta) \cdot |D_x \gamma'|^2
-2(D_x \gamma', R(\delta, \gamma') \gamma' + D^2_x \delta) \cdot |D_x \gamma'|^2
-2(D_x R)(\gamma', D_x \gamma') \gamma', D_x \gamma') - 2(R(D_x \delta, D_x \gamma') \gamma', D_x \gamma')
-2(R(\gamma', R(\delta, \gamma') \gamma' + D^2_x \delta) \gamma', D_x \gamma').
$$
By Lemma 5.1, the assumption implies that \( \|y'\|_{n+3} \leq C_1\). Hence,

\[
(\text{the } H^n \text{ norm of the last expression}) \leq C_2 \cdot \{\|\delta\| + \|D_x^{n+3}\delta\|\}.
\]

Therefore, Lemma 1.3 implies that

\[
\|\partial_t v\|_1 \leq C_3 \cdot \{\|\delta\| + \|D_x^3\delta\|\},
\]

\[
\|\partial_t v\|_{n+1} \leq C_4 \cdot \{\|\delta\| + \|D_x^{n+2}\delta\|\} \quad \text{(when } n \geq 1).\]

Moreover, from

\[
\partial_t \{[D_x y']^2\} = 2(D_x y', R(\delta, y')y' + D_x^2 \delta),
\]

we have

\[
\|\partial_t \{[D_x y']^2\}\|_{n+1} \leq C_5 \cdot \{\|\delta\| + \|D_x^{n+3}\delta\|\}.
\]

Thus the claim holds for any non-negative integer \( n \).

**Lemma 5.3.** The norm \( \|\delta\| \) tends to 0 when \( t \to \infty \). The integrals

\[
\int_0^\infty \|\delta\|^2 dt, \quad \int_0^\infty \|D_x^2 \delta\|^2 dt
\]

are finite.

**Proof.** We have

\[
\int_0^\infty \|\delta\|^2 dt = \int_0^\infty \frac{1}{2} \left( \frac{d}{dt} \|D_x y'\|^2 \right) dt = -\frac{1}{2} \left[ \|D_x y'\|^2 \right]_0^\infty < \infty.
\]

Moreover,

\[
\frac{d}{dt} \|\delta\|^2 = 2(\delta, D_x \delta)
\]

\[
= 2(\delta, -D_x D_x^2 y' + D_x (R(\delta', D_x y')y') + D_x D_x (w y'))
\]

\[
= 2(\delta, -R(\delta, y')D_x^2 y' - D_x (R(\delta, y')D_x y')
\]

\[
- D_x^2 (R(\delta, y')y') - D_x^4 \delta
\]

\[
+ (D_x R)(y', D_x y')y' - R(D_x, D_x y')y'
\]

\[
+ R(y', R(\delta, y')y' + D_x^2 \delta)y'
\]

\[
+ R(\delta, D_x y')D_x \delta
\]

\[
+ R(\delta, y')(w y') + D_x (-\partial_t w \cdot y' + w \cdot D_x \delta)).
\]
Here, from Lemmas 4.6 and 4.2, we know
\[ \| D_\gamma' \| \leq C_1 \cdot (1 + \| \delta \|), \]
\[ \| w \|_1 \leq C_2 \cdot (1 + \| D_\gamma' \|^2) \leq C_3 \cdot (1 + \| D_\gamma' \|) \leq C_4 \cdot (1 + \| \delta \|). \]

Thus, using equation: \( (D_\gamma \delta, \gamma') = 0, \)
\[
\frac{d}{dt} \| \delta \|^2 \leq -2\| D_\gamma^2 \delta \|^2 - 2(D_\gamma \delta, \partial_t w \cdot \gamma') \\
+ C_5 \cdot \| \delta \| \cdot (1 + \| \delta \|^N) \cdot (\| \delta \| + \| D_\gamma^2 \delta \|) \\
\leq -\| D_\gamma^2 \delta \|^2 + C_6 \cdot \| \delta \|^2 \cdot (1 + \| \delta \|^N),
\]

where \( N \) is an absolute constant.

Thus \( \| \delta \| \) tends to 0. In particular,
\[
\frac{d}{dt} \| \delta \|^2 \leq -\| D_\gamma^2 \delta \|^2 + C_7 \cdot \| \delta \|^2.
\]

Therefore,
\[
\int_0^\infty \| D_\gamma^2 \delta \|^2 \, dt \leq C_7 \int_0^\infty \| \delta \|^2 \, dt - \int_0^\infty \frac{d}{dt} \| \delta \|^2 \, dt \\
< \infty.
\]

**Lemma 5.4.** For any non-negative even integer \( n \), \( \| D_\gamma^n \delta \| \) tends to 0 when \( t \to \infty \).

Proof. Suppose
\[
\| D_\gamma^n \delta \| \to 0, \quad \int_0^\infty \| D_\gamma^{n+2} \delta \|^2 \, dt < \infty
\]
for a non-negative even integer \( n \). This holds for \( n = 0 \) by Lemma 5.3.

As in the proof of Lemma 5.3, we have
\[
\frac{d}{dt} \| D_\gamma^{n+2} \delta \|^2 = 2(D_\gamma^{n+2} \delta, D_\gamma D_\gamma^{n+2} \delta) \\
= 2 \left( D_\gamma^{n+2} \delta, \sum_{i=0}^{n+1} D_\gamma^i (R(\delta, \gamma') D_\gamma^{n+1-i} \delta) + D_\gamma^{n+2} D_\gamma \delta \right) \\
= 2(D_\gamma^{n+2} \delta, R(\delta, \gamma') D_\gamma^{n+1} \delta) - 2(D_\gamma^{n+3} \delta, R(\delta, \gamma') D_\gamma^n \delta) \\
+ 2 \sum_{i=2}^{n+1} (D_\gamma^{n+4} \delta, D_\gamma^{i-2} (R(\delta, \gamma') D_\gamma^{n+1-i} \delta))(\#3) + 2(D_\gamma^{n+4} \delta, D_\gamma^n \delta),
\]
and $D_x^n\delta$ in the last term is expanded as
\[ D_x^n \{- R(\delta, y')D_x^2 y' - D_x(R(\delta, y')D_x y') - D_x^2(R(\delta, y')y') - D_x^4 \delta \\
+ (D_x R)(y', D_x y')y' + R(D_x \delta, D_x y')y' + R(y', R(\delta, y')y' + D_x^2 \delta)y' \\
+ R(y', D_x y')D_x \delta + R(\delta, y')(w y') + D_x \{ \partial_t w \cdot y' + w D_x \delta \} \}. \]

Form the assumption, Lemma 5.1 implies that $\|y\|_{n+3} \leq C_1$. Therefore, Lemma 4.2 implies that $\|w\|_{n+2} \leq C_2$, and Lemma 5.2 implies that $\|\delta_t w\|_{n+1} \leq C_3 \cdot \{ \|D_x^{n+3} \delta\| + \|\delta\| \}$. Moreover, we know that $\max |\delta| \leq C_4 \cdot (1 + \|D_x \delta\|^{1/2})$ and $\|D_x^{n+1} \delta\| \leq C_5 \cdot (1 + \|D_x^{n+2} \delta\|^{1/2})$. Thus all terms in the last expression except the term
\[ 2(D_x^{n+4} \delta, -D_x^{n} D_x^4 \delta) = -2\|D_x^{n+4} \delta\|^2 \]
are bounded by the form $C_6 \cdot \|D_x^{n+4} \delta\| \cdot (\|\delta\| + \|D_x^{n+3} \delta\|)$. Therefore,
\[ \frac{d}{dt} \|D_x^{n+2} \delta\|^2 \leq -\|D_x^{n+4} \delta\|^2 + C_7 \cdot (\|\delta\|^2 + \|D_x^{n+3} \delta\|^2) \]
\[ \leq -\frac{1}{2} \|D_x^{n+4} \delta\|^2 + C_8 \cdot \|\delta\|^2. \]

Thus we have $\|D_x^{n+2} \delta\| \to 0$ and $\int_0^\infty \|D_x^{n+4} \delta\|^2 dt$ is finite.

Note that Lemma 5.4 holds on any compact $C^\infty$ riemannian manifold satisfying the assumption of this section. In particular, $\delta$ converges to 0 in $C^\infty$ topology when $t$ tends to $\infty$. Combining it with Lemma 5.1, we have the boundedness of the solution $y$.

**Theorem 5.5.** Let $M$ be a compact riemannian manifold, and let $\gamma_0(x)$ be a closed curve with unit line element and length $L$. If there are no closed geodesics of length $L$ in the manifold $M$, then (EP) has a unique solution $\gamma(x, t)$ for all time, and the solution has a subsequence converging to an elastica.

If the metric is real analytic, we have the main result.

**Theorem 5.6.** Let $M$ be a compact real analytic riemannian manifold, and let $\gamma_0(x)$ be a closed curve with $|\gamma_0'| = 1$ and length $L$. If there are no geodesics of length $L$ in the manifold $M$, then (EP) has a unique solution $\gamma(x, t)$ for all time, and the solution converges to an elastica when $t \to \infty$.

Proof. The proof of Theorem 8.6 of [3] remains valid. We use Simon’s real analytic implicit function theorem. For detail, see [3].
REMARK 5.7. We have an example of almost oscillate solution on a $C^\infty$ riemannian manifold. See [2].

References


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